Evil Deeds in Urban Economics*

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Abstract

The purpose of this note is to update an ancient controversy over the comparison between discrete and continuous agent models of land use and agent location in urban economics. Berliant (1985) shows that that the following statement is self-contradictory: "There is a continuum of agents, each of whom owns or is endowed with a positive Lebesgue measure of land." A corollary follows: "As the number of agents tends to infinity, the set of agents who own a positive Lebesgue measure of land shrinks to measure zero." The basic question is this: Under what circumstances can the discrete agent model be used to justify the continuous agent model in the sense that the equilibria of the logically consistent discrete agent model are close to those of the logically inconsistent continuous agent model? In other words, under what circumstances can the two models be reconciled? They can be reconciled exactly when commuting cost tends to zero at a rate inversely proportional to population size. JEL classifications: D51, R13, R14 Keywords: Large urban economies, Continuous and discrete agent models

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You just asked me, when will it end?
Hahahahaha, well let me tell you
Once an evil deed is done, then it never ends
It goes on, and it will go on forever... (Wu Tang Clan, Evil Deeds, *Wu Tang Chamber Music*)

1 Introduction

The purpose of this note is to update an ancient controversy over the comparison between discrete and continuous agent models of land use and agent location in urban economics. A mathematically trivial result in Berliant (1985) shows that the following statement is self-contradictory: "There is a continuum of agents, each of whom owns or is endowed with a parcel of positive Lebesgue measure of land." A corollary follows: "As the number of agents tends to infinity, the set of agents who own a parcel of positive Lebesgue measure of land shrinks to measure zero." Thus, the urban economics models with a continuum of consumers has a logical inconsistency built into it. This has consequences.[1]

An alternative, based on a finite number of agents consuming measurable subsets of a fixed land mass, is detailed in Berliant (1985) and references therein. However, this model is not quite analogous to the standard model of urban economics. Another alternative is to use the model first proposed by Alonso (1964) that has agents consuming intervals of land in the real line instead of arbitrary measurable sets. This model was developed further by Berliant and Fujita (1992) and exposited for undergraduates by Berliant and LaFountain (2006).

One way to justify use of the logically inconsistent model with a continuum of agents is to show that its equilibria are close to those of a logically consistent model, namely one with a finite number of agents where land consumption is not close to zero. Thus, the basic argument is that the standard continuum of consumers model employed throughout mainstream urban economics is justified as an approximation to an analogous model with a finite number of consumers. This should hold for (at

[1] We note in passing that existence of equilibrium and the welfare theorems can fail in the standard model with a continuum of consumers; see Berliant, Papageorgiou and Wang (1990).

least) equilibrium objects, such as prices and the population distribution. After all, the real world has only a finite number of consumers, although the model with a continuum of consumers is mathematically convenient.

Therefore, the basic question we address here is this: *Under what circumstances, if any, can we reconcile the two models?*

The literature proceeds with Papageorgiou and Pines (1990), Asami, Fujita and Smith (1991), and Kamecke (1993). McLean and Muench (1981) to a certain degree anticipate the controversy. Berliant (1991) and Berliant and ten Raa (1991) provide critiques of these earlier approaches that we shall not repeat here. Berliant and Sabarwal (2008) (henceforth BS) show that, in the versions of the discrete and continuous models used below, *an empirically relevant comparative static differs in the two types of models.* We shall illustrate this with an example. In that sense, there is no hope for reconciliation.

In this note, we begin by comparing equilibrium price and population densities of the linear (one dimensional) monocentric city models for a continuum of agents and a finite number of agents. The models will feature the same utility functions and endowments for all consumers. The finite model and the continuum model will have the same "number" of consumers. In the finite model this is the actual integer number. In the continuum model, this is interpreted as the (Lebesgue) measure of consumers on the real line. There is an absentee landlord. The city boundary is endogenous, and determined by agricultural land rent. We note that as population goes to infinity, in both models equilibrium per capita land consumption tends to zero whereas equilibrium land prices tend to infinity.

To make our ideas concrete and to simplify the calculations, we first employ a log-linear example, though we have worked out analogous calculations in many variations of the model. The price densities and population distributions for the two models are indeed qualitatively similar, though as noted above, a key comparative static in commuting cost differs. The first conclusion from our analysis is that if we substitute the equilibrium price density for the model with a continuum of consumers into the corresponding finite model, we find that the utility levels induced in the identical consumers of the corresponding finite model differ substantially from one another. Thus, the equilibrium prices of the continuum of consumers model do not come close to equilibrating the finite model, and thus cannot be considered approximations to a finite model equilibrium. The reason for this is not the difference in the prices of the two models at the boundaries of the parcels (front and back locations) in the finite model, but rather the (integrated) price differences on the interiors of the parcels. We refer to such differences as *infra-marginal* differences.

The main result contained in this paper is that as the "number" of consumers tends to infinity in *both* models, if the marginal commuting cost tends to zero at a rate that is inversely proportional to the "number" of consumers, then the equilibria converge. Otherwise, they do not. Is the glass half full (Masa) or half empty (Marcus)? We shall address this question at the end.

This note is organized as follows. Section 2 analyzes the linear city for both the model with a continuum of consumers and the model with a finite number of consumers. Section 2.1 introduces the continuous agent model. Section 2.2 does the same for the discrete agent model. Section 2.3 graphs the equilibrium prices for the two models for comparison. Section 2.4 compares equilibrium population distributions of the two models. Section 2.5 provides the key comparison, namely utility levels of identical consumers in the finite consumer model when equilibrium prices from the continuum consumer model are applied. Section 2.6 summarizes the section. At long last, Section 3 provides precise conditions for convergence of the two models to hold. Section 4 provides our conclusions and directions for future research.

2 Log Linear Utility

2.1 The Continuous Agent Model

2.1.1 Notation

Here we introduce a canonical model of urban residential location, described for example in Mills (1972), that is often called the "New Urban Economics." Interpretation and explanation can be found in Fujita (1989).

r	Distance from CBD
z	Consumption of composite commodity
s	Land consumption
t	Per kilometer commuting cost
$w > 0$	Endowment of composite good per capita
$u(s, z)$	Utility function
N	Measure of consumers (exogenous)
$n(r)$	Density of consumers (per kilometer)
$\overline{r}$	City boundary (endogenous)
$\overline{u}$	Utility level (endogenous)
$p(r)$	Per unit land rent at distance r from the CBD (endogenous)
1	Agricultural land rent (exogenous)
1	Land supply density (exogenous)

2.1.2 Equilibrium

Take $u(s, z) = z + \ln(s)$. This is classic, as the resulting rent gradient (derived below) is often used for empirical estimation; see for example Mills (1972, p. 247, Table 1). Quasi-linear utility is useful for solving the finite model explicitly. With a bit of effort, it is possible to solve the finite model when utility is Cobb-Douglas or other CES.

Consumer optimization problem:

$$\max_{z,s} z + \ln(s)$$

subject to:

$$z + p(r) \cdot s + r \cdot t = w$$

Substituting the budget into the objective function,

$$
\begin{aligned}
& \max_s w - p(r)\cdot s - r\cdot t + \ln(s) \\
p(r)\cdot s(r) &= 1 \\
z(r) &= w - p(r)\cdot s(r) - r\cdot t = w - 1 - r\cdot t \\
\ln(s(r)) &= \overline{u} - z(r) = \overline{u} - w + 1 + r\cdot t \\
s(r) &= \exp(\overline{u} - w + 1)\cdot \exp(r\cdot t) \\
p(r) &= n(r) = \frac{1}{s(r)} = \exp(w - 1 - \overline{u})\cdot \exp(-r\cdot t) \qquad (1)
\end{aligned}
$$

Moreover, we know that rent at the urban boundary must be 1:

$$p(\overline{r}) = 1 = \exp(w - 1 - \overline{u})\cdot \exp(-\overline{r}\cdot t)$$

Hence,

$$\exp(w - 1 - \overline{u}) = \exp(\overline{r}\cdot t)$$

Next,

$$
\begin{aligned}
N &= \int_0^{\overline{r}} n(r)dr \\
&= \exp(w - 1 - \overline{u})\cdot \int_0^{\overline{r}} \exp(-r\cdot t)dr \\
&= \exp(w - 1 - \overline{u})\cdot \left[-\frac{1}{t}\exp(-r\cdot t)\Big|_0^{\overline{r}} \right] \\
&= \exp(w - 1 - \overline{u})\cdot \frac{1}{t}\left[1 - \exp(-\overline{r}\cdot t)\right]
\end{aligned}
$$

Therefore,

$$\exp(w - 1 - \overline{u}) = \frac{Nt}{1 - \exp(-\overline{r}\cdot t)}$$

and

$$p(r) = \frac{Nt}{1 - \exp(-\overline{r}\cdot t)}\cdot \exp(-r\cdot t)$$

Since $p(\overline{r}) = 1$,

$$
\begin{aligned}
1 &= \frac{Nt}{1 - \exp(-\overline{r}\cdot t)}\cdot \exp(-\overline{r}\cdot t) \\
1 - \exp(-\overline{r}\cdot t) &= \exp(-\overline{r}\cdot t)Nt \\
1 &= \exp(-\overline{r}\cdot t)\left[Nt + 1\right] \\
\overline{r} &= \frac{\ln(Nt + 1)}{t}
\end{aligned}
$$

Therefore,

$$p(r) = \exp(-rt)\,[Nt + 1] \tag{2}$$

For later use, note that:

$$\ln[p(r)] = \ln(Nt + 1) - rt \tag{3}$$

2.2 The Discrete Agent Model

After introducing the notation for the finite model, we shall verify that in the case of log-linear utility used in this note, the equilibrium utility levels achieved by consumers are indeed identical.

2.2.1 Notation

We will use analogous notation for the finite model. There are just a few alterations. First, the number of consumers is N, an integer. Second, we index bundles (x_i, s_i, z_i) by consumer $i = 1, 2, ..., N$, where consumers (who are *ex ante* identical) are ordered from the CBD outward, and where the location of the front of a parcel is denoted by x_i, so consumer i owns the interval $[x_i, x_i + s_i)$. Finally, we call the price density for this model $P(r)$ to distinguish it from the continuum of agents model.

2.2.2 First Order Conditions

For completeness, we summarize here the first order conditions for consumer optimization subject to the budget constraint for the model with a finite number of agents in the linear city. These conditions are useful for solving for equilibrium. For a city that isn't linear, these conditions will be different.

Assuming local non-satiation of preferences and differentiability of utility, the consumer's problem is:

$$\max_{x,s,z} u(s, z) \text{ subject to}$$

$$w = z + \int_x^{x+s} P(r)dr + t \cdot x$$

Setting

$$z = w - \int_x^{x+s} P(r)dr - t \cdot x$$

to eliminate z, we have:

$$\max_{x,s} u(s, w - \int_x^{x+s} P(r)dr - t \cdot x)$$

with first order conditions:

$$\begin{aligned} \frac{\partial u}{\partial s} - \frac{\partial u}{\partial z} \cdot P(x+s) &= 0 \\ -\frac{\partial u}{\partial z} \cdot (P(x+s) - P(x) + t) &= 0 \end{aligned}$$

The first equation yields:

$$\frac{\partial u / \partial s}{\partial u / \partial z} = P(x+s) \tag{4}$$

This is the standard price equals marginal rate of substitution equation at the end of the parcel. The second equation yields:

$$P(x+s) - P(x) + t = 0$$

or

$$P(x) - P(x+s) = t. \tag{5}$$

The intuition behind condition (5) has been explained in detail in Berliant and LaFountain (2006). If the difference between front and back prices of the parcel is less than t, then the consumer would want to shift the parcel inward (retaining the same size), pocketing the difference in cost of the parcel. If the difference were greater than t, then the consumer would want to shift the parcel outward, pocketing the difference.

Note that this condition does not apply to the parcel and price of the first consumer. The first consumer is special, since their parcel cannot be shifted inward beyond 0. The consumer can pay a constant price for land, or a price density everywhere below or equal to their marginal willingness to pay for land beginning at zero land consumption.

2.2.3 Equilibrium

With quasi-linear utility, it's easiest to solve for the equilibrium parcels first. The first order conditions for the model with finitely many agents yield:[2]

$$\begin{aligned} \frac{1}{s_i} &= 1 + (N - i) \cdot t \\ s_i &= \frac{1}{1 + (N - i) \cdot t} \end{aligned} \tag{6}$$

The price density $P(r)$ can be defined as follows. For consumer 1, the consumer closest to the CBD with the smallest parcel, the price density is $(N-1)t+1$. This applies for r such that $\frac{1}{r} \geq (N-1)t+1$ or $0 \leq r \leq \frac{1}{(N-1)t+1}$. Then, $u(s_1, z_1) = w - 1 - \ln(1 + (N-1) \cdot t)$.

For consumer 2, $P(r)$ is defined as follows. For r with $\frac{1}{(N-2)t+1} \geq r \geq \frac{1}{(N-1)t+1}$, $P(r) = \frac{1}{r}$. For r with $\frac{1}{(N-2)t+1} \leq r \leq \frac{1}{(N-1)t+1} + \frac{1}{(N-2)t+1}$, $P(r) = (N-2)t+1$. Then

$$\begin{aligned} u(s_2, z_2) &= w - \frac{t}{(N-1)t+1} - \ln(1 + (N-2) \cdot t) - \int_{\frac{1}{1+(N-1)\cdot t}}^{\frac{1}{1+(N-1)\cdot t} + \frac{1}{1+(N-2)\cdot t}} P(r)dr \\ &= w - \frac{t}{(N-1)t+1} - \ln(1 + (N-2) \cdot t) \\ &\quad - \int_{\frac{1}{1+(N-1)\cdot t}}^{\frac{1}{(N-2)t+1}} \frac{1}{r} dr - \int_{\frac{1}{(N-2)t+1}}^{\frac{1}{(N-1)t+1} + \frac{1}{(N-2)t+1}} \left[(N-2)t+1\right] dr \\ &= w - \frac{t}{(N-1)t+1} - \ln(1 + (N-2) \cdot t) \\ &\quad + \ln((N-2)t+1) - \ln(1 + (N-1) \cdot t) - \frac{(N-2)t+1}{(N-1)t+1} \\ &= w - 1 - \ln(1 + (N-1) \cdot t) \\ &= u(s_1, z_1) \end{aligned}$$

Thus, the equilibrium utility levels for consumers 1 and 2 are the same. Similar calculations should apply for the other consumers, verifying the claim in Berliant and Sabarwal (2008, p. 441) that this is an equilibrium for the finite model with identical consumers for this particular functional form of utility.[3]

[2] This calculation is obviously not as easy with, say, Cobb-Douglas or CES utility. But we have done these calculations, and the results are qualitatively the same as for log linear utility.

[3] There are likely other equilibria, with for example a non-constant price density for the first consumer. The one we examine appears to be the equilibrium that maximizes consumer utility and minimizes landlord income among all equilibria.

Proceeding to compute the price density for all locations,[4]

$$P(r) = \begin{cases} (N-1)t+1 \text{ for } \frac{1}{(N-1)t+1} \geq r \geq 0 \\ \frac{1}{r-\sum_{j=1}^{i-2}\frac{1}{1+(N-j)\cdot t}} \text{ for } \sum_{j=1}^{i-2}\frac{1}{(N-j)t+1} + \frac{1}{(N-i)t+1} \geq r \geq \sum_{j=1}^{i-1}\frac{1}{(N-j)t+1} \text{ and } i \geq 2 \\ (N-i)t+1 \text{ for } \sum_{j=1}^{i}\frac{1}{(N-j)t+1} \geq r \geq \sum_{j=1}^{i-2}\frac{1}{(N-j)t+1} + \frac{1}{(N-i)t+1} \text{ and } i \geq 2 \end{cases}$$

2.3 Price Graphs

Note that for the finite model, the price density $P(r)$ must be defined piecewise, segment by segment. This is quite labor intensive.

Let's try $N = 10$, $t = 1$ to get things going. The graph of both the discrete agent model land price density, $P(r)$, and the continuous agent model land price density, $p(r)$, are given in Figure 1.

Notice that when more consumers are added by increasing N, due to the quasi-linear utility function, the $P(r)$ function is only modified from $i = 1$ leftward. That is, $P(r)$ is the same for $i = 2, 3, ..., 10$. What this means is that the graph is extended to the left of $r = 0$, but remains almost the same as the graph below for $r > 0$. The only difference is in the first consumer, who is special.

$$p(r) = \begin{cases} \exp(-r)\cdot 11 & 0 \leq r \leq \ln(11) \\ 1 & \ln(11) \leq r \end{cases}$$

[4] The slight differences between this expression and the one in BS are a type-o in that paper $(\zeta_s(s - \sum_{k=1}^{n-1} s_k, u))$ and the way we index consumers.

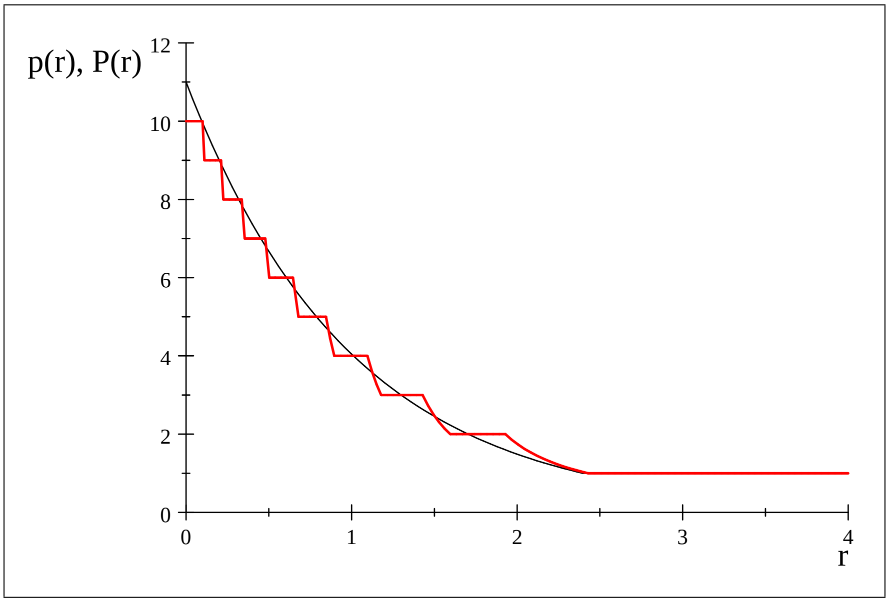

Figure 1: Linear city price densities: p(r) for continuum model (black), P(r) for finite model (red); $t = 1$

Now let's try $t = 10$. The land price densities are given in Figure 2.

$$p(r) = \begin{cases} \exp(-r \cdot 10) \cdot 101 & 0 \leq r \leq \ln(101)/10 \\ 1 & \ln(101)/10 \leq r \end{cases}$$

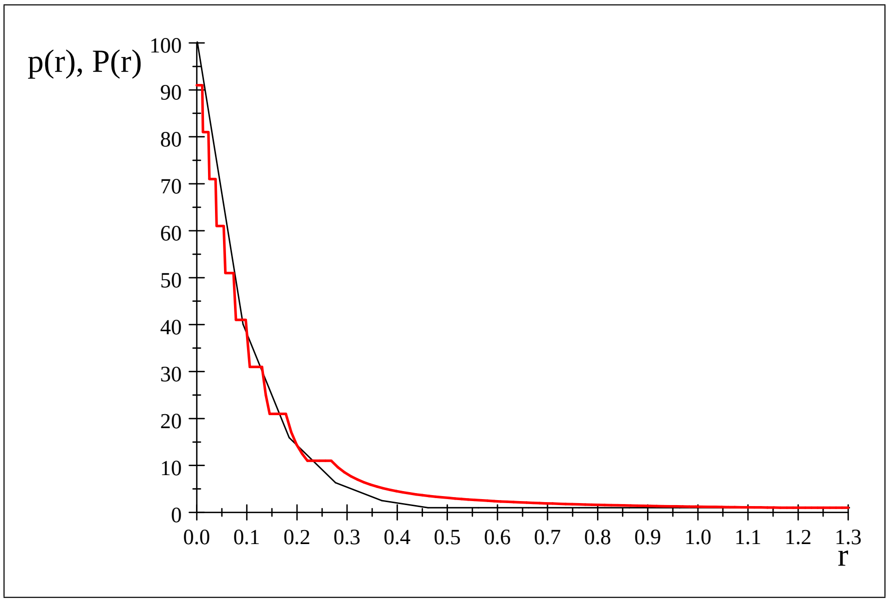

Figure 2: Linear city price densities: p(r) for continuum model (black), P(r) for finite model (red); $t = 10$

Notice that the price graphs for $t = 1$ and $t = 10$ are consistent with the comparative static calculations in Berliant and Sabarwal (2008). They state that in the city with a flexible boundary, increasing t will rotate the rent density clockwise in the continuum model, but increase all land prices in the finite model.

2.4 Population Graphs

We will graph the cumulative population at distance r from the CBD. For the continuous population model, with our functional form, we can explicitly compute from (1) its integral, namely the cumulative population distribution in the case $N = 10$, $t = 1$:

$$\int_0^{r'} n(r)dr = 11 - 11\exp(-r')$$

For the discrete model, we cumulate agents at the back ends of their parcels. For this purpose, we utilize (6), and call the cumulative $N(r)$. Figure 4 illustrates the cumulative population as a function of distance from the CBD.

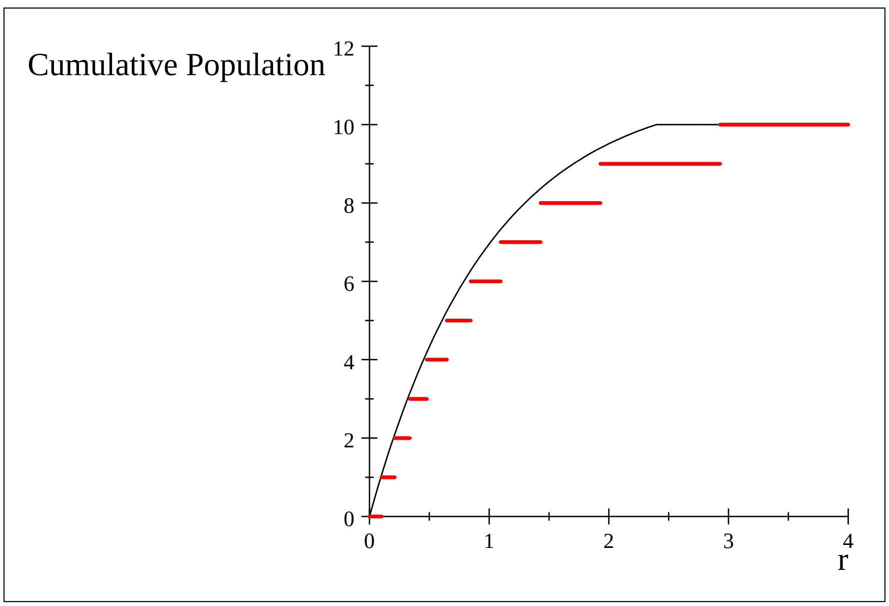

Figure 3: Cumulative population: $\int$n(r)dr for continuum model (black), $N(r)$ for finite model (red); $t = 1$

It is obvious that deviations between the models get larger farther from the CBD.

Next, we try $N = 10$ and $t = 10$.

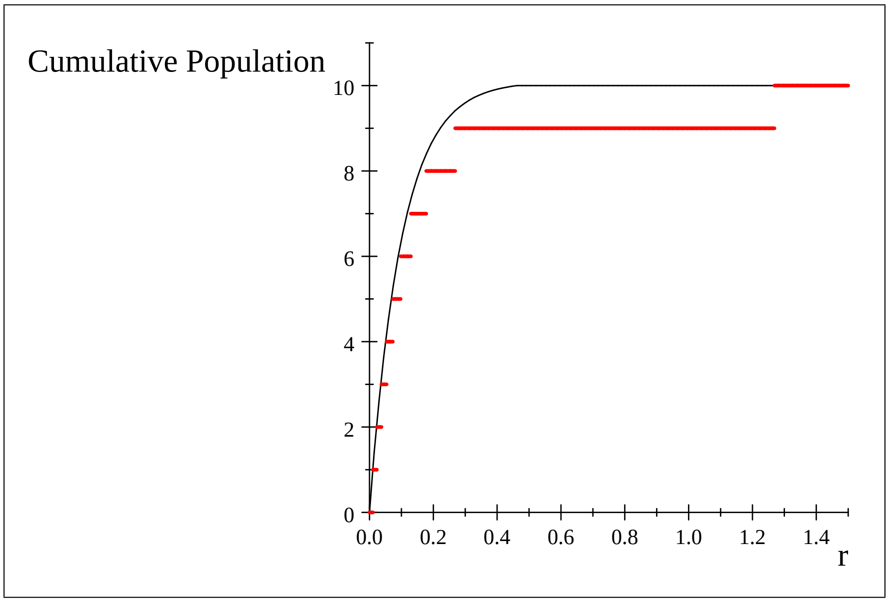

Figure 4: Cumulative population: $\int$n(r)dr for continuum model (black), $N(r)$ for finite model (red); $t = 10$

2.5 Infra-Marginal Analysis

Until now, we have compared price or rent densities in the finite and continuum models. The graphs are reasonably close. The crux of the matter here is actually what happens infra-marginally. More specifically, since all agents are identical, what is required in both models is that utility be equalized across agents, for otherwise the unhappy agents will imitate the happy ones, and a potential equilibrium allocation will be disturbed. In the finite model, for this purpose it is not the point by point comparison of prices that matters, but rather the integral of the differences, which is the difference in total amount paid in land rent by a consumer.

One way to address this phenomenon is to try to use the continuum model price density as an approximation to the finite model price density by calculating the utility level of each consumer in the finite model when the continuum model price density is used. Since the difference between land price at the front and back locations of each parcel (except the one closest to the CBD) must be t, which we set to 1, it is not hard to calculate finite model equilibrium lots and utility for prices set to the continuum model equilibrium rent density.

We set endowments $w = 0$ for convenience, but if this is bothersome, a positive constant w can be added to all utility levels. The calculations in Table 1 incorporate utility of land consumption, commuting cost, and cost of the land parcel.

Consumer	Land Consumption (s)	Cost	Utility
1	.09531	1	−3.3506
2	.10536	1	−3.3457
3	.11778	.99998	−3.3396
4	.13354	1.0001	−3.3319
5	.15415	.99999	−3.3218
6	.18232	.99999	−3.3081
7	.22314	.99998	−3.2884
8	.2877	1.0001	−3.2575
9	.4054	.99985	−3.202
10	.6932	1.0001	−3.0712

Table 1: Using Equilibrium Land Prices from the Continuum Model in the Finite Model[5]

What conclusions should we draw from this calculation? Using the continuum model prices in the finite model, the marginal prices for land at the boundaries of each parcel are not bad approximations to the finite model prices. The issue is actually the infra-marginal continuum model prices, namely those on the interior of the parcel. These must be integrated to obtain the total cost of the parcel. They do not come close to the equilibrium condition that identical consumers in the finite

[5]Define $L(r)$ to be the supply of land at distance r from the CBD. For the linear city, $L(r) = 1$ for all r. Why is the equilibirum cost of each consumer's parcel equal to 1 (Column 3)? Because in the quasi-linear continuum model, $p(r) = n(r)$, and the following theorem from the book applies:

Proposition 1 *Assume that* $\int_{x_i}^{x_i+s_i} n(r)dr = 1$*, so that one consumer in the finite model is represented by measure* 1 *of consumers in the continuum model. Suppose that* $L'(r) = 0$ *a.e. Then the Muth-Mills condition for the continuum model implies that the analogous condition for the finite model must hold for the continuum model prices. Suppose that* $L'(r) > 0$ *(e.g.* $L(r) = 2\pi r$*) or* $L'(r) < 0$ *a.e. Then the Muth-Mills condition for the continuum model implies that the analogous condition for the finite model* ***cannot hold*** *for the continuum model prices.*

model must have the same utility level as every other consumer. In this sense, the continuum model equilibrium prices do not serve as a good proxy for the finite model equilibrium. *Notice that this comparison does not employ finite model equilibrium prices, so it doesn't depend on which finite model equilibrium price is selected.*

2.6 Summary

Here we have computed equilibria of the model with a continuum of identical consumers and of the model with a finite number of identical consumers in the context of the same set of locations, the same commuting cost, and the same log-linear utility function in a linear city. The price densities look similar. They can be made close by taking the number of consumers to be large, although the price densities must then diverge to infinity and the size of parcels will converge to zero. Thus, the rate of convergence of prices in the number of consumers is crucial.

Another comparison, that discouraged us from pursuing this difficult problem further for many years, is that if we try to use the equilibrium price density for the model with a continuum of consumers as the price density for the model with a finite number of consumers, there are substantial equilibrium utility differences between the identical consumers. This derives from the differences in the prices of inframarginal units of land. In this sense, the equilibrium price density for the model with a continuum of consumers does not represent well the equilibrium of the model with a finite number of consumers, where equilibrium utility must be the same across consumers.

3 Resurrection - The Phoenix

3.1 Log-Linear Utility

Let's recast the problem, beginning with the log-linear utility model. We shall continue to choose the price of agricultural land to be 1, though this will be immaterial. We shall also continue to use the model where the boundary of the city is determined by agricultural land rent since this simplifies calculations, though it's rather apparent that we could use an exogenous city boundary by picking the agricultural land price appropriately.

Next, let us calculate the equilibrium values of various endogenous objects that will be useful. Notice that if we use continuum model equilibrium prices in the finite model as in the previous section (Table 1), utility of the *ex ante* identical consumers is monotone increasing from the CBD.[6]

Inverting $p(r)$ from (2), for the continuum model,

$$\begin{aligned} r(p) &= -\frac{1}{t}\ln\left(\frac{p}{Nt+1}\right) \\ &= \frac{1}{t}\ln\left(\frac{Nt+1}{p}\right) \end{aligned}$$

For the finite model, using the equilibrium price density from the continuum model,

$$s_1 = r((N-1)t+1) = \frac{1}{t}\ln\left(\frac{Nt+1}{(N-1)t+1}\right)$$

Therefore, the utility of land for consumer 1 is:

$$\ln(s_1) = \ln\left(\ln\left(\frac{Nt+1}{(N-1)t+1}\right)\right) - \ln(t)$$

From (2), the cost of land for consumer 1 under the continuum model price density is:

$$\begin{aligned} \int_0^{s_1} p(r)dr &= \int_0^{s_1} [Nt+1]\exp(-rt)dr \\ &= -\frac{1}{t}\exp(-rt)\,[Nt+1]\bigg|_0^{s_1} \\ &= -\frac{1}{t}\left[\frac{(N-1)t+1}{Nt+1}\right]\cdot[Nt+1] + \frac{1}{t}[Nt+1] \\ &= \frac{1}{t}[Nt+1-((N-1)t+1)] = 1 \end{aligned}$$

The commuting cost for consumer 1 is 0. Hence, the net utility for consumer 1 using the continuum model land price density is:

$$\ln\left(\ln\left(\frac{Nt+1}{(N-1)t+1}\right)\right) - \ln(t) - 1$$

[6]This won't be used in the general analysis in the next subsection, but makes the illustrative example in this subsection easier.

Next, we compute the net utility under the continuum model equilibrium price density for the consumer farthest from the CBD, namely consumer N.

$$\begin{aligned} r(1) &= \frac{1}{t}\ln\left(Nt+1\right) \\ r(1+t) &= \frac{1}{t}\ln\left(\frac{Nt+1}{1+t}\right) \\ s_N &= r(1)-r(1+t) \\ &= \frac{1}{t}\ln(1+t) \end{aligned}$$

Therefore, the utility of consumer N from land is:

$$\ln(s_N)=\ln(\ln(1+t))-\ln(t)$$

The commuting cost of consumer N is:

$$t\cdot r(1+t)=\ln\left(\frac{Nt+1}{1+t}\right)$$

The cost of land for consumer N is:

$$\begin{aligned} \int_{r(1+t)}^{r(1)} p(r)dr &= \int_{r(1+t)}^{r(1)} [Nt+1]\exp(-rt)dr \\ &= -\frac{1}{t}\exp(-rt)\left[Nt+1\right]\Big|_{r(1+t)}^{r(1)} \\ &= -\frac{1}{t}\left[\frac{1}{Nt+1}\right]\cdot[Nt+1]+\frac{1}{t}\left[t+1\right] \\ &= 1 \end{aligned}$$

Hence the net utility of consumer N at the continuum model equilibrium price density is:

$$\ln(\ln(1+t))-\ln(t)-\ln\left(\frac{Nt+1}{1+t}\right)-1$$

Taking the difference in net utility between consumer 1 and consumer N:

$$\begin{aligned} &\ln\left(\ln\left(\frac{Nt+1}{(N-1)t+1}\right)\right)-\ln(t)-1-\ln(\ln(1+t))+\ln(t)+\ln\left(\frac{Nt+1}{1+t}\right)+1 \\ =\ &\ln\left(\ln\left(\frac{Nt+1}{(N-1)t+1}\right)\right)+\ln\left(\frac{Nt+1}{1+t}\right)-\ln(\ln(1+t)) \end{aligned}$$

Therefore, we are looking for parameters such that:

$$
\begin{aligned}
\ln\left(\ln\left(\frac{Nt+1}{(N-1)t+1}\right)\right)+\ln\left(\frac{Nt+1}{1+t}\right) &= \ln(\ln(1+t)), \text{ or} \\
\ln\left(\frac{Nt+1}{(N-1)t+1}\right)\cdot\left(\frac{Nt+1}{1+t}\right) &= \ln(1+t), \text{ or} \\
\ln\left(\frac{Nt+1}{(N-1)t+1}\right)\cdot\left(\frac{Nt+1}{1+t}\right)-\ln(1+t) &= 0
\end{aligned}
$$

The crucial step is as follows. For any $\gamma \geq 0$, take

$$t = \frac{\gamma}{N}.$$

Then the last expression can be written as:

$$\ln\left(\frac{\gamma+1}{\gamma-\frac{\gamma}{N}+1}\right)\cdot\left(\frac{\gamma+1}{1+\frac{\gamma}{N}}\right)-\ln(1+\frac{\gamma}{N})=0 \tag{7}$$

For the purpose of graphing this function, set $\gamma = 1$, obtaining:

$$\ln\left(\frac{2}{2-\frac{1}{N}}\right)\cdot\left(\frac{2}{1+\frac{1}{N}}\right)-\ln(1+\frac{1}{N})=0 \tag{8}$$

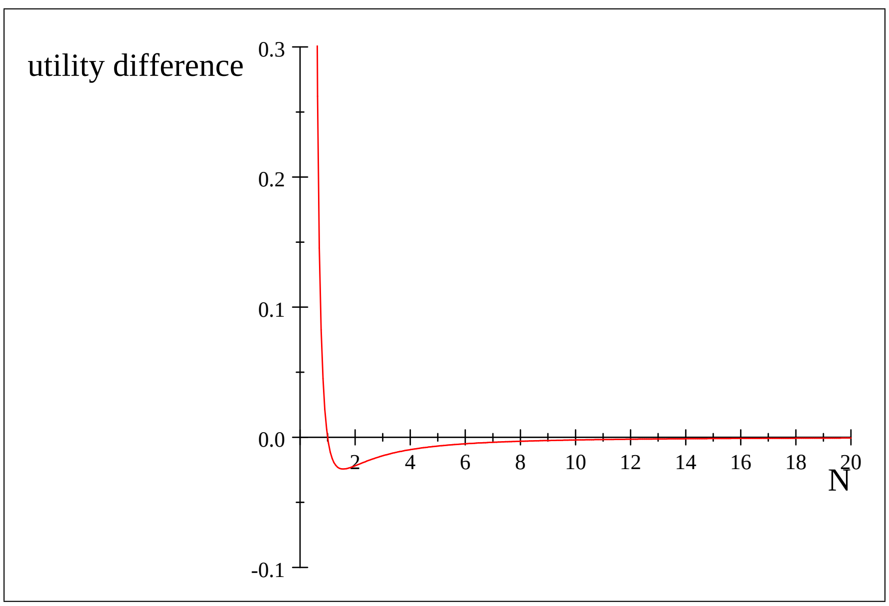

Figure 5: Utility difference between first and last consumers as a function of N where $t = \frac{1}{N}$

The solutions in the case $\gamma = 1$ in equation (8) are $N = 1$ and $N = \infty$. This corresponds to $t = 1$ and $t = 0$, respectively. The same holds more generally for equation (7).

For now, set aside the case $N = 1$, since it is not realistic. The bottom line here is that *to obtain correspondence between the equilibrium price density of the continuum model and an equilibrium of the appropriate finite model, population in both models must be sent to infinity* ***and marginal commuting cost must be sent to zero at the same rate***.

The goal of the next subsection is to make the result for this example general.

3.2 General Utility[7]

In an attempt to keep notation simple, we use the superscript N to denote the equilibrium quantities for an economy with N consumers, e.g. $p^N(r)$ denotes the continuum model equilibrium price of land at distance r from the CBD with measure N consumers.

We generalize the result in the previous subsection:

Theorem 2 *Let u be continuous, locally non-satiated and strictly quasi-concave, satisfying the boundary condition $u(0, z) = \inf_{s,z} u(s, z)$. Fix the number of consumers at positive integer N in the finite agent model and the measure of consumers at N in the continuum of agent model. Let $\gamma > 0$. Take any two consumers in the finite agent model, say i and j. Let the bundles (x_i^N, s_i^N, z_i^N) and (x_j^N, s_j^N, z_j^N) satisfy the first order conditions and budget constraint for the demand problems of consumers i and j at the continuum of agents model equilibrium prices, and let $t^N = \frac{\gamma}{N}$ be marginal commuting cost.*[8] *Then the utility levels of i and j will converge as $N \to \infty$: $\lim_{N\to\infty} \left| u\left(s_i^N, w - \int_{x_i^N}^{x_i^N+s_i^N} p^N(r)dr - t^N \cdot x_i^N\right) - u\left(s_j^N, w - \int_{x_j^N}^{x_j^N+s_j^N} p^N(r)dr - t^N \cdot x_j^N\right) \right| = 0$.*

---

[7] Modern theorems on existence of equilibrium and the welfare theorems for the continuous model can be found in LaFountain (2008) and for the discrete model in Berliant and Fujita (1992). The book contains literature reviews of these topics.

[8] It would be possible to replace the assumption $t = \frac{\gamma}{N}$ with the assumption that t is of order $\mathcal{O}\left(\frac{1}{N}\right)$, but at the cost of complicating matters.

Since utility equalization is the missing piece of competitive equilibrium in the finite agent model beyond the first order conditions and budget satisfaction, we can say in this sense that the continuum of agent equilibrium prices are close to clearing the markets of the finite agent model, so the two models converge. More formally, the continuous model equilibrium price system comes close to a finite model equilibrium price system in the sense that consumers in the finite model come close to optimizing utility subject to the budget constraint whereas markets clear. Aside from standard assumptions on utility, the unusual assumption here is that marginal commuting cost must tend to zero at a rate proportional to the number of consumers in both economies, which tends to infinity.

Proof. Define $t^N = \frac{\gamma}{N}$. Notice first that the individual consumers in the continuum model with measure N consumers all attain the same equilibrium level of utility. Hence, at the front of each parcel in the model with a finite number of agents,

$$u\left(s^N(x_i^N), w - p^N(x_i^N) \cdot s^N(x_i^N) - t^N \cdot x_i^N\right) = u\left(s^N(x_j^N), w - p(x_j^N) \cdot s^N(x_j^N) - t^N \cdot x_j^N\right) \tag{9}$$

Next we calculate the utility levels of consumers i and j in the finite agent model, where we use the first order conditions for agent optimization of utility subject to the budget constraint at the equilibrium price density of the continuum model. We will show that:

$$\lim_{N\to\infty} \left| u\left(s_i^N, w - \int_{x_i^N}^{x_i^N + s_i^N} p^N(r)dr - t^N \cdot x_i^N\right) - u\left(s^N(x_i^N), w - p^N(x_i^N) \cdot s^N(x_i^N) - t^N \cdot x_i^N\right)\right| = 0 \tag{10}$$

The same will hold for consumer j, so once we have these facts in hand, using (9), we are done.

The next step is to show $\lim_{N\to\infty} \left|s_i^N - s^N(x_i^N)\right| = 0$. Define $s^* = \arg\max_s u(s, w-(\gamma+1)\cdot s)$. Since this is simply land demand at land price $\gamma + 1$ with $t = 0$, strict quasi-concavity implies it is unique. We prove $\lim_{N\to\infty} \left|s_i^N - s^N(x_i^N)\right| = 0$ by showing $\lim_{N\to\infty} s_i^N = s^* = \lim_{N\to\infty} s^N(x_i^N)$.

Suppose first that there is a subsequence $\left\{s^{N_k}(x_i^{N_k})\right\}_{k=1}^{\infty}$ that does not converge to s^*. Since for each N and x, $p^N(x) \geq 1$, $s^N(x_i) \leq w$. So without loss of generality, we can pass to a converging subsequence. Call its limit s'. Now using the first order conditions for consumer i optimization in the finite model,

$$\lim_{N\to\infty} p^N(x_i^N) = \lim_{N\to\infty} \left[(N+1-i)\cdot t^N + 1\right] = \gamma + 1 \tag{11}$$

For the continuum model, the budget constraint for any k is $w \geq z^{N_k} + \left[(N_k+1-i)\cdot t^{N_k}+1\right]\cdot s^{N_k}(x_i^{N_k}) + t^{N_k}\cdot x_i^{N_k}$. Notice that $x_i^{N_k}$ is bounded by $i\cdot w$, so the last term tends to zero. Using the budget constraint, we can pass to a converging subsequence in composite good as well, so call its limit z'. The budget is also satisfied at the limit: $w \geq z' + (1+\gamma)\cdot s'$. Hence, since $\left\{s^{N_k}\right\}_{k=1}^{\infty}$ converges to $s' \neq s^*$, $u(s', z') < u(s^*, w-(\gamma+1)s^*)$. The boundary condition and $w > 0$ ensure that $s^* > 0$. So for sufficiently large k as well as sufficiently small $\epsilon > 0$, $u(s^{N_k}(x_i^{N_k}), z^{N_k}) < u(s^*-\epsilon, w-(\gamma+1)s^*)$, and $(s^*-\epsilon, w-(\gamma+1)s^* - t\cdot x_i^{N_k})$ satisfies the budget for N_k, so we have a contradiction.

Turning to the discrete model, suppose next that there is a subsequence $\left\{s_i^{N_k}\right\}_{k=1}^{\infty}$ that does not converge to s^*. Since for each N and x, $p^N(x) \geq 1$, as before, $s_i^N \leq w$. So without loss of generality, we can pass to a converging subsequence. Call its limit s''.

The budget constraint for any k is $w \geq z^{N_k} + \int_{x_i^{N_k}}^{x_i^{N_k}+s_i^{N_k}} p^{N_k}(r)dr + t^{N_k}\cdot x_i^{N_k}$. Notice that since prices are bounded below by 1, agricultural rent, the maximum number of units of land that a consumer can buy is w. Thus, in equilibrium of the finite model, $x_i^{N_k}$ is bounded by $i\cdot w$, so the last term tends to zero. Pass to a converging subsequence in $x_i^{N_k}$. Notice that for each $r \in \left[x_i^{N_k}, x_{i+1}^{N_k}\right]$, since $p(r)$ is non-increasing in r, (11) implies $p^N(r) \to \gamma+1$ pointwise.[9] In addition, for $r \in \left[x_i^{N_k}, x_i^{N_k}+s_i^{N_k}\right]$, $p^{N_k}(r) \leq (N_k+1-i)\cdot t^{N_k}+1 \leq N_k\cdot t^{N_k}+1 = \gamma+1$, and $\int_{x_i^{N_k}}^{x_i^{N_k}+s_i^{N_k}} (\gamma+1)dr \leq w\cdot(\gamma+1) < \infty$. The indicator function of the set $\left[x_i^{N_k}, x_i^{N_k}+s_i^{N_k}\right]$ is also converging pointwise. By Lebesgue's dominated convergence theorem, $\int_{x_i^{N_k}}^{x_i^{N_k}+s_i^{N_k}} p^{N_k}(r)dr \to s''\cdot(1+\gamma)$. We can pass to a converging subsequence in composite good as well, so call its limit z''. The budget is also satisfied at the limit: $w \geq z''+(1+\gamma)\cdot s''$. Hence $u(s'', z'') < u(s^*, w-(\gamma+1)s^*)$. And for sufficiently large k as well as sufficiently small $\epsilon > 0$, $u(s_i^{N_k}, z_i^{N_k}) < u(s^*-\epsilon, w-(\gamma+1)s^*-t\cdot x_i^{N_k})$, and

[9] As $N\to\infty$, the equilibrium price of land in the continuum model tends pointwise to a constant greater than the cost of agricultural land. That is because eventually, any land finite distance from the CBD is urban.

$(s^* - \epsilon, w - (\gamma + 1)s^* - t \cdot x_i^{N_k})$ satisfies the budget for N_k, so we have a contradiction.

Hence $\lim_{N\to\infty} s_i^N = s^* = \lim_{N\to\infty} s^N(x_i^N)$. Notice that the arguments above imply that

$$\lim_{N\to\infty} \int_{x_i^N}^{x_i^N + s_i^N} p^N(r)dr = s^* \cdot (1+\gamma) = \lim_{N\to\infty} p(x_i^N) \cdot s^N(x_i^N)$$

This yields (10). Repeating these arguments for consumer j, we obtain that the utility of the consumers i and j converge as $N \to \infty$. ■

At this point, there are many remarks to be made about the result and its proof.

The *rate* of convergence is an open question. The substitution effect seems to be key. For example, perfect substitutes imply slow convergence of s_i and $s(s_i)$ for the following reason. The marginal price of land in the finite model is $p(x_i) - t$, whereas it is $p(x_i)$ for the finite model. If this price difference, which tends to zero as $N \to \infty$ and $t \to 0$, causes a big difference in demand for land, for example the two corner solutions in the case of perfect substitutes, then the allocations and indirect utility won't converge quickly.

An alternative approach to the problem is to assume that as $N \to \infty$, $w \to \infty$, whereas t is fixed. This seems a bit less reasonable than the approach we have taken, since commuting cost does generally decrease with time, whereas the wealth of all consumers typically does not tend to infinity.

To what degree is the assumption that $t^N = \frac{\gamma}{N}$ (asymptotically) necessary? The key equation is (11). Independent of the other pieces of the economy, $\lim_{N\to\infty} p^N(x_i^N)$ will tend to a finite constant greater than 1 (the agricultural land rent) only if this condition holds. The cases where $\lim_{N\to\infty} p^N(x_i^N) = 1$ or ∞ do not really represent convergence in an economic sense. In the first case, the equilibrium rent gradient tends pointwise to the constant agricultural rent, so the population distribution in both models becomes flat (pointwise). In the case where all prices tend to infinity, equilibrium land consumption tends to zero in both models (pointwise).

4 Conclusion

Is the assumption that $t^N = \frac{\gamma}{N}$ reasonable? Is the glass half empty or half full? In the author's opinion, it depends on what liquid is in the glass. Requiring that commuting cost tend to zero is by itself worrisome. The condition that $N \cdot t^N$ tend to a constant seems to be a restriction on model parameters of measure zero itself.

The reader should consider this an invitation to think and write about the ideas we have presented. Many authors, including this one, use models with land and a continuum of consumers for reasons of pragmatism and expediency. However, the economic and mathematical foundations of such models are, to say the least, subject to doubt. In this context, when the broad conclusions of urban economics are drawn from such models, in the view of this author, the inconsistencies will bubble to the surface in finite time with probability one.

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