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Is Public Debt Growth-Enhancing or Growth-Reducing?

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Abstract

To understand mixed evidence provided by empirical studies for the relationship between the accumulation of public debt and economic growth, it is necessary to consider not only the *crowd-out* effect of public debt on economic growth but also the growth-enhancing *crowd-in* effect that cannot be uncovered by the traditional theoretical achievements. We develop a dynamic general equilibrium model with infinitely lived agents and derive an inverted U-shaped relationship between the accumulation of public debt and economic growth. The analysis focuses on both *crowd-out* and *crowd-in* effects that public debt has on private investment in a financially constrained economy and clarifies the mechanism inducing the inverted U-shaped relationship in the growth process. When the public debt-to-GDP ratio is below a certain threshold level, the *crowd-in* effect dominates the *crowd-out* effect and the accumulation of public debt promotes economic growth. When the public debt-to-GDP ratio exceeds the threshold level, the accumulation of public debt begins to hinder economic growth with the *crowd-out* effect dominating the *crowd-in* effect.

Keywords: Economic growth; Public debt; Crowd-in effect; Financial market imperfections.

JEL Classification Numbers: O41; E62.

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1 Introduction

Many developed countries have accumulated significant amounts of public debt in recent decades, and the accumulation of public debt is one of the most important contemporary policy issues. In traditional neo-classical growth theory, public debt has long been considered to hinder capital accumulation, *crowding out* private investment (e.g., Diamond 1965; Phelps and Shell 1969; Blanchard 1985; Weil 1989). Similarly, in endogenous growth models, the *crowd-out* effect impedes capital accumulation and reduces long-run growth rates, as clarified by Saint-Paul (1992). Despite these theoretical achievements, it remains unclear whether the accumulation of public debt promotes or impedes economic growth because empirical studies provide mixed evidence. Reinhart and Rogoff (2010) suggest the possibility of a non-linear correlation between real GDP growth and the public debt-to-GDP ratio (abbreviated PDG ratio henceforth). They show that real GDP growth tends to decline if the PDG ratio is very high, but there is no significant link between the accumulation of public debt and economic growth if the PDG ratio is low, although they do not clarify the causality between them. The panel data analyses by Baum et al. (2013) and Checherita-Westphal and Rother (2012) find evidence supporting an inverted U-shaped relationship between the accumulation of public debt and economic growth. They perform regressions and address the endogeneity problem associated with the reverse causality from real GDP growth to the PDG ratio. These authors indicate that an increase in the PDG ratio boosts the growth rate of per capita GDP before beginning to reduce it at a certain threshold level of the PDG ratio.[1] The instrumental variable approach employed by Panizza and Presbitero (2013) finds no clear non-linear relationship between real GDP growth and the public debt-to-GDP ratio. Instead, these authors demonstrate that the negative effect of the accumulation of public debt on economic growth does not appear in OECD member countries.

To understand the mixed evidence provided by these studies, it is necessary to consider not only the *crowd-out* effect of public debt on economic growth but also the growth-enhancing *crowd-in* effect of public debt that cannot be uncovered by the aforementioned

[1]According to estimations by Baum et al. (2013) and Checherita-Westphal and Rother (2012), the threshold level is approximately 90%-100% in EU member countries. See also Kumar and Woo (2010) and Cecchetti et al. (2011) for evidence regarding the non-linear relationship between public debt and economic growth.

theoretical achievements. We present a dynamic general equilibrium model in which public debt has both *crowd-in* and *crowd-out* effects on economic growth. We demonstrate that the accumulation of public debt promotes economic growth when the *crowd-in* effect dominates the *crowd-out* effect, but impedes economic growth when the *crowd-out* effect dominates the *crowd-in* effect.

In the extant literature on public debt and growth, Woodford (1990) is a notable exception in the sense that he investigates the *crowd-in* effect that public debt has on private investment, noting that the issuance of public debt may stimulate private investment in a financially constrained economy.[2] Somewhat surprisingly, however, there are few studies that focus on public debt's *crowd-in* effect when analyzing the relationship between the accumulation of public debt and economic growth. Along the same lines as Woodford (1990), constructing a growth model incorporating public debt's *crowd-in* effect and the *crowd-out* effect, we derive an inverted U-shaped relationship between the accumulation of public debt and economic growth.

Our model is outlined as follows. The governmental sector is introduced into the dynamic general equilibrium growth model developed by Kunieda and Shibata (2012). The government covers its expenditures using taxation and/or the issuance of public debt. Each agent is infinitely lived and has access to investment projects in each time period to produce general goods that are used for both consumption and investment. The productivity of investment projects differs across agents because they receive uninsured idiosyncratic productivity shocks in each time period. These productivity differences are the only source of the agents' heterogeneity in the sense that they are ex-ante homogeneous as in the model developed by Angeletos (2007). Each agent faces credit constraints because of agency problems in the financial market when initiating investment projects. Due to the productivity differences across agents, less productive agents become savers and more highly productive agents become producers in equilibrium, namely, savers and producers appear endogenously.

When the financial market is imperfect in an economy without public debt, a lower

[2] Some researchers such as Greiner and Semmler (2000) and Futagami et al. (2008) develop models in which the issuance of public debt may promote economic growth when government spending is productive as in Barro (1990). For instance, Futagami et al. (2008) develop an endogenous growth model with productive government spending and derive multiple steady states. In the low-growth steady state of their model, public debt has a *crowd-in* effect on the economy. Differing from their models, productive government spending is not assumed in our model. We focus on public debt's liquidity effect in a financially constrained economy.

interest rate is obtained in equilibrium than when the financial market is perfect. This is because the demand for borrowing is smaller than in an economy with a perfect financial market. Under these circumstances, the issuance of public debt increases the equilibrium interest rate because the supply of financial resources in the financial market is reduced by the issuance of public debt.[3] The issuance of public debt has two effects on private investment. On the one hand, the higher interest rate induced by the issuance of public debt reduces the number of agents who initiate investment projects because the opportunity cost for investment projects is increased. This is public debt's *crowd-out* effect on private investment. On the other hand, because the higher interest rate excludes less productive agents from production activities, production resources are intensively used by more highly productive investors. Additionally, the higher interest rate makes public bonds a more beneficial vehicle for saving and increases the net worth of agents. The increased net worth relaxes credit constraints, and more highly productive investors are able to increase their investments. This is public debt's *crowd-in* effect on private investment. Our growth model demonstrates that when the PDG ratio is small, the *crowd-in* effect dominates the *crowd-out* effect and the accumulation of public debt promotes economic growth, whereas when the PDG ratio is large, the *crowd-out* effect dominates the *crowd-in* effect and the accumulation of public debt slows down economic growth. In other words, a threshold level of the PDG ratio exists at which the *crowd-in* and *crowd-out* effects are balanced and the economic growth rate is maximized. The issuance of public debt is therefore growth-enhancing below the threshold, whereas it is growth-reducing above the threshold. Our model derives an inverted U-shaped relationship between the accumulation of public debt and economic growth.

In our model, depending on fiscal policy rules, economies follow different growth paths even though the share of government expenditures in GDP is constant across the economies. In an economy engaging in expansionary fiscal policy that relies more on the issuance of public debt than on taxation to finance government expenditures, the steady-state PDG ratio is above the threshold level at which the economic growth rate is maximized. Proceeding toward the steady state on the transition path, the economy experiences a growth process described by an inverted U-shaped relationship, provided that the initial PDG ratio is below

[3]Laubach (2009) provides empirical evidence indicating that public debt's long-horizon forward rate in the United States rises when the PDG ratio increases.

the threshold. In contrast, in an economy that exhibits more severe fiscal restraint and follows a less expansionary fiscal policy, the steady-state PDG ratio is below the threshold level, and both the economic growth rate and PDG ratio continue to rise synchronously until the economy converges to its steady state.

This paper contributes to the literature dealing with the effects that public debt has on financially constrained economies (e.g., Woodford 1990; Aiyagari and McGrattan 1998; Holmström and Tirole 1998; Heathcote 2005; Challe and Ragot 2011; Angeletos et al. 2013). Among these, Woodford (1990) and Aiyagari and McGrattan (1998) focus on the liquidity effect of public debt and conclude that public concerns over the rapid growth of public debt in the United States from the mid-1980s to the mid-1990s might have been misplaced.[4] In an economy modeled by Aiyagari and McGrattan (1998), without public debt, the welfare level attained in equilibrium is sub-optimal because of the agents' precautionary saving motives caused by uninsured idiosyncratic shocks and borrowing constraints. Once the government introduces public debt into this economy, public and private bonds become perfect substitutes with respect to their returns. The moderate issuance of public debt improves welfare, but if an excessive amount of public debt is issued, welfare begins to decline because too much physical capital is *crowded out* by public debt. In Aiyagari and McGrattan's model, however, public debt's *crowd-in* effect is not considered.

In contrast, as mentioned above, Woodford (1990) demonstrates the possibility that public debt may *crowd in* private investment. In his model economy, due to a higher interest rate induced by the issuance of public debt, public debt becomes a beneficial vehicle for storing value, and resources in the economy are efficiently used in production. Although the mechanism in Woodford's model inducing public debt's *crowd-in* effect is similar to that of our model, his investigation of *crowd-in* effect focuses exclusively on the steady state, and his model does not obtain the transitional dynamics of either the PDG ratio or the economic growth rate. Additionally, Woodford's model does not derive an inverted U-shaped relationship between the accumulation of public debt and economic growth. Differing from

[4]Holmström and Tirole (1998) and Angeletos et al. (2013) also consider the liquidity effect of public debt. Holmström and Tirole (1998) demonstrate that the issuance of public debt improves welfare in the three-period setting. Angeletos et al. (2013) investigate the optimal policy regarding the issuance of public debt. Heathcote (2005) and Challe and Ragot (2011) quantitatively study business fluctuations caused by aggregate fiscal shocks. In contrast with all these studies, we consider the non-linear growth effect of public debt.

Woodford (1990), however, our model analytically derives the transitional dynamics of both the PDG ratio and the economic growth rate and obtains an inverted U-shaped relationship between the accumulation of public debt and economic growth.

The remainder of this paper is organized as follows. In section 2, we present an infinitely-lived-agent model that generates both the *crowd-in* and *crowd-out* effects of public debt. Section 3 provides the formula for a growth rate associated with the PDG ratio and obtains a dynamical system in equilibrium. In section 4, we derive the inverted U-shaped relationship between the accumulation of public debt and economic growth and discuss how the fiscal policy rule affects the dynamic patterns of both the PDG ratio and the growth rate.

2 Model

To investigate growth effects of the accumulation of public debt, we incorporate the government into the dynamic general equilibrium growth model developed by Kunieda and Shibata (2012), who study a financial crisis caused by an asset bubble burst.

The economy consists of infinitely lived agents, an infinitely lived government, and an infinitely lived representative financial intermediary. Time is discrete and expands from 0 to ∞. The population of the infinitely lived agents is constant and normalized to one. The infinitely lived agents receive uninsured idiosyncratic productivity shocks in each time period. Due to the idiosyncratic productivity shocks, only more highly productive agents engage in general goods production. General goods at time t can be used interchangeably for consumption and investment in physical capital at time t. General goods are perishable in one period and physical capital entirely depreciates in one period.

Although agents are able to borrow financial resources when beginning investment projects, they face credit constraints imposed by the financial intermediary when they borrow financial resources. Agency problems associated with asymmetric information between borrowers and lenders cause credit market imperfections, and each agent can pledge only a partial amount of their net worth when they borrow financial resources. Because the financial market is competitive, the financial intermediary does not acquire any profits or incur any losses from its business. In addition to imposing credit constraints, the financial intermediary accepts deposits from savers and accommodates borrowers with loans to balance its balance sheet. The government spends a constant proportion of GDP, and public spending covers the costs to maintain, for instance, the rule of law, property rights, and/or other functions of markets, although none of these services explicitly appears in our model. Public spending is financed by the issuance of public bonds and/or taxation on the agents' incomes.

2.1 Agents

There are two types of saving methods. The first is to deposit their net worth with the financial intermediary. Depositing one unit of general goods in the financial intermediary at time $t-1$ yields r_t units of general goods at time t. r_t is the gross interest rate at time t. The second method is to begin an investment project. The investment project transforms one unit of general goods at time $t-1$ into $A\Phi_{t-1}$ units of general goods at time t, where A is a certain constant. Productivity Φ_{t-1} is a random variable received at time $t-1$ and is a function of a stochastic event ω_{t-1}, where $\{\omega_{t-1} \in \Omega | \Phi_{t-1}(\omega_{t-1}) \leq \Phi\}$ is an element of a σ-algebra $\mathcal{F}$ of a probability space $(\Omega, \mathcal{F}, P)$. It should be noted that an agent acquires information regarding her productivity $\Phi_{t-1}(\omega_{t-1})$ at time t before beginning her investment project. There are no insurance markets for idiosyncratic productivity shocks, and low productivity cannot be insured against before its realization. As assumed in Angeletos (2007), the idiosyncratic productivity shocks $\Phi_0(\omega_0), \Phi_1(\omega_1), ...$ (or equivalently, the stochastic events $\omega_0, \omega_1, ...$) are independent and identically distributed across time and agents. The productivity shock Φ has support over $[0, h]$, where $h \in (0, \infty)$, and its cumulative distribution function is $G(\Phi)$, which is continuous, differentiable, and strictly increasing on the support. The history of stochastic events until time $t-1$ is denoted $\omega^{t-1} = \{\omega_0, \omega_1, ..., \omega_{t-1}\}$. Similarly, the history of the idiosyncratic productivity shocks until time $t-1$ is given by $\Phi^{t-1} = \{\Phi_0, \Phi_1, ..., \Phi_{t-1}\}$. Φ^{t-1} is a vector function of ω^{t-1} on $(\Omega^t, \mathcal{F}^t, P^t)$, which is a Cartesian product of t copies of $(\Omega, \mathcal{F}, P)$.

As mentioned above, an agent faces a credit constraint when borrowing financial resources from the financial intermediary. Idiosyncratic productivity $\Phi_{t-1}(\omega_{t-1})$ is private information. Although each agent knows her own productivity before investing in a project, other agents do not know her productivity. The future output produced by the agent, therefore, cannot be pledged. Additionally, agents may choose not to consistently repay their obligations. For these reasons, the financial intermediary imposes credit constraints associated with the agents' at-hand net worth, and each agent is able to borrow financial resources only up to $\lambda \in [0, \infty)$ times her net worth as assumed in the models developed by Aghion and Banerjee (2005), Aghion et al. (2005), or Antrás and Caballero (2009).[5]

An agent at time t maximizes her expected lifetime utility:

$$U_t = E\left[\sum_{s=t}^{\infty} \beta^{s-t} \ln c_s(\omega^s) \middle| \Phi^t(\omega^t)\right],$$

subject to:

$$k_s(\omega^s) + b_s(\omega^s) = [A\Phi_{s-1}(\omega_{s-1})k_{s-1}(\omega^{s-1}) + r_s b_{s-1}(\omega^{s-1})](1-\tau_s) - c_s(\omega^s) \tag{1}$$

$$b_s(\omega^s) \geq -\lambda a_s(\omega^s) \tag{2}$$

$$k_s(\omega^s) \geq 0, \tag{3}$$

for $s \geq t \geq 0$, where β is the subjective discount factor, and $E[.|\Phi^t(\omega^t)]$ is an expectation operator given the information on the history of the idiosyncratic productivity shocks until time t. It is assumed that the budget constraint at time $t = 0$ is given by $k_0 + b_0 = w_0 - c_0$, where w_0 is the initial endowment of each agent at birth, which is common to all agents. Eq.(1) is a flow budget constraint at time s, where $c_s(\omega^s)$ denotes consumption, $k_s(\omega^s)$ is investment, and $b_s(\omega^s)$ is credit if positive and a private debt if negative at time s. At time s, the agent produces general goods $A\Phi_{s-1}(\omega_{s-1})k_{s-1}(\omega^{s-1})$. As discussed above, r_s is the gross interest rate. The government imposes taxes on the agents' incomes exempting of loan repayments, and the tax rate at time s is denoted $\tau_s \leq 1$. Eq.(2) is a credit constraint that an agent faces at time s. $a_s(\omega^s)$ is the agent's saving, defined as $a_s(\omega^s) := [A\Phi_{s-1}(\omega_{s-1})k_{s-1}(\omega^{s-1}) + r_s b_{s-1}(\omega^{s-1})](1-\tau_s) - c_s(\omega^s)$, which is the right-hand side of Eq.(1). We call $a_s(\omega^s)$ net worth in this model because it represents funds for investments and deposits. λ represents the extent of financial market imperfections. As λ goes from 0 to infinity, the financial market approaches perfection. Eq.(3) is the non-negativity

[5]This type of credit constraint is often employed in the literature. See also Aghion et al. (1999), Caballé et al. (2006), and Antrás and Caballero (2010). Regarding the microfoundation for the credit constraints, see, for instance, the appendices in Aghion et al. (2005) and Kunieda and Shibata (2012).

constraint of investment.

From the utility maximization problem, agents with $A\Phi_t > r_{t+1}$ optimally choose to begin investment projects and engage in general goods production, borrowing financial resources up to the limit of the credit constraint. Agents with $A\Phi_t \leq r_{t+1}$ optimally choose to lend their net worth in the financial market through the financial intermediary and acquire the interest rate r_{t+1}.[6] Define ϕ_t such that $A\phi_t = r_{t+1}$ and it is a cutoff for the productivity shocks that divides agents into lenders and borrowers at time t. Using the cutoff ϕ_t, we obtain a lending-investment-borrowing program for an agent who has net worth $a_t(\omega^t)$ as follows:

$$k_t(\omega^t) = \begin{cases} 0 & \text{if } \Phi_t(\omega_t) \leq \phi_t \\ \frac{a_t(\omega^t)}{1-\mu} & \text{if } \Phi_t(\omega_t) > \phi_t, \end{cases} \tag{4}$$

and

$$b_t(\omega^t) = \begin{cases} a_t(\omega^t) & \text{if } \Phi_t(\omega_t) \leq \phi_t \\ -\frac{\mu}{1-\mu} a_t(\omega^t) & \text{if } \Phi_t(\omega_t) > \phi_t, \end{cases} \tag{5}$$

where $\mu := \lambda/(1+\lambda) \in [0,1)$ also measures the extent of financial market imperfections. The higher the value of μ, the more the credit constraint relaxes. Given this lending-investment-borrowing program, the flow budget constraint at time s can be rewritten as follows:

$$a_s(\omega^s) = \tilde{R}_s(1-\tau_s)a_{s-1}(\omega^{s-1}) - c_s(\omega^s), \tag{6}$$

where $\tilde{R}_s := \max\{r_s, \frac{A\Phi_{s-1} - r_s\mu}{1-\mu}\}$. Under the lending-investment-borrowing program given by eqs.(4) and (5), each agent maximizes her lifetime utility subject to Eq.(6). The Euler equation is obtained as follows:

$$\frac{1}{c_t(\omega^t)} = \beta E\left[\tilde{R}_{t+1}(1-\tau_{t+1})\frac{1}{c_{t+1}(\omega^{t+1})} \,\middle|\, \Phi^t(\omega^t)\right]. \tag{7}$$

From eqs.(6), (7), and the transversality condition, we obtain the law of motion of an agent's

[6] It is assumed that agents with $A\Phi_t = r_{t+1}$, who are indifferent to these two choices, lend their net worth in the financial market. This assumption does not affect our results because the productivity shocks are continuously distributed.

net worth $a_t(\omega^t)$ as follows:

$$a_{t+1}(\omega^{t+1}) = \beta \tilde{R}_{t+1}(1 - \tau_{t+1}) a_t(\omega^t). \tag{8}$$

The derivations of both eqs.(6) and (8) are provided in the appendix.

2.2 Government

The government budget constraint is given by:

$$B_t = r_t B_{t-1} + E_t - T_t, \tag{9}$$

where B_t is the public debt issued at time t and redeemed at time $t+1$. E_t and T_t are total government spending and total tax revenues at time t, respectively. Throughout our analysis, it is assumed that the government maintains a constant public-spending-to-GDP ratio such that:

$$\theta = \frac{E_t}{Y_t}, \tag{10}$$

where $\theta \in [0, 1)$. The government finances public spending and interest payments by issuing public debt and/or collecting taxes on the agents' incomes.

The government actively adjusts the tax rate τ_t in each time period such that the PDG ratio does not diverge. As the public-spending-to-GDP ratio is constant, controlling the tax rate to maintain fiscal sustainability is equivalent to controlling the primary surplus $T_t - E_t$. Empirical evidence obtained by Bohn (1998), Mendoza and Ostry (2008), and Greiner and Fincke (2009) indicates that the primary surplus positively reacts to the lagged PDG ratio in many developed and developing countries. Based on such empirical evidence, the tax rate at time t is assumed to be an increasing function of the lagged PDG ratio B_{t-1}/Y_{t-1}. Furthermore, we will find in subsection 3.1 that B_{t-1}/Y_{t-1} is an increasing function of the cutoff ϕ_{t-1} in equilibrium. Then, the government is assumed to adopt a fiscal policy such that the tax rate τ_t positively reacts to the lagged cutoff ϕ_{t-1} to maintain fiscal sustainability.

In other words, τ_t is an increasing function of ϕ_{t-1} as follows:

$$\tau_t := \tau\big(\phi_{t-1}\big),$$

where $\tau(\phi_{t-1})$ is continuous in $[G^{-1}(\mu), h]$. Specifically, $\tau(\phi_{t-1})$ is increasing and differentiable with respect to ϕ_{t-1}, namely, $\tau'(\phi_{t-1}) \geq 0$.[7]

2.3 Financial Intermediary

As in the models of Grandmont (1983) and Rochon and Polemarchakis (2006), the financial sector is competitive and the representative financial intermediary gains no profits from its business. In addition to imposing credit constraints on agents, the financial intermediary accepts deposits from depositors and lends financial resources. The financial intermediary purchases government bonds with the excess total saving, which means that the demand for government bonds at time $t-1$ from the representative financial intermediary is $B^d_{t-1} := \int_{\Omega^t} b_{t-1}(\omega^{t-1}) dP^t(\omega^{t-1})$. It follows from the government bond market-clearing condition that $B_{t-1} = B^d_{t-1}$, or equivalently,

$$B_{t-1} = \int_{\Omega^t} b_{t-1}(\omega^{t-1}) dP^t(\omega^{t-1}), \tag{11}$$

where ω^{t-1} is an element of $(\Omega^t, \mathcal{F}^t, P^t)$ as defined previously.

3 Equilibrium

3.1 Aggregation

From Eq.(8), the net worth $a_t(\omega^t)$ of an agent who receives a stochastic event ω_t at time t and has history ω^{t-1} becomes:

$$a_t(\omega^t) = \beta(A\Phi_{t-1}(\omega_{t-1})k_{t-1}(\omega^{t-1}) + r_t b_{t-1}(\omega^{t-1}))(1 - \tau(\phi_{t-1})), \tag{12}$$

[7] Challe and Ragot (2011) assume that taxation responds to public debt in a similar manner as ours in studying business fluctuations caused by aggregate fiscal shocks. Only the assumption that $\tau(\phi_{t-1})$ is an increasing function is important: $\tau(\phi_{t-1})$'s differentiability is assumed for simple expositions.

where $k_{t-1}(\omega^{t-1})$ and $b_{t-1}(\omega^{t-1})$ are given by eqs.(4) and (5), respectively. Note that the stochastic event ω_t and the history ω^{t-1} are independent of each other. By applying the law of large numbers to the agents, the net worth of the agents who receive the stochastic events ω_t at time t is aggregated as follows:

$$\begin{aligned}\tilde{a}_t(\omega_t) &:= \int_{\Omega^t} a_t(\omega^t)dP^t(\omega^{t-1}) \\ &= \beta(1-\tau(\phi_{t-1}))\int_{\Omega^t}(A\Phi_{t-1}(\omega_{t-1})k_{t-1}(\omega^{t-1})+r_t b_{t-1}(\omega^{t-1}))dP^t(\omega^{t-1}). \quad (13)\end{aligned}$$

Additionally, the aggregate output at time t is given by:

$$Y_t := \int_{\Omega^t} A\Phi_{t-1}(\omega_{t-1})k_{t-1}(\omega^{t-1})dP^t(\omega^{t-1}). \quad (14)$$

From eqs.(11) and (14), Eq.(13) becomes:

$$\tilde{a}_t(\omega_t) = \beta(Y_t + r_t B_{t-1})(1-\tau(\phi_{t-1})). \quad (15)$$

From Eq.(4), the aggregate investment $\tilde{k}_t(\omega_t)$ across the agents with stochastic realization ω_t is:

$$\tilde{k}_t(\omega_t) = \begin{cases} 0 & \text{if } \Phi_t(\omega_t) \le \phi_t, \\ \frac{1}{1-\mu}\tilde{a}_t(\omega_t) = \frac{\beta}{1-\mu}(Y_t + r_t B_{t-1})(1-\tau(\phi_{t-1})) & \text{if } \Phi_t(\omega_t) > \phi_t. \end{cases} \quad (16)$$

Similarly, from Eq.(5), the aggregate debt or credit $\tilde{b}_t(\omega_t)$ across the agents with stochastic realization ω_t is presented by:

$$\tilde{b}_t(\omega_t) = \begin{cases} \tilde{a}_t(\omega_t) = \beta(Y_t + r_t B_{t-1})(1-\tau_t(\phi_{t-1})) & \text{if } \Phi_t(\omega_t) \le \phi_t \\ -\frac{\mu}{1-\mu}\tilde{a}_t(\omega_t) = -\frac{\mu\beta}{1-\mu}(Y_t + r_t B_{t-1})(1-\tau_t(\phi_{t-1})) & \text{if } \Phi_t(\omega_t) > \phi_t. \end{cases} \quad (17)$$

Substituting Eq.(17) into Eq.(11) yields

$$\begin{aligned}B_t &= \int_H \tilde{b}_t(\omega_t)dP(\omega_t) + \int_{\Omega/H}\tilde{b}_t(\omega_t)dP(\omega_t) \\ &= \beta(Y_t + r_t B_{t-1})\left[G(\phi_t) - \frac{\mu(1-G(\phi_t))}{1-\mu}\right](1-\tau_t(\phi_{t-1})), \\ &= \beta(Y_t + r_t B_{t-1})\frac{G(\phi_t)-\mu}{1-\mu}(1-\tau_t(\phi_{t-1})), \quad (18)\end{aligned}$$

where $H = \{\omega_t \in \Omega \mid \Phi_t(\omega_t) \leq \phi_t\}$. The left-hand and right-hand sides of Eq.(18) are the aggregate supply of and the aggregate demand for public debt, respectively. The left-hand side of Eq.(18) is determined by the government's budget constraint and the fiscal policy rules. In the right-hand side of Eq.(18), $\beta(Y_t + r_t B_{t-1})(1 - \tau_t(\phi_{t-1}))$ is the total private saving in this economy and $G(\phi_t) - \mu[1 - G(\phi_t)]/(1 - \mu) = [G(\phi_t) - \mu]/(1 - \mu)$ is the proportion of the total private saving spent on holding government bonds. If $G(\phi_t) < \mu$, B_t is negative. In such a case, the government holds claims on loans to the private sector. In this study, however, we are interested in the case in which the government incurs debts as is observed in most developed and developing countries: we focus our analysis on the case in which $G(\phi_t) \geq \mu$ in what follows.[8]

From eqs.(1), (11), and (14), the government obtains tax revenues at time t as follows:

$$T_t = (Y_t + r_t B_{t-1})\tau_t(\phi_{t-1}).$$

Therefore, the government's budget constraint is rewritten as follows:

$$B_t = r_t(1 - \tau(\phi_{t-1}))B_{t-1} - (\tau_t(\phi_{t-1}) - \theta)Y_t. \tag{19}$$

In Eq.(19), $r_t(1 - \tau(\phi_{t-1}))$ is regarded as the interest rate that the government actually faces. Similarly, $(\tau(\phi_{t-1}) - \theta)Y_t$ is regarded as the primary surplus that the government actually acquires.

From eqs.(18) and (19), the PDG ratio becomes

$$\frac{B_t}{Y_t} = \frac{\beta(1 - \theta)[G(\phi_t) - \mu]}{1 - \mu - \beta[G(\phi_t) - \mu]}. \tag{20}$$

Proposition 1 *The PDG ratio, B_t/Y_t, is an increasing function of the cutoff ϕ_t.*

Proof: The claim of Proposition 1 is obvious from Eq.(20). □

As demonstrated in Proposition 1, there is a positive correlation between the PDG ratio

[8]If μ is arbitrarily close to one and thus the financial market is perfect, the economy returns to the Ramsey-type of a one sector growth model in which there is no liquidity effect of public debt. Our discussion of the dynamic property of this economy concerns the case in which μ is somewhat small such that we can investigate the liquidity effect of public debt. In particular, if μ is arbitrarily close to zero as assumed in Woodford (1990), we are always able to investigate the liquidity effect of public debt.

at time t and the cutoff ϕ_t, or equivalently, the equilibrium interest rate r_{t+1}. Proposition 1 states that as the PDG ratio becomes higher, the number of depositors has to increase for the government bond market to clear. Meanwhile, we have assumed in section 2.2 that the tax rate at time t is an increasing function of the cutoff ϕ_{t-1}. The tax rate is, then, an increasing function of the lagged PDG ratio, which is supported by empirical evidence obtained by Bohn (1998), Mendoza and Ostry (2008), and Greiner and Fincke (2009) indicating the positive correlation between primary surplus and the lagged PDG ratio.

3.2 Growth rate

We derive the economic growth rate, Y_{t+1}/Y_t. Multiplying both sides of the second equation of Eq.(16) by $A\Phi_t$, we aggregate the equations across all producers to obtain total output Y_{t+1} as follows:

$$\begin{aligned} & \int_{\Omega/H} A\Phi_t(\omega_t)\tilde{k}_t(\omega_t)dP(\omega_t) = \int_{\Omega/H} \frac{\beta A\Phi_t(\omega_t)}{1-\mu}(Y_t + r_t B_{t-1})(1-\tau_t(\phi_{t-1}))dP(\omega_t) \\ \Longleftrightarrow \quad & Y_{t+1} = AF(\phi_t)\frac{\beta(Y_t + r_t B_{t-1})(1-\tau_t(\phi_{t-1}))}{1-\mu}, \end{aligned} \tag{21}$$

where $F(\phi_t) := \int_{\phi_t}^{h} \Phi_t(\omega_t)dG(\Phi_t)$. From eqs.(19), (20), and (21), the growth rate Y_{t+1}/Y_t is obtained as follows:

$$\frac{Y_{t+1}}{Y_t} = \frac{\beta A(1-\theta)F(\phi_t)}{1-\mu-\beta(G(\phi_t)-\mu)}. \tag{22}$$

It is noted from Eq.(22) that the growth rate is not directly affected by the tax rate $\tau_t(\phi_{t-1})$. This is because the aggregate net worth at time t of the private sector is given by $(Y_t + r_t B_{t-1})(1-\tau_t(\phi_{t-1})) = (1-\theta)(1-\mu)Y_t/[1-\mu-\beta(G(\phi_t)-\mu)]$, which is determined by the output and the number of depositors and borrowers, other things being equal. In equilibrium, the tax rate has an impact on the growth rate through the cutoff ϕ_t.

Assumption 1 $\beta F(G^{-1}(\mu)) > G^{-1}(\mu)(1-\mu)$.

Assumption 1 is a technical assumption that guarantees that a growth-maximizing cutoff ϕ^* exists in $[G^{-1}(\mu), h]$ as demonstrated in Proposition 2:[9]

[9] Assumption 1 holds if β is close to one and/or if μ is relatively small.

Proposition 2 *Define* ϕ^* *such that* $\beta F(\phi^*) - \phi^*[1-\mu-\beta(G(\phi^*)-\mu)] = 0$. *Under Assumption 1, for* $\phi_t \in [G^{-1}(\mu), h]$, *the highest growth rate of* Y_{t+1}/Y_t *is achieved at* $\phi^* \in [G^{-1}(\mu), h]$.

Proof: See the Appendix.

The growth rate has an inverted U-shaped relationship with the cutoff. As the cutoff ϕ_t increases from $G^{-1}(\mu)$, the growth rate Y_{t+1}/Y_t increases until the cutoff ϕ_t reaches its critical value ϕ^*. If the cutoff ϕ_t increases beyond the critical value, the growth rate begins to decrease. From propositions 1 and 2, the inverted U-shaped relationship between the PDG ratio and the growth rate is immediately obtained as follows:

Proposition 3 *Suppose that Assumption 1 holds. There exists a level of the PDG ratio* $(B/Y)^* := \beta(1-\theta)(G(\phi^*)-\mu)/[1-\mu-\beta(G(\phi^*)-\mu)]$ *such that if* $B_t/Y_t \in [0, (B/Y)^*)$, *the growth rate* Y_{t+1}/Y_t *increases with* B_t/Y_t *and if* $B_t/Y_t \in ((B/Y)^*, \beta(1-\theta)/(1-\beta)]$, *the growth rate* Y_{t+1}/Y_t *decreases with* B_t/Y_t. *In other words, the highest growth rate of* Y_{t+1}/Y_t *is achieved when* $B_t/Y_t = (B/Y)^*$.

Proof: See the Appendix.

Considering eqs.(16), (20), and (22), we can understand how the *crowd-out* and *crowd-in* effects work. Suppose that public debt B_{t-1} grows at time $t-1$. In this case, more depositors are necessary for the government bond market to clear, and thus, the cutoff ϕ_{t-1}, or equivalently, the interest rate $r_t (= A\phi_{t-1})$ at time t should be raised. Now that both B_{t-1} and r_t have become large, it follows from the government's budget constraint that B_t will also become large. As a result, the cutoff ϕ_t rises again to clear the public debt market, and the number of investors is reduced at time t. The reduction in the number of investors produces downward pressure on the aggregate investment at time t. This is the *crowd-out* effect, which is embodied in $F(\phi_t)$ in Eq.(22). Because of the *crowd-out* effect, less productive agents are excluded from production, and production resources are intensively used by more highly productive investors at time t. Because the increase in $r_t B_{t-1}$ relaxes the credit constraints at time t, more highly productive investors are able to increase their investments. The expansion of highly productive investment produces upward pressure on the aggregate output at time $t+1$. This is the *crowd-in* effect, which is embodied in $G(\phi_t)$ in Eq.(22). Whether the economy exhibits the inverted U-shaped relationship in the growth process in equilibrium depends on the fiscal policy rules enacted by the government as studied in section 4.

3.3 Equilibrium Dynamics

From eqs.(20) and (22), the dynamic equation with respect to B_t is obtained as follows:

$$B_t = \frac{\beta A(1-\theta)(G(\phi_t)-\mu)}{1-\mu-\beta(G(\phi_t)-\mu)} \frac{F(\phi_{t-1})}{G(\phi_{t-1})-\mu} B_{t-1}. \tag{23}$$

Eq.(23) represents the aggregate demand for public debt, given the amount of public debt in the initial of period t. Meanwhile, the dynamic equation with respect to B_t is also obtained from eqs.(19) with $r_t = A\phi_{t-1}$, (20), and (22) as follows:

$$B_t = \left[(1-\tau_t(\phi_{t-1}))A\phi_{t-1} + \frac{(\theta-\tau_t(\phi_{t-1}))AF(\phi_{t-1})}{G(\phi_{t-1})-\mu}\right] B_{t-1}. \tag{24}$$

Eq.(24) represents the aggregate supply of public debt, which is determined by the fiscal policy rule and the government's budget constraint. Eventually, Eqs.(23) and (24) yield a dynamic equation with respect to the cutoff ϕ_t:

$$\frac{\beta(1-\theta)(G(\phi_t)-\mu)}{1-\mu-\beta(G(\phi_t)-\mu)} = \frac{(1-\tau(\phi_{t-1}))\phi_{t-1}(G(\phi_{t-1})-\mu)}{F(\phi_{t-1})} + \theta - \tau(\phi_{t-1}). \tag{25}$$

Define two continuous functions $\Psi : [G^{-1}(\mu), h] \to \mathcal{R}_+$ and $\Lambda : [G^{-1}(\mu), h] \to \mathcal{R}_+$ as:

$$\Psi(\phi_t) := \frac{\beta(1-\theta)(G(\phi_t)-\mu)}{1-\mu-\beta(G(\phi_t)-\mu)},$$

and

$$\Lambda(\phi_{t-1}) := \frac{(1-\tau(\phi_{t-1}))\phi_{t-1}(G(\phi_{t-1})-\mu)}{F(\phi_{t-1})} + \theta - \tau(\phi_{t-1}),$$

which are the left-hand and right-hand sides of Eq.(25), respectively. It is noted from Eq.(20) that $\Psi(\phi_t)$ is equal to B_t/Y_t and monotonically increases with ϕ_t. In contrast, the configuration of $\Lambda(\phi_{t-1})$ substantially depends on the nature of $\tau(\phi_{t-1})$. As described in section 2.2, we assume that the government enacts a fiscal policy rule expressed by $(\tau(\phi_{t-1}), \theta)$ such that for any initial PDG ratio B_0/Y_0 the economy follows an equilibrium path in which B_t/Y_t

is bounded above.[10] Allowing θ to be a positive constant throughout our investigation, we impose two assumptions on the nature of $\tau(\phi_{t-1})$ such that the economy is able to attain equilibrium for any value of B_0/Y_0.

Assumption 2 *$\tau(G^{-1}(\mu)) = 0$, $\tau(h) = 1$, and $\lim_{\phi \to h} \tau'(\phi)/(\phi G'(\phi)) = a > 0$, where a is a constant.*

$\tau(G^{-1}(\mu)) = 0$ implies that when public debt in the current period is zero, the government finances public spending by issuing public debt instead of taxation. $\tau(h) = 1$ and $\lim_{\phi \to h} \tau'(\phi)/(\phi G'(\phi)) = a > 0$ are conditions for $\Lambda(\phi)$ to be continuous at h.

Assumption 3 $(1-\theta)/(1-\beta) > ah(1-\mu)$.

Lemma 1 *Suppose that assumptions 2 and 3 hold. Then, it follows that $\Psi(h) > \Lambda(h)$ and $0 = \Psi(G^{-1}(\mu)) < \Lambda(G^{-1}(\mu))$.*

Proof: See the Appendix.

In a competitive equilibrium, the dynamic behavior of the economy is recursively given by sequences $\{\phi_{t-1}, B_{t-1}, Y_t\}$ for $t \geq 1$ that satisfy the difference equations (22), (24), and (25). The acquisition of information on the cutoff ϕ_t enables us to obtain the relationship between the growth rate and the PDG ratio from eqs.(20) and (22). It should be noted from Lemma 1 that for a large (small) ϕ_{t-1}, it follows that $\phi_t < \phi_{t-1}$ ($\phi_t > \phi_{t-1}$). As such, Lemma 1 guarantees that for any initial PDG ratio, B_0/Y_0, the PDG ratio is always bounded above. The economy could, however, exhibit multiple steady states and/or various dynamic patterns, depending on the nature of $\tau(\phi)$. Because it is impossible to investigate such exhaustive cases comprehensively, we focus our analysis on the simplest case in which $\Lambda(\phi)$ is monotonically increasing and there is a unique steady state $\bar{\phi}$ in the dynamical system (25).

Fig. 1 provides a phase diagram that embodies the dynamical system (25) in which a unique steady state $\bar{\phi}$ exists under assumptions 2 and 3. An example satisfying these

[10] To consider a fiscal policy rule violating this assumption, suppose that τ is constant throughout the growth process. In this case, for sufficiently large B_0/Y_0, B_t/Y_t diverges to infinity, and thus the constant tax rate rule is unsustainable. Such a divergent path of B_t/Y_t cannot be an equilibrium. The literature on fiscal sustainability, including Chalk (2000), Rankin and Roffia (2003), Bräuninger (2005), Yakita (2008), and Arai (2011), emphasizes the importance of an initial PDG ratio for fiscal sustainability over time under various fiscal policy rules. In contrast, in this paper, the government is assumed to enact a fiscal policy such that the PDG ratio never diverges.

conditions is the case in which Φ follows a uniform distribution over $[0,1]$ with $\tau(\phi_{t-1}) = \phi_{t-1}^2$ and $\mu = 0$. In this case, it follows that $h = 1$ and $a = 2$. Assumption 3 then holds if and only if $1+\theta < 2\beta$ and the dynamical system with respect to ϕ_t becomes $\beta(1-\theta)\phi_t/(1-\beta\phi_t) = \phi_{t-1}^2 + \theta$, where the unique steady state is globally stable.

[Fig. 1 around here]

Proposition 4 *Suppose that the dynamical system (25) has a unique steady state* $\bar{\phi}$ *with* $\Lambda(\phi_{t-1})$ *being monotonically increasing under assumptions 2 and 3. Then, the steady state* $\bar{\phi}$ *is globally stable.*

Proof: See the Appendix.

4 Public debt and the growth process

In this section, we demonstrate that the inverted U-shaped relationship between the PDG ratio and the economic growth rate is generated on a transition path under a certain fiscal policy rule. We then compare growth patterns affected by different fiscal policy rules.

4.1 The inverted U-shaped relationship

Let us consider the situation of Proposition 4. Suppose that the steady-state cutoff $\bar{\phi}$ is greater than the growth maximizing cutoff ϕ^*.[11] Suppose also that the initial PDG ratio is zero: $B_0/Y_0 = 0$. Because $B_0/Y_0 = 0$, the initial cutoff level ϕ_0 is given by $\phi_0 = G^{-1}(\mu)$ and the dynamic behavior of ϕ_t is illustrated in Fig. 1. The cutoff ϕ_t continues increasing on the transition path and eventually converges to the steady-state cutoff $\bar{\phi}$. The growth rate Y_{t+1}/Y_t also continues increasing synchronously with the cutoff before beginning to decrease at the point where the cutoff reaches the threshold value of ϕ^*. Proposition 2 and Proposition 3 therefore imply that in equilibrium, the economy follows a growth process described by the inverted U-shaped relationship between the PDG ratio and the economic growth rate.

[11] We can verify the existence of fiscal policy rules that satisfy $\bar{\phi} > \phi^*$. Whether the steady-state cutoff $\bar{\phi}$ is greater or smaller than the growth-maximizing cutoff ϕ^* depends on fiscal policy rules $(\tau(\phi), \theta)$. We explain this point in subsection 4.2.

The result regarding the inverted U-shaped relationship agrees not only with time series observations of each individual country but also with cross-country observations. The position of the long-run steady-state cutoff in our model depends on fiscal policy rules as seen in Eq.(25) and the growth rate has a one-to-one relationship with the cutoff as seen in Eq.(22), although the growth-maximizing cutoff is constant, other things being equal. This implies that some countries have long-run steady-state cutoffs greater than the growth-maximizing cutoff, whereas others have long-run steady-state cutoffs below it. The different fiscal policy rules implemented by the different countries therefore result in different growth rates and an inverted U-shaped relationship between the PDG ratio and the growth rate should be observed across countries. The outcome regarding the inverted U-shaped relationship between the PDG ratio and the economic growth rate is consistent with empirical evidence from panel data analyses obtained by Baum et al. (2013) and Checherita-Westphal and Rother (2012).

Our theoretical result is obtained from dual nature of public debt. Saint-Paul (1992), among others, demonstrates that the existence of public debt reduces economic growth rates because public debt *crowds out* private investment. This type of *crowd-out* effect of public debt also exists in our model. If the government issued no public debt, it would follow from Eq.(20) that $\phi_t = G^{-1}(\mu)$ for all $t \geq 0$ and the interest rate would reach its lowest level given a certain extent of financial market imperfections. Now that the government issues public debt, however, it follows from Eq.(20) that $\phi_t > G^{-1}(\mu)$ for all $t \geq 1$, and accordingly, the interest rate is greater than it is when public debt is not issued. As a result, the opportunity cost for investment projects increases, implying that the *crowd-out* effect of public debt produces downward pressure on the growth rate.

Interestingly, the increased interest rate induces a *crowd-in* effect of public debt. If there were no public debt, the less productive agents would only obtain a low interest rate from depositing financial resources with the financial intermediary. In fact, public debt becomes a beneficial vehicle for storing value in our model because the existence of public debt yields a higher interest rate when depositing financial resources than does it in an economy without public debt. In other words, public debt provides the less productive agents with better opportunities for saving. Note that because of the idiosyncratic productivity shock, many unproductive agents in the current period who benefit from public debt will become productive agents in the next and initiate investment projects. These productive agents sell their public debt in the financial market in the next period and obtain greater production resources than in the case without public debt. The higher interest rate increases the producers' net worth that is used as production resources, and the increased net worth relaxes credit constraints. Private investment is, then, *crowded in.* This is a liquidity effect of public debt in a financially constrained economy. Public debt generates pressure that promotes economic growth. In our model, the *crowd-in* effect dominates the *crowd-out* effect if the PDG ratio is smaller than $(B/Y)^*$. In contrast, the *crowd-out* effect dominates the *crowd-in* effect if the PDG ratio is greater than $(B/Y)^*$. This is why the inverted U-shaped relationship between the accumulation of public debt and economic growth is obtained in equilibrium.

4.2 Growth patterns subject to fiscal policy rules

Next, we investigate how economic growth patterns are subject to fiscal policy rules, comparing two types of fiscal policy rules. The two types of fiscal policy rules are represented by $\tau_1(\phi)$ and $\tau_2(\phi)$ where a fiscal policy associated with $\tau_1(\phi)$ is more expansionary than a fiscal policy associated with $\tau_2(\phi)$ under a fixed value of θ. Formally, we define an expansionary fiscal policy as:

Definition 1 *Suppose that both $\tau_1(\phi)$ and $\tau_2(\phi)$ satisfy Assumption 2. Given a fixed positive value of θ, a fiscal policy rule associated with $\tau_1(\phi)$ is more expansionary than a fiscal policy rule associated with $\tau_2(\phi)$ if and only if $\tau_1(\phi) < \tau_2(\phi)$ for $\phi \in (G^{-1}(\mu), h)$.*

Although at the corner points $\phi = G^{-1}(\mu)$ and $\phi = h$, the tax rates of the two different fiscal policy rules are the same because of Assumption 2, the tax rates of the more expansionary fiscal policy rule are always strictly less than those of the less expansionary fiscal policy in the interior of $(G^{-1}(\mu), h)$. Comparing the two fiscal policy rules, the fiscal policy rule with $\tau_1(\phi)$ slowly raises the tax rate when ϕ_t increases from a small value, implying that the role of public debt is relatively central to financing the government's spending when the PDG ratio is small. However, the role of public debt in the fiscal policy rule with $\tau_2(\phi)$ is relatively minor in financing the government's spending even when the PDG ratio is small.[12]

Let $\Lambda_i(\phi)$ and $\bar{\phi}_i$ be associated with the fiscal policy rules represented by $\tau_i(\phi)$ $(i = 1, 2)$. Because $\tau_1(\phi)$ is more expansionary than $\tau_2(\phi)$, it holds that $\Lambda_1(\phi) > \Lambda_2(\phi)$ for $\phi \in (G^{-1}(\mu), h)$. Fig. 2 provides a phase diagram for the cases associated with $\Lambda_1(\phi)$ and $\Lambda_2(\phi)$. It is straightforward from Fig. 2 to verify that $\bar{\phi}_1 > \bar{\phi}_2$. Comparing the two steady states under the two different types of fiscal policy rules, we note from Eq.(20) that the steady-state PDG ratio under a more expansionary fiscal policy is greater than that under a less expansionary fiscal policy.

[Fig. 2 around here]

Now, suppose that the fiscal policy rule associated with $\tau_1(\phi)$ produces the steady state $\bar{\phi}_1$ where $\phi^* < \bar{\phi}_1$, whereas the fiscal policy rule associated with $\tau_2(\phi)$ produces the steady state $\bar{\phi}_2$ where $\bar{\phi}_2 < \phi^*$. Fig. 2 illustrates the two different fiscal policy rules associated with $\tau_1(\phi)$ and $\tau_2(\phi)$ in which $\bar{\phi}_2 < \phi^* < \bar{\phi}_1$ under assumptions 1-3. Let us assume that $B_0/Y_0 = 0$, and thus, $\phi_0 = G^{-1}(\mu)$. If the government applies a fiscal policy associated with $\tau_2(\phi)$, ϕ_t monotonically increases and does not go beyond ϕ^*. In this growth process, public debt promotes economic growth. However, because the fiscal policy is less expansionary, the economy never experiences the highest growth rate achieved at ϕ^*. In contrast, if the government applies a fiscal policy associated with $\tau_1(\phi)$, ϕ_t exceeds ϕ^*. In this case, the economy follows the growth process investigated in subsection 4.1. In summary, Proposition 5 is presented below:

Proposition 5 *Consider the two different fiscal policy rules associated with $\tau_1(\phi)$ and $\tau_2(\phi)$ in which $\bar{\phi}_2 < \phi^* < \bar{\phi}_1$. Suppose that $B_0/Y_0 = 0$ under assumptions 1, 2, and 3. Then, the government's expansionary fiscal policy associated with $\tau_1(\phi)$ yields an inverted U-shaped relationship between the PDG ratio and the growth rate in the growth process of the economy, whereas the government's less expansionary fiscal policy associated with $\tau_2(\phi)$ yields a monotonic growth pattern such that both the PDG ratio and the economic growth rate continue rising synchronously until the economy converges to its steady state.*

[12] As noted in footnote 7, Challe and Ragot (2011) assume similar fiscal policy rules to ours. The discussion of our two fiscal rules here partially follows their discussion on page 280 of their paper.

Proof: The claim has been shown by the discussion preceding Proposition 5. □

Proposition 5 implies that the choice of fiscal policy rules produces the synchronization or unsynchronization between economic growth and the accumulation of public debt in our model. From the dynamical system (25), we note that changes in the fiscal policy rules affecting the tax schedule $\tau_i(\phi)$ or the value of θ shift the position of the steady state, $\bar{\phi}_i$, with the growth-maximizing cutoff value of ϕ^* being unchanged. Such changes in the fiscal policy rules therefore yield various dynamic patterns of economic growth and the accumulation of public debt.

There is a caveat to the discussion of whether our model successfully explains the actually observed dynamic patterns of economic growth and the accumulation of public debt. For deeper investigations of our model's explanatory power for the observed dynamic patterns, it is necessary to individually examine how each country applies its fiscal policy rule in each year. Additionally, only *internal* public debt matters in our model, producing a liquidity effect in domestic economies. For instance, a large share of public debt in the Unites States is held by foreign investors. To discuss public debt's domestic liquidity effect, it is more appropriate to observe the relationship between the accumulation of *internal* public debt and economic growth. Although these tasks are beyond the scope of our paper, the issuance of public debt is likely to produce a liquidity effect in domestic economies. Because of the *crowd-in* and *crowd-out* effects of public debt, various dynamic patterns of economic growth and the accumulation of public debt are obtained in equilibrium in our model.

5 Concluding Remarks

We have shown that an inverted U-shaped relationship between the accumulation of public debt and economic growth appears in a financially constrained economy under certain parameter conditions and fiscal policy rules. This finding contributes to the literature on public debt and growth that many researchers have long debated. In these debates, however, little emphasis has been placed on the *crowd-in* effect of public debt on private investment, with the exception of Woodford (1990). Although Woodford (1990) focuses his investigation exclusively on the steady-state equilibrium, our model allows us to study the relationship between the economy's dynamic growth process and the accumulation of public debt. This is because the uninsured idiosyncratic productivity shocks are continuously distributed in our model, whereas in Woodford's model, the high and low states in the agents' productivity alternate in each time period. Moreover, our research considers both *crowd-in* and *crowd-out* effects together. Our theoretical finding regarding the inverted U-shaped relationship is novel to the existing literature. This result agrees not only with time series observations of each individual country but also with cross-country observations because the position of the long-run steady state cutoff is determined by fiscal policy rules in our model.

As public debt has both *crowd-in* and *crowd-out* effects, it is not neutral to macroeconomic variables, namely, Ricardian equivalence fails in our model.[13] Of course, the failure of Ricardian equivalence originates in financial market imperfections. Without financial market imperfections, our model returns to a Ramsey-type one sector growth model in which Ricardian equivalence holds. Financial market imperfections lead to a decrease in the demand for borrowing relative to an economy with a perfect financial market, and the decreased demand for borrowing induces a lower equilibrium interest rate than in an economy with a perfect financial market. In such a situation, public debt is a beneficial vehicle for unproductive agents to save as discussed in subsection 4.1. Although the issuance of public debt *crowds out* private investment, the increased interest rate causes the *crowd-in* effect. Whether the growth rate increases or decreases depends on which effect dominates.

Appendix

Derivation of eqs. (6) and (8)

At time $t-1$, an agent already knows information about her productivity at time t, which is $A\Phi_{t-1}(\omega_{t-1})$. From eqs.(4) and (5), the lending-investment-borrowing program at time $t-1$ of an agent with $\Phi_{t-1}(\omega_{t-1}) > \phi_{t-1} = r_t/A$ is given such that $b_{t-1}(\omega^{t-1}) = -\mu k_{t-1}(\omega^{t-1})$ and $k_{t-1}(\omega^{t-1}) = a_{t-1}(\omega^{t-1})/(1-\mu)$. Her budget constraint at time t therefore is given by:

$$k_t(\omega^t) + b_t(\omega^t) = (A\Phi_{t-1}(\omega_{t-1}) - r_t\mu)k_{t-1}(\omega^{t-1})(1-\tau_t) - c_t(\omega^t),$$

[13]See Barro (1974) for the neutrality of public debt to economic activities.

or equivalently,

$$a_t(\omega^t) = \frac{A\Phi_{t-1}(\omega_{t-1}) - r_t\mu}{1-\mu} a_{t-1}(\omega^{t-1})(1-\tau_t) - c_t(\omega^t). \tag{A.1}$$

Similarly, from eqs.(4) and (5), the lending-investment-borrowing program at time $t-1$ of an agent with $\Phi_{t-1}(\omega_{t-1}) \leq \phi_{t-1} = r_t/A$ is given such that $b_{t-1}(\omega^{t-1}) = a_{t-1}(\omega^{t-1})$ and $k_{t-1}(\omega^{t-1}) = 0$. Her budget constraint at time t therefore is given by:

$$k_t(\omega^t) + b_t(\omega^t) = r_t b_{t-1}(\omega^{t-1})(1-\tau_t) - c_t(\omega^t),$$

or equivalently,

$$a_t(\omega^t) = r_t a_{t-1}(\omega^{t-1})(1-\tau_t) - c_t(\omega^t). \tag{A.2}$$

From eqs.(A.1) and (A.2), the flow budget constraints for $s \geq t$ are given by Eq.(6).

It follows from the flow budget constraint (6) that:

$$E\left[\frac{a_{t+1}(\omega^{t+1})}{c_{t+1}(\omega^{t+1})}\middle|\Phi^t(\omega^t)\right] = a_t(\omega^t) E\left[\frac{\tilde{R}_{t+1}(1-\tau_{t+1})}{c_{t+1}(\omega^{t+1})}\middle|\Phi^t(\omega^t)\right] - 1. \tag{A.3}$$

Substituting Eq.(7) into Eq.(A.3) yields:

$$\frac{a_t(\omega^t)}{c_t(\omega^t)} = \beta E\left[\frac{a_{t+1}(\omega^{t+1})}{c_{t+1}(\omega^{t+1})}\middle|\Phi^t(\omega^t)\right] + \beta.$$

From this equation and the law of iterated expectations, we obtain:

$$\frac{a_t(\omega^t)}{c_t(\omega^t)} = \beta^\tau E\left[\frac{a_{t+\tau}(\omega^{t+\tau})}{c_{t+\tau}(\omega^{t+\tau})}\middle|\Phi^t(\omega^t)\right] + \beta + \beta^2 + ... + \beta^\tau.$$

From the transversality condition, it follows that $\lim_{\tau\to\infty} \beta^\tau E[a_{t+\tau}(\omega^{t+\tau})/c_{t+\tau}(\omega^{t+\tau})|\Phi^t(\omega^t)] = 0$. Then, $a_t(\omega^t)/c_t(\omega^t) = \beta/(1-\beta)$ for all $t \geq 0$ and thus $a_{t+1}(\omega^{t+1}) = \beta\tilde{R}_{t+1}(1-\tau_{t+1})a_t(\omega^t)$ from Eq.(6). This is Eq.(8). □

Proof of Proposition 2

Define a function of $J(\phi_t)$ such that $J(\phi_t) := \beta F(\phi_t)/[1-\mu-\beta(G(\phi_t)-\mu)] = Y_{t+1}/(Y_t A(1-\theta))$. Then, we have $J'(\phi)[1-\mu-\beta(G(\phi)-\mu)]^2/\beta G'(\phi) = \beta F(\phi) - \phi(1-\mu-\beta(G(\phi)-\mu)) =: I(\phi)$. Because $I(\phi)$ is a decreasing function and because $I(0) > 0$ and $I(h) < 0$, ϕ^* is uniquely determined such that if $\phi_t < \phi^*$, then $J'(\phi_t) > 0$ and if $\phi_t > \phi^*$, then $J'(\phi_t) < 0$. The highest growth rate of Y_{t+1}/Y_t, therefore, is achieved at ϕ^*. Moreover, it follows from Assumption 1 that $J'(G^{-1}(\mu)) > 0$ and thus $\phi^* \in [G^{-1}(\mu), h]$. □

Proof of Proposition 3

Because B_t/Y_t has a one-to-one, monotonic relationship with the cutoff ϕ_t, it follows from propositions 1 and 2 that the highest growth rate is achieved when $B_t/Y_t = (B/Y)^*$. To prove that the maximum of B_t/Y_t is $\beta(1-\theta)/(1-\beta)$, we consider the case in which $\phi_t = h$. In this case, $G(\phi_t) = G(h) = 1$, and thus, $B_t/Y_t = \beta(1-\theta)/(1-\beta)$ from Eq.(20). □

Proof of Proposition 4

$\Psi'(\bar{\phi}) > \Lambda'(\bar{\phi})$ holds because both $\Psi(\phi_t)$ and $\Lambda(\phi_{t-1})$ are monotonically increasing and $\Psi(h) > \Lambda(h)$ and $\Psi(G^{-1}(\mu)) < \Lambda(G^{-1}(\mu))$. Therefore, the unique steady state $\bar{\phi}$ is locally stable. For $\phi_{t-1} \in [G^{-1}(\mu), \bar{\phi})$, it follows that $\Psi(\phi_{t-1}) < \Lambda(\phi_{t-1}) = \Psi(\phi_t)$, and thus, $\phi_t > \phi_{t-1}$, whereas for $\phi_{t-1} \in (\bar{\phi}, h]$, it follows that $\Psi(\phi_{t-1}) > \Lambda(\phi_{t-1}) = \Psi(\phi_t)$, and thus, $\phi_t < \phi_{t-1}$. Therefore, the steady state $\bar{\phi}$ is globally stable. □

Proof of Lemma 1

From the definitions of $\Psi(\phi_t)$ and $\Lambda(\phi_{t-1})$, $\Psi(G^{-1}(\mu)) < \Lambda(G^{-1}(\mu))$ is obvious. Regarding $\Psi(h) > \Lambda(h)$, it follows from $\lim_{\phi \to h} \tau'(\phi)/(\phi G'(\phi)) = a$, $\tau(h) = 1$, and L'Hospital's rule that $\lim_{\phi \to h}(1-\tau(\phi))/F(\phi) = a$, and thus, $\lim_{\phi \to h} \Lambda(\phi) = ah(1-\mu) - (1-\theta)$. Being continuous at $\phi = h$, $\Lambda(h) = ah(1-\mu) - (1-\theta)$ holds. $\Psi(h) > \Lambda(h)$ then holds from Assumption 3. □

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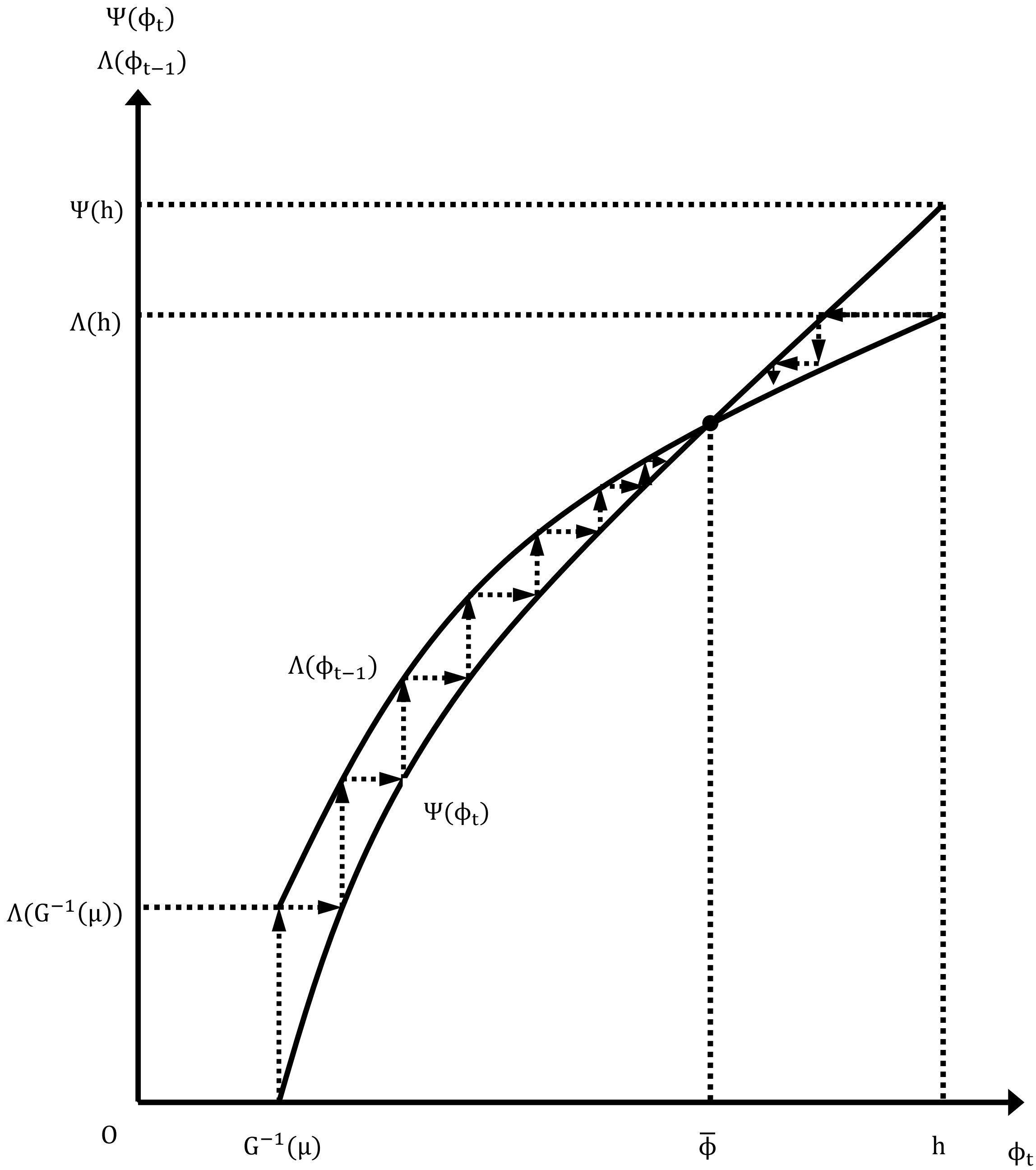

Fig. 1: Dynamic behavior of the cutoff

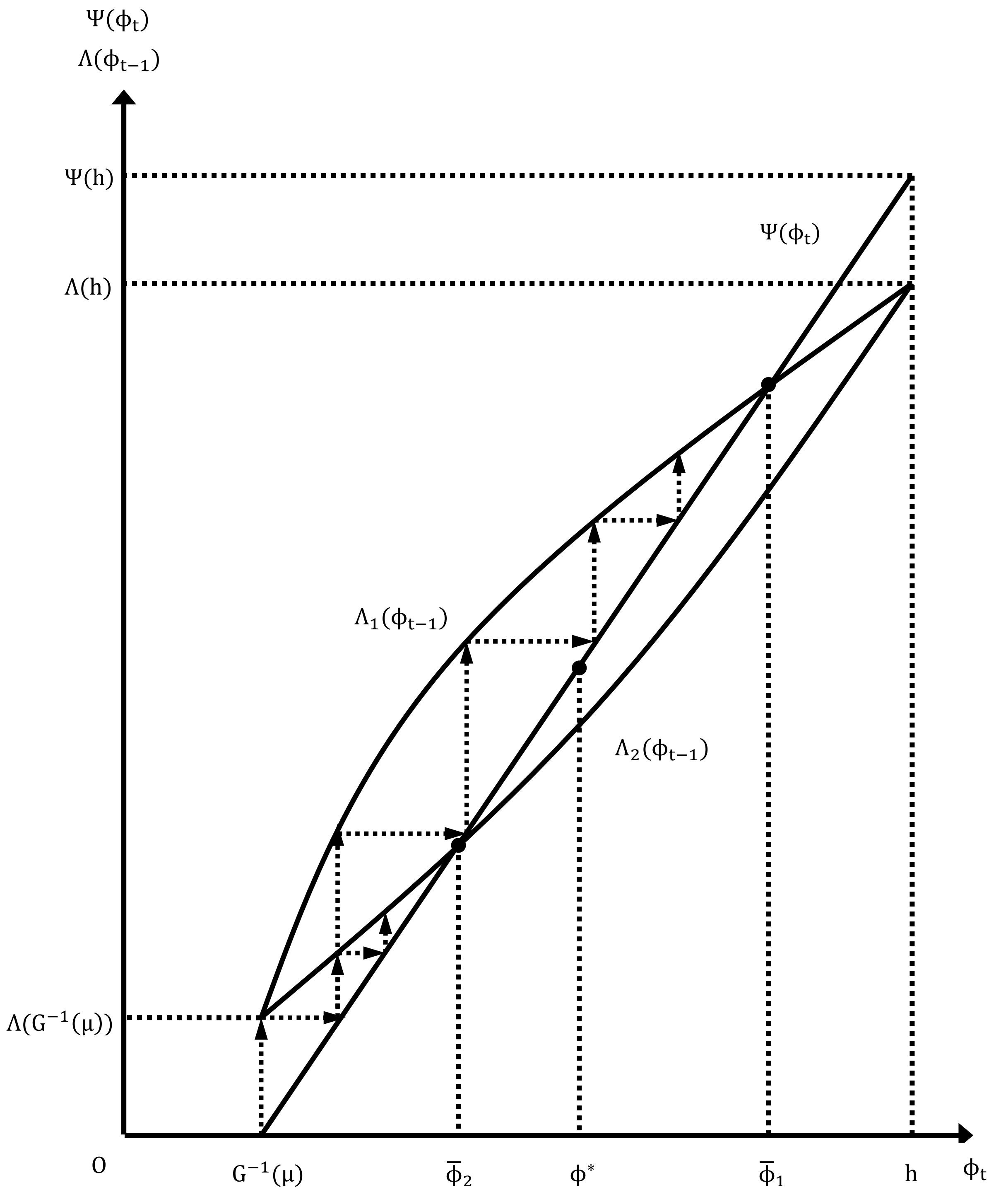

Fig. 2: Dynamic behavior of the cutoff for the two fiscal policy rules