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"Relative Risk Aversion and Business Fluctuations"

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Relative Risk Aversion and Business Fluctuations

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Abstract

By applying a simple dynamic general equilibrium model without exogenous shocks inhabited by infinitely lived capitalists and workers, we show that a higher degree of relative risk aversion can destabilize an economy. In traditional real business cycle (RBC) theory, a higher degree of relative risk aversion dampens the amplitude of the consumption fluctuations caused by exogenous shocks through consumption smoothing. However, a higher degree of relative risk aversion combined with a high degree of elasticity of the marginal product of capital can also lead to the emergence of a nonlinear mechanism that causes endogenous business fluctuations. The nontrivial steady state loses stability due to the higher degree of relative risk aversion; thus, endogenous business fluctuations can occur. This result suggests that for a deeper understanding of boom-bust cycles, researchers should merge exogenous and endogenous business fluctuations when investigating economies.

Keywords: endogenous business fluctuations, relative risk aversion, dynamic general equilibrium, instability.

JEL Classification Numbers: E1, E2, E3.

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1 Introduction

In real business cycle (RBC) theory pioneered by Kydland and Prescott (1982), the origin of business fluctuations is exogenous shocks. In theory, a higher degree of relative risk aversion in consumer preferences dampens the amplitude of the consumption fluctuations caused by exogenous shocks through consumption smoothing.[1] In calibration analyses of dynamic stochastic general equilibrium (DSGE) models, consumers are often assumed to be endowed with the constant-relative-risk-aversion (CRRA) utility function. In the Euler equation, a higher degree of relative risk aversion activates consumption smoothing, mitigating the impact of exogenous shocks on consumption volatility. Figure 1 in Havranek et al. (2015) illustrates the simulated impulse responses of changes in consumption and investment to an increase in the monetary policy rate, which shows that those economic fluctuations caused by the monetary shock diminish as the degrees of elasticity of intertemporal substitution (EIS) become lower (or equivalently, degrees of relative risk aversion become high with the CRRA utility function).

Given the theoretical insights from modern macroeconomic theory, we expect the degree of relative risk aversion to mitigate economic volatility. We then regress the variance of consumption growth on the squared degree of relative risk aversion and the variance of interest rates (logarithms all) using cross-country data.[2] The result of ordinary least squares (OLS) regression is as follows:

$$\log(vac_i) = \underset{\substack{(0.160)\\ [2.127,\, 2.777]\\ [2.121,\, 2.784]}}{2.452} \; \underset{\substack{(0.088)\\ [0.177,\, 0.536]\\ [0.181,\, 0.532]}}{+0.357} \times \log(vai_i) \; \underset{\substack{(0.045)\\ [-0.096,\, 0.088]\\ [-0.103,\, 0.095]}}{-0.003} \times \log(\gamma_i^2) + \nu_i,$$

where vac_i is the variance of country i's consumption growth, vac_i is the variance of the interest rate, γ_i is the degree of relative risk aversion, and ν_i is the error term.

[1]Concerning consumption smoothing, Barro and Sala-i-Martin's (2004, Figure 2.2 in Ch.2) textbook illustrates that the degree of risk aversion affects the slope of the equilibrium path.

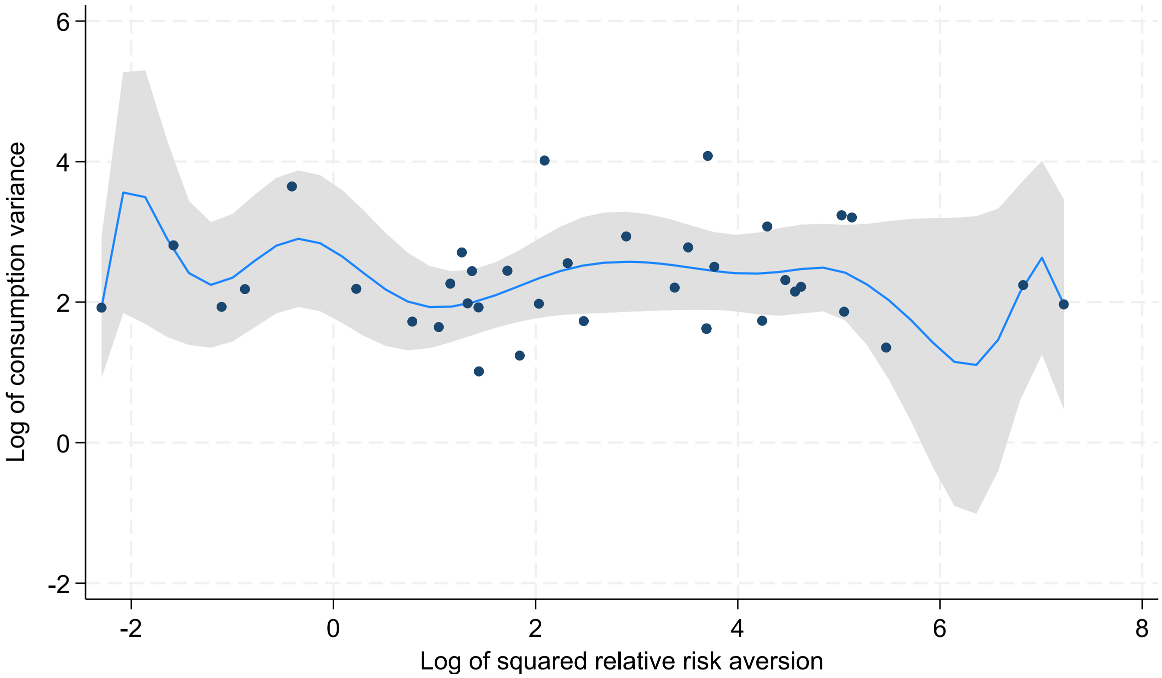
[2]Many empirical studies produce evidence that the degree of relative risk aversion differs across countries (Chiappori and Paiella, 2011; Gandelman and Hernandez-Murillo, 2014; Havranek et al., 2015; Banerjee, 2020). For our regression, we prepare data on the degree of relative risk aversion computed from the data for the EIS obtained from Havranek et al. (2015). We collect macroeconomic data from Penn World Table, version 10.01 (PWT 10.01; Feenstra et al., 2015). The data description and the estimation equation specified by the Euler equation are provided in the Appendix.

The figures in parentheses are standard errors. Those figures inside the brackets in the upper row indicate the 95% confidence intervals from the OLS estimation, and those in the lower row indicate the intervals from bootstrapping. The effect of the variance of the interest rate on the variance of consumption growth is positive and significant, which is consistent with theory. The point estimate of the coefficient of $\log(\gamma_i^2)$ is -0.003, the sign of which agrees with theory. However, this estimate is insignificant, and its absolute impact is too small relative to theory because we expect the coefficient to be -1 according to the Euler equation. Furthermore, Figure 1 provides the result of the semiparametric estimation by replacing $\log(\gamma_i^2)$ in the linear estimation equation with an unspecified function of $\log(\gamma_i^2)$. As seen in the figure, there is no negative relationship between $\log(\gamma_i^2)$ and $\log(vac_i)$, whereas the impact of the linear term of $\log(vai_i)$ remains significant (although we do not report it here).

These puzzling empirical observations lead us to the following natural questions: Do other mechanisms produce business fluctuations in economies? Does a higher degree of relative risk aversion only play a role in consumption smoothing? Is there not the possibility of a higher degree of relative risk aversion destabilizing an economy? To answer these questions, we presumably consider that an economy's nonlinearity originating from a higher degree of relative risk aversion causes business fluctuations. If the effect of nonlinearity outweighs consumption smoothing, then business fluctuations may be more amplified. Thus far, the literature has not sufficiently investigated the mechanism produced by an economy's nonlinearity resulting from a higher degree of relative risk aversion. In this work, we explore this mechanism.

Traditionally, there are two kinds of microfounded theoretical models for explaining business cycles in macroeconomics. One is based on the model of Kydland and Prescott (1982), followed by subsequent vast literature. We may refer to it as RBC theory, but more widely, this can be referred to as the DSGE model because it involves both monetary and nonmonetary components. In this context, researchers have focused on how exogenous shocks propagate across the economy, thus driving business cycles. For example, regarding the relation of shock-driven business cycles to the degree of risk aversion, Chen et al. (2020) presented the DSGE framework to investigate economic fluctuations with state-dependent risk aversion. The other

Figure 1: Consumption instability versus degree of relative risk aversion



Notes. This graph shows the result of the semiparametric estimation of the log of the variance of consumption growth regressed on the log of the squared degree of relative risk aversion. The shaded area indicates the 95% confidence interval given at each squared degree of relative risk aversion. The number of sample countries is 38. The data for the degree of relative risk aversion are assembled from Havranek et al. (2015). We collected macroeconomic data from Penn World Table, version 10.01 (PWT 10.01; Feenstra et al., 2015). See the Appendix for the data description and estimation specification.

originates from the models of Benhabib and Nishimura (1979,1985) and Grandmont (1985), again followed by subsequent vast literature. This strand stresses that the internal force of an economy generates business cycles endogenously. Even though shocks are absent in an economy, business cycles can arise because of an economy's nonlinearity in equilibrium. Beaudry et al. (2015, 2017, 2020) provided supportive evidence of endogenous business cycles. Our study belongs to the second abovementioned stream. We aim to investigate how a higher degree of relative risk aversion causes the nonlinearity of an economy and generates endogenous business cycles. Standard RBC theory ignores this mechanism.

We apply a simple dynamic general equilibrium model without any exogenous shocks inhabited by infinitely lived capitalists and workers, which is otherwise fairly similar to the standard Ramsey model. In each period, capitalists and workers earn capital and wage income, respectively. Workers consume all their income in each period: they are hand-to-mouth consumers. Capitalists optimally consume or save their income in each period, and their savings become capital stock due to the assumption of a closed economy. In equilibrium, we derive the two-dimensional dynamical system with respect to the consumption-to-capital-income ratio (i.e., capitalists' marginal propensity to consume) and capital stock. We obtain a unique nontrivial steady state. Given the high level of elasticity of the marginal product of capital, if the degree of relative risk aversion is also high, then the steady state loses stability, and endogenous business fluctuations can appear. Intuitively, if an investment boom occurs with a given capital income in a certain period, say, period t, the consumption-to-capital-income ratio decreases in this period. Since the interest rate becomes lower under a high level of elasticity of the marginal product of capital, capital income decreases in period $t + 1$. In this situation, if the degree of relative risk aversion is low, the consumption-to-capital-income ratio does not change significantly or even decreases because consumption smoothing does not work sufficiently, with consumption decreasing greatly in response to the decrease in capital income. The invariability of the consumption-to-capital-income ratio mitigates the negative impact of a reduction in capital income on investment in period $t + 1$. In such a case, endogenous business fluctuations do not occur. However, when the degree of relative risk aversion is high, consumption smoothing works sufficiently; thus, the consumption-to-capital-income ratio becomes large (because the denominator becomes small). An increased consumption-to-capital-income ratio increases the negative impact of decreased capital income on investment in period $t+1$. In this case, endogenous business fluctuations occur. The unique characteristic of this mechanism is that consumption smoothing creates the nonlinearity that generates endogenous fluctuations.

The current paper belongs to the literature on (deterministic) endogenous business cycles in the dynamic general equilibrium model with infinitely lived agents; however, it is too exhaustive to list all the papers because many researchers have addressed this topic over the past forty years. For example, Woodford (1989) developed a model with infinitely lived capitalists and workers and derived conditions under which endogenous business cycles can occur. Moreover, Hashimoto et al. (2022) considered an economy with capitalists and workers and demonstrated that endogenous business cycles occur as credit constraints are relaxed. Although Benhabib and Nishimura (1979, 1985), Boldrin and Deneckere (1990), and Nishimura and Yano (1995) also employed models of infinitely lived agents to obtain endogenous business cycles, they assumed two production sectors. Additionally, Aghion et al. (2004), Pintus (2011), and Kunieda and Shibata (2017) employed models of infinitely lived agents but studied economies with financial frictions to derive endogenous business cycles. In contrast to the current paper, these studies do not analyze the nonlinearity created by risk aversion, which generates endogenous business fluctuations. The following overlapping generations models have studied the endogenous business cycles caused by a high degree of relative risk aversion: Benhabib and Day (1982), Bertocchi and Wang (1995), Grandmont (1985), and Reichlin (1992). However, these works did not consider a production sector with capital stock, assuming a pure-exchange economy, and derived oscillations with money.[3] Our model assumes a production economy with capital stock but without money by employing a Ramsey-type model of infinitely lived agents. By doing so, we can elucidate the unidentified role of the degree of relative risk aversion, directly compare it to RBC models, and stimulate a discussion on the deficit of those models.

The remainder of the current paper is organized as follows. In the next section, we present the basic structure of the model. In section 3, we derive a dynamical system in equilibrium, and in section 4, we investigate period-two cycles. Section 5 concludes the paper. In the Online Appendix, bifurcation diagrams present the concrete occurrence of endogenous business cycles.

2 Model

The economy consists of infinitely lived firms and two types of infinitely lived agents: capitalists and workers. Time is discrete and indexed by t, ranging from $t = 0$ to ∞. In each period, capitalists lend their physical assets to firms and earn capital

[3]Many other studies also employed overlapping generations models, such as Farmer (1986), Reichlin (1986), Benhabib and Laroque (1988), Grandmont et al. (1998), Rochon and Polemarchakis (2006), and Yokoo (2000). Although these studies introduced production sectors, they still derived oscillations with money.

income. Workers supply labor inelastically to firms and receive wage income. Firms, capitalists, and workers are homogeneous, and we then consider the representative for each entity. The populations of firms and capitalists are one, and that of workers is L.

2.1 Capitalist

In each period, the representative capitalist obtains its utility from consumption c_t with the CRRA utility function $u(c_t)$, which is given by

$$u(c_t) = \begin{cases} \frac{c_t^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \text{ and } \gamma > 0 \\ \log c_t & \text{if } \gamma = 1, \end{cases}$$

where γ represents the degree of relative risk aversion. The capitalist maximizes lifetime utility

$$U_t := \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to

$$a_{t+1} + c_t = r_t a_t, \tag{1}$$

where $\beta \in (0, 1)$ is the capitalist's subjective discount factor, a_t denotes the asset holdings in period t, and r_t is the gross market interest rate. The initial budget constraint is given by $a_1 + c_0 = \tilde{a}_0$, where $\tilde{a}_0 > 0$ is initial wealth. For exposition, we let $\tilde{a}_0 = r_0 a_0$, although the interest rate is determined from $t = 1$ onward. The necessary and sufficient conditions for the optimality of the lifetime utility maximization problem consist of the Euler equation,

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta r_{t+1}, \tag{2}$$

and the transversality condition,

$$\lim_{t\to\infty} \beta^t \left(\frac{a_{t+1}}{c_t^{\gamma}}\right) = 0. \tag{3}$$

The ratio of consumption to capital income is defined as $\theta_t := c_t/(r_t a_t)$. Then, Eqs. (1) and (2), respectively, yield the following dynamic equations:

$$\theta_{t+1} = \frac{\beta^{\frac{1}{\gamma}} r_{t+1}^{\frac{1}{\gamma}-1}}{1-\theta_t}\theta_t, \tag{4}$$

$$a_{t+1} = (1-\theta_t) r_t a_t, \tag{5}$$

where $1-\theta_t$ captures the marginal propensity to save.

2.2 Workers

In each period, each worker earns wage income w_t. Workers are hand-to-mouth consumers and consume their wage income entirely in each period.[4] Since the population of workers is L, workers' total consumption C_t^w is given by

$$C_t^w = w_t L. \tag{6}$$

2.3 Production

The representative firm produces general goods from capital and labor with production technology: $Y_t = F(K_t, N_t)$, where Y_t is the total output. K_t and N_t are capital and labor inputs, respectively, where capital depreciates entirely in one period. Let us assume that $F(K_t, N_t)$ is standard neoclassical technology—continuous, constant returns to scale with respect to K_t and N_t and positive and diminishing marginal product with respect to both inputs. Additionally, $F(0, N_t) = F(K_t, 0) = 0$ holds. Let us define per worker capital by $k_t := K_t/N_t$. Then, it follows that $f''(k_t) < 0 < f'(k_t)$ and $f(0) = 0$, where $f(k_t) := F(k_t, 1)$. The capital and labor markets are competitive. Therefore, we obtain the following conditions from the firm's profit maximization problem:

$$r_t = f'(k_t) \tag{7}$$

and

$$w_t = f(k_t) - f'(k_t)k_t. \tag{8}$$

[4] The empirical evidence obtained by King and Leape (1998) and Guiso et al. (2003) strongly supports the existence of hand-to-mouth consumers.

3 Equilibrium dynamics

The competitive equilibrium is defined as trajectories of quantities $\{c_t, a_t, \theta_t, K_t, C_t^w, N_t, Y_t\}$ and prices $\{r_t, w_t\}$, such that (i) the representative capitalist solves the lifetime utility maximization problem so that Eqs. (4) and (5) hold, (ii) the representative firm solves the profit maximization problem so that Eqs. (7) and (8) hold, (iii) the representative worker consumes the wage income entirely in each period so that Eq. (6) holds, and (iv) all markets clear.

From the labor and capital market clearing conditions, it follows that $L = N_t$ and $a_t = K_t$. Therefore, from Eqs. (4), (5), (7), and $a_t = K_t = k_t L$, the dynamical system with respect to (k_t, θ_t) is characterized by

$$k_{t+1} = (1 - \theta_t) f'(k_t) k_t =: \Gamma(k_t, \theta_t) \tag{9}$$

and

$$\theta_{t+1} = \frac{\beta^{\frac{1}{\gamma}} f'(\Gamma(k_t, \theta_t))^{\frac{1}{\gamma}-1}}{1 - \theta_t} \theta_t =: H(k_t, \theta_t). \tag{10}$$

With the initial condition $k_0 > 0$ given, Eqs. (9) and (10) obtain equilibrium sequences $\{k_{t+1}, \theta_t\}_{t=0}^{\infty}$, where $(k_t, \theta_t) \in [0, \infty) \times [0, 1]$ for all t. When $\gamma = 1$, the transversality condition (3) has the above two dynamic equations reduced to a single dynamic equation, $k_{t+1} = \beta f'(k_t) k_t$.

3.1 Steady state

Define (k, θ) as the steady-state values of k_t and θ_t. Then, from Eqs.(9) and (10), we have

$$\beta f'(k) = 1, \tag{11}$$

and

$$\theta = 1 - \beta.$$

To ensure the existence of the steady state, we impose the following assumption:

Assumption 1. $\lim_{k_t \to \infty} f'(k_t) < 1/\beta < \lim_{k_t \to 0} f'(k_t)$.

Assumption 1 and $f''(k_t) < 0$ guarantee the existence and uniqueness of the steady-state value of k.

3.2 Local dynamics

Linearizing Eqs. (9) and (10) around the steady state (k, θ) yields the local dynamical system as follows:

$$\begin{pmatrix} k_{t+1} - k \\ \theta_{t+1} - \theta \end{pmatrix}$$
$$= \begin{pmatrix} 1 - \eta(k) & -\frac{k}{\beta} \\ -(1-\beta)(\frac{1}{\gamma} - 1)\,(1-\eta(k))\,\frac{\eta(k)}{k} & \frac{1}{\beta}\Big(1 + (1-\beta)(\frac{1}{\gamma} - 1)\eta(k)\Big) \end{pmatrix} \begin{pmatrix} k_t - k \\ \theta_t - \theta \end{pmatrix}, \tag{12}$$

where $\eta(k) := -f''(k)k/f'(k) > 0$ is an elasticity of the marginal product of capital in the steady state. Then, it follows from Eq.(11) that $\eta(k) = -\beta f''(k)k$. Therefore, the value of $\eta(k)$ depends on not only the configuration of the production function but also the capitalist's subjective discount factor β. Eq.(11) can rewrite $\eta(k)$ as a function of β, but we use $\eta(k)$ throughout the analysis for convenience.

Let λ_1 and $\lambda_2 (< \lambda_1)$ be the eigenvalues of the Jacobian matrix in Eq. (12). Then, λ_1 and λ_2 are solutions of the characteristic equation given by

$$\Xi(\lambda) := \lambda^2 - T\lambda + D = 0, \tag{13}$$

where T and D are the trace and determinant, respectively, of the Jacobian matrix, where

$$T := 1 - \eta(k) + \frac{1}{\beta}\left(1 + (1-\beta)\left(\frac{1}{\gamma} - 1\right)\eta(k)\right) \tag{14}$$

and

$$D := \frac{1 - \eta(k)}{\beta}. \tag{15}$$

Lemma 1. *Both λ_1 and λ_2 are real valued. Furthermore, it holds that $\lambda_2 < 1 < \lambda_1$.*

Proof. The claim of Lemma 1 follows from $\Xi(1) = -(1-\beta)\eta(k)/(\beta\gamma) < 0$. □

Lemma 1 implies that the local stability of the steady state of the dynamical system depends on whether the value of λ_2 is less than -1.

Proposition 1. *The steady state of the dynamical system given by Eqs.* (9) *and* (10) *exhibits the following properties:*

(i) *If* $1+\beta+[(1-\beta)/(2\gamma)-1]\eta(k)>0$*, the steady state* (k,θ) *is a saddle point.*

(ii) *If* $1+\beta+[(1-\beta)/(2\gamma)-1]\eta(k)<0$*, the steady state* (k,θ) *is unstable.*

Proof. From Eqs. (13)-(15), we have $\Xi(-1)=(2/\beta)[1+\beta+\{(1-\beta)/(2\gamma)-1\}\eta(k)]$. If $\Xi(-1)>(<)0$, it follows that $-1<\lambda_2<1$ ($\lambda_2<-1$). Then, the claims of Proposition 1 hold. □

When $\gamma=1$ (log preferences), the condition for the steady state to be unstable is $\eta(k)>2$. Woodford (1989) and Hashimoto et al. (2022) investigate this case and derive endogenous business cycles.

3.3 Local stability

The combination of $\eta(k)$ and γ determines the local stability of the steady state, with their values being independent of one another. Remark 1 below, which immediately follows from Proposition 1, is helpful for understanding what value of λ_2 is obtained under what conditions.

Remark 1. *Define* $\Omega:=\gamma(1+\beta)/[\gamma-(1-\beta)/2]$*. Then, the following hold:*

(i) *If* $0<\eta(k)<1$ *holds, the first claim of Proposition 1 holds (i.e., the steady state* (k,θ) *is a saddle point). In this case, it follows that* $\Xi(0)=D>0$*, and thus,* $0<\lambda_2<1$*.*

(ii) *If* $\gamma\le(1-\beta)/2$*, the first claim of Proposition 1 holds, regardless of the value of* $\eta(k)$ *(i.e., the steady state* (k,θ) *is a saddle point), but in particular, if* $\eta(k)>1$ *under the same condition, it follows that* $\Xi(0)<0<\Xi(-1)$*, and thus,* $-1<\lambda_2<0$*.*

(iii) *If* $\gamma>(1-\beta)/2$ *and* $1<\eta(k)<\Omega$*, the first claim of Proposition 1 holds (i.e., the steady state* (k,θ) *is a saddle point). In this case, it follows that* $\Xi(0)<0<\Xi(-1)$*, and thus,* $-1<\lambda_2<0$*.*

(iv) *If* $\gamma>(1-\beta)/2$ *and* $\Omega<\eta(k)<\infty$*, the second claim of Proposition 1 holds (i.e., the steady state,* (k,θ)*, is unstable). In this case, it follows that* $\lambda_2<-1$*.*

Regardless of the stability of the steady state, equilibrium in this economy is on a one-dimensional invariant manifold that is nonlinear globally but associated with λ_2 in the vicinity of the steady state (henceforth referred to as "equilibrium manifold" unless stated otherwise). The reason for this is the transversality condition, which degenerates the effect of $\lambda_1 > 1$ by letting θ_0 on it with k_0 given. In particular, if the steady state is a saddle point, equilibrium is uniquely determined on the saddle path and converges to the steady state. If the production technology is of the Cobb-Douglas type, we have $0 < \eta(k) < 1$. Then, Remark 1-(i) implies that the Cobb-Douglas production technology shapes the steady state into a saddle point. If $-1 < \lambda_2 < 0$ holds as in Remarks 1-(ii) and (iii), the equilibrium converges to the steady state while oscillating over the steady state. If the steady state is unstable, as shown in Remark 1-(iv), endogenous business cycles can occur in equilibrium. We elaborate on this point in the numerical analysis in section 4.

In the cases of Remarks 1-(ii), (iii), and (iv), the equilibrium oscillates, sliding over the steady state because $\lambda_2 < 0$, regardless of the convergence property of the equilibrium. The intuition behind why the equilibrium oscillates is as follows. Capitalists' income is $f'(k_t)k_t$, and they accumulate capital by using part of their income. Whether capitalists' income increases or decreases with capital stock k_t depends on the level of capital stock. An increase in the amount of capital has two conflicting effects on capital income. The increase in k_t directly increases the source of capital income, which is k_t, but the increased capital decreases the marginal product of capital, $r_t = f'(k_t)$, which places reducing pressure on capital income. The latter negative effects dominate the former positive effects if $\eta(k_t) > 1$ holds because

$$\frac{d[f'(k_t)k_t]}{dk_t} = (1 - \eta(k_t))\, f'(k_t) < 0 \quad \text{if} \quad \eta(k_t) > 1. \tag{16}$$

Consider the vicinity of the steady state. Eq.(16) implies that increased (decreased) capital causes lower (higher) capital income when $\eta(k) > 1$. The low (high) level of capital income means that capitalists accumulate less (more) capital in the next period, which yields oscillations around the steady state. When the coefficient of relative risk aversion is sufficiently large, λ_2 decreases (i.e., the absolute value of λ_2 becomes greater) as $\eta(k)$ becomes large. The reason for this is that if the coefficient of relative risk aversion is sufficiently large, consumption smoothing works sufficiently, and accordingly, the consumption-to-capital-income ratio becomes large (small) when capital income changes to a low (high) level. The increased (decreased) consumption-to-capital-income ratio increases the negative (positive) impact of decreased (increased) capital income on capital accumulation. As such, if $\eta(k)$ becomes large, with the coefficient of relative risk aversion being sufficiently large, then the amplification of capital oscillations becomes larger. Eventually, when $\eta(k)$ is sufficiently high such that $1 < \Omega < \eta(k)$, the steady state becomes unstable, and endogenous business fluctuations can occur.

3.4 Flip bifurcation with respect to γ

Since the value of γ has no effect on $\eta(k)$, in turn, Proposition 1 implies that when $\eta(k)$ is sufficiently large, the size of γ alters the stability of the steady state. More concretely, Remark 2 below immediately follows from Proposition 1.

Remark 2. *Define* $\bar{\gamma} := (1-\beta)/\left[2\{1-(1+\beta)/\eta(k)\}\right]$ *and suppose that* $\eta(k) > 1+\beta$. *Then, the following hold:*

(i) If $0 < \gamma < \bar{\gamma}$, *the steady state is a saddle point.*

(ii) If $\bar{\gamma} < \gamma$, *the steady state is unstable.*

We clarify that since $\eta(k) > 1+\beta$, it follows that $\Xi(0) < 0$. Therefore, the value of λ_2 is $-1 < \lambda_2 < 0$ if the condition of Remark 2-(i) holds or $\lambda_2 < -1$ if the condition of Remark 2-(ii) holds. In particular, if $\gamma = \bar{\gamma}$, we have $\lambda_2 = -1$.

Our interest is in the effect of γ, which is the degree of relative risk aversion, on the dynamic property in equilibrium. Again, equilibrium is on the invariant manifold that is nonlinear globally but associated with λ_2 in the vicinity of the steady state. Although it is difficult to investigate equilibrium on the equilibrium manifold analytically because it is nonlinear globally, we can analyze the local dynamics. More concretely, from Eqs.(9) and (10) and the transversality condition, we are certain that there is a policy function, $\theta_t = \theta(k_t)$, which is assumed to be differentiable at least three times. The policy function $\theta(k_t)$ lets the economy on the equilibrium manifold. Inserting this function into Eq.(9) yields

$$k_{t+1} = (1-\theta(k_t))f'(k_t)k_t =: \tilde{\Gamma}(k_t;\gamma), \tag{17}$$

where k_{t+1} is a function of k_t, which displays the equilibrium manifold. The configuration of the right-hand side of Eq. (17) is affected by γ, which is our parameter of interest, with other parameters (including the characteristics of the production function) remaining unchanged. The linearization of Eq.(17) around the steady state is given by

$$k_{t+1} - k = \frac{\partial \tilde{\Gamma}(k;\gamma)}{\partial k_t}(k_t - k),$$

where $\partial \tilde{\Gamma}(k;\gamma)/\partial k = \lambda_2(k;\gamma)$. Let us define $\tilde{\Gamma}^2 = \tilde{\Gamma} \circ \tilde{\Gamma}$. We consider a supercritical flip bifurcation with respect to γ that occurs at $\gamma = \bar{\gamma}$, in which (i) if $0 < \gamma < \bar{\gamma}$, the dynamical system given by Eq.(17) has a unique steady state that is asymptotically stable and (ii) if $\bar{\gamma} < \gamma$, the dynamical system has an unstable steady state and a period-two cycle exists in the neighborhood of the steady state, which is asymptotically stable. Because $\lambda_2(k;\bar{\gamma}) = -1$, the two other conditions for a supercritical flip bifurcation to occur are $\partial \lambda_2(k;\bar{\gamma})/\partial \gamma < 0$ and $\partial^3 \tilde{\Gamma}^2(k;\bar{\gamma})/\partial k^3 < 0$ (Grandmont, 2008). Then, it follows from Eqs. (13), (14), and (15) that

$$\frac{\partial \lambda_2(k;\gamma)}{\partial \gamma} = \frac{1}{2}\left(1 - \frac{T}{\sqrt{T^2 - 4D}}\right)\left(\frac{\partial T}{\partial \gamma}\right). \tag{18}$$

The inside of the first parentheses on the right-hand side of Eq.(18) is positive, and $\partial T/\partial \gamma$ is negative. Hence, it holds that $\partial \lambda_2(k;\bar{\gamma})/\partial \gamma < 0$.

It is difficult to directly investigate whether $\partial^3 \tilde{\Gamma}^2(k;\bar{\gamma})/\partial k^3 < 0$ holds. The reason for this is that the sign of $\partial^3 \tilde{\Gamma}^2(k;\bar{\gamma})/\partial k^3$ depends on not only the explicit parameter values but also the configuration of the production function. Even though we specify the production function, analyzing the sign of $\partial^3 \tilde{\Gamma}^2(k;\bar{\gamma})/\partial k^3$ is still complicated because it requires us to compute $\partial^3 \tilde{\Gamma}^2(k;\bar{\gamma})/\partial k^3$ in the steady state of the dynamical system given by Eq. (17), while $\tilde{\Gamma}^2$ is the twice iteration of the dynamical system. In the next section, we employ a more specific strategy to verify the presence of a stable period-two cycle. Concretely, we first identify a stationary period-two cycle by using the dynamical system given by Eqs. (9) and (10), and then, we numerically confirm that the period-two cycle is stable by investigating the local stability of the twice-iterated dynamical system around the stationary period-two cycle with constant-elasticity-of-substitution (CES) technology and plausible parameter values.

4 Period-two cycles

Suppose that the dynamical system given by Eqs.(9) and (10) has a stationary period-two solution, denoted by $\{(k_a, \theta_a), (k_b, \theta_b)\}$. According to Eqs. (9) and (10), these variables satisfy

$$
\begin{aligned}
k_a &= \Gamma(k_b, \theta_b), \quad \theta_a = H(k_b, \theta_b) \\
k_b &= \Gamma(k_a, \theta_a), \quad \theta_b = H(k_a, \theta_a).
\end{aligned}
$$

From Eq. (9), we have $\theta_t = 1 - k_{t+1}/(f'(k_t)k_t)$. By substituting this into Eq. (10) and rearranging it, we obtain

$$
k_{t+2} = f'(k_{t+1})k_{t+1} - \beta^{\frac{1}{\gamma}} f'(k_{t+1})^{\frac{1}{\gamma}} \left(f'(k_t)k_t - k_{t+1} \right). \tag{19}
$$

Substituting $k_{t+2} = k_t = k_a$ and $k_{t+3} = k_{t+1} = k_b$ into Eq.(19) yields the following two equations that determine the period-two solution:

$$
k_a = f'(k_b)k_b - \beta^{\frac{1}{\gamma}} f'(k_b)^{\frac{1}{\gamma}} \left(f'(k_a)k_a - k_b \right) \tag{20}
$$

$$
k_b = f'(k_a)k_a - \beta^{\frac{1}{\gamma}} f'(k_a)^{\frac{1}{\gamma}} \left(f'(k_b)k_b - k_a \right). \tag{21}
$$

If k_a and k_b are well defined, we have θ_a and θ_b, respectively, as follows:

$$
\theta_a = 1 - \frac{k_b}{f'(k_a)k_a} \tag{22}
$$

$$
\theta_b = 1 - \frac{k_a}{f'(k_b)k_b}. \tag{23}
$$

4.1 Parameterization

To make the analysis as lucid as possible, we assume that the economy is endowed with the CES production function such that $Y_t = A\left((1-\alpha)K_t^{-\sigma} + \alpha N_t^{-\sigma}\right)^{-\frac{1}{\sigma}}$ with $\alpha \in (0,1)$ and $\sigma \in (0, \infty)$, where A is the total factor productivity (TFP). Then, it follows that $f(k_t) = A((1-\alpha)k_t^{-\sigma} + \alpha)^{-\frac{1}{\sigma}}$. In the Appendix, we verify the existence of k and $\eta(k)$ under the assumption that $A(1-\alpha)^{-1/\sigma} > 1/\beta$. We base our parameterization on Remark 2. Following standard RBC theory, we set $\beta = 0.96$ and $\alpha = 0.67$. By conducting a meta-analysis with 3,186 observations in total of the elasticity of

Table 1: Parameter values

Parameter	**Value**	**Source/Target**
Subjective discount factor	$\beta = 0.96$	RBC literature
Labor share when $\sigma \to 0$	$\alpha = 0.67$	RBC literature
Elasticity of substitution between capital and labor	$\sigma = 2.33$ $(1/(1+\sigma) = 0.3)$	Gechert et al. (2022)
TFP	$A = 2.5$	based on Remark 2

substitution between capital and labor obtained from 121 prior studies, Gechert et al. (2022) report that the mean elasticity of substitution should be 0.3. Then, we examine the case in which the elasticity of substitution is equal to $1/(1+\sigma) = 0.3$, which solves approximately $\sigma = 2.33$. In Remark 2, we assume that $\eta(k) > 1 + \beta$. Accordingly, the lower limit of A is approximately 2.303; thus, we employ $A = 2.5$. Under the CES production function with the assumption that $\eta(k) > 1 + \beta$, we can numerically verify that k_a, k_b, θ_a, and θ_b are well defined by Eqs. (20)-(23) and unique. Table 1 summarizes the parameter values that we employ.

We display the relationship between A and $\bar{\gamma}$ in Figure 2, which indicates a negative relationship between the two. More concretely, as A approaches 2.303 from above, $\bar{\gamma}$ goes to infinity, and as A goes to infinity, it converges to approximately 0.0486. Thus, Remark 2 implies that as A becomes large, the steady state is more likely to be unstable with the range of γ that produces the unstable steady state widening.

4.2 Local stability of the period-two cycle

Eqs. (9) and (10) yield the twice-iterated dynamical system as follows:

$$k_{t+1} = \Gamma(\Gamma(k_{t-1}, \theta_{t-1}), H(k_{t-1}, \theta_{t-1})) \tag{24}$$

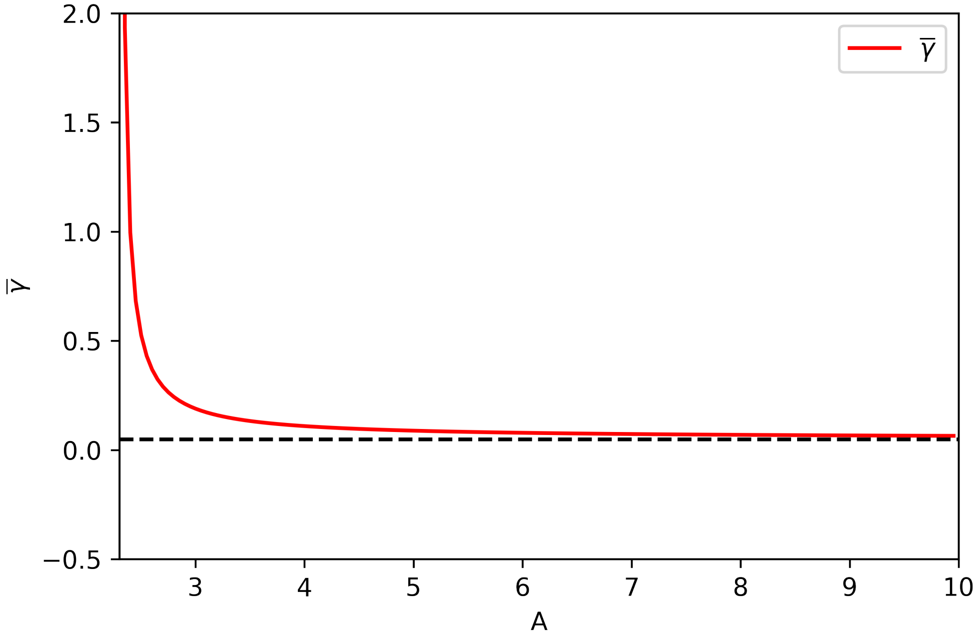
and

$$\theta_{t+1} = H(\Gamma(k_{t-1}, \theta_{t-1}), H(k_{t-1}, \theta_{t-1})). \tag{25}$$

By linearizing Eqs. (24) and (25) around (k_j, θ_j), we have

$$\begin{pmatrix} k_{t+1} - k_j \\ \theta_{t+1} - \theta_j \end{pmatrix} = \begin{pmatrix} \Gamma_k^i & \Gamma_\theta^i \\ H_k^i & H_\theta^i \end{pmatrix} \begin{pmatrix} \Gamma_k^j & \Gamma_\theta^j \\ H_k^j & H_\theta^j \end{pmatrix} \begin{pmatrix} k_{t-1} - k_j \\ \theta_{t-1} - \theta_j \end{pmatrix},$$

Figure 2: A versus $\bar{\gamma}$ (likelihood of the occurrence of an unstable steady state)



Notes. As A approaches 2.303 from above, $\bar{\gamma}$ goes to infinity, and as A goes to infinity, $\bar{\gamma}$ converges approximately to 0.0486. Then, Remark 2 implies that as A becomes large, the steady state is more likely to be unstable with the range of γ that produces the unstable steady state widening.

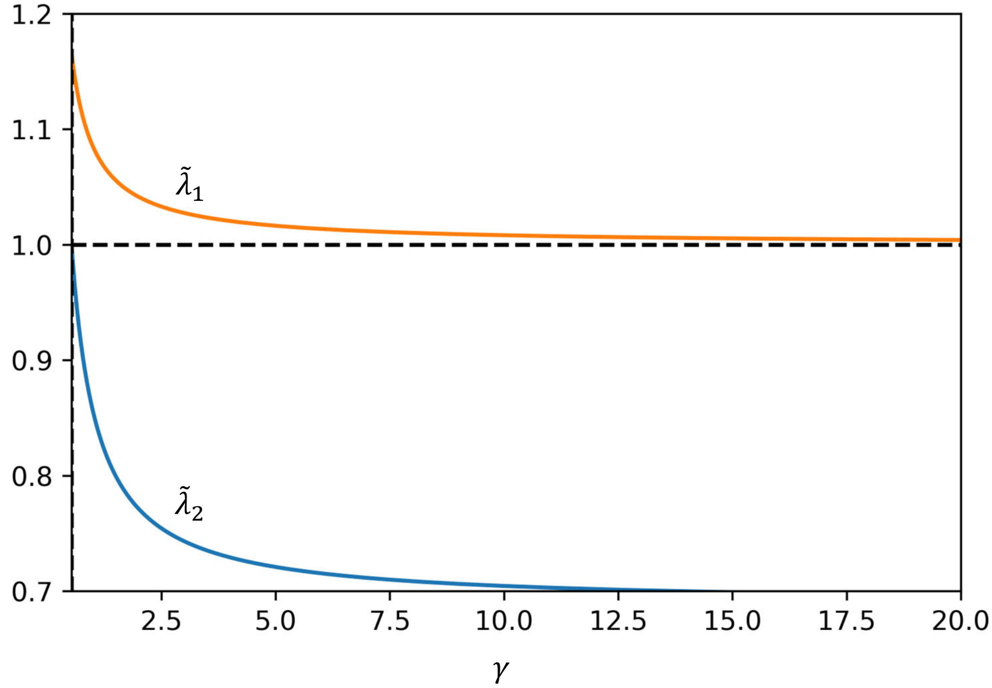
where $(i, j) = (a, b)$ or (b, a), $\Gamma_k^i = \partial\Gamma(k_i, \theta_i)/\partial k$, $\Gamma_\theta^i = \partial\Gamma(k_i, \theta_i)/\partial\theta$, $H_k^i = \partial H(k_i, \theta_i)/\partial k$, and $H_\theta^i = \partial H(k_i, \theta_i)/\partial\theta$.

The stability of the period-two cycle depends on the product of two matrices:

$$\tilde{J} := \begin{pmatrix} \Gamma_k^i & \Gamma_\theta^i \\ H_k^i & H_\theta^i \end{pmatrix} \begin{pmatrix} \Gamma_k^j & \Gamma_\theta^j \\ H_k^i & H_\theta^j \end{pmatrix}.$$

Let $\tilde{\lambda}_1$ and $\tilde{\lambda}_2 (< \tilde{\lambda}_1)$ be eigenvalues of $\tilde{J}$. Under our parameter setting with the CES production function but varying the value of γ, we numerically obtain the eigenvalues such that $0 < \tilde{\lambda}_2 < 1 < \tilde{\lambda}_1$ when $\gamma > \bar{\gamma}$, as shown in Figure 3. As γ increases from $\bar{\gamma}$, both $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ decrease, and $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ converge to 1 and 0.7 as γ increases, respectively. If the period-two solution $\{(k_a, \theta_a), (k_b, \theta_b)\}$ is well defined, it is on the equilibrium manifold. Otherwise, the stationary period-two cycle cannot be stationary because one of the eigenvalues of the dynamical system of Eqs.(9) and (10) is

Figure 3: Eigenvalues of the twice-iterated dynamical system ($\gamma > \bar{\gamma}$)



Notes. As γ increases from $\bar{\gamma}$, both $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ decrease, and $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ converge to 1 and 0.7 as γ becomes big, respectively.

greater than 1; i.e., $\lambda_1 > 1$. In terms of the twice-iterated dynamical system given by (24) and (25), equilibrium is on the equilibrium manifold that is nonlinear globally but associated with $\tilde{\lambda}_2$ in the vicinity of the stationary period-two solution so that the transversality condition should hold. As such, the stationary period-two solution is stable under our parameter setting with the CES production function. By inserting the policy function $\theta_{t-1} = \theta(k_{t-1})$ into Eq. (24), we obtain

$$k_{t+1} = \Gamma(\Gamma(k_{t-1}, \theta(k_{t-1})), H(k_{t-1}, \theta(k_{t-1}))) =: \Psi(k_{t-1}).$$

Moreover, we perform twice iterations of $\tilde{\Gamma}$ in Eq. (17) as follows:

$$k_{t+1} = \tilde{\Gamma}(\tilde{\Gamma}(k_{t-1}; \gamma); \gamma) =: \tilde{\Psi}(k_{t-1}; \gamma).$$

The two functions Ψ and $\tilde{\Psi}$ should be identical.

In the Appendix, we perform two numerical exercises under our parameter setting—one aims to investigate an equilibrium path via Brunner and Strulik's (2004) method, and the other aims to produce bifurcation diagrams for macroeconomic variables when varying γ. The first exercise finds equilibrium paths of k_t and θ_t. To compare two typical situations, we employ two values—$\gamma = 0.2$ and $= 2$. When $\gamma = 0.2$, k_t and θ_t monotonically converge to the steady state, and when $\gamma = 2$, they converge to the stationary period-two cycle. Notably, 32 out of 38 countries have a degree of relative risk aversion greater than 2 in the empirical observation in the introduction. In the second exercise, we produce bifurcation diagrams of capital stock, output, and the consumption levels of workers' and capitalists'. We find that the amplitude of the period-two cycles becomes wider as γ increases, except for the case of capitalists' total consumption. The amplitude of capitalists' total consumption first expands and then begins to shrink as γ increases.

5 Conclusion

One of the crucial theoretical insights from modern macroeconomic theory is that a higher degree of relative risk aversion promotes consumption smoothing and mitigates the economic volatility caused by exogenous shocks. In contrast to this conventional wisdom, our simple dynamic general equilibrium model has shown that a higher degree of relative risk aversion combined with a higher degree of elasticity of the marginal product of capital can destabilize economies, producing a nonlinear mechanism that causes endogenous business fluctuations. Our simple macroeconomic model suggests that one cannot identify the cause of business fluctuations if only the propagation of exogenous shocks is investigated, as has been done in traditional RBC theory. To understand boom-bust cycles deeply, researchers should merge exogenous and endogenous business fluctuations when investigating economies.

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Appendix A: Empirical observations

Estimation equation

The specification of the estimation equation is based on the Euler equation. Let us consider Eq. (2) in section 2, which is a typical Euler equation of macroeconomic models. If this equation is subject to exogenous shocks associated with production, an expectation operator given the information until period t accompanies this equation such that

$$1 = E_t\left[\left(\frac{c_t^i}{c_{t+1}^i}\right)^{\gamma_i}\beta r_{t+1}^i\right], \tag{A1}$$

where i denotes the country. Then, it follows from Eq.(A1) that

$$\epsilon_{t+1}^i = \left(\frac{c_t^i}{c_{t+1}^i}\right)^{\gamma_i}\beta r_{t+1}^i,$$

or equivalently,

$$\frac{c_{t+1}^i}{c_t^i} = \left(\frac{\beta r_{t+1}^i}{\epsilon_{t+1}^i}\right)^{\frac{1}{\gamma_i}}, \tag{A2}$$

where r_{t+1}^i and ϵ_{t+1}^i are stochastic variables with $E_t[\epsilon_{t+1}^i] = 1$. Taking the logarithm of both sides of Eq. (A2) yields

$$\log\left(\frac{c_{t+1}^i}{c_t^i}\right) = \frac{1}{\gamma_i}\left[\log(\beta) + \log(r_{t+1}^i) - \log(\epsilon_{t+1}^i)\right]. \tag{A3}$$

From the linear approximation of both sides of Eq.(A3) around $c_{t+1}^i/c_t^i = r_{t+1}^i = \epsilon_{t+1}^i = 1$, we have

$$\frac{c_{t+1}^i - c_t^i}{c_t^i} \approx \frac{1}{\gamma_i}\left[\log(\beta) + r_{t+1}^i - \epsilon_{t+1}^i\right]. \tag{A4}$$

Now, we assume that ϵ_{t+1}^i is a function such that $\epsilon_{t+1}^i := g(r_{t+1}^i) + \tilde{\epsilon}_{t+1}^i$, where $g(\cdot)$ is a differentiable function and r_{t+1}^i and $\tilde{\epsilon}_{t+1}^i$ are independent. Since the linear approximation of $g(r_{t+1}^i)$ around $r_{t+1}^i = 1$ is given by $g(r_{t+1}^i) \approx g(1) + g'(1)(r_{t+1}^i - 1)$, it follows that

$$\epsilon_{t+1}^i \approx g(1) - g'(1) + g'(1) r_{t+1}^i + \tilde{\epsilon}_{t+1}^i. \tag{A5}$$

Substituting Eq. (A5) into Eq.(A4) yields

$$\frac{c_{t+1}^i - c_t^i}{c_t^i} \approx \frac{1}{\gamma_i} \left[\log(\beta) + g'(1) - g(1) + (1 - g'(1)) r_{t+1}^i - \tilde{\epsilon}_{t+1}^i \right]. \tag{A6}$$

Taking the variances of both sides of Eq.(A6), it follows that

$$Var\left(\frac{c_{t+1}^i - c_t^i}{c_t^i} \right) = \frac{\bar{\Phi} Var(r_{t+1}^i) + \Phi}{\gamma_i^2}, \tag{A7}$$

where $Var(\cdot)$ represents the variance, $\bar{\Phi} := (1 - g'(1))^2$, and $\Phi := Var(\tilde{\epsilon}_{t+1}^i)$. Note that Φ is assumed to be constant across countries. Furthermore, from the log-linearization of $\log(\bar{\Phi} Var(r_{t+1}^i) + \Phi)$ around the country-mean of $Var(r_{t+1}^i)$, we have

$$\begin{aligned} \log(\bar{\Phi} Var(r_{t+1}^i) + \Phi) &= \log(\bar{\Phi} \overline{Var(r_{t+1}^i)} + \Phi) \\ &+ \left(\frac{\bar{\Phi} \overline{Var(r_{t+1}^i)}}{\bar{\Phi} \overline{Var(r_{t+1}^i)} + \Phi} \right) [\log(Var(r_{t+1}^i)) - \log(\overline{Var(r_{t+1}^i)})], \end{aligned} \tag{A8}$$

where $\overline{Var(r_{t+1}^i)}$ is the country mean of $Var(r_{t+1}^i)$. Taking the logarithm of both sides of Eq. (A7) and using Eq. (A8) yield

$$\log\left(Var\left(\frac{c_{t+1}^i - c_t^i}{c_t^i} \right) \right) = \tilde{\Phi} + \left(\frac{\bar{\Phi} \overline{Var(r_{t+1}^i)}}{\bar{\Phi} \overline{Var(r_{t+1}^i)} + \Phi} \right) \log(Var(r_{t+1}^i)) - \log(\gamma_i^2), \tag{A9}$$

where $\tilde{\Phi} := \log(\bar{\Phi} \overline{Var(r_{t+1}^i)} + \Phi) - \bar{\Phi} \overline{Var(r_{t+1}^i)} \log(\overline{Var(r_{t+1}^i)}) / (\bar{\Phi} \overline{Var(r_{t+1}^i)} + \Phi)$ is a constant. Eq. (A9) is the basis of the estimation equation.

Data

We draw the data for the EIS from Table A1 of Havranek et al. (2015). In the process of the meta-analysis, Havranek et al. (2015) collect 2735 estimates of the EIS from published studies and display the means of the EIS country by country for 45 countries, excluding estimates larger than 10 in absolute value, in Table A1 of their paper. We omit 5 countries that have negative EIS means, thereby obtaining

40 EIS data points from the abovementioned table. From the EIS data collected, we compute the squared degree of relative risk aversion to obtain $\log(\gamma_i^2)$. To obtain per capita consumption growth, we use real consumption at constant 2017 national prices (rconna) and population (pop) assembled from Penn World Table, version 10.01 (PWT 10.01, Feenstra et al., 2015) for the period 1950-2019 at maximum depending on each country's availability. We also collect the real internal rate of return (irr) for the period 1950-2019 from Penn World Table, version 10.01. We then take the sample variances of per capita consumption and the real internal rate of return to obtain $\log(vac_i)$ and $\log(vai_i)$, respectively. The real internal rate of return for Myanmar and Pakistan is not available; therefore, the sample size for the estimation is 38.

Appendix B: Steady state with the CES production function

The output per capita under the CES production function given in section 3 is $f(k_t) = A((1-\alpha)k_t^{-\sigma} + \alpha)^{-1/\sigma}$. Then, it follows that $f'(k_t) = A(1-\alpha)/[1-\alpha+\alpha k_t^{\sigma}]^{1+1/\sigma}$. Furthermore, we have $\lim_{k_t \to 0} f'(k_t) = A(1-\alpha)^{-1/\sigma}$ and $\lim_{k_t \to \infty} f'(k_t) = 0$ for $\sigma \in (0, \infty)$. Under the parameter assumption, $A(1-\alpha)^{-1/\sigma} > 1/\beta$, which satisfies Assumption 1, the steady state k exists, and from Eq. (11), the steady-state value of k_t is given by the following equation:

$$\alpha k^{\sigma} = (\beta A(1-\alpha))^{\frac{\sigma}{1+\sigma}} - (1-\alpha).$$

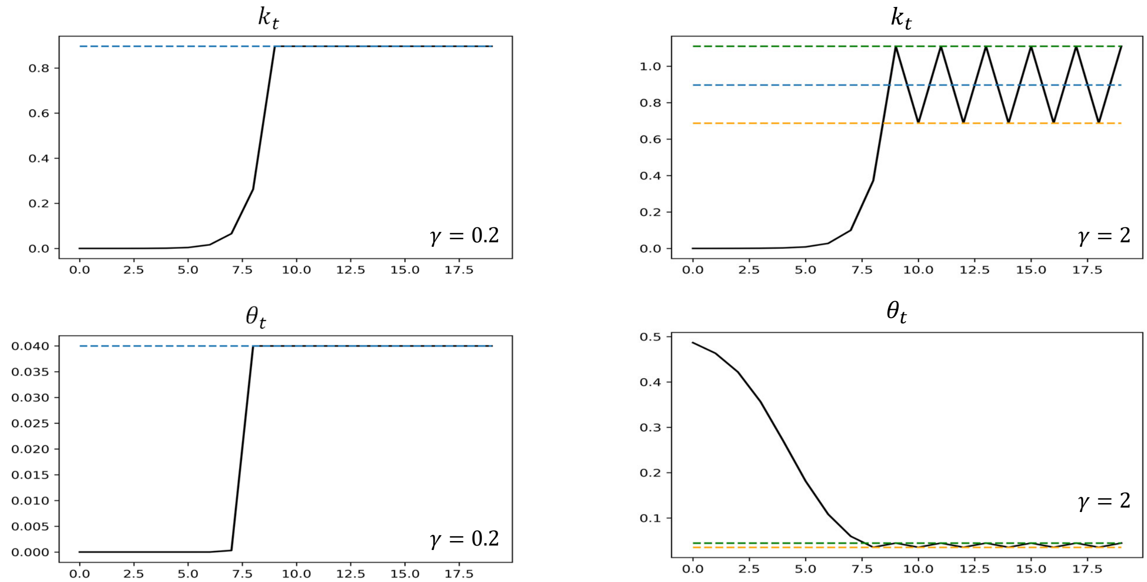
The elasticity of marginal product $\eta(k_t) := -f''(k_t)k_t/f'(k_t)$ is obtained as

$$\eta(k_t) = (1+\sigma)\frac{\alpha k_t^{\sigma}}{\alpha k_t^{\sigma} + 1 - \alpha}.$$

Appendix C: Equilibrium path

To find an equilibrium path, we employ Brunner and Strulik's (2002) method. Following their approach, we perform backward iterations of the dynamical system of Eqs.(9) and (10) from the vicinity of the steady state when the steady state is a saddle point or from the vicinity of the stationary period-two solution if the steady state is unstable. The crucial point of the abovementioned method is that when starting from the vicinity of the steady state or the stationary period-two solution, the backward

Figure C1: Time course of k_t and θ_t in equilibrium



Notes. We produce an equilibrium path for 20 periods. When $\gamma = 0.2$, k_t and θ_t converge to the steady state (left panels) and when $\gamma = 2$, they converge to the stationary period-two cycle (right panels).

iteration tends toward the direction of the eigenvector associated with λ_2 or $\tilde{\lambda}_2$. Then, we can obtain the economy approximately on the equilibrium manifold embodied by Eq. (17).

Under our parameter setting, $\sigma = 2.33$ computes $\eta(k) \approx 2.04 > 1+\beta$ and $\bar{\gamma} \approx 0.53$. Therefore, from Remark 2, the steady state is a saddle point when $\gamma = 0.2$ and is unstable when $\gamma = 2$. Brunner and Strulik's method finds equilibrium paths of k_t and θ_t when $\gamma = 0.2$ and 2, respectively. As observed in Figure C1, when $\gamma = 0.2$, k_t and θ_t converge to the steady state (left panels), and when $\gamma = 2$, they converge to the stationary period-two cycle (right panels).

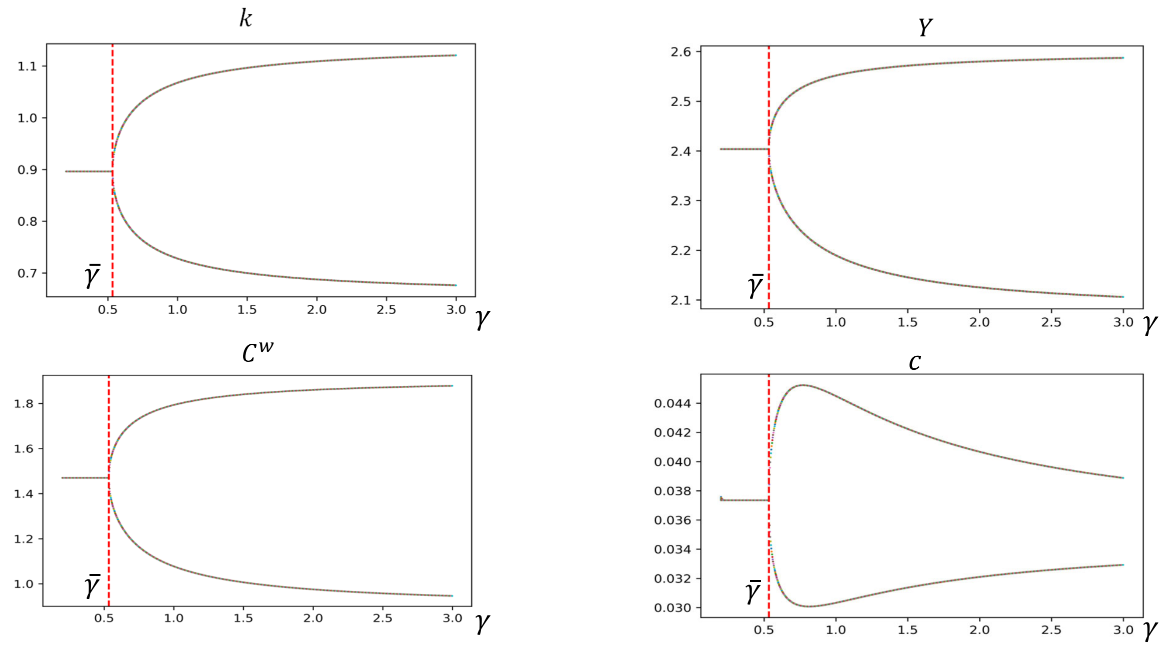
Appendix D: Bifurcation diagram

Let us produce bifurcation diagrams by varying γ from 0.2 to 3.0 for macroeconomic variables such as total output, capital, workers' total consumption, and capitalists' total consumption. We assume that $L = 100$. The way to produce these diagrams is different than usual. Whereas one usually iterates the dynamical system forward many times and finds the convergence or nonconvergence terminal, we cannot employ such an ordinary method in our study. The reason for this is that we must analyze the equilibrium dynamics on the equilibrium manifold that we cannot analytically derive. Nevertheless, our analytical investigation thus far has sufficient information to produce bifurcation diagrams. Focusing on the case in which $\gamma(k) > 1+\beta$, we find from Remark 2 and the numerical analysis of the stability of the period-two solution that when γ increases from 0.2, a supercritical flip bifurcation occurs at $\gamma = \bar{\gamma}$ and the stable period-two cycle appears, which means that there are no more period-doubling bifurcations as γ increases under our parameter setting (see Figure 3).

Figure D1 provides the bifurcation diagrams. In each panel, the horizontal line is γ, and the vertical line is the macroeconomic variable. As expected, all panels in Figure 5 show that bifurcation occurs at $\gamma = \bar{\gamma}$ and endogenous business fluctuations (period-two cycles) appear. The amplitude of these period-two cycles becomes wider as γ increases, except for the case of capitalists' total consumption, in which the amplitude first expands and then shrinks as γ increases from $\bar{\gamma}$ to 3.0.

Compared with standard RBC theory, we obtain some takeaways from analyzing the bifurcation diagrams. First, despite our simple model setting such that workers are hand-to-mouth consumers, capitalists are endowed with the CRRA utility function, and the representative firm employs the CES production technology, the larger (but plausible) coefficient of relative risk aversion produces the nonlinearity of equilibrium, causing endogenous business fluctuations. Standard RBC theory has ignored this mechanism; according to this theory, the origin of business cycles is only exogenous productivity shocks. Second, as the coefficient of relative risk aversion becomes large, the amplitude of macroeconomic variables widens in our model. This outcome has never been considered in standard RBC theory. In particular, in standard RBC theory, it is more likely that the amplitude of consumption fluctuations caused by exogenous productivity shocks shrinks as the coefficient of relative risk aversion becomes larger due to consumption smoothing. In contrast, the amplitude of consumption in our model becomes larger as the coefficient of relative risk aversion increases beyond the critical value of $\bar{\gamma}$, although the amplitude of capitalists' consumption starts to shrink when the coefficient of relative risk becomes too large because the effect of consumption smoothing surpasses the nonlinear effect that causes endogenous busi-

Figure D1: Bifurcation diagrams for macroeconomic variables when γ varies from 0.2 to 3



Notes. In each panel, bifurcation occurs at $\gamma = \bar{\gamma}$ and endogenous business fluctuations (period-two cycles) appear. The amplitude of the period-two cycles becomes wider as γ increases, except for the case of capitalists' total consumption, in which the amplitude first expands and then shrinks as γ increases from $\bar{\gamma}$ to 3.0.

ness fluctuations. Third, the outcome of our study proposes an empirical research question regarding whether the economy is more likely to fluctuate as the coefficient of relative risk aversion increases. Although we provide a preliminary result of such an empirical study in the introduction, further investigations are necessary.

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