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Shareholder Engagement in an ESG-CAPM with Incomplete Markets: Much ado about nothing?

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Abstract

We give a general equilibrium model of incomplete asset markets in which investors care not only about risk and return but also have ESG concerns. We consider two notions of equilibrium, a market value maximization equilibrium and a Dreze equilibrium. While the firms simply maximize profit with respect to common state prices at a market value maximization equilibrium, each firm maximizes profit with respect to the weighted average of its shareholders' subjective state prices at a Dreze equilibrium. We take the difference in social welfare between the two as the impact of shareholder engagement. We establish the existence of these equilibria. We give an equivalent condition for the two to coincide, which means that shareholder engagement makes no difference. We show, moreover, that even when it makes a difference, it is at most of second order, hence negligible, in a sense that can be made precise.

Keywords: ESG, CAPM, incomplete markets, shareholder engagement, representative investor, Grassmann manifold

1 Introduction

ESG (Environment, Social, and Governance) attracts a lot of attention in the global economy. On the first one, environment, it is often argued, most notably Kölbel et al (2020), that investors, institutional or individual, should actively engage in the firm's production decision making to induce its managers to employ greener

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technologies. While it seems nice, at first glance, to encourage green activists to engage in firms' management, the social value of such engagement should ultimately be judged by its welfare consequences. Since such consequences, in turn, depend on how the investors and the asset markets react to the firms' production changes, any sound theoretical analysis would require a general equilibrium model that takes all repercussion effects into consideration. The purpose of this paper is to provide a general equilibrium model of asset markets where shareholders' views on environment are respected in firms' decision making.

Let us take a moment to elaborate on the findings of Kölbel et al (2020). They surveyed 64 contributions to identify what kind of mechanisms would make a significant difference in greenness of firms' production activities. They singled out shareholders engagement as the most effective mechanism, while the effectiveness of capital allocation, whereby investors shift their capital from less to more green firms, is only partially supported. In particular, when it comes to the extent to which capital allocation affects asset prices and, then, asset prices incentivize firms towards greener production activities, the conclusions in the literature are rather mixed. They also emphasized, as topics of future research, the importance of quantifying the impact of shareholder engagement and clarifying the nature of firms' reactions to changes in asset prices caused by capital allocation. In this paper, we give theoretical answers to these question: We quantify the impact of engagement as the representative investor's utility function and derive production plans at equilibrium where investors maximize their ESG-sensitive utility functions. All these results are made possible by our general equilibrium model with ESG-conscious investors and endogenous production decisions.

In our model, investors have the same mean-variance utility functions as in the Capital Asset Pricing Model, except that they have possibly heterogeneous views on environment, embedded in their utility functions. We use two notions of equilibrium. One is a market-value maximization equilibrium, where the firms maximize their profit without paying special attention to the shareholders' ESG concerns. The other is Dreze equilibrium, named after Jacques Dreze (1974), where the firms maximize profit with respect to the weighted average of its shareholders' utility gradients. This latter criterion respects the shareholders' environment concerns and can thus be considered as a proper, albeit condensed, formulation of shareholder engagement in this context.

While there are many ways to introduce ESG concerns into a general equilibrium model, we employ the following assumptions to capture the equilibrium implications of shareholder engagement in the simplest possible manner.

First, we opt for the notion of an equilibrium due to Dreze (1974) rather than that due to Grossman and Hart (1974). The two notions differ as regards to who can determine the firm's production plan. Dreze's notion assumes that ex-post (after trade) shareholders can determine the firm's production plan, while Grossman and Hart's notion assumes that ex-ante (before trade) shareholders can do so. We use Dreze's notion because we would like to model a situation where environmentally conscious investors buy shares, being aware of the possibility that they could overturn, whenever necessary, the existing production decision (made, presumably, by ex-ante shareholders) in their favor. We also assume that they bear the cost of production.

Second, we allow for short sales and, moreover, assume that short-sellers are also involved in the firm's decision making. The second assumption is admittedly unrealistic and deviates drastically from Dreze's original formulation, who assumed that no short sales is allowed (and, hence, there was no need to specify whether short-sales are involved in the firm's decision making). Yet, we can justify our inclusion of short-sellers on the normative ground: To attain an efficient allocation, the firm's objective needs to take not only shareholders' welfare but also short-sellers' welfare into consideration.

Third, we assume that the risk-free lending is in fixed supply and the risk-free interest rate is endogenously determined at equilibrium. In the finance literature, it is often assumed that the risk-free interest rate is exogenously given and there is no limit on the risk-free lending. While this assumption may be plausible especially where the analysis is based on the CAPM, it blurs the welfare comparison between Dreze equilibrium and market value maximization equilibrium because the advantage of one over the other may be due to a larger supply of risk-free lending at equilibrium, rather than shareholder engagement.

With these model specifications, we present three results in this paper.

First, we prove that both a market value maximization equilibrium and a Dreze equilibrium exist. The mathematical techniques needed to prove their existence are quite different. To establish the existence of a market-value maximization equilibrium, it suffices to use the well known fact that every continuous function on a compact set attains a maximum. For the existence of a Dreze equilibrium, we need to introduce an auxiliary notion of an equilibrium, called a pseudo Dreze equilibrium, and construct a vector bundle whose base space is a Grassmann manifold and apply a theorem on its mod 2 Euler number. Our existence theorem of a (pseudo) Dreze equilibrium is particularly noteworthy because most papers in general equilibrium theory with incomplete asset markets and production, such as Dreze (1974) and Geanakoplos, Magill, Quinzii, and Dreze (1990), assumed that no short sales are allowed, thereby circumventing some discontinuity problems that can only be dealt with by using Grassmann manifolds.

Second, we give an equivalent condition for the two equilibria to coincide. We also give a novel interpretation to this equivalent condition in terms of ESG integration. ESG integration stipulates that ESG concerns should be taken into investment decisions. In our model, the integrated part of ESG concerns is nothing but the deviation of investors' portfolios from the mutual fund theorem. The non-integrated part of ESG concerns is left in the heterogeneity in shareholders' subjective state price density (or marginal rates of substitution) due to market incompleteness. The equivalent condition is that the integrated and non-integrated parts of ESG concerns must be uncorrelated. It means, in particular, that the knowledge of shareholders' portfolios gives no information on their view on which production activity they would like the firm to implement. Thus, shareholder engagement makes a difference if and only if the integrated and the non-integrated parts of ESG concerns are (positively or negatively) correlated. For this condition to hold, two things are necessary. First, shareholders' ESG concerns are heterogenous; that is, some shareholders need to be more environmentally conscious than others, possibly in different dimensions. Second, asset markets are incomplete in the sense that the investors' ESG concerns cannot be fully diversified via asset trades. This result tells us that any model that assumes homogenous ESG concern or complete markets cannot capture the impact of shareholder engagement.

Third, we show that shareholder engagement has at most a second-order impact on social welfare. More precisely, starting at a profile of investors' ESG concerns under which the market value maximization equilibrium and Dreze equilibrium coincide, we change the profile to induce the two to diverge. We prove that the induced difference in the sum of investors' utilities between the two equilibria is of at most second order with respect to the size of the change, formulated as probability distortions, in ESG profiles. We can thus say that shareholder engagement may make a difference in social welfare, but the magnitude is negligible. An important implication of this result is that a shareholder may be able to increase his own welfare, in the magnitude of first order, by engaging himself in the firm's decision making, but he can do so only at the sacrifice of some other investors. This tradeoff seems to have been left unnoticed in the literature. In our accompanying paper (Hara, Hens, and Trutwin (2024)), we give numerical examples for these effects to happen.

This paper is organized as follows. We present the setup, the notions of equilibrium, and some technical results in Sections 2 through 5. We prove the existence of a market value maximization equilibrium in Section 6 and the existence of a Dreze equilibrium in Section 7. We give an equivalent condition for a market value maximization equilibrium and a Dreze equilibrium to coincide in Section 8. We show in Section 9 that there is no first-order impact of shareholder engagement on social welfare. We conclude in Section 10. Proofs and some supplementary materials are gathered in the appendices.

2 Setup

There are two time periods, 0 and 1. There is a state space $S = \{1, \dots, S\}$, endowed with a probability measure $p = (p_1, \dots, p_S) \in \boldsymbol{R}^S_{++}$, that describes uncertainty on period 1. Denote by $\Lambda(p)$ is the $S \times S$ diagonal matrix with its s-th diagonal entry equal to p_s.

Denote the vector of 1's in $\boldsymbol{R}^S$ by $\mathbf{1}$, and also by D^0.

An economy of the ESG-inclusive CAPM is defined by the pairs of Arrow-Pratt measure of absolute risk aversion and ESG scores, (ψ^i, γ^i), for $i = 1, \dots, I$, and the production possibility sets, Y^k, for $k = 1, \dots, K$.

To accommodate ESG concerns, as we assume that investor i's utility function over stochastic consumption on period 1 is

$$U^i(c^i) = E[c^i] - \frac{\psi^i}{2}\mathrm{Var}[c^i] - \gamma^i \cdot c^i. \tag{1}$$

This utility function differs from the mean-variance utility function in that it has a third term $-\gamma^i \cdot c^i$. We can assume without loss of generality that $\mathbf{1} \cdot \gamma^i = 0$, because, otherwise, we can replace γ^i by $\gamma^i - E[\gamma^i]\mathbf{1}$ and change ψ^i to represent the same utility function. Thus, the ESG score γ^i can be thought of as a probability distortion: the distorted probability distribution assigns probability $1 - \gamma^i_s$ to state s. Write $\delta^i = \Lambda(p)^{-1}\gamma^i$. The utility gradient is $\nabla U^i(c^i) = \Lambda(p)\pi^i(c^i)$, where

$$\pi^i(c^i) = \mathbf{1} - \psi^i(c^i - E[c^i]\mathbf{1}) - \delta^i.$$

Since $\nabla U^i(c^i)z = E[\pi^i(c^i)z]$ for every $z \in \boldsymbol{R}^S$, $\pi^i(c^i)$ is the density, also known as the Riesz representation, of the linear function defined by $\nabla U^i(c^i)$. Here and throughout this paper, we identify a vector in $\boldsymbol{R}^S$ with a random variable defined on S, by dividing each coordinate of the former by the probability. The former is useful when we use Hessians, while the latter is useful when we take expectations.

Note that $E[\pi^i(c^i)] = 1$. Write $\tau_i = \psi_i^{-1}$, $\bar{\tau} = \sum_i \tau_i$, $\bar{\psi} = \bar{\tau}^{-1}$, and $\bar{\delta} = \sum_i (\tau_i/\bar{\tau})\delta^i$. Then, τ^i is investor i's risk tolerance and, as we will later see, $\bar{\tau}$ is the representative investor's risk tolerance and $\bar{\delta}$ is her ESG score.

The production set of firm k is given by

$$Y^k = \{(-F^k(D^k), D^k) \in \boldsymbol{R} \times \boldsymbol{R}^S_+ \mid D^k \in \boldsymbol{R}^S_+\} - (\boldsymbol{R}_+ \times \boldsymbol{R}^S_+)$$

where $F^k : \boldsymbol{R}^S_+ \to \boldsymbol{R}_+$ is strictly increasing, continuous, and convex, and satisfies $F^k(0) = 0$. It is nothing but the cost function: $F^k(D^k)$ is the amount of input at period zero that is needed to produce D^k_s units in each state s on period one. The continuity and convexity imply that for every $w > 0$, the set $\{D^k \in \boldsymbol{R}^S_+ \mid F^k(D^k) \leq w\}$ is closed and convex. Indeed, if it were not bounded, then its asymptotic cone has a vector $z \in \boldsymbol{R}^S_+ \setminus \{0\}$. But, then, $w \geq F^k(D^k) = F^k(D^k + z)$, which contradicts the assumption that F^k is strictly increasing. Hence, the set $\{D^k \in \boldsymbol{R}^S_+ \mid F^k(D^k) \leq w\}$ is bounded. Note also that by definition, Y^k satisfies free disposal. Hence, the profit maximization conditions, to be defined later, imply that the equilibrium state prices are non-negative.

The aggregate production set $\bar{Y}$ is defined as the sum $\sum_{k=1}^K Y^k$ of the firms' production sets. Define $\bar{F} : \boldsymbol{R}^S_+ \to \boldsymbol{R}_+$ by letting

$$\bar{F}(\bar{D}) = \min \left\{ \sum_k F^k(D^k) \mid D^k \in \boldsymbol{R}^S_+ \cap M \text{ for every } k \text{ and } \sum_k D^k = \bar{D} \right\}$$

for every $\bar{D} \in \boldsymbol{R}^S_+$. Then,

$$\bar{Y} = \{(-\bar{F}(\bar{D}), \bar{D}) \in \boldsymbol{R} \times \boldsymbol{R}^S_+ \mid \bar{D} \in \boldsymbol{R}^S_+\} - (\boldsymbol{R}_+ \times \boldsymbol{R}^S_+).$$

That is, $\bar{F}$ is the cost function that represents the aggregate production set $\bar{Y}$.

Example 1 Imagine that each state s corresponds to a global average temperature t_s and assume that $t_1 < \cdots < t_S$. That is, the state with a larger index s is the state where the global average temperature is higher.

We take any investor i with $\gamma^i_s = 0$ for every s as environmentally neutral. When it comes to evaluating expected utility levels, an environmentally conscious investor would assign probabilities lower than the reference (natural) probability p_s to high temperatures and probabilities higher than the reference (natural) probability p_s to lower temperatures. The idea behind this restriction is that an environmentally conscious investor would feel guilty of consumption in environmentally bad states and this tendency is reflected by a reduction of probabilistic assessments on such states in evaluating expected utility levels. In particular, the lower probabilities assigned to environmentally bad states should not be interpreted as the investor's probabilistic assessments of environmentally bad states being lower than the reference probabilities. Indeed, the reality of environmental activism seems to suggest the contrary.

To formalize this idea, we rely on the notion of monotone likelihood ratio property. We could alternatively use the notion of first-order stochastic dominance, but we skip its formalization to simplify the exposition. We say that investor i is environmentally conscious if

$$\frac{p_s - \gamma^i_s}{p_s},$$

is decreasing in s, or, equivalently, δ^i_s is increasing in s, because, having known that either one of two temperatures will be realized, investor i puts a higher conditional probability on the higher temperature than is calculated from the reference probabilities.

We can extend the notion of environmental consciousness to the notion of more-environmentally-conscious-than relation. We say that investor i is more environmentally conscious than investor j if

$$\frac{p_s - \gamma_s^i}{p_s - \gamma_s^j},$$

is decreasing in s or, equivalently,

$$\frac{1 - \delta_s^i}{1 - \delta_s^j}$$

is decreasing in s.

Example 2 As for the firms, a firm is green if it produces less in higher temperatures. Equivalently, a firm is brown if it produces more in higher temperatures, thereby contributing to the higher temperatures. To formalize this idea, we impose a particular functional form on the cost functions. Assume that for each firm k, there is a $\nu^k = (\nu_1^k, \dots, \nu_S^k) \in \boldsymbol{R}_{++}^S$ such that

$$F^k(D^k) = \sum_s \frac{p_s}{2\nu_s^k}(D_s^k)^2 = E\left[\frac{(D^k)^2}{2\nu^k}\right].$$

It can be shown that if π is the state price density, then the profit-maximizing (state-contingent) output under π is a positive multiple of ν^k. Thus, in line with the more-environmentally-conscious-than relation for investors, we can say that firm k is greener than firm h if ν_s^k/ν_s^h is decreasing in s.

More generally, let R_f and π be the risk-free rate and the state price density. Let D^k and D^h maximize profit under (R_f, π) for firms k and h, the precise definition of which will be introduced later. Then, by its first-order condition, $\pi^\top \Lambda(p) = R_\mathrm{f} \nabla F^k(D^k)$ and $\pi^\top \Lambda(p)\pi = R_\mathrm{f} \nabla F^h(D^h)$. Thus, $\nabla F^k(D^k) = \nabla F^h(D^h)$. Hence, we can say that firm k is greener than firm h if D_s^k/D_s^h is decreasing in s, whenever $\nabla F^k(D^k) = \nabla F^h(D^h)$.

3 Stocks, bond, and portfolios

Assume that, besides the stocks, the risk-free bond with payoff $\mathbf{1}$ is tradable in period zero. We take the risk-free bond as the numeraire, so that its price is equal to one.

For each i, investor i has initial endowments of the bond and the stocks, $(\bar{\theta}^{i0}, \bar{\theta}^{i1}, \dots, \bar{\theta}^{iK})$ and non-financial (such as labor) income w^i on period zero. We can think of $\bar{\theta}^{i0}$ as investor i's endowment in the risk-free consumption in period 1 and w^i as his endowment in input in period 0. Assume, as a normalization, that $\sum_i \bar{\theta}^{ik} = 1$ for every $k \geq 1$ (but we impose no such condition on $\bar{\theta}^{i0}$ as we want to allow for both $\sum_i \bar{\theta}^{i0} = 0$ and $\sum_i \bar{\theta}^{i0} > 0$). We write $\bar{w} = \sum_i w^i$, $\bar{\theta}^0 = \sum_i \bar{\theta}^{i0}$.

The stock entitles its ex-post (after trade) holder the output D^k it generates on period one. As regards to who pays for the input $F^k(D^k)$, there are two possibilities: the ex-post (after trade) holder versus the ex-ante (before trade) holder. We will formulate, in Appendix B, these two possibilities in turn and show that they are, in fact, equivalent. For the simplicity of exposition, however, we shall assume in Section 4 that the ex-post shareholders pay for the input.

4 Two notions of equilibrium

Denote the bond and stock prices by $q = (q^0, q^1, \dots, q^k) \in \boldsymbol{R}^{1+K}$ and the (one-plus) risk-free rate by R_f. We take the bond as the numeraire so that $q^0 = 1$. Then, the price of the period-0 input is R_f. To define two notions of equilibrium, we, first, give the following two conditions.

Since the after-trade ownership of the stock come with the obligation of paying for the input $F^k(D^k)$ on period zero in proposition to her stock holding, the utility maximization problem of investor i is

$$\begin{aligned} \max_{(\theta^{i0}, \theta^{i1}, \dots, \theta^{iK})} \quad & U^i\left(\sum_{k=0}^{K} \theta^{ik} D^k\right) \\ \text{subject to} \quad & \sum_{k=0}^{K} q^k \theta^{ik} + R_\mathrm{f} \sum_{k=1}^{K} F^k(D^k)\theta^{ik} \leq \sum_{k=0}^{K} q^k \bar{\theta}^{ik} + R_\mathrm{f} w^i. \end{aligned}$$

At the solution of the utility maximization problem, the weak inequality holds as an equality.

The market clearing condition is that $\sum_i \theta^{i0} = \bar{\theta}^0$, $\sum_i \theta^{ik} = 1$ for every $k \geq 1$, and $\sum_k F^k(D^k) = \bar{w}$.

Remark 1 It is often assumed, in the finance literature, that the risk-free rate R_f is exogenously given and the market-clearing condition is not required for the risk-free bond. Such a formulation is possible in our setting but would require us to introduce a non-standard notion of an efficient allocation.

We say that a state price density π is consistent with the asset price vector q under $(D^1, \dots, D^K)$ if $q^k = E\left[\pi D^k\right] - R_\mathrm{f} F^k(D^k)$ for every k.

Definition 1 Suppose that $((\theta^{ik})_{i,k}, R_\mathrm{f}, q)$ satisfies the utility maximization condition and the market-clearing condition are met under $(D^1, \dots, D^K)$. Suppose, in addition, that there is a consistent state price density π under $(D^1, \dots, D^K)$ such that for every k, D^k is a solution to the problem of maximizing

$$E\left[\pi D^k\right] - R_{\mathrm{f}} F^k(D^k).$$

Then, we say that $((D^1, \dots, D^K), (\theta^{ik})_{i,k}, R_{\mathrm{f}}, q)$ is a *market-value-maximization equilibrium* (MVE for short).

Definition 2 Suppose that $((\theta^{ik})_{i,k}, R_{\mathrm{f}}, q)$ satisfies the utility maximization condition and the market-clearing condition are met under $(D^1, \dots, D^K)$. Suppose, in addition, that for every k, D^k is a solution to the problem of maximizing

$$E\left[\left(\sum_i \theta^{ik}\pi^i(c^i)\right) D^k\right] - R_{\mathrm{f}} F^k(D^k),$$

where $c^i = \sum_{k=0}^{K} \theta^{ik} D^k$. Then, we say that $((D^1, \dots, D^K), (\theta^{ik})_{i,k}, R_{\mathrm{f}}, q)$ is a *Dreze equilibrium* (DE for short).

The two definitions differ in terms of the state price densities with respect to which the profit maximization condition is defined. At the MVE, the state price density π is required to be consistent with the stock prices q and all the firms maximize profit with respect to the common π. As for DE, by the utility maximization condition, the $\pi^i(c^i)$ all coincide on the market span $\langle 1, D\rangle$, which is defined as the linear subspace of $\boldsymbol{R}^S$ spanned by $(\mathbf{1}, D^1, \dots, D^K)$. Outside the market span, however, they may differ from each other. Thus, different firms may maximize profit with respect to different state price densities.

The notion of Dreze equilibrium can be justified by the following fact, which differs from Proposition 31.5 of Magill and Quinzii (1996), only in that short sales are allowed, and can be similarly proved. If D^k did not satisfy the Dreze criterion, then there would be a $\hat{D}^k$ and $(\hat{\theta}^{10}, \dots, \hat{\theta}^{I0})$ with $\sum_i \hat{\theta}^{i0} = \sum_i \theta^{i0}$ such that for every i with $\theta^{ik} \neq 0$,

$$U^i\left(\sum_{h\geq 1, h\neq k} \theta^{ih} D^h + \hat{\theta}^{i0}\mathbf{1} + \theta^{ik}\left(\hat{D}^k - R_{\mathrm{f}}\left(F^k(\hat{D}^k) - F^k(D^k)\right)\mathbf{1}\right)\right) > U^i\left(\sum_{k=0}^{K} \theta^{ik} D^k\right). \tag{2}$$

This can be understood as saying that if the Dreze criterion is not met, then, at the shareholders' assembly, via proxy fight, they would unanimously vote in favor of $\hat{D}^k$ over D^k, thereby kicking out the incumbent managers. Thus, the Dreze criterion is a necessary condition for the production plan D^k to be carried out.

Remark 2 Some caveats are in order on the inclusion of short sellers in the firm's profit maximization problem of a Dreze equilibrium. First and foremost, it is necessary (but not sufficient) to include short sellers ($\theta^{ik} < 0$) for this criterion to achieve a Pareto-efficient allocation. Instead of defining a Pareto improvement by the inequality (2) for every i with $\theta^{ik} \neq 0$, it is possible, and, in fact, more realistic, to require the inequality (2) to hold only for the shareholders ($\theta^{ik} > 0$). In this alternative criterion, the short sellers' welfare is still impacted by the change of production plans from D^k to $\hat{D}^k$, but the change in their welfare is not taken into consideration. Thus, even if the shareholders do not find any Pareto-improving production plan among themselves, there may well be a Pareto-improving production plan once the short sellers are included and transfers in terms of the risk-free bond are feasible between shareholders and short sellers. Second, Dreze (1974) himself assumed that investors cannot short-sell ($\theta^{ik} \geq 0$ for every $k \geq 1$), thereby excluding short sellers from the firm's maximization problem, and this assumption seems reasonable given the associated complex transactions in reality, but it is inconsistent with the theory of optimal portfolios and asset pricing, especially in the CAPM, which has been developed mainly under the assumption that short sales are possible with no extra transaction costs. Third, it is possible to allow for short sales and, yet, involve only long buyers in the firm's maximization problem. One such example is to assume that firm k maximizes

$$E\left[\left(\sum_i \eta^{ik}\pi^i\right) D^k\right] - R_\mathrm{f} F^k(D^k),$$

where $\Theta^k = \sum_i \max\{\theta^{ik}, 0\}$ and

$$\eta^{ik} = \frac{\max\{\theta^{ik}, 0\}}{\Theta^k}.$$

The corresponding equilibrium could be called a truncated Dreze equilibrium, as the short sales are truncated to zero. This criterion was used by Momi (2002) and can be justified by assuming that stocks cannot be sold short but there are derivatives that have the same payoffs as the stocks except that they do not come with the voting right and can be sold short. Appendix C formalizes this fact.

Remark 3 Grossman and Hart (1979) introduced an alternative objective of the firm by which the profit is maximized with respect to the weight average of the before-trade shareholders' utility gradients. This means, in our notation, that the firm maximizes

$$E\left[\left(\sum_i \bar{\theta}^{ik}\pi^i(c^i)\right) D^k\right] - R_\mathrm{f} F^k(D^k).$$

While this criterion would be more compelling in a dynamic model of multiple trading periods, we opt for Dreze criterion, as we would like to formulate the situation where investors (activists) buy shares to affect firms' production decision making.

5 Constrained equilibrium

In this section, we give an auxiliary notion of equilibrium, called a constrained equilibrium, that covers both MVE and DE and, yet, allows us to make the difference between the two mathematically tractable. It is given in terms of consumption plans and state price densities, rather than portfolios and stock prices. In the sequel, we explore the existence, uniqueness, and characterization of the constrained equilibrium. Recall that the condition $\sum_k F^k(D^k) \leq \bar{w}$ is equivalent to $(-\sum_k F^k(D^k), \sum_k D^k) \in \bar{Y}$.

Definition 3 Let M be a linear subspace of $\boldsymbol{R}^S$ that contains $\mathbf{1}$. Let $((D^k)_k, (c^i)_i)$ be a profile of output and consumption plans. We say that it is *M-constrained feasible* if $D^k \in M$ for every k, $c^i \in M$ for every i, $\sum_k F^k(D^k) \leq \bar{w}$, and $\sum_i c^i = \bar{\theta}^0 \mathbf{1} + \sum_k D^k$.

Note that $\boldsymbol{R}^S$-constrained feasibility is nothing but the standard (unconstrained) feasibility.

Definition 4 Let M be a linear subspace of $\boldsymbol{R}^S$ that contains $\mathbf{1}$. Then, $((D^k)_k, (c^i)_i, R_{\mathrm{f}}, \pi)$ with $E[\pi] = 1$ is an *M-constrained equilibrium* if

1. $((D^k)_k, (c^i)_i)$ is M-constrained feasible.
2. For every i, c^i is a solution to

$$\begin{aligned} \max_{c^i} \quad & U^i(c^i) \\ \text{subject to} \quad & E[\pi c^i] \leq \bar{\theta}^{i0} + \sum_{k=1}^{K} (E[\pi D^k] - R_{\mathrm{f}} F^k(D^k) \bar{\theta}^{ik} + R_{\mathrm{f}} w^i, \\ & c^i \in M. \end{aligned}$$

3. For every k, D^k is a solution to

$$\begin{aligned} \max_{D^k} \quad & E\left[\pi D^k\right] - R_{\mathrm{f}} F^k(D^k) \\ \text{subject to} \quad & D^k \in M. \end{aligned}$$

The second condition is the utility maximization condition under the spanning constraint $c^i \in M$ in addition to the (standard) budget constraint. The third condition is the profit maximization condition under the spanning constraint $D^k \in M$. Note that an $\boldsymbol{R}^S$-constrained equilibrium is nothing but a standard (unconstrained) equilibrium in terms of consumption plans and state prices.

The next lemma shows that MVE's and DE's are constrained equilibria.

Lemma 1 *Let* $((D^1,\dots,D^K),(\theta^{ik})_{i,k},R_{\mathrm{f}},q)$ *be a MVE or a Dreze equilibrium. For each* i*, write* $c^i=\sum_{k=0}^K \theta^{ik}D^k$*. Let* π *be a state price density that is consistent with* q*. Then,* $((D^k)_k,(c^i)_i,R_f,\pi)$ *is a* $\langle \mathbf{1}, D\rangle$*-constrained equilibrium.*

Remark 4 It is true that $M \supseteq \langle \mathbf{1}, D\rangle$ for every M-constrained equilibrium $((D^k)_k,(c^i)_i,R_f,\pi)$. By construction, the $\langle \mathbf{1}, D\rangle$-constrained equilibrium in Lemma 1 satisfies this inclusion as an equality. There are, however, M-constrained equilibria for which M is strictly larger than $\langle \mathbf{1}, D\rangle$.

We now define a notion of constrained efficiency.

Definition 5 Let M be a linear subspace of $\boldsymbol{R}^S$ that contains $\mathbf{1}$. Let $((D^k)_k,(c^i)_i)$ be M-constrained feasible. We say that it is *M-constrained efficient* if there is no other profile that is M-constrained feasible and improves upon $(c^i)_i$ in the sense of Pareto.

Note that $\boldsymbol{R}^S$-constrained efficiency is nothing but standard (unconstrained) efficiency.

Thanks to the quasi-linearity of the U^i, the M-constrained efficient allocations can be characterized as the maximum of the (unweighted) sum of utilities. The next characterizes the maximum.

Lemma 2 *Let* M *be a linear subspace of* $\boldsymbol{R}^S$ *that contains* $\mathbf{1}$*. If* $((D^k)_k,(c^i)_i)$ *is* M*-constrained efficient, then*

$$\sum_i U^i\left(c^i\right)=\bar{\theta}^0+E[\bar{D}]-\frac{1}{2\bar{\tau}}\mathrm{Var}[\bar{D}]-E[\bar{\delta}\bar{D}]+\sum_i\frac{\tau_i}{2}\mathrm{Var}[A^i_M], \tag{3}$$

where $\bar{D}=\sum_{k=1}^K D^k$ *and, for each* i*,* A^i_M *is the* p*-orthogonal projection of* $\delta^i-\bar{\delta}$ *onto* M*.*

The following lemma characterizes constrained efficient allocations and their uniqueness in two senses that will be clarified after its statement.

Lemma 3 *Let* M *be a linear subspace of* $\boldsymbol{R}^S$ *that contains* $\mathbf{1}$*. Let* $((D^k)_k,(c^i)_i)$ *be an* M*-constrained efficient allocation.*

1. *For every* i*,*

$$c^i-E[c^i]\mathbf{1}=\frac{\tau^i}{\bar{\tau}}(\bar{D}-E[\bar{D}]\mathbf{1})-\tau^i A^i_M. \tag{4}$$

Moreover, if $((D^k)_k,(c^i)_i,R_{\mathrm{f}},\pi)$ *is an* M*-constrained equilibrium, then* π *coincides with*

$$\mathbf{1}-\frac{1}{\bar{\tau}}(\bar{D}-E[\bar{D}]\mathbf{1})-\bar{\delta}. \tag{5}$$

on M*.*

2. *For any other M-constrained efficient allocation* $\left((\hat{D}^k)_k, (\hat{c}^i)_i\right)$, $\sum_{k\geq 1} D^k = \sum_{k\geq 1} \hat{D}^k$ *and* $\sum_{i\geq 1} c^i = \sum_{i\geq 1} \hat{c}^i$.
3. *For every* $k \geq 1$, *if* F^k *is strictly convex, then* $D^k = \hat{D}^k$. *If* F^k *is strictly convex and differentiable for some* k *and if* $((D^k)_k, (c^i)_i, R_{\mathrm{f}}, \pi)$ *is an* M*-constrained equilibrium, then* $R_{\mathrm{f}} = \hat{R}_{\mathrm{f}}$ *and* $c^i = \hat{c}^i$ *for every* i.

Part 1 of this lemma characterizes the M-constrained efficient allocations and the associated state price densities. (4) implies that the mean-zero part of each investor's consumption plan is uniquely determined, and (5) implies that the state price density is uniquely determined up to the residuals of its p-orthogonal projection on the market span M. Part 2 establishes the uniqueness relative to the weaker notion, which claims that the aggregate production plan and the aggregate consumption plan are uniquely determined. Together with part 1, it implies that between any two M-constrained efficient allocations, each investor's consumption plans differ only in the direction of $\mathbf{1}$ and, since the aggregate consumption plan is uniquely determined, these differences add up to zero. Part 3 gives sufficient conditions for a firm's output plan, an investor's consumption plan, and the risk-free rate to be uniquely determined. Note that if $K = 1$, then part 2 implies that $D^1 = \hat{D}^1$, without relying on these conditions.

Remark 5 Since a DE is also a constrained equilibrium, an important corollary of Lemma 3 is that whenever two DE's involve different aggregate output plans, the corresponding market spans are different. Recall that Dreze gave three examples to illustrate possible inefficiency of Dreze equilibrium allocations. The examples involve two states, two investors, and two firms, and, yet, production plans (state-contingent outputs) depend on who owns the firms. Lemma 3 seems to contradict his examples, because the market span is unchanged in his examples. In fact, it does not, because our setting allows consumption levels to be negative (that is, the consumption set is the entire $\boldsymbol{R}^S$, rather than the non-negative orthant $\boldsymbol{R}^S_+$) and our notion of a Dreze equilibrium requires all the investors' subjective state price densities, $\pi^i(c^i)$'s, to be always equated on the entire market span M, which Dreze's own notion does not require when the consumption levels for some states are zero.

The following lemma is the first welfare theorem for constrained equilibrium allocations. It can be proved by applying the first welfare theorem (Proposition 16.C.1 of Mas-Colell, Whinston, and Green (1995), for example) to the constrained consumption sets $\boldsymbol{R}_+ \times M$ and the constrained production possibility sets $Y^k \cap (\boldsymbol{R} \times M)$.

Lemma 4 *Let* M *be a linear subspace of* $\boldsymbol{R}^S$ *that contains* $\mathbf{1}$. *If* $((D^k)_k, (c^i)_i, R_{\mathrm{f}}, \pi)$ *is an* M*-constrained equilibrium, then* $((D^k)_k, (c^i)_i)$ *is* M*-constrained efficient.*

Since both MVE and DE are M-restricted equilibria, the above lemma implies that they are M-restricted efficient.

The following lemma is the second welfare theorem for constrained efficient allocations. It can be proved by applying the first welfare theorem (Proposition 16.D.1 of Mas-Colell, Whinston, and Green (1995), for example) to the constrained consumption sets $\boldsymbol{R}_+ \times M$ and the constrained production possibility sets $Y^k \cap (\boldsymbol{R} \times M)$, and noting that $\boldsymbol{R}_+ \times M$ is not bounded from below.

Lemma 5 *Let M be a linear subspace of $\boldsymbol{R}^S$ that contains $\mathbf{1}$. If $((D^k)_k, (c^i)_i)$ is M-constrained efficient, then there is a (R_f, π) with $R_\mathrm{f} > 0$ and $E[\pi] = 1$ such that $((D^k)_k, (c^i)_i, R_\mathrm{f}, \pi)$ is an M-constrained equilibrium.*

While this lemma is also a consequence of the separating hyperplane theorem, to guarantee that $R_\mathrm{f} > 0$ and $E[\pi] = 1$, we need a modification to one of the two sets to which the theorem is applied. Yet, we can only guarantee that $((D^k)_k, (c^i)_i, R_\mathrm{f}, \pi)$ is an M-constrained equilibrium: it need not be a MVE or a DE.

The last lemma of this section establishes the existence of an M-constrained equilibrium.

Lemma 6 *For every linear subspace M of $\boldsymbol{R}^S$ that contains $\mathbf{1}$, there is an M-constrained equilibrium.*

This lemma can be proved by applying the equilibrium existence theorem (Proposition 17.BB.2 of Mas-Colell, Whinston, and Green (1995), for example) to the constrained consumption sets

$$\boldsymbol{R}_+ \times \{c^i \in M \mid U^i(c^i) \geq U^i(\bar{\theta}^{i0})\}.$$

and the constrained production possibility sets $Y^k \cap (\boldsymbol{R} \times M)$. Note that $U^i(\bar{\theta}^{i0})$ is the utility level that consumer i attains when not trading any asset or input. Hence, at any constrained equilibrium, she attains a utility level that is at least as high as this level, which justifies the use of the above constrained consumption set with the additional benefit of being bounded from below.

6 Canonical market value maximization equilibrium

In this section, we prove that there always exists a MVE with the state price density under which all the firms maximize profit admits an easy-to-grasp expression.

Proposition 1 *There is a market value maximization equilibrium at which the consistent state price density under which the firms maximize profit is*

$$\mathbf{1} - \frac{1}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta}. \tag{6}$$

Definition 6 A MVE $((D^1, \ldots, D^K), (\theta^{ik})_{i,k}, R_{\mathrm{f}}, q)$ is called the *canonical MVE* if all firms maximize profit under the state price density (6).

The notion of a MVE puts no restriction, outside the market span M, the state price density π with respect to which all the firms maximize profit. In particular, the definition of a MVE (Definition 1) does not even require the state price density to be a convex combination of the density of the investors' utility gradients. In this respect, we follow Duffie and Shafer (1986), and they, in fact, gave an example of a single-firm economy having a continuum of equilibria, some of which are even Pareto-ranked. In Appendix D, we show that the multiplicity may as well persist in our setting.

Yet, the state price density (6) is the most natural candidate among the multiple MVE state price densities. To see this point, suppose, for simplicity, that all investors have the same (p-discounted) ESG score $\bar{\delta}$. In this homogeneous case, the social welfare is measured by the representative investor's utility function

$$\bar{U}(\bar{c}) = E[\bar{c}] - \frac{1}{2\bar{\tau}}\mathrm{Var}[\bar{c}] - E[\bar{\delta}\bar{c}], \tag{7}$$

and the canonical state price density (6) coincides with the (p-discounted) gradient of the above utility function. In the aggregate, therefore, the firms would not maximize social welfare whenever they maximize profit under any non-canonical state price density. Thus, any improvement of Dreze equilibrium allocations over MVE allocation may then be attributed to this erroneous choice of the state price density. In other words, by focusing on the canonical state price density (6), we are being most demanding when it comes to assessing whether shareholder engagement may lead to Pareto-superior allocations.

Since the utility function (7) is strictly quasi-concave, even when there are multiple MVE's, the aggregate output plan $\bar{D}$ is uniquely determined. Under conditions similar to those in parts 2 and 3 of Lemma 3, the uniqueness of the canonical MVE is guaranteed, which justifies our terminology of "the canonical MVE" rather than "a canonical MVE".

Remark 6 Although Proposition 1 is concerned with a single economy, it has an important implication when comparing the canonical MVE's of two economies. The proof of Lemma 6 and Proposition 1 together reveal that the aggregate output plan $\bar{D}$ and the state price density π depend on the aggregated parameters $\bar{Y}$, $\bar{w}$, $\bar{\theta}^0$,

$\bar{\tau}$, and $\bar{\delta}$ but not directly on the individual parameters Y^k, w^i, θ^{i0}, and τ^i, or δ^i. Underlying this fact is the quasi-linearity of the utility functions U^i in the direction of the payoff $\mathbf{1}$ of the risk-free bond, which implies that there is no income effect on the demand for the shares.

7 Existence of a Dreze equilibrium

In this section, we establish the existence of a Dreze equilibrium. Unlike some earlier contributions, such as Dreze (1974) and Geanakoplos, Magill, Quinzii, and Dreze (1990), we do not impose exogenous bounds on portfolios. Duffie and Shafer (1986) imposed no such constraint but only established the existence of a MVE. A notable exception is Momi (2001), who gave a robust example of the non-existence of a Dreze equilibrium. We will elaborate on the difference between our formulation and his formulation towards the end of this section.

7.1 Additional assumptions

We use the following assumptions in this section. For each k, F^k is twice continuously differentiable at all points except for 0. $\nabla F^k(D^k) \in \boldsymbol{R}^S_+ \setminus \{0\}$ and $\nabla^2 F^k(D^k)$ is positive definite (even along the direction of D^k) for every $D^k \in \boldsymbol{R}^S_+ \setminus \{0\}$. $\nabla F^k(D^k) \in \boldsymbol{R}^S_{++}$ for every $D^k \in \boldsymbol{R}^S_{++}$. Then, the positive multiples of $(1, \nabla F^k(D^k))$ are the normal vectors of Y^k at $(-F^k(D^k), D^k)$. Write

$$\xi^k(D^k) = \Lambda(p)^{-1} \nabla F^k(D^k),$$

where $\Lambda(p)$ is the $S \times S$ diagonal matrix of which the s-th diagonal element is equal to p_s. Then, $(F^k(D^k), D^k)$ maximizes profit under (R_f, π) (a risk-free rate $R_\mathrm{f} > 0$ and a state price density π satisfying $E[\pi] = 1$) if and only if $(1, \xi^k(D^k))$ is a positive multiple of (R_f, π), which is equivalent to $\pi - R_\mathrm{f}\xi^k(D^k) = 0$. More generally, for every linear subspace M of $\boldsymbol{R}^S$ that contains $\mathbf{1}$, $(F^k(D^k), D^k)$ maximizes profit under (R_f, π) on $Y^k \cap (\boldsymbol{R} \times M)$ (the M-constrained profit maximization condition) if and only if $\pi - R_\mathrm{f}\xi^k(D^k) \in M^\perp$.

7.2 Pseudo Dreze equilibrium

Recall that $E[\delta^i] = 0$ for every i and $E[\bar{\delta}] = 0$. For each linear subspace M of $\boldsymbol{R}^S$ that contains $\mathbf{1}$, write $\delta^i - \bar{\delta} = A^i_M + B^i_M$ with $A^i_M \in M$ and $B^i_M \in M^\perp$, where the orthogonality is defined with respect to the probability p on $\boldsymbol{R}^S$. Write

$A_M = (A^1_M, \dots, A^I_M) \in \boldsymbol{R}^{S\times I}$ and $B_M = (B^1_M, \dots, B^I_M) \in \boldsymbol{R}^{S\times I}$. Moreover, write

$$A_M = \begin{pmatrix} \tilde{A}_M \\ \check{A}_M \end{pmatrix},$$

where $\tilde{A}_M \in \boldsymbol{R}^{K\times I}$ and $\check{A}_M \in \boldsymbol{R}^{(S-K)\times I}$.

For any give profile $(D^k)_k$ of state-contingent outputs, write $\hat{D}^k = D^k - E[D^k]\mathbf{1}$ for every k, and, then, $\hat{D} = (\hat{D}^1, \dots, \hat{D}^K) \in \boldsymbol{R}^{S\times K}$. Moreover, write

$$\hat{D} = \begin{pmatrix} \tilde{D} \\ \check{D} \end{pmatrix},$$

where $\tilde{D} \in \boldsymbol{R}^{K\times K}$ and $\check{D} \in \boldsymbol{R}^{(S-K)\times K}$.

Definition 7 Let M be a linear subspace of $\boldsymbol{R}^S$ that contains $\mathbf{1}$ and $((D^k)_k, (c^i)_i, R_\mathrm{f}, \pi)$ with $E[\pi] = 1$ be an M-constrained equilibrium. It is a *pseudo Dreze equilibrium* if, in addition,

$$\left(\pi - R_\mathrm{f}\xi^1(D^1), \dots, \pi - R_\mathrm{f}\xi^K(D^K)\right) \tilde{D}^\top = -B_M\Lambda(\tau)\tilde{A}_M^\top, \tag{8}$$

where $\left(\pi - R_\mathrm{f}\xi^1(D^1), \dots, \pi - R_\mathrm{f}\xi^K(D^K)\right) \in \boldsymbol{R}^{S\times K}$.

A Dreze equilibrium is a pseudo Dreze equilibrium, as the following proposition shows.

Proposition 2 *Suppose that $((D^k)_k, ((\theta^{i0}, \theta^i))_i, R_\mathrm{f}, (q^0, q^1, \dots, q^K))$, with $q^0 = 1$, is a Dreze equilibrium. Let M be the linear subspace of $\boldsymbol{R}^S$ spanned by $(\mathbf{1}, D^1, \dots D^K)$. Let $c^i = \sum_{k\geq 0} \theta^{ik} D^k$ for each i and*

$$\pi = \mathbf{1} - \frac{1}{\bar{\tau}}\left(\bar{D} - E\left[\bar{D}\right]\mathbf{1}\right) - \bar{\delta},$$

where $\bar{D} = \sum_{k\geq 1} D^k$. Then, π is consistent with q (that is, $q^0 = 1$ and $q^k = E[\pi D^k] - R_\mathrm{f} F^k(D^k)$ for every $k \geq 1$) and $((D^k)_k, (c^i)_i, R_\mathrm{f}, \pi)$ is a pseudo Dreze equilibrium.

The following proposition gives the converse of Proposition 2 under the additional assumption that $(\mathbf{1}, D^1, \dots, D^K)$ is linearly independent. The two propositions, together, mean that when $(\mathbf{1}, D^1, \dots, D^K)$ is linearly independent, (8) is a necessary and sufficient condition for a Dreze equilibrium.

Proposition 3 *Let M be a linear subspace of $\boldsymbol{R}^S$ that contains $\mathbf{1}$. Suppose that $((D^k)_k, (c^i)_i, R_\mathrm{f}, \pi)$ is the M-constrained equilibrium that satisfies*

$$\pi = \mathbf{1} - \frac{1}{\bar{\tau}}\left(\bar{D} - E\left[\bar{D}\right]\mathbf{1}\right) - \bar{\delta},$$

where $\bar{D} = \sum_{k\geq 1} D^k$*, and that it is also a pseudo Dreze equilibrium. Assume that* $\operatorname{rank}\hat{D} = \operatorname{rank}\tilde{D} = K$*. Let* q *be the asset price vector with which* π *is consistent (that is,* $q^0 = 1$ *and* $q^k = E[\pi D^k] - R_\mathrm{f} F^k(D^k)$ *for every* $k \geq 1$*). Then, for each* i *and* $k \geq 0$*, there is a* θ^{ik} *such that* $c^i = \sum_{k\geq 0} \theta^{ik} D^k$ *for every* i *and* $((D^k)_k, (\theta^{ik})_{i,k\geq 0}, R_\mathrm{f}, q)$ *is a Dreze equilibrium.*

This proposition states that under the additional assumption that $\operatorname{rank}\hat{D} = \operatorname{rank}\tilde{D} = K$, the converse of Proposition 2 holds. Of the two equalities in the assumption, only the first one, $\operatorname{rank}\hat{D} = K$, which means that $(\hat{D}^1, \dots, \hat{D}^K)$ is linearly independent, is critical. The other assumption, $\operatorname{rank}\tilde{D} = K$, means that the first K rows of $\hat{D}$ constitute a basis of its row space, but choosing the first K rows is apparently ad hoc. In fact, we could dispense with this latter assumption by modifying Definition 7 as follows: there is a $K \times K$ submatrix $\tilde{D}$ of $\hat{D}$ such that $\operatorname{rank}\hat{D} = \operatorname{rank}\tilde{D}$ and

$$\left(\pi - R_\mathrm{f}\xi^1(D^1), \dots, \pi - R_\mathrm{f}\xi^K(D^K)\right) \tilde{D}^\top = -B_M \Lambda(\tau) \tilde{A}_M^\top,$$

where the $K \times I$ submatrix $\tilde{A}_M$ consists of the same K rows as $\tilde{D}$ does. In this proposition, then, we would only need to assume that $\operatorname{rank}\hat{D} = K$. Indeed, take a $K \times K$ submatrix $\tilde{D}$ of $\hat{D}$ as stipulated in the modified definition. Since $\operatorname{rank}\hat{D} = K$, $\operatorname{rank}\tilde{D} = K$ and the above equality holds for this $\tilde{D}$; and the proof below is still valid. This modification, therefore, provides an alternative (but not stronger) notion of a pseudo Dreze equilibrium, and, at the same time, a weaker sufficient condition for it to be a Dreze equilibrium.

7.3 Vector bundle approach

In this subsection, we prove that there is a pseudo Dreze equilibrium. As we saw in Proposition 3, a pseudo Dreze equilibrium is a Dreze equilibrium if $(\mathbf{1}, D^1, \dots, D^K)$ is linearly independent. We will later give a sufficient condition for this linear independence condition to be met.

Theorem 1 *There is a subspace* M *that contains* $\mathbf{1}$ *such that the* M*-restricted equilibrium is a pseudo Dreze equilibrium.*

To prove Theorem 1, write $N \equiv \{z \in \boldsymbol{R}^S \mid E[z] = 0\}$ and $\mathscr{L}$ be the set of all K-dimensional linear subspaces of N. Then, $\mathscr{L}$ can be identified with the Grassmann manifold $\mathscr{G}^{J,S-1}$, and endowed with the topology induced by the topology of $\mathscr{G}^{J,S-1}$. Denote by $L^\perp$ the p-orthogonal complement of L in M_0. By Proposition 6, for each $L \in \mathscr{L}$, there is a unique (up to the p-orthogonal projection of π onto the p-orthogonal complement of $\langle \mathbf{1} \rangle + L$) $(\langle \mathbf{1} \rangle + L)$-constrained equilibrium, which we denote by $\left((D_L^k)_k, (c_L^i)_i, R_{\mathrm{f}L}, \pi_L\right)$. Among the multiple π_L, we take

$$\pi_L = \mathbf{1} - \bar{\psi}(\bar{D}_L - E[\bar{D}_L]\mathbf{1}) - \bar{\delta}.$$

By a sight abuse of notation, we write A_L and B_L for $A_{\langle \mathbf{1} \rangle + L}$ and $B_{\langle \mathbf{1} \rangle + L}$. Define $\eta : \mathscr{L} \to (\boldsymbol{R}^S)^K$ by

$$\eta(L) = \left(\pi_L - R_{\mathrm{f}L}\xi^1(D_L^1), \ldots, \pi_L - R_{\mathrm{f}L}\xi^K(D_L^K)\right) \tilde{D}_L^\top + B_L \Lambda(\tau) \tilde{A}_L^\top.$$

By definition, the $(\langle \mathbf{1} \rangle + L)$-constrained equilibrium is a pseudo Dreze equilibrium if and only if $\eta(L) = 0$. Thus, the problem of establishing the existence of a pseudo Dreze equilibrium is nothing but showing that there is an $L \in \mathscr{L}$ such that $\eta(L) = 0$.

Define $\sigma : \mathscr{L} \to \mathscr{L} \times (\boldsymbol{R}^S)^K$ by letting $\sigma(L) = (L, \eta(L))$ for every $L \in \mathscr{L}$. Define

$$\Xi = \{(L, z^1, \ldots, z^K) \in \mathscr{L} \times N^K \mid z^k \in L^\perp \text{ for every } k\}.$$

This is a vector bundle with the base space $\mathscr{L}$ and fiber $(L^\perp)^K$ above each $L \in \mathscr{L}$. Both have dimension $K(S - 1 - K)$.

Lemma 7 *The mapping σ is a section on Ξ. That is, $\sigma(L) \in \Xi$ for every $L \in \mathscr{L}$.*

Then, the $(\langle \mathbf{1} \rangle + L)$-constrained equilibrium is a pseudo Dreze equilibrium if and only if the section σ intersects the zero section at L. Thus, the problem of establishing the existence of a pseudo Dreze equilibrium is nothing but showing that σ intersects the zero section at some point in $\mathscr{L}$.

Lemma 8 *The section σ is continuous.*

We can now prove Theorem 1 as follows. By Hirsch, Magill, and Mas-Colell (1990, page 100), the mod 2 Euler number of Ξ is nonzero. Hence, every continuous section on Ξ intersects the zero section at least once. By Lemma 8, therefore, there is an $L \in \mathscr{L}$ such that the section σ intersects the zero section at L. The $(\langle \mathbf{1} \rangle + L)$-constrained equilibrium is a pseudo Dreze equilibrium.

When the rank condition in Proposition 3 is not met, the pseudo Dreze equilibrium need not be a Dreze equilibrium. Hence, we fall short of proving the existence of a Dreze equilibrium. We will give a (much less general) sufficient condition for the existence of a Dreze equilibrium towards the end of Section 9, as we need a theorem and a lemma that will appear in Sections 8 and 9.

Remark 7 Momi (2001) gave an example of an economy where there is no Dreze equilibrium. The notion of a Dreze equilibrium he employed is, in fact, the truncated Dreze equilibrium, which we introduced in Remark 2. Moreover, his non-existence example is robust with respect to perturbations of endowments. We now argue that the nature of non-existence of his example is quite different from the nature of non-existence of a Dreze equilibrium of our model.

Recall that when defining a pseudo Dreze equilibrium in (8) of Definition 7, we used the first K-coordinates $\tilde{A}^i_M$ of the diversifiable part A^i_M of ESG scores. The diversifiable part can be written as $\hat{D}a^i_M$ where a^i_M is, in essence, the deviation of investor i's portfolio θ^i from the mutual fund theorem, $(\tau^i/\bar{\tau})\mathbf{1}$. To define a pseudo truncated Dreze equilibrium, we need to replace a^i_M by the deviation of the positive part of investor i's portfolio, $\theta^i \vee 0 = (\max\{\theta^{ik}, 0\})_{k\geq 1}$. This modification may induce discontinuity of the section σ. To see this point, suppose that there are two firms and two investors. Let M_0 be a market span such that the profile $(D^1_{M_0}, D^2_{M_0})$ of output plans at the M_0-constrained equilibrium is linearly dependent, spanning just a line. Unless the A^i_M happen to lie on this line, at any other market span M near M_0 with a linearly independent profile (D^1_M, D^2_M), one investor takes a large long position and the other investor takes a large short position of the share of each firm. Hence, with the truncated shareholdings, $\max\{\theta^{ik}, 0\}$, each firm' production plan is determined solely by one investor. Moreover, there may well be a sequence of market spans converging to M_0 along which the sole owner of each firm alternates between the two investors; and this was the case in Momi's (2001) example. Then, the modified diversifiable part would be fluctuating as M converges to M_0, and the section σ fails to be continuous at M_0, in whatever way $\sigma(M_0)$ is defined. This discontinuity does not occur in our model because the diversifiable part A^i_M changes continuously with respect to M. This also explains why his example is robust: a small perturbation of endowments would not eliminate the possibility that there is a sequence of market spans along which the sole owner of each firm alternates between the two investors. On the other hand, since the diversifiable part A^i_M changes continuously with respect to M in our model, we conjecture that the non-existence of a Dreze equilibrium cannot be robust.

In concluding this remark, we mention that the robust non-existence may occur for modified Dreze equilibria in the model of long-only shares and derivatives, mentioned in Remark 2 and formulated in Appendix C. This is because a modified Dreze equilibrium in the markets of shares and derivatives (and the risk-free bond) is a truncated Dreze equilibrium in Momi's (2001) example, which robustly fails to exist.

8 When a MVE is also a Dreze equilibrium

The following theorem gives necessary and sufficient conditions for the canonical MVE to coincide with a Dreze equilibrium. This is the case where shareholder engagement makes no difference to social welfare.

Theorem 2 *Let $((D^k)_k, (\theta^{ik})_{i,k\geq 0}, R_{\mathrm{f}}, q)$ be the canonical MVE. Let M is the linear subspace spanned by $(\mathbf{1}, D^1, \dots, D^K)$.*

1. *If $((D^k)_k, (\theta^{ik})_{i,k\geq 0}, R_{\mathrm{f}}, q)$ is also a DE, then $B_M^\top \Lambda(\tau) A_M = 0$.*
2. *Suppose that $(\mathbf{1}, D^1, \dots, D^K)$ is linearly independent. If $B_M^\top \Lambda(\tau) A_M = 0$, then $((D^k)_k, (\theta^{ik})_{i,k\geq 0}, R_{\mathrm{f}}, q)$ is also a DE.*

Theorem 2 shows that the key condition for the canonical MVE to be also a DE is the equality $B_M^\top \Lambda(\tau) A_M = 0$. Denote the s-th coordinate of A_M^i by A_{Ms}^i and the t-th coordinate of B_M^i by B_{Mt}^i. Then, the equality is equivalent to

$$\sum_i \frac{\tau^i}{\bar{\tau}} A_{Ms}^i B_{Mt}^i = 0. \tag{9}$$

for all s and t. Since $\sum_i (\tau^i/\bar{\tau}) A_{Ms}^i = 0$ and $\sum_i (\tau^i/\bar{\tau}) B_{Mt}^i = 0$, it can be considered as the zero covariance condition between the coordinates of the A_M^i and B_M^i with respect to the weights $\tau^i/\bar{\tau}$. As the proof of this theorem shows, when $(\mathbf{1}, D^1, \dots, D^K)$ is linearly independent, this is equivalent to

$$\sum_i \frac{\tau^i}{\bar{\tau}} a^{ik} B_{Mt}^i = 0. \tag{10}$$

for every $k = 1, 2, \dots, K$ and $t = 1, 2, \dots, S$. Since $\sum_i (\tau^i/\bar{\tau}) a^{ik} = 0$, it can be considered as the zero covariance condition between the coordinates of the a^i and B_M^i with respect to the weights $\tau^i/\bar{\tau}$.

These equalities are best interpreted in terms of ESG integration. ESG integration stipulates that ESG concerns should be taken into investment decisions. In our setting, this means that the investors' portfolios deviate from what they would choose in the standard CAPM without ESG concerns. The latter is equal to $\tau^i/\bar{\tau}$, because the mutual fund theorem would then hold and the investor holds each stock in proportion to his own risk tolerance. In contrast, the investor's optimal portfolio in our setting is θ^{ik}, and the difference

$$\theta^{ik} - \frac{\tau^i}{\bar{\tau}} = -\tau^i a^{ik} \tag{11}$$

can be attributed to his integrated ESG concern, multiplied by his risk tolerance. On the other hand, B_M^i represents the part of his ESG scores that cannot be hedged or diversified and can be addressed only by shareholder engagement, as embodied by the notion of a Dreze equilibrium. As such, it can be interpreted as his non-integrated ESG concerns. Therefore, the equalities (9) and (10) can be considered as the zero correlation between the integrated and the non-integrated ESG concerns. In other words, the ESG scores may not be fully integrated, but, on average, the non-integrated parts cannot be inferred from the (presumably observable) investment decisions.

There are two special, easier-to-interpret, cases where (9) and (10) hold. The first case is where $B^i_M = 0$ for every i. In this case, the ESG scores are fully diversifiable and the asset markets are (effectively) complete. The second case is where all investors' ESG scores are the same, which implies that $a^{ik} = 0$ and $B^i_M = 0$ for all i and k. Putting these two conditions together, we can conclude that for shareholder engagement to matter, markets must be incomplete and ESG scores must be heterogeneous. Neither condition is dispensable, but even when these two conditions are met, shareholder engagement is irrelevant if the integrated and non-integrated parts of ESG scores are uncorrelated.

9 No first-order impact on social welfare

In this section, we show that the difference in social welfare between the canonical MVE and DE is of at most second order with respect to ESG scores. The lesson to be learned from this result is that shareholder engagement may make a difference in firms' production plans, but the difference it makes in social welfare may well be small.

For each i, write $\delta^i - \bar{\delta} = A^i(D, \delta) + B^i(D, \delta)$, the projections onto the linear subspace spanned by $(\mathbf{1}, D^1, \dots, D^K)$ and its p-orthogonal complement, to make its dependence on the profile $D \equiv (D^1, \dots, D^K)$ of output plans and the profile $\delta \equiv (\theta^i)_i$ of ESG scores explicit. We define the portfolio $a^i(D, \delta)$ that generate $A^i(D, \delta)$ in the analogous manner. Denote the right-hand side of the social welfare function (3) by $U_{\mathrm{RE}}(D, \delta)$. Then,

$$U_{\mathrm{RE}}(D, \delta) = \sum_i \bar{\theta}^{i0} + E[\bar{D}] - \frac{1}{2\bar{\tau}} \mathrm{Var}[\bar{D}] - E[\bar{\delta}\bar{D}] + \sum_i \frac{\tau_i}{2} \mathrm{Var}[A^i(D, \delta)].$$

As we will see in Lemma 11, this is differentiable at (D, δ) if $(\mathbf{1}, D^1, \dots, D^K)$ is linearly independent, but not even continuous otherwise, because so is $\mathrm{Var}[A^i(D, \delta)]$ at (D, δ).

We also define the representative investor's utility function U_{RI} by

$$U_{\mathrm{RI}}(\bar{D}, \delta) = \sum_i \bar{\theta}^{i0} + E\left[\bar{D}\right] - \frac{1}{2\bar{\tau}} \mathrm{Var}\left[\bar{D}\right] - E\left[\bar{\delta}\bar{D}\right].$$

As the name suggests, it is equal to the representative investor's utility level (7) when the aggregate output plan is D:

$$\bar{U}\left(\bar{\theta}^0\mathbf{1}+\bar{D}\right)=E\left[\bar{\theta}^0\mathbf{1}+\bar{D}\right]-\frac{1}{2\bar{\tau}}\mathrm{Var}\left[\bar{\theta}^0\mathbf{1}+\bar{D}\right]-E\left[\bar{\delta}\left(\bar{\theta}^0\mathbf{1}+\bar{D}\right)\right]=U_{\mathrm{RI}}(\bar{D},\delta).$$

Note that $U_{\mathrm{RI}}(\bar{D},\delta)$ is different from $U_{\mathrm{RE}}(D,\delta)$ in that it misses the welfare evaluation $\sum_i \frac{\tau_i}{2}\mathrm{Var}[A^i(D,\delta)]$ arising from the hedging opportunities provided by the stocks. As such, the former depends only on the aggregate output plan $\bar{D}$, while the latter depends also on how it is distributed among the K firms, $D=(D^1,\dots,D^K)$.

The first-order necessary and sufficient condition to the problem of maximizing $U_{\mathrm{RI}}\left(\sum_k D^k,\delta\right)$ subject to the constraint that $\sum_k F^k(D^k)\le\bar{w}$ is that there is a $R_{\mathrm{f}}>0$ such that

$$\mathbf{1}-\frac{1}{\bar{\tau}}(\bar{D}-E[\bar{D}]\mathbf{1})-\bar{\delta}-R_{\mathrm{f}}\xi^k(D^k)=0 \tag{12}$$

for every k. The next lemma shows that this equality is equivalent to the condition that (D^k) is the profile of output plans at the canonical MVE.

Lemma 9 *Let $D\equiv(D^1,\dots,D^K)$ be a profile of output plans. Write $\bar{D}=\sum_{k\ge1}D^k$. Then, the following two conditions are equivalent.*

1. *There is a profile $((\theta^{ik})_{i,k\ge0},R_{\mathrm{f}},q)$ of portfolios, risk-free rate, and stock prices such that $((D^1,\dots,D^K),(\theta^{ik})_{i,k},R_{\mathrm{f}},q)$ is the canonical MVE.*
2. *$\sum_k F^k(D^k)=\bar{w}$ and there is a $R_{\mathrm{f}}>0$ such that* (12) *holds for every k.*

The next lemma characterizes a Dreze equilibrium.

Lemma 10 *Let $D\equiv(D^1,\dots,D^K)$ be a profile of output plans. Write $\bar{D}=\sum_{k\ge1}D^k$. Then, the following two conditions are equivalent.*

1. *There is a profile $((\theta^{ik})_{i,k\ge0},R_{\mathrm{f}},q)$ of portfolios, risk-free rate, and stock prices such that $((D^1,\dots,D^K),(\theta^{ik})_{i,k},R_{\mathrm{f}},q)$ is a DE.*
2. *$\sum_k F^k(D^k)=\bar{w}$. Moreover, there is a $R_{\mathrm{f}}>0$ and, for each i, an $a^i=(a^{ik})_{k\ge1}\in\boldsymbol{R}^K$ such that $A^i(D,\delta)=\sum_{k\ge1}a^{ik}(D^k-E[D^k]\mathbf{1})$, $\sum_i\tau^i a^i=0$, and*

$$\mathbf{1}-\frac{1}{\bar{\tau}}(\bar{D}-E[\bar{D}]\mathbf{1})-\bar{\delta}+\sum_{k\ge1}\tau^i a^{ik}B^i(D,\delta)-R_{\mathrm{f}}\xi^k(D^k)=0 \tag{13}$$

for every k.

The following lemma relates the first-order condition for the problem of maximizing $U_{\mathrm{RE}(D,\delta)}$ to the profit maximization condition (13) of a Dreze equilibrium.

Lemma 11 *For every $\delta \in N^I$ and $D \equiv (D^1, \ldots, D^K)$, if $(\mathbf{1}, D^1, \ldots, D^K)$ is linearly independent, then, for every i,* $\mathrm{Var}[A^i(D,\delta)]$ *is a differentiable function at D and*

$$\frac{\mathrm{d}}{\mathrm{d}D^k}\mathrm{Var}[A^i(D,\delta)] = 2a^{ik}(D,\delta)B^i(D,\delta)^\top \Lambda(p)$$

for every k.

By Proposition 1, for each profile $\delta \in N^I$ of ESG scores, there is a canonical MVE. We also stated after the proposition that conditions similar to Parts 2 and 3 of Lemma 3 are sufficient to guarantee its uniqueness. The next lemma shows that under these assumptions, the profile of output plans at the canonical MVE is a continuously differentiable function of the ESG profile δ.

Lemma 12 *Assume that for every k, F^k is twice continuously differentiable and $\nabla^2 F^k(D^k)$ is positive definite at every $D^k \neq 0$. Let $\delta_0 \in N^I$ and $D_0 \in \boldsymbol{R}_+^{SK}$ be the output profile at the canonical MVE under δ_0. Then, there is a continuously differentiable mapping D_{MVE} of some neighborhood of δ_0 into some neighborhood of D_0 such that for every pair (D,δ) in the product of these two neighborhoods, $D_{\mathrm{MVE}}(\delta) = D$ if and only if D is the profile of output plans at the canonical MVE of δ.*

While no equally general result is available for the continuous differentiability of D_{DE}, the following lemma guarantees it under the additional assumptions of linear independence and effective completeness.

Lemma 13 *Assume that for every k, F^k is twice continuously differentiable and $\nabla^2 F^k(D^k)$ is positive definite at every $D^k \neq 0$. Let $\delta_0 \in N^I$ and $D_0 \in \boldsymbol{R}_+^{SK}$ be the output profile at a DE under δ_0. Suppose that $(\mathbf{1}, D_0^1, \ldots, D_0^K)$ is linearly independent and $B^i(D_0,\delta_0) = 0$ for every i. Then, there is a continuously differentiable mapping D_{DE} of some neighborhood of δ_0 into some neighborhood of D_0 such that for every pair (D,δ) in the product of these two neighborhoods, $D_{\mathrm{DE}}(\delta) = D$ if and only if D is the profile of output plans at the DE of δ.*

For a given profile δ_0 of ESG scores, Lemma 12 showed that the profiles of output plans at the canonical MVE under profiles δ close δ_0 can be written as a continuously differentiable mapping of δ, which we denote by $D_{\mathrm{MVE}}(\delta)$. As for DE, if the assumptions of Lemma 13 are satisfied, then the profiles of output plans at the DE under profiles δ close δ_0 can be written as a continuously differentiable mapping of δ, which we denote by $D_{\mathrm{MVE}}(\delta)$. If these assumptions are violated, however, we simply assume that the profiles of output plans at DE under profiles δ close δ_0 can also be written as a differentiable mapping of δ, which we denote by

$D_{\mathrm{DE}}(\delta)$. Then, define $U_{\mathrm{MVE}}(\delta) = U_{\mathrm{RE}}(D_{\mathrm{MVE}}(\delta), \delta)$ and $U_{\mathrm{DE}}(\delta) = U_{\mathrm{RE}}(D_{\mathrm{DE}}(\delta), \delta)$. They are the levels of social welfare attained at the canonical MVE and the DE under the ESG score δ.

Theorem 3 *Assume that for every k, F^k is twice continuously differentiable and $\nabla^2 F^k(D^k)$ is positive definite at every $D^k \neq 0$. Let $\delta \in N^I$ and suppose that $D_{\mathrm{MVE}}(\delta) = D_{\mathrm{DE}}(\delta)$, which we denote simply by $D \equiv (D^1, \ldots, D^K)$. Assume that $(\mathbf{1}, D^1, \ldots, D^K)$ is linearly independent. If D_{DE} are differentiable at δ, then U_{MVE} and U_{DE} are differentiable at δ and $\nabla U_{\mathrm{MVE}}(\delta) = \nabla U_{\mathrm{DE}}(\delta)$.*

The equality $D_{\mathrm{MVE}}(\delta) = D_{\mathrm{DE}}(\delta)$ means that the profiles of output plans at MVE and DE are the same. It is, thus, no surprise that they attain the same social welfare, $U_{\mathrm{MVE}}(\delta) = U_{\mathrm{DE}}(\delta)$. What is surprising is that the first-order change in social welfare caused by changes in the ESG profiles, $\nabla U_{\mathrm{MVE}}(\delta)$ and $\nabla U_{\mathrm{DE}}(\delta)$, are also the same. To see that this theorem implies that shareholder engagement can only make a small difference in social welfare, let's take an arbitrary profile δ of ESG scores. We would like to determine the sign and size of the difference $U_{\mathrm{DE}}(\delta) - U_{\mathrm{MVE}}(\delta)$. If the sign is positive, it means that shareholders' engagement does good to the economy, while if it is negative, then it means that their engagement does harm to the economy. The size also makes economic sense, because U is quasi-linear in the direction of the risk-free bond, and, thus, the difference is equal to the gain from or loss by shareholder engagement in terms of certainty equivalents. The quasi-linearity also guarantees that the difference is equal to both compensated variation and equivalent variation before and after shareholder engagement.

Among other possibilities to evaluate the difference $U_{\mathrm{DE}}(\delta) - D_{\mathrm{MVE}}(\delta)$, one way is to take an "initial" or "reference" profile of ESG scores, denoted by δ_0, under which shareholder engagement makes no difference, and then use linear approximation

$$
\begin{aligned}
&U_{\mathrm{DE}}(\delta) - U_{\mathrm{MVE}}(\delta) \\
&= (U_{\mathrm{DE}}(\delta_0) - U_{\mathrm{MVE}}(\delta_0)) + (\nabla U_{\mathrm{DE}}(\delta) - \nabla U_{\mathrm{MVE}}(\delta))(\delta - \delta_0) + o(\|\delta - \delta_0\|),
\end{aligned}
$$

where $\|\delta - \delta_0\|$ be equal to the L^2 norm $\left(\sum_i \frac{\tau^i}{\bar{\tau}} \mathrm{Var}\left[\delta^i - \delta_0^i\right] \right)^{1/2}$ or any other norm on N^I that converges to zero whenever all the $\mathrm{Var}\left[\delta^i - \delta_0^i\right]$ converge to zero. The following corollary formalizes the idea that shareholder engagement has no first-order impact on social welfare.

Corollary 1 *Let $\delta_0 \in N^I$ and suppose that $D_{\mathrm{MVE}}(\delta_0) = D_{\mathrm{DE}}(\delta_0)$. If D_{DE} are* *differentiable at* δ_0*, then, as* $\delta \to \delta_0$*,*

$$\frac{U_{\mathrm{DE}}(\delta) - U_{\mathrm{MVE}}(\delta)}{\|\delta - \delta_0\|} \to 0.$$

This corollary only gives a limit result on the difference $U_{\mathrm{DE}}(\delta) - U_{\mathrm{MVE}}(\delta)$. How small it really is hinges on two aspects. First, how much nonlinear the difference $U_{\mathrm{DE}}(\delta) - U_{\mathrm{ME}}(\delta)$ is with respect to the ESG profile δ. Second, how much different the two profiles δ and δ_0 are.

The second point, in our view, deserves more serious consideration than the first. To see this point, notice that to apply the theorem, we need to find, for any given ESG profile δ, another profile δ_0 that satisfies $D_{\mathrm{MVE}}(\delta_0) = D_{\mathrm{DE}}(\delta_0)$. The most obvious candidate is the homogeneous profile $\delta_0 = (\delta_0^i)_i$ where $\delta_0^i = \bar{\delta}$ for every i. Then, by Theorem 2, $D_{\mathrm{MVE}}(\delta_0) = D_{\mathrm{DE}}(\delta_0)$ and, by Theorem 3, $U_{\mathrm{DE}}(\delta) - D_{\mathrm{MVE}}(\delta) = o\left(\|\delta - \delta_0\|\right)$ (because, then, $B_M^i = 0$ for δ_0^i). We could, instead, take $\delta_0^i = \bar{\delta} + B_M^i$ for every i to guarantee that $D_{\mathrm{MVE}}(\delta_0) = D_{\mathrm{DE}}(\delta_0)$. The second choice has the advantage over the first because, for every i, $\mathrm{Var}[\delta^i - \delta_0^i]$ is equal to $\mathrm{Var}[A_M^i] + \mathrm{Var}[B_M^i]$ in the first case, but to $\mathrm{Var}[B_M^i]$ in the second. Yet another possibility is to take $\delta_0^i = \bar{\delta} + A_M^i$ for every i, but $\mathrm{Var}[\delta^i - \delta_0^i]$ would then be equal to $\mathrm{Var}[A_M^i]$, which may be larger or smaller than $\mathrm{Var}[B_M^i]$. If the degree of non-linearity of $U_{\mathrm{DE}} - U_{\mathrm{MVE}}$ were constant on N^I we would choose the δ_0 that minimizes $\|\delta - \delta_0\|$ small among all δ_0's that satisfy $D_{\mathrm{MVE}}(\delta_0) = D_{\mathrm{DE}}(\delta_0)$. We saw in Theorem 2 that $D_{\mathrm{MVE}}(\delta_0) = D_{\mathrm{DE}}(\delta_0)$ if and only if $\sum_i \tau^i a^i(D_{\mathrm{MVE}}(\delta_0), \delta_0) B^i(D_{\mathrm{MVE}}(\delta_0), \delta_0) = 0$. It is, therefore, necessary to find how prevalent the set of the ESG profiles δ_0 in N^I is that satisfy $\sum_i \tau^i a^i(D_{\mathrm{MVE}}(\delta_0), \delta_0) B^i(D_{\mathrm{MVE}}(\delta_0), \delta_0) = 0$. However, there seems to be no systematic study to answer this question. We note, also, that in all three cases above, $\bar{\delta}_0 = \bar{\delta}$ and, hence, $D_{\mathrm{MVE}}(\delta_0) = D_{\mathrm{MVE}}(\delta)$. Thus, $U_{\mathrm{DE}}(\delta) - U_{\mathrm{MVE}}(\delta) = U_{\mathrm{DE}}(\delta) - U_{\mathrm{DE}}(\delta_0)$. That is, by restricting ESG profiles to those which satisfy $\bar{\delta} = \bar{\delta}_0$, the difference $U_{\mathrm{DE}}(\delta) - U_{\mathrm{MVE}}(\delta)$ can be derived from $D_{\mathrm{DE}}(\delta)$, a fact that may well facilitate our calculation.

As we will see in the following proof, the crux of the theorem lies in the fact that the coincidence of DE and MVE implies not only the coincidence of output plans but also the coincidence of the (firms', and, then, the society's) objectives embedded in the notions of DE and MVE. Since the common output plans at DE and MVE then satisfies the same first-order condition, any divergence between the two generated by a deviation from the initial ESG profile (under which DE and MVE coincides) has no first-order impact. The consequence of this fact is that $\nabla U_{\mathrm{MVE}}(\delta) = \nabla_\delta U_{\mathrm{RE}}(D, \delta) = \nabla U_{\mathrm{DE}}(\delta)$, which is reminiscent of the envelope theorem, but Theorem 3 is not a consequence of the envelope theorem, because the Dreze equilibrium $D_{\mathrm{DE}}(\delta)$ need not maximize the social welfare $U_{\mathrm{RE}}(D, \delta)$.

Remark 8 Theorem 3 and Corollary 1 concerned with the sum of all investors' utilities, not with individual investors' utilities. It is, thus, conceivable that shareholders' engagement has a positive first-order impact on a particular shareholder's utility. They imply that whenever there is such a shareholder, some other investors who suffer from negative first-order impacts. In other words, any first-order utility gain from shareholders' engagement is attained only at the sacrifice of other investors.

Remark 9 Theorem 3 (and Corollary 1) compares social welfare in two case, one where all firms maximize market value and the other where all firms follow the Dreze criterion. We could consider the intermediate case where some firms maximize market value and other firms follow the Dreze criterion. In this case, this theorem, as well as the existence theorem (Theorem 1) and the coincidence theorem (Theorem 2), is still valid.

In concluding this section, we give a sufficient condition for the existence of a Dreze equilibrium. In Theorem 1, we only showed that there is a pseudo Dreze equilibrium. We now give a sufficient condition, in the special case of Example 1, under which the existence of a Dreze equilibrium is guaranteed. The proof relies on Theorem 2 and Lemma 13.

Proposition 4 *In Example 1, suppose* $(\mathbf{1}, \nu^1, \dots, \nu^K)$ *is linearly independent. Let* $\delta_0 = (\delta_0^i)_i$ *be a profile of ESG scores. Let* $((D^k)_k, (\theta^{ik})_{i,k\geq 0}, R_{\mathrm{f}}, q)$ *be a MVE under* δ_0 *and let* $M = \langle \mathbf{1}, D^1, \dots, D^K \rangle$*. If* $B_M^i = 0$ *for every* i *under* δ_0*, then, for every ESG profile* δ *close to* δ_0*, there is a Dreze equilibrium under* δ*.*

Since $(\mathbf{1}, \nu^1, \dots, \nu^K)$ is linearly independent, $(\mathbf{1}, D^1, \dots, D^K)$ is linearly independent. The assumption of effectively complete markets, $B_M^i = 0$ for every i, implies that the MVE is also a Dreze equilibrium, and that the Dreze equilibria under δ close to δ_0 can be written as a continuously differentiable function of δ, which, in turn, implies its existence. The assumption of effectively complete markets is met when the ESG scores are homogeneous, that is, $\delta_0^1 = \cdots = \delta_0^I$, but, even in that case, we do not require $\delta^1 = \cdots = \delta^I$ as long as δ is close to δ_0.

10 Conclusion

In this paper, we presented a variant of the Capital Asset Pricing Model that incorporates the investors' ESG consciousness to study whether shareholder engagement may make a difference in social welfare. Our model is a minimal deviation from the CAPM, but investor heterogeneity and market incompleteness turned out to be crucial for the role of shareholder engagement. In this sense, our model sits nicely at the juncture of the CAPM and general equilibrium theory with incomplete asset markets.

We formulated shareholder engagement as a Dreze equilibrium, and its absence as a market value maximization equilibrium. We proved the existence of a market-value-maximization equilibrium and a Dreze equilibrium. While the proof of the former was elementary, the proof of the latter relied on a suitably constructed vector bundle and its mod 2 Euler number. We then identified an equivalent condition for the two equilibria to coincide. The equivalent condition can be interpreted as the zero correlation between the investors' integrated ESG concerns (which are reflected by their optimal portfolios) and non-integrated ESG concerns (which can be reflected om the firms' production activities only by shareholder engagement). Necessary conditions for this equivalent condition are investor heterogeneity and incomplete markets. While the literature tends to impose at least one of these two conditions on their models, our result shows that such models cannot capture the impact of shareholder engagement. Last but not least, we showed that while shareholder engagement may make a difference, it has no first-order impact on social welfare. This justifies the subtitle of our paper, "Much ado about nothing."

Here are some suggested topics for future research. First, we should look into how prevalent the set of ESG profiles under which MVE and DE coincide is, as the size of the residual term of the first-order approximation hinges on how far the (heterogenous) profile of ESG scores under consideration is from this set of ESG profiles. Second, while we dealt with the two polar cases where all firms employ the Dreze criterion or no firm employs the Dreze criterion, we should extend the analysis to the case the case where some, but not all, firms employ the Dreze criterion. By doing so, we can see whether a firm has an incentive to unilaterally shift from the market value criterion to the Dreze criterion. Third, while we concentrate on the social welfare, we should clarify the impact of shareholder engagement on individual investors' (shareholders') welfare, and also on the share prices and the cost of capital. A companion paper (Hara, Hens, and Trutwin (2024)) conducts some numerical analysis on portfolios, output plans, cost of capital, and individual welfare, as well as social welfare. It shows that while a quantitative difference in social welfare between MVE and DE which is, indeed, small, there may be large difference in portfolios and individual welfare. Analytical results that would underlie these numerical results are much needed.

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Appendices

A Proofs

Proof of Lemma 1 The market-clearing condition is met by construction. The constrained utility maximization condition follows from the consistency of π. In the case of a MVE, π coincides, on M, with the state price density with respect to which all the firms maximize profit at the MVE. In the case of a Dreze equilibrium, since π^i coincide with π on M for every i, $\sum_i \theta^{ik}\pi^i$ coincides with π on M for every k. The restricted profit maximization condition follows from these facts. ///

Proof of Lemma 2 Since the U^i are quasi-linear in $\mathbf{1}$, an M-restricted feasible allocation is M-restricted efficient if and only if it maximizes the sum $\sum_i U^i(c^i)$ of utilities subject to the M-constrained feasibility. As for its maximized value, by Lemma 3,

$$
\begin{aligned}
\mathrm{Var}[c^i] &= \left(\frac{\tau^i}{\bar{\tau}}\right)^2 \mathrm{Var}[\bar{D}] - 2\frac{(\tau^i)^2}{\bar{\tau}}\mathrm{Cov}\left[\bar{D}, A^i_M\right] + (\tau^i)^2\mathrm{Var}[A^i_M],\\
E[\delta^i c^i] &= E\left[\left(\bar{\delta} + A^i_M\right)\left(\frac{\tau^i}{\bar{\tau}}\bar{D} - \tau^i A^i_M\right)\right]\\
&= \frac{\tau^i}{\bar{\tau}}E[\bar{\delta}\bar{D}] - E\left[\bar{\delta}\left(\tau^i A^i_M\right)\right] + E\left[\bar{D}\left(\frac{\tau^i}{\bar{\tau}}A^i_M\right)\right] - \tau^i\mathrm{Var}[A^i_M],
\end{aligned}
$$

where $\bar{D} = \sum_k D^k$. Hence,

$$
\begin{aligned}
U^i(c^i) =& E[c^i] - \frac{\tau^i}{2(\bar{\tau})^2}\mathrm{Var}[\bar{D}] + \frac{\tau^i}{\bar{\tau}}\mathrm{Cov}\left[\bar{D}, A^i_M\right] - \frac{\tau^i}{2}\mathrm{Var}[A^i_M]\\
&- \frac{\tau^i}{\bar{\tau}}E[\bar{\delta}\bar{D}] + E\left[\bar{\delta}\left(\tau^i A^i_M\right)\right] - E\left[\bar{D}\left(\frac{\tau^i}{\bar{\tau}}A^i_M\right)\right] + \tau^i\mathrm{Var}[A^i_M]\\
=& E[c^i] - \frac{\tau^i}{2(\bar{\tau})^2}\mathrm{Var}[\bar{D}] - \frac{\tau^i}{\bar{\tau}}E[\bar{\delta}\bar{D}] + E\left[\bar{\delta}\left(\tau^i A^i_M\right)\right] + \frac{\tau^i}{2}\mathrm{Var}[A^i_M].
\end{aligned}
$$

Since $\sum_i E[c^i] = \bar{\theta}^0\mathbf{1} + E[\bar{D}]$, $\sum_i \tau^i = \bar{\tau}$, and $\sum_i \tau^i A^i_M = 0$, by summing the left- and right-hand sides of this equality over i, we obtain (3). ///

Proof of Lemma 3 1. Thanks to (1), this characterization can be derived in the same way as the characterization of the equilibrium of the CAPM with incomplete markets for the aggregate endowment $\bar{\theta}^0\mathbf{1}+\bar{D}$ as in Koch-Medina and Wenzelburger (2018) and Hara (2022).

2. Since F^k is convex for every k, the allocation $\left(((1/2)D^k + (1/2)\hat{D}^k)_k, ((1/2)c^i + (1/2)\hat{c}^i)_i\right)$

is M-constrained feasible. Since U^i is concave,

$$U^i\left(\frac{1}{2}c^i+\frac{1}{2}\hat{c}^i\right)\geq\frac{1}{2}U^i\left(c^i\right)+\frac{1}{2}U^i\left(\hat{c}^i\right) \tag{14}$$

for every i. Summing each side over i, we obtain

$$\sum_i U^i\left(\frac{1}{2}c^i+\frac{1}{2}\hat{c}^i\right)\geq\sum_i\left(\frac{1}{2}U^i\left(c^i\right)+\frac{1}{2}U^i\left(\hat{c}^i\right)\right)=\frac{1}{2}\sum_i U^i\left(c^i\right)+\frac{1}{2}\sum_i U^i\left(\hat{c}^i\right). \tag{15}$$

By Lemma 2, the left-hand side is less than or equal to each of the two terms on the far right-hand side. Hence, (15) holds as an equality. Thus, the weak inequality $\geq$ in (14) holds as an equality for every i. By (1), for every i, there is a $\rho^i\in\boldsymbol{R}$ such that $c^i-\hat{c}^i=\rho^i\mathbf{1}$. Hence,

$$0=\sum_i U^i(c^i)-\sum_i U^i(\hat{c}^i)=\sum_i\left(U^i(c^i)-U^i(\hat{c}^i)\right)=\sum_i\rho^i.$$

Thus, $\sum_{i\geq1}c^i=\sum_{i\geq1}\hat{c}^i$. Since $\sum_{i\geq1}c^i=\bar{\theta}^0\mathbf{1}+\sum_{k\geq1}D^k$ and $\sum_{i\geq1}\hat{c}^i=\bar{\theta}^0\mathbf{1}+\sum_{k\geq1}\hat{D}^k$, $\sum_{k\geq1}D^k=\sum_{k\geq1}\hat{D}^k$.

3. Since F^k is convex for every k,

$$F^k\left(\frac{1}{2}D^k+\frac{1}{2}\hat{D}^k\right)\geq\frac{1}{2}F^k\left(D^k\right)+\frac{1}{2}F^k\left(\hat{D}^k\right), \tag{16}$$

and if F^k is strictly convex and $D^k\neq\hat{D}^k$, then the weak inequality $\geq$ holds as a strict inequality. Summing each side over k, we obtain

$$\sum_k F^k\left(\frac{1}{2}D^k+\frac{1}{2}\hat{D}^k\right)\geq\sum_k\left(\frac{1}{2}F^k\left(D^k\right)+\frac{1}{2}F^k\left(\hat{D}^k\right)\right)=\frac{1}{2}\sum_k F^k\left(D^k\right)+\frac{1}{2}\sum_k F^k\left(\hat{D}^k\right). \tag{17}$$

By constrained feasibility, the left-hand side is less than or equal to $\bar{w}$. By constrained efficiency, each of the two terms on the far right-hand side is equal to $\bar{w}$. Hence, (17) holds as an equality. Thus, the weak inequality $\geq$ in (16) holds as an equality for every k. Hence, for every k, if F^k is strictly convex, then $D^k=\hat{D}^k$. Suppose that F^k is strictly convex and differentiable. By the profit maximization condition of each M-constrained equilibrium, π and $R_{\mathrm{f}}\Lambda(p)^{-1}\nabla F^k(D_k)$ coincide on M, and $\hat{\pi}$ and $\hat{R}_{\mathrm{f}}\Lambda(p)^{-1}\nabla F^k(\hat{D}^k)$ coincide on M. Since $\pi=\hat{\pi}$ on M and $D^k=\hat{D}^k$, this implies that $R_\mathrm{f} = \hat{R}_\mathrm{f}$. Thus,

$$\begin{aligned} E[\pi c^i] &= \bar{\theta}^{i0} + \sum_{k\geq 1} \left(E[\pi D^k] - R_\mathrm{f} F^k(D^k) \right) \bar{\theta}^{ik} + R_\mathrm{f} w^i \\ &= \bar{\theta}^{i0} + \sum_{k\geq 1} \left(E[\hat{\pi} \hat{D}^k] - \hat{R}_\mathrm{f} F^k(\hat{D}^k) \right) \bar{\theta}^{ik} + \hat{R}_\mathrm{f} w^i = E[\hat{\pi} \hat{c}^i] \end{aligned}$$

for every i. Since $\pi = \hat{\pi}$ coincide on M and $c^i - \hat{c}^i$ is a scalar multiple of $\mathbf{1}$, this equality implies that $c^i = \hat{c}^i$ for every i. ///

Proof of Lemma 5 For each $k \geq 1$, define $Y^k_M = Y^k \cap (\boldsymbol{R} \times M)$ and $\bar{Y}_M = \sum_{k\geq 1} Y^k_M$. Define

$$\begin{aligned} X &= \left(\boldsymbol{R}_+ \times \left\{ \sum_i \hat{c}^i \in \boldsymbol{R}^S \mid \hat{c}^i \in M \text{ and } U^i(\hat{c}^i) > U^i(c^i) \text{ for every } i \right\} \right) + (\{0\} \times M^\perp), \\ Z &= \bar{Y}_M + \left\{ \left(\bar{w}, \bar{\theta}^0 \mathbf{1} \right) \right\}. \end{aligned}$$

where $M^\perp$ is the p-orthogonal complement of M. Then, $X \cap Z = \varnothing$. Hence, by the separating hyperplane theorem, there is a $(R_\mathrm{f}, \pi) \in \boldsymbol{R} \times \boldsymbol{R}^S$ such that at least one of R_f and π is not zero and separates X and Y at $\left(0, \bar{\theta}^0 \mathbf{1} + \sum_k D^k\right)$, because $\sum_k F^k(D^k) = \bar{w}$ by constrained efficiency. Since $X \supseteq X + (\{0\} \times M^\perp)$, $(R_\mathrm{f}, \pi) \in \boldsymbol{R} \times M$. Since the U^i are strictly increasing in the direction of $\mathbf{1}$, the (relative) interior of X in $\boldsymbol{R} \times M$ is nonempty, and $E[\pi] > 0$. If $R_\mathrm{f} \leq 0$, then, by replacing the production plan $\bar{D}$ by $\bar{D} + \mathbf{1}$, the firm would increase its profit by 1 or more (because $E[\pi] > 0$), which is a contradiction. Thus, $R_\mathrm{f} > 0$. Hence, by dividing R_f and π by $E[\pi]$, we obtain (R_f, π) as needed. ///

Proof of Lemma 6 For each $k \geq 1$, define $Y^k_M = Y^k \cap (\boldsymbol{R} \times M)$ and $\bar{Y}_M = \sum_{k\geq 1} Y^k_M$. Consider the auxiliary economy in which there is only one firm with the production possibility set $\bar{Y}_M$ and only one investor with the utility function

$$\bar{U}(\bar{c}) = E[\bar{c}] - \frac{\bar{\psi}}{2} \mathrm{Var}[\bar{c}] - \bar{\gamma} \cdot \bar{c}$$

and endowments for the assets $\bar{\theta}^{ik}$ and for the period-zero input $\bar{w}$.

The Pareto-efficient allocations of this auxiliary economy are nothing but the solutions to the following problem:

$$\begin{aligned} \max_{(\bar{c}_0, \bar{c}) \in \boldsymbol{R}_+ \times \boldsymbol{R}^S} \quad & \bar{U}(\bar{c}) \\ \text{subject to} \quad & (\bar{c}_0, \bar{c}) \in \bar{Y}_M + \left\{ \left(\bar{w}, \bar{\theta}^0 \mathbf{1} \right) \right\}. \end{aligned} \tag{18}$$

In this formulation, we included the consumption level $\bar{c}_0$ on period 0 so that the consumption set of the representative investor is $\boldsymbol{R}_+ \times \boldsymbol{R}^S$, not just $\boldsymbol{R}^S$, although the period-0 consumption does not affect utility. Since $\bar{U}$ is strictly monotone in the direction of $\mathbf{1}$, at every solution (if any) of this problem, $\bar{c}_0 = 0$. Hence, this maximization problem can be rewritten as

$$\begin{aligned} \max_{\bar{D} \in \boldsymbol{R}_+^S} \quad & \bar{U}\left(\bar{D} + \bar{\theta}^0 \mathbf{1}\right) \\ \text{subject to} \quad & (-\bar{w}, \bar{D}) \in \bar{Y}_M. \end{aligned}$$

Since the objective function is continuous in $\bar{D}$ and the set of $\bar{D}$ that satisfy the constraint $(-\bar{w}, \bar{D}) \in \bar{Y}_M$ is nonempty and compact, the problem has a solution. Since $\bar{U}$ is strictly quasi-concave and the set of $\bar{D}$ that satisfy the constraint $(-\bar{w}, \bar{D})$ is convex, the solution is unique. By an abuse of notation, we denote it by $\bar{D}$. Write $\bar{c} = \bar{D} + \bar{\theta}^0 \mathbf{1}$. By Lemma 5, there is a vector (R_f, π) with $R_\mathrm{f} > 0$ and $E[\pi] = 1$ such that $(\bar{D}, \bar{c^i}, R_\mathrm{f}, \pi)$ is an M-constrained equilibrium of the auxiliary economy. Thus, π coincides, on M, a positive multiple of the density of the utility gradient (in $\boldsymbol{R}^S$) of the representative investor,

$$\bar{\pi} \equiv \nabla \bar{U}\left(\bar{D} + \bar{\theta}^0 \mathbf{1}\right) \Lambda(p)^{-1} = \mathbf{1} - \frac{1}{\bar{\tau}}\left(\bar{D} - E[\bar{D}]\mathbf{1}\right) - \bar{\delta}.$$

Since $E[\bar{\pi}] = 1$, $\pi = \bar{\pi}$ on M.

Since $\bar{Y}_M = \sum_k Y_M^k$, for each k, there is a $D^k \in \boldsymbol{R}_+^S \cap M$ such that D^k maximizes profit in Y_M^k under (R_f, π) and $\sum_k D^k = \bar{D}$. For each i, define

$$c^i = \rho^i \mathbf{1} + \frac{\tau^i}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \tau^i A_M^i,$$

where ρ^i is chosen to satisfy

$$E[\pi c^i] = \bar{\theta}^{i0} + \sum_{k=1}^{K} \left(E[\pi D^k] - R_\mathrm{f} F_M^k(D^k)\right) \bar{\theta}^{ik} + R_\mathrm{f} w^i.$$

We now prove that $((D^k)_k, (c^i)_i, R_\mathrm{f}, \pi)$ is an M-constrained equilibrium. By definition, $c^i \in M$ (by $A_M^i \in M$) for every i and $D^k \in M$ for every k. By the definition of the ρ^i and $E[\pi\bar{c}] = \sum_i E[\pi c^i]$, $\sum_i \rho^i = E[\bar{D}] + \bar{\theta}^0$. This, together with $\sum_i \tau^i A_M^i = 0$, implies that the market clearing condition, $\sum_i c^i = \bar{c} = \bar{D} + \bar{\theta}^0 \mathbf{1}$, is met. The profit maximization condition is met by the definition of D^k. By the definition of c^i, $\pi^i(c^i) = \bar{\pi} - B_M^i$ for all i. Since they all coincide on M (because $B_M^i \in M^\perp$), the utility maximization condition is met. ///

Proof of Proposition 1 By Lemma 6, there is an $\boldsymbol{R}^S$-constrained (unconstrained) equilibrium $((D^k)_k, (\tilde{c}^i)_i, R_\mathrm{f}, \pi)$. By Lemma 3,

$$\tilde{c}^i - E[\tilde{c}^i]\mathbf{1} = \frac{\tau^i}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \tau^i(\delta^i - \bar{\delta})$$

for every i, and

$$\pi = \mathbf{1} - \frac{1}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta}.$$

For each $k \geq 1$, write $q^k = E[\pi D^k] - R_\mathrm{f} F^k(D^k)$. Write $M = \langle \mathbf{1}, D \rangle$. For each i, let

$$c^i = \rho^i \mathbf{1} + \frac{\tau^i}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \tau^i A^i_M,$$

where ρ^i is chosen to satisfy

$$E[\pi c^i] = \bar{\theta}^{i0} + \sum_{k=1}^{K} \left(E[\pi D^k] - R_\mathrm{f} F^k_M(D^k) \right) \bar{\theta}^{ik} + R_\mathrm{f} w^i.$$

Then, $\pi^i(c^i) = \pi - B^i_M$. Thus, $\pi^i(c^i) = \pi$ on M. Moreover, $\sum_i c^i = \bar{\theta}^0 \mathbf{1} + \bar{D}$. Hence, for each i and $k \geq 0$, there is a θ^{ik} such that $\sum_{k \geq 0} \theta^{ik} D^k = c^i$ for every i, $\sum_i \theta^{i0} = \bar{\theta}^0$, and $\sum_i \theta^{ik} = 1$ for every k. Then, $((D^k)_k, (\theta^{ik})_{i,k}, R_\mathrm{f}, q)$ is a MVE. Indeed, the market-clearing condition is met by construction. The profit maximization condition follows from that of the $\boldsymbol{R}^S$-constrained equilibrium. The utility maximization condition follows from that of the $\boldsymbol{R}^S$-constrained equilibrium and $\pi^i(c^i) = \pi$ on M. ///

Proof of Proposition 2 Since $((D^k)_k, (\theta^{i0}, \theta^i)_i, R_\mathrm{f}, (q^0, q^1, \ldots, q^K))$ is a Dreze equilibrium, if we define c^i as in the statement of this proposition but let π coincide with π^i on M for any i, then, by Lemma 1, $((D^k)_k, (c^i)_i, R_\mathrm{f}, \pi)$ is the M-constrained equilibrium.

By Lemma 6,

$$c^i - E[c^i]\mathbf{1} = \frac{\tau^i}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \tau^i A^i_M,$$

for every i. Thus,

$$\pi^i(c^i) = \mathbf{1} - \bar{\psi}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta} - B^i_M$$

for every i. Since this coincides with $\mathbf{1} - \bar{\psi}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta}$ on M, we can, in fact, take $\pi = \mathbf{1} - \bar{\psi}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta}$. Then, π is consistent with q; that is, $q^k = E[\pi D^k] - R_\mathrm{f} F^k(D^k)$ for every $k \geq 1$, and $((D^k)_k, (c^i)_i, R_\mathrm{f}, \pi)$ is an M-constrained equilibrium (which is unique up to the p-orthogonal projection of π onto $M^\perp$).

Since $c^i = \sum_{k \geq 0} \theta^{ik} D^k$,

$$\sum_{k \geq 1} \theta^{ik}(D^k - E[D^k]\mathbf{1}) = \frac{\tau^i}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \tau^i A^i_M,$$

Define $a^i = (a^{i1}, \dots, a^{iK}) \in \boldsymbol{R}^K$ by letting, for each $k \geq 1$,

$$a^{ik} = \frac{1}{\tau^i}\left(\frac{\tau^i}{\bar{\tau}} - \theta^{ik}\right),$$

then

$$\sum_{k\geq 1} a^{ik}(D^k - E[D^k]\mathbf{1}) = \frac{1}{\tau^i}\left(\frac{\tau^i}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \sum_{k\geq 1}\theta^{ik}(D^k - E[D^k]\mathbf{1})\right) = A^i_M.$$

By the profit maximization condition of a Dreze equilibrium, $R_{\mathrm{f}}\xi^k(D^k) = \sum_i \theta^{ik}\pi^i(c^i)$. Since $\sum_i \tau^i a^{ik} = 0$ and $\sum_i \tau^i B^i_M = 0$,

$$\sum_i \theta^{ik}\pi^i(c^i) = \sum_i \left(\frac{\tau^i}{\bar{\tau}} - \tau^i a^{ik}\right)(\pi - B^i_M) = \pi + \sum_i \tau^i a^{ik} B^i_M.$$

Hence,

$$\pi - R_{\mathrm{f}}\xi^k(D^k) = \pi - \sum_i \theta^{ik}\pi^i = -\sum_i \tau^i a^{ik} B^i_M = -B_M\Lambda(\tau)\begin{pmatrix} a^{1k} \\ \vdots \\ a^{Ik}\end{pmatrix} \in \boldsymbol{R}^S. \tag{19}$$

Write $a = (a^1, \dots, a^I) \in \boldsymbol{R}^{K\times I}$, then, by gathering the above equality over all k, we obtain

$$\left(\pi - R_{\mathrm{f}}\xi^1(D^1), \dots, \pi - R_{\mathrm{f}}\xi^K(D^K)\right) = -B_M\Lambda(\tau)a^\top.$$

By multiplying $\hat{D}^\top$ to each side of the above equality from right, we obtain

$$\left(\pi - R_{\mathrm{f}}\xi^1(D^1), \dots, \pi - R_{\mathrm{f}}\xi^K(D^K)\right)\hat{D}^\top = -B_M\Lambda(\tau)(\hat{D}a)^\top = -B_M\Lambda(\tau)A^\top_M.$$

By eliminating the last $S-K$ columns of each side, we complete the proof. ///

Proof of Proposition 3 Since $((D^k)_k, (c^i)_i, R_{\mathrm{f}}, \pi)$ is the M-constrained equilibrium that satisfies

$$\pi = \mathbf{1} - \frac{1}{\bar{\tau}}\left(\bar{D} - E\left[\bar{D}\right]\mathbf{1}\right) - \bar{\delta},$$

Lemma 6 implies that

$$\begin{aligned} c^i - E[c^i]\mathbf{1} &= \frac{\tau^i}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \tau^i A^i_M, \\ \pi^i(c^i) &= \mathbf{1} - \bar{\psi}(\bar{D}_1 - E[\bar{D}_1]\mathbf{1}) - \bar{\delta} - B^i_M = \pi - B^i_M \end{aligned}$$

for every i.

Since $\operatorname{rank}\hat{D} = K$, for each i and $k \geq 0$, there is a unique θ^{ik} such that $c^i = \sum_{k\geq 0} \theta^{ik} D^k$; and they together satisfy $\sum_i \theta^{i0} = \sum_i \bar{\theta}^{i0}$ and $\sum_i \theta^{ik} = 1$ for every $k \geq 1$. Moreover, as we saw in the proof of Proposition 2, if we define

$$a^{ik} = \frac{1}{\tau^i}\left(\frac{\tau^i}{\bar{\tau}} - \theta^{ik}\right),$$

then $\sum_{k\geq 1} a^{ik}\hat{D}^k = A^i_M$.

We need to show that $((D^k)_k, (\theta^{ik})_{i,k\geq 0}, R_{\mathrm{f}}, (q^0, q^1, \dots q^K))$ is a Dreze equilibrium. It follows from the standard equivalence on equilibria in terms of asset prices and portfolios versus state price densities and consumption plans that the utility maximization condition is met. By the definition of $(\theta^{ik})_{i,k\geq 0}$, the market clearing condition is met. We shall prove the profit maximization condition by using the definition of a pseudo Dreze equilibrium and the assumption that $\operatorname{rank}\hat{D} = \operatorname{rank}\tilde{D} = K$.

Define $a = (a^1, \dots, a^I) \in \boldsymbol{R}^{K\times I}$. Since $\operatorname{rank}\tilde{D} = K$, there is an $H \in \boldsymbol{R}^{(S-K)\times K}$ such that $\check{D} = H\tilde{D}$. Thus,

$$\begin{pmatrix} \tilde{A}_M \\ \check{A}_M \end{pmatrix} = A_M = \hat{D}a = \begin{pmatrix} \tilde{D}a \\ \check{D}a \end{pmatrix} = \begin{pmatrix} \tilde{D}a \\ H\check{D}a \end{pmatrix}.$$

Thus, $\check{A}_M = H\tilde{A}_M$.

By the definition of a pseudo Dreze equilibrium,

$$\left(\pi - R_{\mathrm{f}}\xi^1(D^1), \dots, \pi - R_{\mathrm{f}}\xi^K(D^K)\right)\tilde{D}^\top = -B_M\Lambda(\tau)\tilde{A}_M^\top,$$

By multiplying $H^\top$ each side from right, we obtain

$$\left(\pi - R_{\mathrm{f}}\xi^1(D^1), \dots, \pi - R_{\mathrm{f}}\xi^K(D^K)\right)\check{D}^\top = -B_M\Lambda(\tau)\check{A_M}^\top,$$

By combining these two equalities together, we obtain

$$\left(\pi - R_{\mathrm{f}}\xi^1(D^1), \dots, \pi - R_{\mathrm{f}}\xi^K(D^K)\right)\hat{D}^\top = -B_M\Lambda(\tau)A_M^\top.$$

Again by $A_M = \hat{D}a$, this equality can be rewritten as

$$\left(\pi - R_{\mathrm{f}}\xi^1(D^1), \dots, \pi - R_{\mathrm{f}}\xi^K(D^K)\right)\hat{D}^\top = -B_M\Lambda(\tau)a^\top\hat{D}^\top.$$

Since $\operatorname{rank}\hat{D} = K$, this equality is equivalent to

$$\left(\pi - R_{\mathrm{f}}\xi^1(D^1), \dots, \pi - R_{\mathrm{f}}\xi^K(D^K)\right) = -B_M\Lambda(\tau)a^\top,$$

that is,

$$\pi - R_{\mathrm{f}}\xi^k(D_1^k) = -B_M\Lambda(\tau)\begin{pmatrix} a^{1k} \\ \vdots \\ a^{Ik} \end{pmatrix} = -\sum_i \tau^i a^{ik} B_M^i$$

for every $k \geq 1$. Finally, as we saw in the proof of Proposition 2,

$$\sum_i \theta^{ik}\pi^i = \sum_i \left(\frac{\tau^i}{\bar{\tau}} - \tau^i a^{ik}\right)(\pi - B_M^i) = \pi + \sum_i \tau^i a^{ik} B_M^i.$$

Hence, $\sum_i \theta^{ik}\pi^i = R_{\mathrm{f}}\xi^k(D^k)$. The profit maximization condition of a Dreze equilibrium is thus met. ///

Proof of Lemma 7 Since $\pi_L - R_{\mathrm{f}L}\xi^k(D_L^k) \in L^\perp$ for every k by the profit maximization condition of the $(\langle \mathbf{1}\rangle + L)$-constrained equilibrium and since the first term consists of K linear combination of these K vectors, it belongs to $(L^\perp)^K$. Since $B_L^i \in L^\perp$ and since the second term consists of K linear combination of these I vectors, it belongs to$(L^\perp)^K$ as well. Thus, $\eta(L) \in (L^\perp)^K$ and, hence, $\sigma(L) \in \Xi$ for every $L \in \mathscr{L}$. ///

Proof of Lemma 8 Since the mapping defined on $\mathscr{L}$, $L \mapsto \tilde{A}_L$, is continuous, it suffices to show that $L \mapsto \big((D_L^k)_k, (c_L^i)_i, R_{\mathrm{f}L}, \pi_L\big)$ is continuous. For each L, define $\bar{F}_L : \boldsymbol{R}_+^S \cap (\langle \mathbf{1}\rangle + L) \to \boldsymbol{R}_+$ by letting

$$\bar{F}_L(\bar{D}) = \min\left\{\sum_k F^k(D^k) \mid D^k \in \boldsymbol{R}_+^S \cap (\langle \mathbf{1}\rangle + L) \text{ for every } k \text{ and } \sum_k D^k = \bar{D}\right\}$$

for every $\bar{D} \in \boldsymbol{R}_+^S \cap M$. It can be shown that $\bar{F}_L$ is the cost function of the $(\langle \mathbf{1}\rangle + L)$-constrained aggregate production set $\bar{Y}_{\langle \mathbf{1}\rangle+L}$; that is, $\bar{Y}_{\langle \mathbf{1}\rangle+L}$ coincides with

$$\big\{(-\bar{F}_L(\bar{D}), \bar{D}) \in \boldsymbol{R} \times \boldsymbol{R}_+^S \mid \bar{D} \in \boldsymbol{R}_+^S \cap (\langle \mathbf{1}\rangle + L)\big\} - (\boldsymbol{R}_+ \times (\boldsymbol{R}_+^S \cap (\langle \mathbf{1}\rangle + L))).$$

Then, the maximization problem (18), where $M = \langle \mathbf{1}\rangle + L$, can be rewritten as

$$\begin{aligned} \max_{\bar{D} \in \boldsymbol{R}_+^S \cap (\langle \mathbf{1}\rangle + L)} \quad & \bar{U}(\bar{D} + \bar{\theta}^0 \mathbf{1}) \\ \text{subject to} \quad & \bar{F}_L(\bar{D}) \leq \bar{w}. \end{aligned}$$

Write $\bar{D}_L = \sum_{k \geq 1} D_L^k$. Then, $\bar{D}_L$ is the solution to this maximization problem. Since the set $\boldsymbol{R}_+^S \cap (\langle \mathbf{1}\rangle + L)$ depends continuously on (is an upper and lower hemicontinuous correspondence of) L, the mapping $L \mapsto \bar{D}_L$ is continuous.

Since the Hessians of F^k are positive definite for every k, $\bar{F}_L$ is continuously differentiable. Since $\bar{D}_L$ maximizes profit $E[\pi_L \bar{D}] - R_{fL}\bar{F}_L(\bar{D})$ on $\boldsymbol{R}_+^S \cap (\langle \mathbf{1} \rangle + L)$ under (R_{fL}, π_L), the first-order condition implies that R_{fL} is a continuous function of L. As for π_L, since $\pi_L = \mathbf{1} - \bar{\psi}(\bar{D}_L - E[\bar{D}_L]\mathbf{1}) - \bar{\delta}$, π_L is a continuous function of L. Thus, by the constrained profit maximization condition, D_L^k is a continuous mapping of L for every k. Hence, the wealth level of investor i, $\bar{\theta}^{i0} + \sum_{k \geq 1}(E[\pi_L D_L^k] - R_{fL}F^k(D_L^k))\bar{\theta}^{ik} + R_{fL}w_i$, is a continuous function of L for every i. By the utility maximization condition, c_L^i is a continuous mapping of L for every i. ///

Proof of Theorem 2 1. The profit maximization condition for the MVE implies that $\pi - R_f\xi^k(D^k) = 0$ for every k. Since $\pi = \mathbf{1} - \bar{\psi}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta}$, as was shown in the proof of Proposition 2, the profit maximization condition for the Dreze equilibrium implies that

$$\left(\pi - R_f\xi^1(D^1), \ldots, \pi - R_f\xi^K(D^K)\right)\hat{D}^\top = -B_M\Lambda(\tau)A_M^\top.$$

Hence, $B_M\Lambda(\tau)A_M^\top = 0$.

2. For each $k \geq 1$, write $\hat{D}^k = D^k - E[D^k]\mathbf{1}$ and $\hat{D} = (\hat{D}^1, \ldots, \hat{D}^K) \in \boldsymbol{R}^{S\times K}$. As in the proof of Proposition 2, define $a^i = (a^{i1}, \ldots, a^{iK}) \in \boldsymbol{R}^K$ by letting, for each $k \geq 1$,

$$a^{ik} = \frac{1}{\tau^i}\left(\frac{\tau^i}{\bar{\tau}} - \theta^{ik}\right),$$

and $a = (a^1, \ldots, a^I) \in \boldsymbol{R}^{K\times I}$. Then, $\hat{D}a = A_M$. Since $B_M\Lambda(\tau)A_M^\top = 0$, this implies that $\left(B_M\Lambda(\tau)a^\top\right)\hat{D}^\top = 0$. Since $(\mathbf{1}, D^1, \ldots, D^K)$ is linearly independent, the row vectors of $\hat{D}^\top$ (the column vectors of $\hat{D}$) constitute a linearly independent set. Hence, $B_M\Lambda(\tau)a^\top = 0$, that is, $\sum_i \tau^i a^{ik}B_M^i = 0$ for every $k \geq 1$. As we showed in the proof of Proposition 2,

$$\sum_i \theta^{ik}\pi^i = \pi + \sum_i \tau^i a^{ik}B_M^i.$$

Hence, $\pi = \sum_i \theta^{ik}\pi^i$ for every k. Since the profit maximization condition for the MVE is met under π, the profit maximization condition for a Dreze equilibrium is met as well. ///

Proof of Lemma 9 Suppose, first, that Condition 1 is met. Then, $\pi \equiv \mathbf{1} - \frac{1}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta}$ is the (canonical) state price density under which D^k maximizes profit $E[\pi D^k] - R_f F^k(D^k)$ for every k. The first-order condition for profit maximization is nothing but (12). The equality $\sum_k F^k(D^k) = \bar{w}$ follows from the utility maximization condition and the positivity of R_f. Suppose, conversely, that

Condition 2 is met. Let $\pi = \mathbf{1} - \bar{\tau}^{-1}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta}$ and $q^k = E[\pi D^k] - R_\mathrm{f} F^k(D^k)$ for each $k \geq 1$. For each i, write $\delta^i - \bar{\delta} = A^i(D,\delta) + B^i(D,\delta)$, the projections onto the linear subspace spanned by $(\mathbf{1}, D^1, \dots, D^K)$ and its p-orthogonal complement. Let $a^i(D,\delta) \in \boldsymbol{R}^K$ satisfy $A^i(D,\delta) = \sum_{k\geq 1} a^{ik}(D,\delta)(D^k - E[D^k]\mathbf{1})$ and $\sum_i \tau^i a^{ik}(D,\delta) = 0$. Then, define $(\theta^{ik})_{i,k\geq 1}$ by letting $\theta^{ik} = \tau^i/\bar{\tau} - \tau^i a^{ik}(D,\delta)$ for each i and k. Define

$$\theta^{i0} = \bar{\theta}^{i0} + \sum_{k\geq 1} q^k \bar{\theta}^{ik} + R_\mathrm{f} w^i - \left(\sum_{k\geq 1} q^k \theta^{ik} + R_\mathrm{f} \sum_{k=1}^{K} F^k(D^k)\theta^{ik} \right).$$

Since the after-trade shareholders bear the costs $R_\mathrm{f} F^k(D^k)$ for production, $\theta^i = (\theta^{ik})_{k\geq 0}$ satisfies the budget constraint for each i. The feasibility $\sum_i \theta^{ik} = 1$ is met by construction for every $k \geq 1$, and the feasibility $\sum_i \theta^{i0} = \sum_i \bar{\theta}^{i0}$ is met by $\sum_k F^k(D^k) = \bar{w}$. Write $c^i = \sum_{k\geq 0} \theta^{ik} D^k$ and his p-discounted utility gradient (density) $\nabla U^i(c^i)\Lambda(p)^{-1}$ by π^i. Then,

$$\begin{aligned} c^i - E[c^i]\mathbf{1} &= \frac{\tau^i}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \tau^i A^i(D,\delta), \\ \pi^i &= \pi - B^i(D,\delta) \end{aligned}$$

for every i. Since π and π^i coincide on the subspace spanned by $(\mathbf{1}, D^1, \dots, D^K)$, the utility maximization condition is met. By (12), the profit maximization condition is met under π for every k. Hence, Condition 1 is met. ///

Proof of Lemma 10 Suppose, first, that Condition 1 is met. Denote the canonical state price density by π:

$$\pi = \mathbf{1} - \frac{1}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta}.$$

Then, $k \geq 1$, let $q^k = E[\pi D^k] - R_\mathrm{f} F^k(D^k)$ for each $k \geq 1$. For each i, write $c^i = \sum_{k\geq 0} \theta^{ik} D^k$ and denote its p-discounted utility gradient (density) $\nabla U^i(c^i)\Lambda(p)^{-1}$ by π^i. By the utility maximization and feasibility conditions,

$$\begin{aligned} c^i - E[c^i]\mathbf{1} &= \frac{\tau^i}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \tau^i A^i(D,\delta), \\ \pi^i &= \pi - B^i(D,\delta) \end{aligned}$$

for every i. Define $a^i \in \boldsymbol{R}^K$ by letting $a^{ik} = 1/\bar{\tau} - \theta^{ik}/\tau^i$ for each $k \geq 1$. Then, $A^i(D,\delta) = \sum_{k\geq 1} a^{ik}(D^k - E[D^k]\mathbf{1})$, $\sum_i \tau^i a^i = 0$, and

$$\sum_i \theta^{ik}\pi^i = \sum_i \left(\frac{\tau^i}{\bar{\tau}} - \tau^i a^{ik} \right)(\pi - B^i(D,\delta)) = \pi + \sum_i \tau^i a^{ik} B^i(D,\delta). \qquad (20)$$

The first-order necessary necessary condition for profit maximization is that

$$\sum_i \theta^{ik}\pi^i = R_\mathrm{f}\xi^k(D^k)$$

for every k. By (20), this is equivalent to (13). Hence, Condition 2 is met.

Suppose, conversely, that Condition 2 is met. For each $k \geq 1$, let $q^k = E[\pi D^k] - R_\mathrm{f}F^k(D^k)$. For each i and $k \geq 1$, define

$$\theta^{ik} = \frac{\tau^i}{\bar{\tau}} - \tau^i a^{ik}.$$

Then, define

$$\theta^{i0} = \bar{\theta}^{i0} + \sum_{k\geq 1} q^k\bar{\theta}^{ik} + R_\mathrm{f}w^i - \left(\sum_{k\geq 1} q^k\theta^{ik} + R_\mathrm{f}\sum_{k=1}^{K} F^k(D^k)\theta^{ik}\right).$$

We now prove that $((D^1,\dots,D^K),(\theta^{ik})_{i,k},R_\mathrm{f},q)$ is a DE. Since the after-trade shareholders bear the costs $R_\mathrm{f}F^k(D^k)$ for production, $\theta^i = (\theta^{ik})_{k\geq 0}$ satisfies the budget constraint for each i. For every $k \geq 1$, the feasibility $\sum_i \theta^{ik} = 1$ is met by construction. For $k = 0$, the feasibility $\sum_i \theta^{i0} = \sum_i \bar{\theta}^{i0}$ is met by $\sum_k F^k(D^k) = \bar{w}$. Write $c^i = \sum_{k\geq 0}\theta^{ik}D^k$ and his p-discounted utility gradient (density) $\nabla U^i(c^i)\Lambda(p)^{-1}$ by π^i. Then,

$$\begin{aligned} c^i - E[c^i]\mathbf{1} &= \frac{\tau^i}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \tau^i A^i(D,\delta),\\ \pi^i &= \pi - B^i(D,\delta) \end{aligned}$$

for every i. Since π and π^i coincide on the subspace spanned by $(\mathbf{1}, D^1,\dots,D^K)$, the utility maximization condition is met. As for the profit maximization condition,

$$\sum_i \theta^{ik}\pi^i = \sum_i \left(\frac{\tau^i}{\bar{\tau}} - \tau^i a^{ik}\right)(\pi - B^i(D,\delta)) = \pi + \sum_i \tau^i a^{ik} B^i(D,\delta),$$

Hence, by (13),

$$\sum_i \theta^{ik}\pi^i = R_\mathrm{f}\xi^k(D^k)$$

for every k. Thus, the profit maximization condition under $\sum_i \theta^{ik}\pi^i$ is met for every k. ///

Proof of Lemma 11 We will prove this lemma by applying the envelope theorem to a constrained maximization problem to which $B^i(D,\delta)$ is a solution. Write

$\hat{D}^k = D^k - E[D^k]\mathbf{1} \in N$ and $\hat{D} = \left(\hat{D}^1, \dots, \hat{D}^K\right) \in N^K$. Then, consider

$$\begin{aligned} \max_{(B,a)\in \boldsymbol{R}^S \times \boldsymbol{R}^K} \quad & -B^\top \Lambda(p) B, \\ \text{subject to} \quad & \delta^i - \bar{\delta} - B - \hat{D}a = 0. \end{aligned}$$

Since the objective function is quadratic and strictly concave, the constraint functions are affine, and $\hat{D}$ has full rank (K), there is a unique solution to this problem. Moreover, the first-order condition for a solution is necessary and sufficient, which is that there is a $\lambda = (\lambda_s)_s \in \boldsymbol{R}^S$ such that

$$\begin{aligned} -2B^\top \Lambda(p) - \lambda^\top &= 0, \\ -\lambda^\top \hat{D} &= 0. \end{aligned} \tag{21}$$

We write, for a moment, the solution as $(B(\hat{D}), a(\hat{D}))$, taking it as a function of $\hat{D}$. By the envelope theorem, the value function $\hat{D} \mapsto -B(\hat{D})^\top \Lambda(p) B(\hat{D})$ is differentiable and its partial derivative with respect to $\hat{D}^k_s$ is equal to $-\lambda_s a^k(\hat{D})$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}\hat{D}^k} \left(-B(\hat{D})^\top \Lambda(p) B(\hat{D})\right) = -2a^k(\hat{D})\lambda^\top,$$

which is, by (21), equal to $2a^k(\hat{D})B(\hat{D})^\top \Lambda(p)$.

If $(B, a) \in \boldsymbol{R}^S \times \boldsymbol{R}^K$ satisfies the constraint, then $B = (\delta^i - \bar{\delta}) - \hat{D}a$. Since δ^i, $\bar{\delta}$, and all the $\hat{D}^k$ belong to N, $B \in N$. Thus, the objective function is equal to $-\mathrm{Var}[(\delta^i - \bar{\delta}) - \hat{D}a]$. Hence, if (B, a) is the solution, then $\hat{D}a$ and B constitute the p-orthogonal projection of $\delta^i - \bar{\delta}$. Thus, in fact, $a(\hat{D}) = a^i(D, \delta)$ and $B(\hat{D}) = B^i(D, \delta)$ according to the notation of the proof of Lemma 9. Moreover, since $\mathrm{Var}[B^i(D, \delta)]$ is differentiable with respect to D in all the directions on N and addition to or subtraction from D^k of any scalar multiple of $\mathbf{1}$ does not affect $A^i(D, \delta)$ or $B^i(D, \delta)$, $\mathrm{Var}[B^i(D, \delta)]$ is, in fact, differentiable with respect to D in all the directions on $\boldsymbol{R}^S$. Since $z - E[z]\mathbf{1} = (I_S - \mathbf{1}\mathbf{1}^\top \Lambda(p))z \in N$ for every $z \in \boldsymbol{R}^S$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}D^k} \left(-\mathrm{Var}[B^i(D, \delta)]\right) &= \frac{\mathrm{d}}{\mathrm{d}\hat{D}^k} \left(-B(\hat{D})^\top \Lambda(p) B(\hat{D})\right) (I_S - \mathbf{1}\mathbf{1}^\top \Lambda(p)) \\ &= 2a^{ik}(D, \delta) B^i(D, \delta)^\top \Lambda(p)(I_S - \mathbf{1}\mathbf{1}^\top \Lambda(p)) \\ &= 2a^{ik}(D, \delta) B^i(D, \delta)^\top \Lambda(p), \end{aligned}$$

where the last equality follows from $B^i(D, \delta)^\top \Lambda(p)\mathbf{1} = 0$ (that is, $B^i(D, \delta) \in N$). Finally, since $\mathrm{Var}[A^i(D, \delta)] + \mathrm{Var}[B^i(D, \delta)] = \mathrm{Var}[\delta^i - \bar{\delta}]$ for every D,

$$\frac{\mathrm{d}}{\mathrm{d}D^k} \mathrm{Var}[A^i(D, \delta)] = \frac{\mathrm{d}}{\mathrm{d}D^k} \left(-\mathrm{Var}[B^i(D, \delta)]\right) = 2a^{ik}(D, \delta) B^i(D, \delta)^\top \Lambda(p).$$

///

Proof of Lemma 12 Define $\Pi^0 : \boldsymbol{R}_{++} \times \boldsymbol{R}_+^{SK} \times N^I \to \boldsymbol{R}$ by

$$\Pi^0(R_{\rm f}, D, \delta) = \bar{w} - \sum_{k \geq 1} F^k(D^k).$$

For each $k \geq 1$, define $\Pi^k : \boldsymbol{R}_{++} \times \boldsymbol{R}_+^{SK} \times N^I \to \boldsymbol{R}^S$ by

$$\begin{aligned} \Pi^k(R_{\rm f}, D, \delta) &= \Lambda(p) \left(\mathbf{1} - \frac{1}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta} - R_{\rm f}\xi^k(D^k) \right) \\ &= p - \frac{1}{\bar{\tau}} H\bar{D} - \Lambda(p)\bar{\delta} - R_{\rm f}\nabla F^k(D^k), \end{aligned} \tag{22}$$

where $H = \Lambda(p) - \Lambda(p)\mathbf{1}\mathbf{1}^\top\Lambda(p) \in \boldsymbol{R}^{S \times S}$. Then, $\nabla_{D^k}\Pi^0(R_{\rm f}, D, \delta) = -\nabla F^k(D^k)$ for each k and $\nabla_{R_{\rm f}}\Pi^0(R_{\rm f}, D, \delta) = 0$. Moreover,

$$\begin{aligned} \frac{\partial \Pi^k}{\partial D^\ell}(R_{\rm f}, D, \delta) &= \begin{cases} -\frac{1}{\bar{\tau}}H - R_{\rm f}\nabla^2 F^k(D^k) & \text{if } k = \ell, \\ -\frac{1}{\bar{\tau}}H & \text{otherwise,} \end{cases} \\ \frac{\partial \Pi^k}{\partial R_{\rm f}}(R_{\rm f}, D, \delta) &= -\nabla F^k(D^k) \end{aligned}$$

Define $\Pi : \boldsymbol{R}_+^{SK} \times \boldsymbol{R}_{++} \times N^I \to \boldsymbol{R} \times \boldsymbol{R}^{SK}$ by $\Pi(R_{\rm f}, D, \delta) = (\Pi^k(R_{\rm f}, D, \delta))_{k \geq 0}$. We now show that

$$\frac{\partial \Pi}{\partial(R_{\rm f}, D)}(R_{\rm f}, D, \delta) \in \boldsymbol{R}^{(1+SK)\times(1+SK)} \tag{23}$$

is invertible. By the above derivations, by suppressing the variables $(R_{\rm f}, D, \delta)$, we can write

$$\begin{aligned} &\frac{\partial \Pi}{\partial(R_{\rm f}, D)}(R_{\rm f}, D, \delta) \\ &= \begin{pmatrix} \frac{\partial \Pi^0}{\partial R_{\rm f}} & \frac{\partial \Pi^0}{\partial D^1} & \cdots & \frac{\partial \Pi^0}{\partial D^K} \\ \frac{\partial \Pi^1}{\partial R_{\rm f}} & \frac{\partial \Pi^1}{\partial D^1} & \cdots & \frac{\partial \Pi^1}{\partial D^K} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Pi^K}{\partial R_{\rm f}} & \frac{\partial \Pi^K}{\partial D^1} & \cdots & \frac{\partial \Pi^K}{\partial D^K} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\nabla F^1(D^1) & \cdots & \cdots & -\nabla F^K(D^K) \\ -\nabla F^1(D^1)^\top & -\frac{1}{\bar{\tau}}H - R_{\rm f}\nabla^2 F^1(D^1) & & & \\ \vdots & & \ddots & -\frac{1}{\bar{\tau}}H & \\ \vdots & & -\frac{1}{\bar{\tau}}H & \ddots & \\ -\nabla F^K(D^K)^\top & & & & -\frac{1}{\bar{\tau}}H - R_{\rm f}\nabla^2 F^K(D^K) \end{pmatrix}. \end{aligned}$$

To show that this matrix is invertible, multiply a row vector $(y, (z^1)^\top, \ldots, (z^K)^\top) \in \boldsymbol{R}^{1+SK}$ from left and assume that the vector-matrix product coincides with the zero vector. The first column (scalar) of the product is

$$-\sum_{k\geq 1} \nabla F^k(D^k) z^k = 0 \tag{24}$$

and the k-th group of S columns coincides with

$$-y\nabla F^k(D^k) - \frac{1}{\bar{\tau}} \left(\sum_{\ell} z^{\ell} \right)^\top H - R_{\mathrm{f}} (z^k)^\top \nabla^2 F^k(D^k) = 0 \tag{25}$$

for every $k \geq 1$. Multiply y to (24), multiply z^k to (25) from right, and sum over k as well as y times (24), then we obtain the quadratic form of (23):

$$\begin{aligned} & -\frac{1}{\bar{\tau}} \left(\sum_k z^k \right)^\top H \left(\sum_k z^k \right) - R_{\mathrm{f}} \sum_k (z^k)^\top \nabla^2 F^k(D^k) z^k - 2y \sum_{k \geq 1} \nabla F^k(D^k) z^k \\ = & -\frac{1}{\bar{\tau}} \left(\sum_k z^k \right)^\top H \left(\sum_k z^k \right) - R_{\mathrm{f}} \sum_k (z^k)^\top \nabla^2 F^k(D^k) z^k = 0 \end{aligned}$$

Since H is positive semidefinite and the $\nabla^2 F^k(D^k)$ are positive definite, this equality implies that $z^k = 0$ for every k. By (25), $y = 0$. Hence, the matrix (23) is invertible.

Let $\delta_0 \in N^I$ and $(R_{\mathrm{f}0}, D_0)$ be the profile of the risk-free rate and the outputs at the canonical MVE under δ_0. Since the matrix (23) is invertible, there is a continuously differentiable mapping D_{MVE} of some neighborhood of δ_0 into some neighborhood of $(R_{\mathrm{f}0}, D_0)$ such that for every $(\delta, R_{\mathrm{f}}, D)$ in the product of these two neighborhoods, $D_{\mathrm{MVE}}(\delta) = (R_{\mathrm{f}}, D)$ if and only if $\Pi(R_{\mathrm{f}}, D, \delta) = 0$. By Lemma 9, D is the output profile at the canonical MVE under δ if and only if $\Pi(R_{\mathrm{f}}, D, \delta) = 0$ for some $R_{\mathrm{f}} > 0$. Thus, by restricting the value taken by $D_{\mathrm{MVE}}(\delta)$ onto D (rather than keeping (R_{f}, D), we complete the proof. ///

Proof of Lemma 13 We follow the same method of proof as for Lemma 12, except that, for each k, we replace $\Pi^k(R_{\mathrm{f}} D, \delta)$ by

$$\Phi^k(R_{\mathrm{f}}, D, \delta) = \Lambda(p) \left(\mathbf{1} - \frac{1}{\bar{\tau}} (\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta} + \sum_{k \geq 1} \tau^i a^{ik} B^i(D, \delta) - R_{\mathrm{f}} \xi^k(D^k) \right)$$

because, by Lemma 10, the equalities $\Phi^k(R_{\mathrm{f}} D, \delta) = 0$ for all k define the DE. Since $(\mathbf{1}, D_0^1, \ldots, D_0^K)$ is linearly independent, $D \mapsto \left(\sum_i \frac{\tau^i}{2} \mathrm{Var}[A^i(D)] \right)$ is twice continuously differentiable at $D = D_0$. Hence, Φ^k is continuously differentiable at $(R_\mathrm{f}, D_0, \delta_0)$. Write $\Pi^{-0}(R_\mathrm{f}, D, \delta) = \big(\Pi^k(R_\mathrm{f}, D, \delta)\big)_{k\geq 1}$ and $\Phi^{-0}(R_\mathrm{f}, D, \delta) = \big(\Phi^k(R_\mathrm{f}, D, \delta)\big)_{k\geq 1}$. Then, by Lemma 11,

$$\frac{\partial \Phi^{-0}}{\partial D}(R_\mathrm{f}, D_0, \delta_0) = \frac{\partial \Pi^{-0}}{\partial D}(R_\mathrm{f}, D_0, \delta_0) + \nabla_D^2 \left(\sum_i \frac{\tau^i}{2} \mathrm{Var}[A^i(D)] \right). \tag{26}$$

Since $B^i(D, \delta) = 0$ for every i, $\sum_i \frac{\tau^i}{2} \mathrm{Var}[A^i(D)]$ is maximized and equal to $\sum_i \frac{\tau^i}{2} \mathrm{Var}[\delta^i - \bar{\delta}]$. Thus, $\nabla_D^2 \left(\sum_i \frac{\tau^i}{2} \mathrm{Var}[A^i(D)] \right)$ is negative semidefinite. On the other hand, we showed in the proof of Lemma 12 that $\frac{\partial \Pi^{-0}}{\partial D}(R_\mathrm{f}, D_0, \delta_0)$ is negative definite (on the entire $\boldsymbol{R}^{SK}$) and this fact guarantees the invertibility of the $(1+SK) \times (1+SK)$ matrix $\frac{\partial \Pi}{\partial (R_\mathrm{f}, D)}(R_\mathrm{f}, D_0, \delta_0)$, which, in turn, implies the continuous differentiability of D_MVE at D_0. By (26), $\frac{\partial \Phi^{-0}}{\partial D}(R_\mathrm{f}, D_0, \delta_0)$ is also negative definite (on the entire $\boldsymbol{R}^{SK}$). As in the proof of Lemma 12, this guarantees the invertibility of the $(1+SK) \times (1+SK)$ matrix $\frac{\partial \Pi}{\partial (R_\mathrm{f}, D)}(R_\mathrm{f}, D_0, \delta_0)$, where $\frac{\partial \Pi^{-0}}{\partial D}(R_\mathrm{f}, D_0, \delta_0)$ is replaced by $\frac{\partial \Phi^{-0}}{\partial D}(R_\mathrm{f}, D_0, \delta_0)$, which implies the continuous differentiability of D_DE at δ_0. ///

Proof of Theorem 3 By Lemma 12, D_MVE is differentiable at δ. By assumption, D_DE is differentiable at δ. As we noted after Lemma 2, since $(\mathbf{1}, D^1, \dots, D^K)$ is linearly independent, U_RE is differentiable at (D, δ). Hence, by chain rule,

$$\nabla U_\mathrm{MVE}(\delta) = \nabla_D U_\mathrm{RE}(D, \delta) \frac{\mathrm{d}D_\mathrm{MVE}}{\mathrm{d}\delta}(\delta) + \nabla_\delta U_\mathrm{RE}(D, \delta), \tag{27}$$

$$\nabla U_\mathrm{DE}(\delta) = \nabla_D U_\mathrm{RE}(D, \delta) \frac{\mathrm{d}D_\mathrm{DE}}{\mathrm{d}\delta}(\delta) + \nabla_\delta U_\mathrm{RE}(D, \delta). \tag{28}$$

For each i, write $\delta^i - \bar{\delta} = A^i(D) + B^i(D)$, the decomposition onto the linear subspace spanned by $(\mathbf{1}, D^1, \dots, D^K)$ and its p-orthogonal complement. Then, there is a unique $a^i(D) = (a^{ik}(D))_k \in \boldsymbol{R}^K$ such that $A^i(D) = \sum_k a^{ik}(D) D^k$. (At each of the DE and the MVE, $\theta^{ik} = \tau^i / \bar{\tau} - \tau^i a^{ik}(D)$ for all i and $k \geq 1$.) By Lemma 11,

$$\frac{\mathrm{d}}{\mathrm{d}D^k} \sum_i \frac{\tau^i}{2} \mathrm{Var}[A^i(D)] = \left(\sum_i \tau^i a^{ik}(D) B^i(D) \right)^\top \Lambda(p)$$

for every k. Write $\bar{D} = \sum_{k\geq 1} D^k$. Then,

$$\nabla_{D^k} U_\mathrm{RE}(D, \delta) = \left(\mathbf{1} - \frac{1}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta} + \sum_i \tau^i a^i(D) B^i(D) \right)^\top \Lambda(p)$$

for every k. Since $D_{\text{MVE}}(\delta) = D = D_{\text{DE}}(\delta)$, the proof of Theorem 2 shows that $\sum_i \tau^i a^{ik}(D) B^i(D) = 0$ for every k. Thus,

$$\nabla_{D^k} U_{\text{RE}}(D,\delta) = \left(\mathbf{1} - \frac{1}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta}\right)^{\top} \Lambda(p)$$

for every k. By the definition of U_{RI},

$$\nabla_{D^k} U_{\text{RI}}(D,\delta) = \left(\mathbf{1} - \frac{1}{\bar{\tau}}(\bar{D} - E[\bar{D}]\mathbf{1}) - \bar{\delta}\right)^{\top} \Lambda(p).$$

Thus,

$$\nabla_{D^k} U_{\text{RE}}(D,\delta) = \nabla_{D^k} U_{\text{RI}}(D,\delta)$$

for every k. By Lemma 9, D is a solution to the maximization problem in part 3 of the lemma. The first-order condition is that there is a $R_{\text{f}} > 0$ (the risk-free rate at the MVE) such that

$$\nabla_{D^k} U_{\text{RI}}(D,\delta) = R_{\text{f}} \nabla F^k(D^k)$$

for every k. Thus,

$$\nabla_D U_{\text{RE}}(D,\delta) \frac{\mathrm{d}D_{\text{MVE}}}{\mathrm{d}\delta}(\delta) = \sum_{k\geq 1} \nabla_{D_k} U_{\text{RE}}(D,\delta) \frac{\mathrm{d}D^k_{\text{MVE}}}{\mathrm{d}\delta}(\delta) = R_{\text{f}} \sum_{k\geq 1} \nabla F^k(D^k) \frac{\mathrm{d}D^k_{\text{MVE}}}{\mathrm{d}\delta}(\delta),$$
$$\nabla_D U_{\text{RE}}(D,\delta) \frac{\mathrm{d}D_{\text{DE}}}{\mathrm{d}\delta}(\delta) = \sum_{k\geq 1} \nabla_{D_k} U_{\text{RE}}(D,\delta) \frac{\mathrm{d}D^k_{\text{DE}}}{\mathrm{d}\delta}(\delta) = R_{\text{f}} \sum_{k\geq 1} \nabla F^k(D^k) \frac{\mathrm{d}D^k_{\text{DE}}}{\mathrm{d}\delta}(\delta)$$

Since the functions $\delta \mapsto \sum_k F^k(D^k_{\text{MVE}}(\delta))$ and $\delta \mapsto \sum_k F^k(D^k_{\text{MVE}}(\delta))$ are constantly equal to $\bar{w}$,

$$\sum_{k\geq 1} \nabla F^k(D^k) \frac{\mathrm{d}D^k_{\text{MVE}}}{\mathrm{d}\delta}(\delta) = 0,$$
$$\sum_{k\geq 1} \nabla F^k(D^k) \frac{\mathrm{d}D^k_{\text{DE}}}{\mathrm{d}\delta}(\delta) = 0.$$

Hence,

$$\nabla_D U_{\text{RE}}(D,\delta) \frac{\mathrm{d}D_{\text{MVE}}}{\mathrm{d}\delta}(\delta) = 0,$$
$$\nabla_D U_{\text{RE}}(D,\delta) \frac{\mathrm{d}D_{\text{DE}}}{\mathrm{d}\delta}(\delta) = 0.$$

Thus, by (27) and (28),

$$\nabla U_{\mathrm{MVE}}(\delta) = \nabla_\delta U_{\mathrm{RE}}(D, \delta),$$
$$\nabla U_{\mathrm{DE}}(\delta) = \nabla_\delta U_{\mathrm{RE}}(D, \delta).$$

Thus, $\nabla U_{\mathrm{MVE}}(\delta) = \nabla U_{\mathrm{DE}}(\delta)$. ///

Proof of Corollary 1 Since $U_{\mathrm{DE}}(\delta_0) = U_{\mathrm{MVE}}(\delta_0)$,

$$\frac{U_{\mathrm{DE}}(\delta) - U_{\mathrm{MVE}}(\delta)}{\|\delta - \delta_0\|} = \frac{U_{\mathrm{DE}}(\delta) - U_{\mathrm{DE}}(\delta_0)}{\|\delta - \delta_0\|} - \frac{U_{\mathrm{MVE}}(\delta) - U_{\mathrm{MVE}}(\delta_0)}{\|\delta - \delta_0\|}$$

and, by Theorem 3, the right-hand side converges to zero as $\delta \to \delta_0$. ///

Proof of Proposition 4 As we saw in Example 1, at a MVE, D^k is a scalar multiple of ν^k. Thus, $(\mathbf{1}, D^1, \dots, D^K)$ is linearly independent. Since $B^i_M = 0$ for every i, by Theorem 2, the MVE is also a DE. Moreover, by Lemma 13, the Dreze equilibria under δ close to δ_0 can be written as a continuously differentiable function of δ, which implies the existence of a DE under such δ. ///

B Who bears the cost of inputs?

The ex-post (after trade) shareholders are entitled to the (state-contingent) output D^k on period 1 but we need to specify who pays for the input $F^k(D^k)$ on period 0. On this, there are two possibilities: the ex-post (after trade) holder versus the ex-ante (before trade) holder. We will formulate these two possibilities in turn and show that they are, in fact, equivalent, in the sense that in both cases, the budget set in terms of a consumption plan c^i is given by

$$E[\pi c^i] \leq \bar{\theta}^{i0} + \sum_{k=1}^{K} (E[\pi D^k] - R_{\mathrm{f}} D^k)\bar{\theta}^{ik} + R_{\mathrm{f}} w^i,$$

and the weak inequality holds as an equality at any solution c^i to the utility maximization problem.

B.1 When the after-trade holder pays for the input

Since the risk-free bond is the numeraire and the risk-free rate is R_{f}, the price for the period-zero good is equal to R_{f}. The budget constraint of consumer i on period zero in terms of the portfolio $(\theta^{i0}, \theta^{i1}, \dots, \theta^{iK})$ is

$$\sum_{k=0}^{K} q^k \theta^{ik} + R_{\mathrm{f}} \sum_{k=1}^{K} F^k(D^k)\theta^{ik} \leq \sum_{k=0}^{K} q^k \bar{\theta}^{ik} + R_{\mathrm{f}} w^i,$$

and the resulting consumption plan on period one is equal

$$c^i = \sum_{k=0}^{K} \theta^{ik} D^k$$

Since the after-trade ownership of the stock come with the obligation of paying for the input D_0^k on period zero, the stock price q^k is given by its profit

$$q^k = E[\pi D^k] - R_\text{f} F^k(D^k).$$

Thus, the budget constraint can be rewritten as

$$\theta^{i0} + \sum_{k=1}^{K} \theta^{ik} E[\pi D^k] = E[\pi c^i] \leq \bar{\theta}^{i0} + \sum_{k=1}^{K} (E[\pi D^k] - R_\text{f} F^k(D^k))\bar{\theta}^{ik} + R_\text{f} w^i.$$

At the solution of the utility maximization problem, this weak inequality is satisfied as an equality. By summing each side over i, the market-clearing condition for the stocks (but not for the bond) implies that

$$\sum_i \theta^{i0} + E[\pi \bar{D}] = \bar{\theta}^0 + E[\pi \bar{D}] - R_\text{f} \sum_k F^k(D^k) + \bar{w}.$$

This can be rewritten as

$$\sum_i \theta^{i0} - \sum_i \bar{\theta}^{i0} = R_\text{f} \left(\sum_i w^i - \sum_k F^k(D^k) \right).$$

Since $R_\text{f} > 0$, the bond market clears, along with the stock markets, if and only if $\sum_i w^i - \sum_{k \geq 1} F^k(D^k) = 0$.

B.2 When the before-trade holder pays for the input

Since the risk-free bond is the numeraire and the risk-free rate is R_f, the price for the period-zero good is equal to R_f. The budge constraint of consumer i on period zero in terms of the portfolio $(\theta^{i0}, \theta^{i1}, \ldots, \theta^{iK})$ is

$$\sum_{k=0}^{K} q^k \theta^{ik} + R_\text{f} \sum_{k=1}^{K} F^k(D^k)\bar{\theta}^{ik} \leq \sum_{k=0}^{K} q^k \bar{\theta}^{ik} + R_\text{f} w^i,$$

and the resulting consumption plan on period one is equal, as before, to

$$c^i = \sum_{k=0}^{K} \theta^{ik} D^k$$

Since the after-trade ownership of the stock only entitles its owner the output D^k on period one, the stock price q^k is given by the revenue

$$q^k = E[\pi D^k].$$

Thus, the budget constraint can be rewritten as

$$\begin{aligned} E[\pi c^i] = \theta^{i0} + \sum_{k=1}^{K} \theta^{ik} E[\pi D^k] &\leq \bar{\theta}^{i0} + \sum_{k=1}^{K} E[\pi D^k]\bar{\theta}^{ik} + R_{\mathrm{f}} w^i - R_{\mathrm{f}} \sum_{k=1}^{K} F^k(D^k)\bar{\theta}^{ik} \\ &= \bar{\theta}^{i0} + \sum_{k=1}^{K} (E[\pi D^k] - R_{\mathrm{f}} F^k(D^k)\bar{\theta}^{ik} + R_{\mathrm{f}} w^i. \end{aligned}$$

Thus, the budget constraint in the case where the before-trade shareholder bears the cost of input is, in terms of a consumption plan c^i and the state price density π, identical to that in the case where the after-trade shareholder bears the cost of input. The resulting consumption or asset allocation, or cum- or ex-input-payment stock prices at any exchange equilibrium would not depend on the choice of who pays for the input once the production plans $(D^k)_k$ are fixed; and since the investors' portfolio and consumption choices are not affected by this choice, the firms' choices would not be affected either.

C Short sales versus derivatives

An unrealistic aspect of a Dreze equilibrium is that the firm's decision making is affected not only by shareholders but also by short-sellers, as the coefficients θ^{ik} may well be negative in the weighted sum of utility gradients, $\sum_i \theta^{ik} \Lambda(p) \pi^i(c^i)$. Although the first-order necessary condition for a production plan to be efficient (in the sense stated in Section 5) must involve short-sellers, their inclusion in the firm's decision making is hardly realistic. We could circumvent this difficulty simply by assuming, as Dreze (1974) did, that no short-sales are allowed. But this (and, indeed, any) short-sales restriction comes with the cost of sacrificing the convenience of writing equilibria in terms of state prices.

In this appendix, we give an alternative formulation of asset markets, where shares cannot be sold short but there are also derivatives, one for each firm, which has the same payoff as the firm's (state-contingent) output, does not come with the voting right, and, yet, can be sold short. The benefit of introducing short-sales restrictions on shares is that we can eliminate negative coefficients from Dreze criterion while still using state prices to define all notions of equilibrium (market value maximization equilibrium, Dreze equilibrium, and constrained equilibrium). The downside is that those who trade the derivatives are impacted by changes in the firms' production plans, but their welfare change is not taken into consideration in the firms' production choice. The resulting allocation is, therefore, unlikely to be Pareto efficient.

In this alternative formulation, we assume that the investors can buy the shares of the K firms but cannot sell them more than they initially own. We also introduce another set of assets, or derivatives with zero-net supply, one for each firm, which pays off the firm's (state-contingent) output, does not come with the voting right or the cost-bearing obligation, and can be sold short. Thus, for each $k \geq 1$, the state-contingent payoff of the k-derivative in period 1 is equal to D^k but there is no (positive or negative) payoff in period 0.

Recall that if π is a state price density, then the share price q^k is equal to $E\left[\pi D^k\right] - R_\mathrm{f} F^k(D^k)$. We now assume that the price of the k-th derivative is equal to $E\left[\pi D^k\right]$. Since the payoffs of the share and the derivative differ only in that the former comes with the cost-bearing obligation but the latter does not, and since the cost is equal to $R_\mathrm{f} F^k(D^k)$, this assumption implies that the price for the voting right is equal to zero. We will justify this assumption shortly.

Let

$$\left(\eta^{i0}, \eta^{i1}, \dots, \eta^{iK}, \zeta^{i1}, \dots, \zeta^{iK}\right) \in \boldsymbol{R} \times \boldsymbol{R}_+^K \times \boldsymbol{R}^K$$

be a portfolio of the risk-free bond, shares, and derivatives. Since no investor derives utility from consumption on period 0, it satisfies the budget constraint if and only if

$$\sum_{k \geq 0} q^k \eta^{ik} + \sum_{k \geq 1} E[\pi D^k] \zeta^{ik} + R_\mathrm{f} \sum_{k \geq 1} F^k(D^k) \eta^{ik} \leq R_\mathrm{f} w^i + \sum_{k \geq 0} \bar{\theta}^{ik} q^k,$$

which can be rewritten as

$$\eta^{i0} + \sum_{k \geq 1} E[\pi D^k](\eta^{ik} + \zeta^{ik}) \leq R_\mathrm{f} w^i + \sum_{k \geq 0} \bar{\theta}^{ik} q^k, \tag{29}$$

and the resulting consumption plan in period 1 is

$$\eta^{i0} + \sum_{k \geq 1} (\eta^{ik} + \zeta^{ik}) D^k. \tag{30}$$

Recall, for the purpose of comparison, that in our original formulation, if investor i holds a portfolio of the risk-free bond and shares, $(\theta^{i0}, \theta^{i1}, \dots, \theta^{iK})$, where θ^{ik} may be negative, then the budget constraint, with zero consumption in period 0, is that

$$\sum_{k \geq 0} q^k \theta^{ik} + R_\mathrm{f} \sum_{k \geq 1} F^k(D^k) \theta^{ik} \leq R_\mathrm{f} w^i + \sum_{k \geq 0} \bar{\theta}^{ik} q^k,$$

which can be rewritten as

$$\eta^{i0} + \sum_{k \geq 1} E[\pi D^k]\theta^{ik} \leq R_{\mathrm{f}} w^i + \sum_{k \geq 0} \bar{\theta}^{ik} q^k \tag{31}$$

and the consumption plan in period 1 is

$$\theta^{i0} + \sum_{k \geq 1} \theta^{ik} D^k. \tag{32}$$

By comparing (29) with (31) and (30) with (32), we can see that the two portfolios generate the same consumption plan if

$$\theta^{i0} = \eta^{i0}, \tag{33}$$

$$\theta^{ik} = \eta^{ik} + \zeta^{ik} \text{ for every } k \geq 1, \tag{34}$$

and the converse also holds if $(\mathbf{1}, D^1, \ldots, D^K)$ is linearly independent. Moreover, whenever one satisfies the budget constraint, the other does so too. Hence, whenever one satisfies the utility maximization condition, the other does so too.

We have so far been concerned with an investor's utility maximization in the alternative formulation. We now consider the market-clearing conditions. The market-clearing conditions in the η^{ik} and the ζ^{ik} are that

$$\sum_i \eta^{i0} = \bar{\theta}^0,$$

$$\sum_i \eta^{ik} = 1 \text{ for every } k \geq 1,$$

$$\sum_i \zeta^{ik} = 0 \text{ for every } k \geq 1.$$

The modified Dreze equilibrium is defined as a profile $((\theta^{ik})_{i,k}, R_{\mathrm{f}}, q)$ that satisfies the utility maximization condition and the market-clearing condition, and for every k, D^k is a solution to the problem of maximizing profit under the state price density $\sum_i \eta^{ik}\pi^i(\pi)$. Note that the coefficients η^{ik} are all non-negative. The derivative holdings ζ^{ik} are irrelevant to the firm's production decision.

(30) shows that the resulting period-1 consumption plan depend only on the sum $\eta^{ik}+\zeta^{ik}$. Using this fact and a no-arbitrage argument, we can justify our assumption that the price for the voting right is equal to zero. To do so, recall that the share price q^k satisfies $q^k = E\left[\pi D^k\right] - R_{\mathrm{f}} F^k(D^k)$ and denote the derivative price by t^k. The assumption of zero price for the voting right is nothing but $t^k = E\left[\pi D^k\right]$. In the general case where this equality need not hold, the budget constraint (29)

should be changed into

$$\eta^{i0} + \sum_{k\geq 1} E[\pi D^k]\eta^{ik} + \sum_{k\geq 1} t^k \zeta^{ik} \leq R_{\mathrm{f}} w^i + \sum_{k\geq 0} \bar{\theta}^{ik} q^k,$$

If $E\left[\pi D^k\right] < t^k$, then we can increase η^{ik} and decrease ζ^{ik} by the same amount, and increase the bond holding η^{i0} by the same amount as the net reduction in expenditure for the share and the derivative. Thus, there is no solution to the utility maximization problem. If, on the other hand, $E\left[\pi D^k\right] > t^k$, then it is never optimal to have $\eta^{ik} > 0$, because, if so, then it would be possible to decrease η^{ik} down to 0 and increase ζ^{ik} by the same amount, and increase the bond holding η^{i0} by the same amount as the net reduction in expenditure for the share and the derivative. Hence, $\eta^{ik} = 0$ for all i, which would violate the market-clearing condition $\sum_i \eta^{ik} = 1$. At equilibrium, therefore, we must have $t^k = E\left[\pi D^k\right]$. Underlying this argument is the (implicit) assumption that the investors take the production plan D^k as given when making their portfolio choices. In particular, if unanimity is required for any change in production plans, then, at a modified Dreze equilibrium, they are indeed rational in expecting that the voting right is not invoked.

Recall that the market-clearing conditions for the θ^{ik} are that

$$\sum_i \theta^{i0} = \bar{\theta}^0,$$
$$\sum_i \theta^{ik} = 1 \text{ for every } k \geq 1.$$

Suppose that the η^{ik}, the ζ^{ik}, and the θ^{ik} satisfy (33) and (34) for every i. If, in addition, the market-clearing conditions for the η^{ik} and ζ^{ik} are satisfied, then the market-clearing conditions for the θ^{ik} are also satisfied. The converse, however, need not hold. Yet, for every profile of the θ^{ik}, if it satisfies the market-clearing conditions, then there are a profile of the η^{ik} and a profile of the ζ^{ik} that together satisfy the market-clearing conditions. Among many possibilities, one such example can be given as follows. For each $k \geq 1$, write $\Theta^k = \sum_i \max\left\{\theta^{ik}, 0\right\}$, then, for every k, $\Theta^k \geq 1$ and $\Theta^k = 1$ if and only if $\theta^{ik} \geq 0$ for every i. For each i, let $\eta^{i0} = \theta^{i0}$ and, for each $k \geq 1$,

$$\eta^{ik} = \begin{cases} \dfrac{1}{\Theta^k}\theta^{ik} & \text{if } \theta^{ik} \geq 0, \\ 0 & \text{otherwise,} \end{cases}, \tag{35}$$

$$\zeta^{ik} = \begin{cases} \left(1 - \dfrac{1}{\Theta^k}\right)\theta^{ik} & \text{if } \theta^{ik} \geq 0, \\ \theta^{ik} & \text{otherwise,} \end{cases} \tag{36}$$

Then, for all i and k, $\eta^{ik} \geq 0$ and $\eta^{ik} + \zeta^{ik} = \theta^{ik}$, and $\sum_i \eta^{ik} = 1$. Since $\Theta^k + \sum_{\{i|\theta^{ik}<0\}} \theta^{ik} = 1$,

$$\sum_k \zeta^{ik} = \left(1 - \frac{1}{\Theta^k}\right) \Theta^k + \left(1 - \Theta^k\right) = 0.$$

Hence, the market-clearing conditions are met.

We have already seen that whenever one of the two portfolios satisfies the utility maximization condition, so does the other. Together with the equivalence of the market-clearing conditions, therefore, we can conclude that if one of the two portfolios is an M-constrained equilibrium, then so is the other. However, they do not share the same Dreze equilibrium unless $\theta^{ik} \geq 0$ for all i and $k \geq 1$. In fact, if the θ^{ik}, the η^{ik}, and the ζ^{ik}, are related via (35) and (36), then the modified Dreze equilibrium in the markets for the bond, the shares, and the derivatives is a truncated Dreze equilibrium, which was introduced in Remark 2.

D Multiplicity of non-canonical MVE

Proposition 5 *Suppose that $K = 1$. Then, for every linear subspace M that contains $\mathbf{1}$, there is a MVE $(D^1, (\theta^{ik})_{i,k}, R_\mathrm{f}, q)$ such that $D^1 \in M$.*

This proposition does not quite imply that there are multiple MVE's. However, if there is a MVE at which $D^1 \neq \lambda \mathbf{1}$ for any λ, then, by taking any M that contains $\mathbf{1}$ but not D^1, we can show that there is another MVE for which the firm's state-contingent output is different from D^1.

Proof of Proposition 5 By Proposition 1, for every linear subspace M that contains $\mathbf{1}$, there is an M-restricted equilibrium $(D^1, (c^i)_i, \tilde{R}_\mathrm{f}, \tilde{\pi})$. The profit maximization condition implies that Y^1 does not intersect with

$$\left\{(z_0, z) \in \boldsymbol{R} \times M \mid E[\tilde{\pi} z] + \tilde{R}_\mathrm{f} z_0 > E[\tilde{\pi} D^1] - \tilde{R}_\mathrm{f} F^1(D^1)\right\}.$$

Since Y^1 is convex, the separating hyperplane theorem implies that there is a non-zero vector (R_f, π) that separates these two sets. Then, $(-F^1(D^1), D^1)$ maximize profit on Y^1 (not restricted on M) under (R_f, π) and, by multiplying a positive scalar if necessary, we can assume that (R_f, π) coincides with $(\tilde{R}_\mathrm{f}, \tilde{\pi})$ on $\boldsymbol{R} \times M$. As we saw in the proof of Proposition 1, for each i and $k = 0, 1$, there is a θ^{ik} such that $c^i = \sum_{k=0,1} \theta^{ik} D^k$ and $\sum_i \theta^{ik} = \sum_i \bar{\theta}^{ik}$ for every $k = 0, 1$. Let $q^1 = E[\pi D^1] - R_\mathrm{f} F^1(D^1)$. We now show that $(D^1, (\theta^{ik})_{i,k}, \tilde{R}_\mathrm{f}, q)$ is a MVE with $D^1 \in M$.

The profit maximization condition under (R_f, π) has been shown. Since D^1 is a profit maximizing output at the M-restricted equilibrium, $D^1 \in M$. The market-clearing condition follows from the construction of the θ^{ik}. The utility maximization condition follows from the utility maximization condition for the M-restricted equilibrium and the fact that (R_{f}, π) coincides with $(\tilde{R}_{\mathrm{f}}, \tilde{\pi})$ on $\boldsymbol{R} \times M$. ///