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Kazuo Mino and Hiroaki Sasaki

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Population Aging and Income Inequality in a Semi-Endogenous Growth Model*

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Abstract

Using a continuous-time overlapping generations model with semi-endogenous growth, this study examines the impact of population aging on inequality. We characterize the stationary distribution of income and wealth among households and investigate how an increase in life expectancy and a decrease in birth rate affect the distributional profile. The numerical experiments revealed that an increase in life expectancy lowers inequality, whereas a decrease in birth rate increases inequality. We also consider extended models with exogenous productivity growth, agents' retirement from labor participation, and endogenous labor supply.

Keywords: population aging, semi-endogenous growth, overlapping generations model, income and wealth inequality, Pareto distribution

JEL classification: E2, O4

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† *Corresponding author*, Institute of Economic Research, Kyoto University, Yoshida Honmachi, Sakyo-ku, Kyoto 606-8501, Japan. E-mail: mino.kazuo.2u@kyoto-u.ac.jp

‡ Graduate School of Economics, Kyoto University. Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan, E-mail: sasaki.hiroaki.7x@kyoto-u.ac.jp

1 Introduction

Population aging has accelerated in many countries. Figure 1 displays changes in the population share of older adults aged 65 years and above in several countries in Asia and Europe (and the United States) from 1950 to 2020. The graphs also show the predicted population share up to 2040. In Japan, which is one of the most rapidly aging countries worldwide, the population share of older adults aged 65 years and above increased from 10% in 1980 to 29.2% in 2020 and is predicted to reach 35% in 2040. These graphs indicate that an upward trend in population aging could be observed even in emerging countries with large populations such as China and India.

[Figure 1]

Population aging results from the extension of life expectancy and a decrease in the fertility rate. Table 1 shows the changes in the average life expectancy of the Japanese population, and Table 2 displays the total fertility rate in Japan. Table 1 shows that the average life expectancy of the Japanese people has increased substantially in the last 50 years. In addition, according to Table 2, the total fertility rate in Japan was lower than 2.0 in 1975 and continued to decrease to reach 1.32 in 2020[1]. As a result, the total population in Japan has been decreasing since 2008. The figures in Tables 1 and 2 reveal that both the extension of life expectancy and the decline in the fertility rate yield substantial population aging in Japan. Inspecting changes in the life expectancy and the total fertility rate in other aging countries listed in Figure 1, we see that the pattern of population aging in those countries is essentially the same as that in Japan.

Year	1970	1980	1990	2000	2010	2020
Female	74,6	78.6	81.9	84.4	86.4	89.6
Male	69.4	73.3	75,8	77,6	81.2	83.2

Table 1: The average lifee xpectancy in Japan(years)

Year	1970	1980	1990	2000	2010	2020
	2.28	1.81	1.58	1.41	1.39	1.32

Table2: Fertility rate in Japan[2]

[1] The total fertility rate means the average number of children born to women between the ages of 15 and 49.

[2] Sources of Tables 1 and 2 are based on the Report of Vital Statistics 2022 issued by the Japanese Ministry of Health, Labor, and Welfare.

As population aging may yield significant effects on a wide range of economic activities, the issue has been attracting considerable attention. Recent studies explored the impact of demographic change caused by population aging on economic growth (Maestas et al. 2022), labor markets (Abraham and Kearney 2020), structural transformation (Cravino, et al. 2022), social security (Heer et al. 2020, Yakita 2018), and firm dynamics (Peter and Walsh 2021, Hopenhayn et al. 2022)[3]. In this study, we focus on the effect of population aging on income and wealth distribution. To inspect the distribution effect of demographic change in a tractable manner, we use the overlapping generations (OLG) model. Specifically, we construct a continuous-time, perpetual youth model in which each household faces the probability of death. Thus, in our model, households are heterogeneous in terms of their age and wealth holding. We combine this setting with a simple semi-endogenous growth model in which persistent growth in per capita income is sustained by external increasing returns. We allow population changes so that a decrease in the birth rate (i.e., the growth rate of newly born agents) and a reduction in the mortality rate promote population aging.

Given this analytical framework, we characterize the stationary distribution of income and wealth. Then, we examine how population aging alters the distribution profiles. First, we confirm that the stationary distributions of income and wealth exhibit the Pareto profile and the shape parameter of the distribution function decreases with the steady-state rate of return on capital. Because the reciprocal of the shape parameter measures the degree of distributional inequality among households, a higher rate of return increases inequality. More precisely, the inequality index is the product of the growth-adjusted net rate of return on capital and the degree of population aging. In our OLG setting, older households accumulate larger wealth than younger ones; hence, population aging increases capital accumulation, which lowers the steady-state rate of return on capital. Hence, in this respect, population aging decreases distributional inequality. At the same time, it has a direct positive effect on inequality, and thus its total effect on the degree of inequality depends on the relative strength of these opposite effects. To explore this issue further, we examine numerical examples. Our quantitative experiments reveal that given empirically plausible parameter values, population aging caused by the extension of life expectancy lowers inequality, whereas that caused by a decrease in the population growth rate

[3] See Lee (2016) for a broad overview of the impacts of population aging.

increases inequality. This quantitative difference stems from the fact that in our semi-endogenous growth model, a change in the population growth rate affects the steady-state growth rate of income, but changes in the mortality rate do not affect the long-run growth rate of per capita income.

In addition to the baseline analysis mentioned so far, we conduct three extensions to the base model. First, we introduce the exogenous technical progress. Since exogenous productivity growth does not affect the rate of population change, it alters income and wealth distribution without affecting the degree of population aging. We find that a permanent drop in exogamous productivity growth enhances inequality.

In the second extension, we reconsider the labor supply behavior of households. The baseline model assumes that each agent supplies one unit of labor at each moment until he or she dies. By contrast, the second extended model considers the possibility of retirement from labor participation. We assume that each household may retire according to a given probability distribution. Since many people in aging economies tend to postpone their retirement, we consider the distribution effect of a decrease in retirement probability. We can confirm that a lower probability of retirement decreases inequality in stationary equilibrium.

The final extension endogenizes the household labor supply. For analytical simplicity, we assume that agents have the Greenwood-Hercowitz-Huffman (GHH) preferences under which labor supply is independent of the wealth effect. We find that the distributions of income and wealth become more unequal as the elasticity of the labor supply increases. We discuss the relationship between flexibility of labor supply and population aging.

Our study is related to the existing contributions that characterize income and wealth distributions in the context of dynamic models with heterogeneous agents and idiosyncratic shocks. Benhabib et l. (2011) and Benhabib et al. (2016) explore OLG models with a bequest motive in which idiosyncratic income shocks hit individual agents. The authors reveal that the stationary distribution of wealth exhibits a double Pareto distribution. Similarly, based on a simple model with the birth and death of agents, Jones (2014) shows that the stationary income distribution is Pareto. His main concern is to study a recent rise in top income inequality emphasized by Piketty (2014). Using a continuous-time OLG model, Hiraguchi (2019) also considers Piketty's claims by examining the relationship between income growth and inequality. Nirei and Aoki (2016) study a neoclassical growth model with idiosyncratic investment shocks and drive a stationary Pareto distribution of wealth. More recently, Moll et al. (2022) obtain a similar outcome in the neoclassical growth model with idiosyncratic preference shocks[4]. Their primary concern is to study the effect of automation technology on the distribution of income and wealth. The model examined by Moll et al. (2022) is complex, but the mechanics that generate the Pareto distribution of income and wealth are essentially the same as that in the foregoing studies[5]. None of the studies mentioned thus far consider the effect of population aging on income and wealth destitutions.

From an analytical viewpoint, Hiraguchi (2019) is the closest to our study. The author also utilizes the Blanchard-Yaari-type perpetual youth model to characterize the stationary distribution of income. Although our research concern and the analytical framework overlap with Hiraguchi (2019), our study departs from his contribution in three important aspects. First, we employ a semi-endogenous growth model in which the population growth rate affects the steady-state growth rate of income, whereas Hiraguchi (2018) treats a neoclassical growth model in which the steady-state growth rate of per capita income is specified exogenously. Second, Hiratuchi (2019) focuses on the relationship between income growth rate, rate of return on capital, and inequality, whereas our concern is the distributional effect of demographic change. Third, we characterize the stationary distributions of income and wealth in a more general and detailed manner than Hiraguchi (2019).

The remainder of this paper proceeds as follows. Section 2 sets up the baseline model and examines its dynamic properties. Focusing on the steady-state growth equilibrium, Section 3 explores the linkage between population aging and income and wealth inequality. Section 4 examines the extended models mentioned above. Finally, Section 5 concludes the paper.

[4] Moll et al. (2022) assume that when a 'dissipation shock' hits an agent, she consumes all of her wealth instantaneously. As pointed out by the authors, their setting is essentially the same as that of the Blanchard-Yaari-type perpetual youth model in which each agent faces a death shock at every moment.

[5] The models examined by the above-mentioned studies are variants of the dynamic models with random shocks in which the stationary distribution of income follows the power law. Our model also belongs to this class of models. See Gabaix (2009) for a useful survey of this class of models.

2 Model

2.1 Production

The production side of our setting is a simple semi-endogenous growth model in which external increasing returns sustain persistent growth of per capita income. There is a continuum of identical firms with a unit mass. The aggregate production function is

$$Y_t = AK_t \bar{K}_t^{\gamma} N_t^{1-\alpha}, \quad 0 < \alpha < 1, \quad \gamma > 0, \quad \alpha + \gamma < 1, \tag{1}$$

where Y_t, K_t and N_t denote output, capital and labor, respectively. Here, $\bar{K}_t^{\gamma}$ represents external effects associated with the aggregate capital. Since the number of firms is normalized to one, it holds that $\bar{K}_t = K_t$ for all $t \geq 0$. Hence, the social production function that internalizes the external effects is expressed as

$$Y_t = AK_t^{\alpha+\gamma} N_t^{1-\alpha}. \tag{2}$$

The social production function exhibits increasing returns to scale, but the marginal product of capital is diminishing under the assumption of $\alpha + \gamma < 1$.

The factor markets are competitive, and the factor prices are determined by

$$r_t = \alpha \frac{Y_t}{K_t} - \delta, \tag{3}$$

$$w_t = (1 - \alpha) \frac{Y_t}{N_t}, \tag{4}$$

where r_t is the net rate of return on capital, where $\delta \in (0, 1)$ is the capital depreciation rate, and w_t denotes the real wage.

2.2 Households

Population Dynamics and Age Distribution

Time is continuous. At each moment, new households are born and their size is B_t. We assume that B_t changes at a constant rate of b so that $B_t = B_0 e^{bt}$. Each household may die at each moment according to a Poisson process with an intensity denoted by m. Hence, the size of households born at time s and surviving at $t\,(\geq s)$, denoted by $N_{t,s}$, is

$$N_{t,s} = B_t e^{-m(t-s)} = B_0 e^{(b+m)s} e^{-mt},$$

and the total population (total labor force) is given by

$$N_t = \int_{-\infty}^{t} N_{t,s} ds = B_0 e^{-mt} \int_{-\infty}^{t} e^{(b+m)s} ds.$$

Thus, the total population changes according to

$$\dot{N}_t = B_t - mN_t. \tag{5}$$

Suppose that the total population is sufficiently large. Then, the low of large number means that mN_t denotes the number of agents who die at t; hence, (5) expresses the instantaneous change in the total population at t. As a result, if N_t changes at a constant rate, it should hold that $\dot{N}_t/N_t = \dot{B}_t/B_t = b$. We focus on the steady state of the population dynamics, and hence, we obtain the following relation:

$$N_l = \frac{B_0 e^{bt}}{b+m}. \tag{6}$$

We allow a negative population growth rate $(b < 0)$, but we assume that $b + m > 0$ to maintain the total population positive. By setting $B_0 = b + m$, we obtain $N_t = e^{bt}$.

To characterize the age distribution in the steady state of population dynamics, define the complementary cumulative distribution function (tail distribution function) such that

$$G(v, t) = \Pr(\text{age} \geq v).$$

This function represents the share of households with ages above v, and satisfies the following

Kolmogorov forward equation[6]:

$$\frac{\partial G(v,t)}{\partial t} = -(b+m)\,G(v,t) - \frac{\partial G(v,t)}{\partial v}. \tag{7}$$

The stationary distribution is independent of time so that it fulfills

$$G'(v) = -(b+m)\,G(v).$$

The solution of this ordinary differential equation is written as

$$G(v) = e^{-(b+m)v}.$$

Conversely, the stationary cumulative distribution of households with ages lower than v is given by

$$\Pr(\text{age} \leq v) = 1 - G(v) = 1 - e^{-(b+m)v}.$$

The density of this function is $G'(v) = (b+m)\,e^{-(b+m)v}$. Hence, the average age of the households is given by

$$\int_0^\infty v\,(b+m)\,e^{-(b+m)}dv = (b+m)\left[-\frac{e^{-(b+m)v}}{(b+m)^2}\right]_0^\infty = \frac{1}{b+m}.$$

Consequently, when the birth rate b or the mortality rate m decreases, the average age of households increases. Namely, $1/(b+m)$ represents the index of population aging in the steady state of population dynamics.

Blanchard (1985) and many subsequent studies assume that the total population is constant over time. This means that from (5), $b=0,\ m>0$, and it holds that $B_t = mN$ for all $t \geq 0$. To confirm that the key outcomes of OLG models do not stem from the finite horizon of agents,

[6] From t to $t+\Delta t$, the population share of households whose ages are higher than v changes from $G(v,t)$ to $G(v,t+\Delta t)$. Given v, the share of households who die during t and $t+\Delta t$ is approximated by $mG(v,t)\,\Delta t$, whereas the share of households born during that time is $bG(v,t)\,\Delta t$. Additionally, setting $\Delta t = \Delta v$, the change in the population share of households whose ages increase from $v-\Delta v$ to v is expressed as $G(v-\Delta v,t) - G(v,t)$. In sum, we obtain the following relation:

$$\frac{G(v,t+\Delta t) - G(v,t)}{\Delta t} = -(b+m)\,G(v,t) + \frac{G(v-\Delta v,t) - G(v,t)}{\Delta v}.$$

Letting $\Delta t \to 0$ and $\Delta v \to 0$ leads to (7).

Weil (1989) assumes that households live forever, that is, $b > 0$ and $m = 0$. Buiter (1989) treats the general case where $b > 0$ and $m > 0$[7]. We follow Buiter's (1989) setting, but we assume that the growth rate of the number of newborns, b, can be negative.

Households' Optimization Problem

Households' behavior is based on the perpetual youth model developed by Blanchard (1985) and Yaari (1965). Faced with the probability of death, the objective function of the households born at time s is

$$U_s = \int_s^{\infty} e^{-(\rho+m)(t-s)} \log c_{t,s} dt, \tag{8}$$

where $c_{t,s}$ is consumption of cohort s at time t, and $\rho\,(> 0)$ is the time discount rate. The flow budget constraint of the household is

$$\dot{a}_{t,s} = (r_t + m)\, a_{t,s} + w_t - c_{t,s}, \tag{9}$$

where $a_{t,s}$ is the asset holding of cohort s at time t, r_t is the net rate of return on capital determined by (3), and w_t denotes the real wage rate given by (4). Following Blanchard (1985) and Yaari (1965), we assume the presence of fair insurance under which the assets of households who die at t are distributed among the surviving households at that time. Hence, the rate of return on assets includes the risk premium, m. We also assume that households do not have a bequest motive, implying that the initial condition of the asset holding is

$$a_{s,s} = 0. \tag{10}$$

Households maximize U_s by choosing $\{c_{t,s}\}_{t=s}^{\infty}$ subject to (9) and (10), together with the no-

[7] Our formulation of population dynamics follows Buiter (1989) who analyzes a continuous-time OLG model with population change. Buiter (1989) assumes that the number of newly born agents is proportional to the current population in such a way that $B_t = \lambda N_t$, where $\lambda > 0$ is the birth rate. In this formulation, the population growth rate is given by $n = \lambda - m$, indicating that (5) means $\dot{N}_t/N_t = \lambda - m = n$.Thus, if the birth rate, λ, is higher (resp. lower) than the mortality rate, m, then the total population increases (resp. decreases). Notice that the presence of a proportional relation between B_t and N_t means that the population dynamics is in its steady state. In this sense, as Jones (2014) points out, λ represents the long-run birth rate, and it corresponds to $b+m$ in our model. Consequently, there is no substantial difference between our formulation and that of Buiter (1989).

Ponzi-game condition such that

$$\lim_{v\to\infty} \exp\left(-\int_t^v (r_\tau + m)\, d\tau\right) a_{v,s} \geq 0.$$

The optimal consumption must satisfy the following Euler equation:

$$\dot{c}_{t,s} = c_{t,s}\left(r_t - \rho\right). \tag{11}$$

When both the no-Ponzi-game and transversality conditions are fulfilled, the intertemporal budget constraint at t is given by

$$\int_t^\infty \exp\left(-\int_t^v (r_\tau + m)\, d\tau\right) c_{vt,s} dv = \int_t^\infty \exp\left(-\int_t^v (r_\tau + m)\, d\tau\right) w_v dv. \tag{12}$$

Using (11) and (12) , we obtain

$$c_{t,s} = (\rho + m)\left(a_{t,s} + h_t\right), \tag{13}$$

where h_t expresses the human wealth defined by

$$h_t = \int_t^\infty \exp\left(-\int_t^v (r_\tau + m) d\tau\right) w_v dv. \tag{14}$$

2.3 Equilibrium Dynamics

Dynamic System

The aggregate consumption and asset levels are expressed as

$$C_t = \int_{-\infty}^t c_{t,s} N_{t,s} ds, \quad A_t = \int_{-\infty}^t a_{t,s} N_{t,s} ds.$$

Then, (13) presents

$$C_t = (\rho + m)\left(A_t + H_t\right), \tag{15}$$

where $H_t = h_t N_t$. Similarly, summing up the individual flow budget constraint (9) , the aggregate asset changes according to

$$\dot{A}_t = (r_r + m) A_t + w_t N_t - C_t - m A_t = r_t A_t + w_t N_t - C_t. \tag{16}$$

Note that the assets left by households who die at t are mA_t, which are transferred to living households. Thus, the aggregate net revenue from the assets received by the household sector is $r_t A_t$. In a closed economy, the equilibrium condition for the asset market is

$$A_t = K_t. \tag{17}$$

From (3) , (4) , and (17) , we can confirm that (16) also represents the market equilibrium condition for final goods:

$$Y_t = C_t + \dot{K}_t + \delta K_t. \tag{18}$$

Differentiating both sides of (15) with respect to time gives

$$\dot{C}_t = (\rho + m) \left(\dot{A}_t + \dot{H}_t \right). \tag{19}$$

Equation (14) presents $\dot{h}_t = (r_t + m) h_t - w$, which leads to

$$\dot{H}_t = \dot{N}_t h_t + N_t \dot{h}_t = b H_t + (r_t + m) H_t - w_t N_t. \tag{20}$$

Therefore, substituting (16), (16) and (20) into (19) , we obtain the following:As

$$\begin{aligned}
\dot{C}_t &= (\rho + m) \left[r_t A_t + w_t N_t - C_t + (r_t + m + b) H_t - w_t N_t \right] \\
&= (\rho + m) \left[(r_t + m + b) (A_t + H_t) - C_t - (b + m) A_t \right] \\
&= (\rho + m) \left[\frac{r_t + m + b}{\rho + m} C_t - C_t - (\rho + m) (b + m) A_t \right] \\
&= (r_t + b - \rho) C_t - (\rho + m) (b + m) A_t.
\end{aligned}$$

The above equation indicates that If $b = m = 0$ so that households live forever and there are no newborns, the aggregate consumption follows $\dot{C}_t = (r_t - \rho) C_t$, which is the Euler equation of aggregate consumption in the representative-gent economy with fixed population. The above result also means that from $A_t = K_t$, the aggregate consumption changes according to

$$\dot{C}_t = \left(\alpha \frac{Y_t}{K_t} + b - \rho - \delta\right) C_t - (\rho + m)(b + m) K_t. \tag{21}$$

Moreover, the dynamic behavior of the aggregate capital is determined by (18) in such a way that

$$\dot{K}_t = Y_t - C_t - \delta K_t. \tag{22}$$

Now define $\frac{Y_t}{K_t} = x_t$, $\frac{C_t}{K_t} = z_t$. Then, the rate of change of K_t Y_t, and C_t are respectively written as

$$\begin{aligned}
\frac{\dot{K}_t}{K_t} &= x_t - z_t - \delta, \\
\frac{\dot{Y}_t}{Y_t} &= (\alpha + \gamma)(x_t - z_t - \delta) + (1 - \alpha) b, \\
\frac{\dot{C}_t}{C_t} &= \alpha x_t + b - (\rho + \delta) - (\rho + m)(b + m) \frac{1}{z_t}.
\end{aligned}$$

Hence, a complete dynamic system that depicts the equilibrium dynamics of our economy consists of the following differential equations:

$$\frac{\dot{x}_t}{x_t} = (\alpha + \gamma - 1)(x_t - z_t - \delta) + (1 - \alpha) b, \tag{23}$$

$$\frac{\dot{z}_t}{z_t} = (\alpha - 1) x_t + z_t - (\rho + m)(b + m) \frac{1}{z_t} + b - \rho. \tag{24}$$

Note that if $m = b = 0$, then the initial households live forever, and new households will not appear. In this case, the model reduces to a representative agent setting, and the dynamic system becomes the following:

$$\begin{aligned}
\frac{\dot{x}_t}{x_t} &= (\alpha + \gamma - 1)(x_r - z_t - \delta), \\
\frac{\dot{z}_t}{z_t} &= (\alpha - 1) x_t + z_t - \rho.
\end{aligned}$$

Stability of the Steady-State Growth Equilibrium

In the steady-state growth equilibrium, K_t, Y_t, and, C_t change at a common rate, so that x_t and z_t stay constant over time. From (2), the steady-state growth equilibrium establishes

$g_Y = (\alpha + \gamma) g_K + (1 - \alpha) b$, where g_x denotes the steady-growth rate of variable x_t. Using this relation and a steady-state growth condition, $g_Y = g_K = g$, we obtain

$$g_Y = g_K = g_C = \frac{(1-\alpha) b}{1-\alpha-\gamma}. \tag{25}$$

Because $r_t = \alpha x_t$ and $w_t = (1-\alpha) \frac{Y_t}{N_t}$, in the steady-state growth equilibrium, r_t stays constant and w_t changes at the rate of g_y. Hence, it holds that

$$g_y = g_w = g_Y - b = \frac{\gamma b}{1-\alpha-\gamma}. \tag{26}$$

When we express the steady-state value of a variable, we drop the time subscript. The steady-state values of x_t and z_t fulfill the following conditions:

$$\dot{x} = 0 \text{ locus: } x = z + \delta + \frac{(1-\alpha) b}{1-\alpha-\gamma}, \tag{27}$$

$$\dot{z} = 0 \text{ locus: } x = \frac{z}{1-\alpha} - \frac{(\rho+m)(b+m)}{(1-\alpha) z} + \frac{b-\rho}{1-\alpha}. \tag{28}$$

Equations (27) and (28) present

$$\frac{\alpha}{1-\alpha} z^2 - \left[\frac{b-\rho}{1-\alpha} - \delta - \frac{(1-\alpha) b}{1-\alpha-\gamma}\right] z - \frac{(\rho+m)(b+m)}{1-\alpha} = 0.$$

It is easy to confirm that under our restrictions on the parameter values, this quadratic equation of z has real roots, one of which is positive and the other is negative. This implies that the phase diagram of (23) and (24) can be depicted as Figure 2 in which there is a unique feasible steady state with positive x and z.

[Figure 2]

As shown in the figure, the stationary state of the dynamic system has a saddle point property. There are stable saddle paths converging to the steady state, which are expressed as

$$z_t = \phi(x_t), \ \phi'(x_t) > 0 \text{ and } \lim_{x_t \to x} \phi(x_t) = z.$$

Since the initial levels of K_0 and N_0 $(= 1)$ are exogenously specified, $x_0 = Y_0/K_0 = A K_0^{\alpha+\gamma-1} N_0^{1-\alpha}$ is given, whereas $z_0 = C_0/K_0$ is not predetermined. Thus, the initial value of z_t (so the initial value of C_t) is determined by $z_0 = \phi(x_0)$.

Proposition 1 *There exists a unique steady-state growth equilibrium that satisfies saddle-point stability.*

3 Population Aging and Inequality

In this and the next sections, we focus on the steady-state growth equilibrium defined in the previous section.

3.1 Stationary Distribution of Income and Wealth

In the steady-state growth equilibrium, r_t stays constant over time, so that from (14), the per capita human wealth is given by

$$h_t = \int_t^{\infty} e^{-(r+m)(v-t)} w_v dv. \tag{29}$$

In the steady-state growth equilibrium, w_v changes at the rate of g_y, implying that $w_v e^{-g_y v} = \widetilde{w}$ remains constant over time. Hence, (29) yields

$$\widetilde{h} = \frac{\widetilde{w}}{r+m-g_y}, \tag{30}$$

where $\widetilde{h} = e^{-g_y t} h_t$, which is constant over time. We assume that

$$r + m > g_y. \tag{31}$$

to keep $\widetilde{h}$ positive. Similarly, denoting $a_{z,t}e^{-g_y t} = \widetilde{a}_{t,s}$ and $c_{s,t}e^{-g_y t} = \widetilde{c}_{t.s}$, and using (30), on the steady-growth path, the flow budget constraint for households in cohort s is expressed as

$$\begin{aligned}\frac{d}{dt}\widetilde{a}_{t,s} &= (r+m-g_y)\,\widetilde{a}_{t,s} + \widetilde{w} - \widetilde{c}_{t.s} \\ &= (r+m-g_y)\left(\widetilde{a}_{t,s} + \widetilde{h}\right) - (\rho+m)\left(\widetilde{a}_{t,s} + \widetilde{h}\right).\end{aligned}$$

Following Moll et al. (2021), we define the household's 'effective wealth' on the steady-state growth path as follows:

$$\omega_{t,s} = \widetilde{a}_{t,s} + \widetilde{h},$$

which measures the sum of financial assets and human wealth held by the household. Since $\widetilde{h}$ is constant in the steady-state growth equilibrium, we obtain

$$\dot{\omega}_{t,s} = \frac{d}{dt}\widetilde{a}_{t,s} = (r - \rho - g_y)\,\omega_{t,s}, \tag{32}$$

implying that $\omega_{s,t}$ changes at the rate of $r - \rho - g_y$ in the steady-state growth equilibrium.

Now define the complementary cumulative distribution function (CCDF) of $\omega_{s,t}$ as follows:

$$\Phi(\omega, t) = \Pr(\omega_{t,s} \geq \omega) \text{ for } \omega \in [\widetilde{h}, \infty).$$

This function represents the share of households with effective wealth larger than ω. Function $\Phi(\omega, t)$ satisfies the following Kolmogorov forward equation:

$$\frac{\partial \Phi(\omega, t)}{\partial t} = -\frac{\partial}{\partial \omega}\Phi(\omega, t)\left[(r - \rho - g_y)\,\omega\right] - (b + m)\,\Phi(\omega, t). \tag{33}$$

An intuitive implication of this equation is as follows. The change in the share of households with effective wealth larger than ω between t and $t + \Delta t$ is $\Phi(\omega, y + \Delta t) - \Phi(\omega, t)$. This change first stems from the fact that households with $\omega - \Delta\omega$ increase their effective wealth up to ω between t and $t + \Delta t$. This effect is approximated as $[\Phi(\omega - \Delta\omega, t) - \Phi(\omega, t)]\Delta\omega$. Second, the share of households who die between t and $t + \Delta t$ is approximated as $m\Phi(\omega, t)\,\Delta t$, and the population share of new households who are born during that time is $b\Phi(\omega, t)\,\Delta t$. Because the effective wealth held by the new households is less than ω, the population share that is excluded from the household group with $\omega_{t,s} \geq \omega$ is $(b + m)\,\Delta(\omega, t)\,\Delta t$. Consequently, noting that from (32), the change in ω is approximated by $\Delta\omega = (r - \rho - g_y)\,\omega\Delta t$, we see that the rate of change $\Phi(\omega, t)$ from t to $t + \Delta$ can be approximated as follows:

$$\frac{\Phi(\omega, t + \Delta t) - \Phi(\omega, t)}{\Delta t} = \frac{\Phi(\omega - \Delta\omega, t) - \Phi(\omega, t)}{\Delta\omega}(r - \rho - g_y)\,\omega - (m + b)\,\Phi(\omega, t).$$

Then, letting $\Delta t \to 0$ and $\Delta\omega \to 0$, we obtain (33).

Here, we focus on the stationary distribution of effective wealth. The stationary distribution is independent of time, and the Kolmogorov forward equation of the stationary CCDF fulfills

$$\Phi'(\omega)(r-\rho-g_y)\omega+(b+m)\Phi(\omega)=0. \tag{34}$$

Suppose that the solution of (34) is written as $\Phi(\omega)=\chi\omega^{-\zeta}$, where χ and ζ are undetermined constants. Substituting this into (34) yields $\zeta(r-\rho-g_y)=b+m$, meaning that

$$\zeta=\frac{b+m}{r-\rho-g_y}. \tag{35}$$

By definition, $\Phi\left(\widetilde{h}\right)=\chi\left(\widetilde{h}\right)^{-\zeta}=1$, so that $\chi=\widetilde{h}^{\zeta}$. This shows that $\Phi(\omega)$ is specified as $\Phi(\omega)=\left(\frac{\omega}{\widetilde{h}}\right)^{-\frac{b+m}{r-\rho-g_y}}$ for $\omega\in[\widetilde{h},\infty)$. Hence, the stationary cumulative distribution function is $1-\Phi(\omega)$. That is, the stationary distribution of the effective wealth exhibits a Pareto distribution with a shape parameter $\frac{b+m}{r-\rho-g_y}$ and a support $\widetilde{h}$. Moreover, the stationary distribution of the growth-adjusted financial asset, $\widetilde{a}_{t,s}$, satisfies

$$\Pr(\widetilde{a}_{t,s}\geq\widetilde{a})=\Pr\left(\widetilde{a}_{t,s}+\widetilde{h}\geq\widetilde{a}+\widetilde{h}\right)=\Pr(\omega_{t,s}\geq\omega)=\left(\frac{\omega}{\widetilde{h}}\right)^{-\frac{b+m}{r-\rho-g_y}}.$$

Similarly, the stationary distribution of the growth-adjusted income defined as $\widetilde{y}_{t.s}=(r+m-g_y)\widetilde{a}_{t.s}+\widetilde{w}$ follows

$$\begin{aligned}\Pr(\widetilde{y}_{t,s}\geq\widetilde{y})&=\Pr\left(\frac{\widetilde{y}_{t,s}}{r+m-g_y}\geq\frac{\widetilde{y}}{r+m-g_y}\right)=\Pr\left(\widetilde{a}_{t.s}+\frac{\widetilde{w}}{r+m-g_y}\geq\frac{\widetilde{y}}{r+m-g_y}\right)\\&=\Pr\left(\widetilde{a}_{t.s}+\widetilde{h}\geq\omega\right)=\left(\frac{\omega}{\widetilde{h}}\right)^{-\frac{b+m}{r-\rho-g_y}}.\end{aligned}$$

Therefore, the stationary distributions of financial assets and income have the same profile as that of the effective wealth.

Proposition 2 *In the steady-state growth equilibrium, the cumulative distribution function of effective wealth, asset holdings, and income is Pareto with the support* $\widetilde{h}$ *and the shape parameter* $\zeta=\frac{b+m}{r^{*}-\rho-g_y}$.

3.2 Distributional Impact of Population Aging

The reciprocal of the shape parameter (tail index) is given by

$$\frac{1}{\zeta} = \frac{r - \rho - g_y}{b + m} = \text{growth rate of individual wealth} \times \text{level of population aging}$$

Namely, other factors being equal, when individual wealth grows faster and/or population aging progresses, income and wealth distribution becomes more unequal. Specifically, $1/\xi$ is expressed as

$$\frac{1}{\zeta} = \frac{(1 - \alpha - \gamma)(r - \rho) - \gamma b}{(1 - \alpha - \gamma)(b + m)}. \tag{36}$$

A larger $1/\zeta$ means that the stationary distribution function has a fatter and longer tail so that inequality of income and wealth distribution rises. Thus, a higher r means a higher degree of inequality. Moreover, given r, a lower b or a lower m increases inequality. However, since r depends on b and n, we should consider how a change in the degree of population aging affects the steady-state level of the rate of return on capital.

To find the equilibrium level of r, note that K_t changes at $g_y + b$ on the steady-state growth path. Therefore, using $A_t = K_t$ and (22), in the steady-state growth equilibrium, we obtain

$$(g_y + b) K_t = rK_t + w_t N_t - (\rho + m)(K_t + N_t h_t). \tag{37}$$

Now define $q_t = K_t / w_t N_t$, which is constant in the steady-state growth equilibrium. Using this notation, (37) yields

$$q = \frac{1}{g_y + b + \rho + m - r} \left[1 - \frac{\rho + m}{r + m - g_y} \right]. \tag{38}$$

This relationship expresses the supply side of capital per $w_t N_t$. From (3) and (4) , the demand side of capital is given by

$$\frac{w_t}{r_t + \delta} = \left(\frac{1 - \alpha}{\alpha} \right) \frac{K_t}{N_t}.$$

In the steady-state growth equilibrium, the above equation is expressed as

$$q = \frac{\alpha}{(1 - \alpha)(r + \delta)}. \tag{39}$$

Combining (38) and (39) yields the following equation:

$$\frac{\alpha}{(1-\alpha)(r+\delta)} = \frac{1}{g_y + b + \rho + m - r}\left[1 - \frac{\rho + m}{r + m - g_y}\right]. \tag{40}$$

The left-hand side (LHS) of (40) corresponds to the demand side of the capital, while the right-hand side (RHS) expresses the supply side of the capital. Figure 3 shows the graphs of LHS) and RHS, As the figure depicts, if we assume that $\rho + g_y > 0$ and $m > g_y$, (40) has a unique solution denoted by r^*.

[Figure 3]

Moreover, as depicted in Figure 4, if m or b, increases, the graph of RHS in (40) shifts downward, which increases the steady-state level of r. Conversely, acceleration of population aging caused by a decrease in b or m lowers r^*.

[Figure 4]

Population aging increases the population share of older agents who accumulate larger levels of wealth than younger households. Hence, population aging accelerates the accumulation of aggregate capital, which lowers the steady-state rate of return on capital. This indicates that population aging may contribute to lowering the degree of inequality represented by $1/\zeta$ through a decrease in r^*. The total effect of a change in m on $1/\zeta$ is given by

$$\frac{d}{dm}\left(\frac{1}{\zeta}\right) = \frac{1}{(b+m)^2}\left[(b+m)\frac{dr^*}{dm} - (r^* - \rho) + \frac{\gamma b}{1 - \alpha - \gamma}\right]. \tag{41}$$

As $dr^*/dm > 0$ and $r^* - \rho > 0$, the sign of the right-hand side of the above equation is analytically indeterminate. As mentioned previously, the index of inequality, $1/\xi$, is the product of the net growth rate of individual wealth, $r^* - \rho - g_y$, and the degree of population aging, $1/(b+m)$. Thus, population aging caused by a decrease in the mortality rate, m, reduces the net growth rate of individual effective wealth and raises the degree of population aging. The total effect of population aging on inequality depends on the relative strength of those opposite outcomes.

Similarly, the effect of a change in b is shown by

$$\frac{d}{db}\left(\frac{1}{\zeta}\right)=\frac{1}{(b+m)^2}\left[(b+m)\frac{dr^*}{db}-(r^*-\rho)-\frac{\gamma m}{1-\alpha-\gamma}\right]. \tag{42}$$

Again, the sign of the right-hand side of the above is analytically ambiguous. Note that a change in the birth rate, b, directly affects the steady-state growth rate of per capita income, g_y, while a change in the mortality rate, m, will not affect the long-term growth rate of per capita income. Such a difference is captured by the last terms of the right-hand sides of (41) and (42) . In (41) , the last term $\gamma b/(b+m)$ is positive (negative) if the birth rate is positive (negative), whereas the last term in (42) , $-\gamma m/(1-\alpha-\gamma)$, is strictly negative.

Proposition 3 *Population aging lowers the rate of return on capital in the steady-state growth equilibrium. The long-run impact of population aging on inequality depends on the relative strength of its indirect, negative effect on the rate of return on capital and its positive, direct effect on the index of inequality.*

We now examine numerical examples to inspect if population aging raises inequality under plausible parameter values. As the baseline setting, we specify that the income share of capital is $\alpha = 0.35$, the degree of external effect $\gamma = 0.3$, the time discount rate $\rho = 0.02$, and the capital depreciation rate $\delta = 0.075$. The magnitudes of α, ρ, and δ are conventional. The value of γ is selected to make the steady-state growth rate of per capita income is $g_y = \gamma b/(1-\alpha-\gamma)$ $= 0.0174$ under $b = 0.02$. The graph in Figure 5 shows the relationship between m and $1/\zeta$. In this figure, we set b at 0.02 and change m from 0 to 0.05. The graphs indicate that $1/\zeta$ monotonically increases with m. This means that population aging caused by a decline in the mortality rate, m, lowers the degree of inequality in income and wealth under plausible parameter values.

[Figure 5]

Similarly, Figure 6 illustrates the relationship between b and $1/\zeta$. In this case, we fix m at 0.02 and change b from -0.01 to 0.05. The graph reveals that $1/\zeta$ monotonically decreases with b. Hence, population aging caused by a decline in the birth rate enhances inequality in the steady-state growth equilibrium. As shown above, a decrease in b lowers the steady-state level of the rate of return on capital, r^*. At the same time, a lower b reduces the growth rate of per capita income; hence, the detrended rate of return, $r^* - g_y$ may not decrease significantly or may even increase. Therefore, the degree of inequality may increase with a drop in the birth rate.

[Figure 6]

4 Extensions

In this section, we modify assumptions in the baseline model. We inspect how these modifications affect the main outcomes obtained in the base model.

4.1 Exogenous Productivity Growth

In our semi-endogenous growth setting, the steady-state growth rate of income is proportional to the rate of population change. Hence, persistent growth of per capita income fails to hold without population expansion. To relax such a restrictive setting, suppose that in addition to productivity growth sustained by external increasing returns, there is exogenous technical progress. Specifically, we assume that the total productivity in (2) increases at a fixed rate of x, that is, $A_t = A_0 w^{xt}$. In this case, the steady-state growth rate of per capita income is

$$g_y = \frac{x + \gamma b}{1 - \alpha - \gamma}.$$

Thus, even if $b < 0$, the per capita income can grow if $x > -\gamma b$. Now, let us denote $z = r - g_y$. Then (40) is expressed as

$$\frac{\alpha}{(1-\alpha)(z + g_y + \delta)} = \frac{1}{b + \rho + m - z}\left[1 - \left(\frac{\rho + m}{z + m}\right)\right]. \quad (43)$$

Figure 7 shows the graphs of LHS and RHS in (43) We see that a rise in x increases g_y, which leads to a downward shift of the graph of LHS. As a result, the steady-state level of $z = r^* - g_y$ decreases.

[Figure 7]

Because (35) can be expressed as $\frac{1}{\zeta} = \frac{z^* - \rho}{b+m}$, the impact of a change in x on $1/\zeta$ is evaluated by-

$$\frac{d}{dx}\left(\frac{1}{\zeta}\right) = \frac{dz^*/dx}{b+m},$$

which means that a higher x lowers the degree of inequality and that its impact is larger as the degree of population aging, $1/(b\_m)$, has a higher value. Conversely, a reduction in x increases inequality and its effect is larger under a higher $1/(b+m)$. As a numerical example, we set the levels of ρ, α, γ, δ, m, and b at their baseline values mentioned in Section 3.2, we plot the relationship between $1/\zeta$ and x : see Figure 8.

[Figure 8]

Proposition 4 *If the exogenous productivity growth rate increases, then the gap between the rate of return on capital and the growth rate of per capita income is lowered, which reduces inequality.*

There is a large body of empirical studies on the relationship between population aging and productivity growth. Most of the previous investigations suggest that population aging negatively affects productivity growth: see, for example, Daniele et al. (2020) and Maestas et al. (2022). Applying the empirical findings to our model, we may conjecture that a lower x may be associated with a larger $1/(b+m)$, which gives rise to a higher inequality in the steady-state growth equilibrium.

4.2 Retirement of Agents

Thus far, we have assumed that households supply one unit of labor at each moment until they die. Although this assumption simplifies the model analysis, it fails to capture the household life-cycle decisions emphasized in the standard, discrete-time OLG models. Here, we consider the retirement of agents from labor participation. Following Blanchard (1985), we assume that the probability of retirement follows a Poisson process with intensity $\psi\,(>0)$. In this formulation, the labor supply at t offered by the household born at s is

$$l_{s,t} = e^{-\psi(t-s)} \times 1,$$

where $e^{\psi(t-s)}$ denotes the probability that the household born at s remains in the workforce at $t\ (\geq s)$[8]. Thus, noting that $B_t = B_0 e^b t$, we see that the aggregate labor supply at t is given by

$$L_t = \int_{-\infty}^{t} l_{s,t} N_{s,t} ds = \int_{-\infty}^{t} e^{-\psi(t-s)} B_0 e^{(b+m)s} e^{-mt} ds = \frac{b+m}{b+m+\psi} N_t.$$

Hence, it holds that $L_t = N_t$ for $\psi = 0$, and a smaller ψ means an increase in the labor participation rate, L_t/N_t.

The flow budget for each household is given by

$$\dot{a}_{t,s,} = (r_t + m)\, a_{t,s} + e^{-\psi(t-s)} w_t - c_{t,s}.$$

As before, optimal consumption follows (11) and the consumption at t is given by (13), where the human wealth on the steady-state growth path is

$$h_t = \int_t^{\infty} e^{-(g_y - r - m - \psi)(v-t)} w_t dv = \frac{w_t}{r+m+\psi-g_y}.$$

Thus, the aggregate consumption in the balanced-growth equilibrium is expressed as

$$C_t = (\rho+m)\left[K_t + h_t L_t\right] = (\rho+m)\left(K_t + \frac{w_t L_t}{r+m+\psi-g_y}\right).$$

As a result, (37) is replaced with

$$(g_y + b)\, K_t = rK_t + w_t L_t - (\rho+m)\,(K_t + \frac{w_t L_t}{r+m+\psi-g_y}).$$

By denoting $K_t/w_t L_t = q$, which is constant on the balanced growth path, the above equation yields

$$q = \frac{1}{g_y + b + m + \rho - r}\left(1 - \frac{\rho+m}{r+m+\psi-g_y}\right).$$

As before, $\frac{w_t}{r+\delta} = \frac{(1-\alpha)K_t}{\alpha L_t}$ presents (39), meaning that (40) is replaced with

$$\frac{\alpha}{(1-\alpha)(r+\delta)} = \frac{1}{g_y + b + \rho + m - r}\left(1 - \frac{\rho+m}{r+m+\psi-g_y}\right). \tag{44}$$

[8]Blanchard (1985) considers the case in which the labor productivity of an agent decreases exponentially at a constant rate, His formulation can be interpreted as a model in which the retirement opportunity arrives at each moment according to a Poisson process.

Figure 9 depicts the graphs of LHS and RHS of (44). If the labor participation rate increases because of a reduction in ψ, the graph of RHS shifts downward, and thus the steady-state rate of return on capital, r^*, rises.

[Figure 9]

Since the level of ψ does not directly affect $1/\zeta$, if the retirement probability declines, income and wealth inequalities increase in the steady-state growth equilibrium. Intuitively, a decrease in the retirement probability yields a uniform rise in the labor participation of all cohorts, which increases the aggregate labor supply and promotes the substitution of capital with labor. Consequently, the real wage falls, whereas the rate of return on capital increases[9].

Proposition 5 *A lower probability of retirement enhances inequality in the long run.*

In the last 25 years, the labor force participation of older adults has been increasing in many advanced countries. For example, according to the Aging Society White Paper 2017 issued by the Japanese Cabinet Office, the labor force's share of Japanese older persons aged 65 years and above increased from 5.9% in 1980 to 11.8% in 2016. Currently, more than 50% of Japanese male adults aged 65–69 years engage in full-time or part-time jobs. The increase in the older persons' labor force participation may reflect the rises in the life expectancy and health status of the older adults and from changes in the social environment that raise the activeness of the older people. Additionally, many researchers claim that the recent increase in the labor participation of older persons in advanced countries stems from social security reforms conducted in those countries that are less beneficial for older adults[10]. The above proposition indicates that in our setting, a lower probability of retirement associated with population aging increases inequality.

[9] An alternative formulation of the retirement of agents in continuous-time OLG models is to assume that each agent works for a finite length of time. In this case, the lifetime budget constraint for an agent born at s is written as

$$\int_s^\infty \exp\left(-\int_s^t (r_\tau + m)\, d\tau\right) c_{s,t} dt = \int_s^{s+R} \exp\left(-\left(r_t + m\right)\right) w_t dt,$$

where R is the length of the workforce participation. This formulation is used by, for example, Khun and Prettner (2022). We can confirm that the effect of a rise in R on the stationary distributions of income and wealth is essentially the same as that caused by a fall in ψ.

[10] See, for example, Berkel and Börsch-Supan (2004), Shimizutani and Oshio (2013), and Coil (2015).

4.3 Endogenous Labor Supply

The baseline model assumes that each household supplies one unit of labor at each moment. In this subsection, we allow labor-leisure choices of the households. We replace the objective function of the household given by (8) with the following:

$$U_s = \int_s^\infty e^{-(\rho+m)(t-s)} \log\left(c_{t,s} - \frac{n_{t,ss}^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}}\right) dt, \quad \gamma > 0,$$

where $n_{t,s}$ denotes the hours worked at t by the agent in cohort s. Here, we assume that the instantaneous utility function of the household takes the GHH form provided by Greenwood et al. (1998) which has been frequently used in the business cycle literature[11]. As shown below, under the GHH preferences, the households' labor supply is independent of the income (wealth) effect, which substantially simplifies the analytical discussion. The household maximize U_s by choosing $\{c_{s,t}, n_{s,t}\}_{t=s}^\infty$ subject to the flow budget constraint

$$\dot{a}_{t,s} = (r_t + m)\, a_{t,s} + w_t n_{t,s} - c_{t,s}. \tag{45}$$

Denoting the utility value of the assets by $q_{t,s}$, the first-order conditions for an optimum include the following:

$$\left(c_{t,s} - \frac{n_{t,s}^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}}\right)^{-1} = q_{t,s}, \tag{46}$$

$$n_{t,s}^{\frac{1}{\gamma}} \left(c_{t,s} - \frac{n_{t,s}^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}}\right)^{-1} = w_t q_{t,s}, \tag{47}$$

$$\dot{q}_{t,s} = q_{t,s'}\, (\rho - r_t)\, q_{t,s}, \tag{48}$$

together with the transversality condition: $\lim_{t\to\infty} e^{-(\rho+m)t} q_{t,s} a_{t,s} = 0$. Conditions (46) and (47) give

$$n_{t,s} = w_t^\gamma, \tag{49}$$

[11] Ascari and Rankin (2007) point out that the standard instantaneous utility function in which leisure is a normal good, the labor supply of old agents with large wealth could be negative. They confirm that the GHH preferences are free from such a deficiency.

which represents the labor supply of each household. The labor supply is independent of the income effect, and the elasticity of the labor supply is γ. When $\gamma = 0$, each household supplies one unit of labor at each moment.

Using (49) , we define the 'net' consumption in the following manner:

$$\widetilde{c}_{t,s} = c_{t,s} - \frac{\gamma w_t^{1+\gamma}}{1+\gamma}.$$

We restrict our attention to the case where $\widetilde{c}_{s,t} > 0$. From (46) and (48) , the Euler equation of the net consumption is

$$\frac{d}{dt}\widetilde{c}_{t,s} = (r_t - \rho)\,\widetilde{c}_{t,s}. \tag{50}$$

Additionally, the flow budget constraint (45) is rewritten as

$$\dot{a}_{t,s} = (r_t + m)\,a_{t,s} + \frac{1}{1+\gamma} w_t^{1+\gamma} - \widetilde{c}_{t,s}. \tag{51}$$

Hence, when both the no-Ponzi-game and transversality conditions are satisfied, the intertemporal budget constraint at time t can be expressed as

$$\int_t^\infty \exp\left(-\int_t^v (r_\mu + m)\,d\mu\right)\widetilde{c}_{v,s}dv = a_{t,s} + \int_t^\infty \exp\left(-\int_t^v (r_\mu + m)\,d\mu\right)\frac{1}{1+\gamma} w_v^{1+\gamma} dv. \tag{52}$$

From (50) and (52) , we obtain

$$\widetilde{c}_{t,s} = (\rho + m)\,(a_{t,s} + h_t)\,, \tag{53}$$

where h_t is a modified human wealth defined as

$$h_t = \int_t^\infty \exp\left(-\int_t^v (r_\mu + m)\,d\mu\right)\frac{w_v^{1+\gamma}}{1+\gamma} dv.$$

Consequently, the optimal consumption at time t is

$$c_{t,s} = (\rho + m)\,(a_{s,t} + h_t) + \frac{\gamma w_t^{1+\gamma}}{1+\gamma}. \tag{54}$$

Denoting the aggregate labor employment by L_t, the production function of final goods is given by

$$Y_t = A\bar{K}_t^{\gamma} K_t^{\alpha} L_t^{1-\alpha},$$

and the factor prices are determined by $r_t = \alpha Y_t / K_t - \delta$ and $w_t = (1-\alpha) Y_t / L_t$. Equation (49) means that the labor market equilibrium condition is

$$L_t = w_t^{\gamma} N_t. \tag{55}$$

The aggregate capital changes according to

$$\dot{K}_t = r_t K_t + w_t L_t - C_t, \tag{56}$$

and from (54), C_t is given by

$$C_t = (\rho + m)(A_t + h_t N_t) + \frac{\gamma w_t^{1+\gamma}}{1+\gamma} N_t. \tag{57}$$

From (56) and (57), in the steady-state growth equilibrium, (22) presents

$$bK_t = rK_t + w^{1+\gamma} N_t - (\rho + m)\left[K_t + \frac{w^{1+\gamma}}{(1+\gamma)(r+m)} N_t\right] - \frac{\gamma w^{1+\gamma}}{1+\gamma} N_t. \tag{58}$$

Define $K_t / w_t^{1+\gamma} N_t = x_t$, which is constant in the steady-state growth equilibrium. Then (58) can be written as

$$x = \frac{1}{(1+\gamma)(b+m+\rho-r)}\left[1 - \frac{\rho+m}{(r+m)}\right], \tag{59}$$

where x expresses the steady-state value of x_t. Moreover, noting that $\frac{w_t}{r_t+\delta} = \left(\frac{1-\alpha}{\alpha}\right)\frac{K_t}{L_t}$, from (55), we obtain the following relation in the steady-state growth eauilibriu:

$$x = \frac{\alpha}{(1-\alpha)(r+\delta)}. \tag{60}$$

Combining (59) and (60), we obtain

$$\frac{\alpha}{(1-\alpha)(r+\delta)} = \frac{1}{(1+\gamma)(b+m+\rho-r)}\left[1 - \frac{\rho+m}{(r+m)}\right]. \tag{61}$$

Figure 10 displays the graphs of the left-hand side (LHS) and the right-hand side (RHS) of (61). As the figure shows, (61) has a unique solution denoted by r^*. Note that a rise in γ shifts the graph of RHS downward, leading to a higher r^*.

[Figure 10]

The degree of inequality is still given by (35). Since we have found that r^* increases with the elasticity of labor supply γ, flexible labor supply increases inequality in the long run. Intuitively, $\gamma {w_t}^{1+\gamma}/(1+\gamma)$ in (54) plays the same role as subsistence consumption in the Stone-Geary utility function. Thus, its aggregate level, $\gamma w^{1+\gamma}/(1+\gamma) N_t$, involved in the right-hand side of (58) corresponds to the aggregate subsistence consumption. This additional consumption reduces capital accumulation, which yields a higher rate of return on capital in the steady state than the model with a fixed labor supply. Furthermore, other things being equal, a higher γ yields a larger subsistence consumption, meaning that r^* increases with the elasticity of labor supply.

Proposition 6 *If each agent has the GHH preference, a more flexible labor supply increases inequality in the steady-state growth equilibrium.*

As noted in the previous subsection, the labor force participation of older adults has been increasing in many advanced countries. Compared to young people, older persons tend to be sensitive to labor-leisure choices, so the elasticity of the labor supply of old workers would be higher than that of young workers. This fact implies that population aging may increase the average elasticity of the labor supply function. In our model, this suggests that economies with a higher degree of population aging may hold a higher level of the average elasticity of labor supply, which may enhance inequality in income and wealth distribution.

5 Conclusion

This study explores the impact of population aging on income and wealth distribution in the context of a continuous-time OLG model with semi-endogenous growth. We show that the stationary distributions of income and wealth exhibit a Pareto profile and its shape parameter is affected by the degree of population aging. Analytically, both the extension of life expectancy and a fall in the population growth rate decrease the rate of return on capital in the steady-state growth equilibrium, which lowers inequality of income and wealth. Simultaneously, population aging directly increases the long-run inequality; hence the total effect of population aging on inequality depends on the relative strength of these opposite effects. We numerically confirm that the total effect of the extension of life expectancy on inequality is negative, whereas the total effect of the fall in the population growth rate is positive. That is, a smaller mortality rate decreases inequality, but a reduction in the population growth rate enhances inequality. In addition to the baseline analysis, we examine the extended models with exogenous productivity growth, agent retirement, and flexible labor supply. In each extended model, we examine the inequality in the steady-state growth equilibrium and discuss the implications of the outcomes from the prospect of population aging.

This study uses a simple semi-endogenous growth model in which external increasing returns sustain the persistent growth of per capita income. In this setting, the long-run growth rate of per capita income is proportional to the rate of change in population specified exogenously. Introducing the R&D activities of firms and the endogenous population change would enrich the analytical outcomes of this study. In addition, we have not discussed the policy issues in our study. Examination of the distributional effects of income tax and intergenerational transfer programs in our model would deserve further study.

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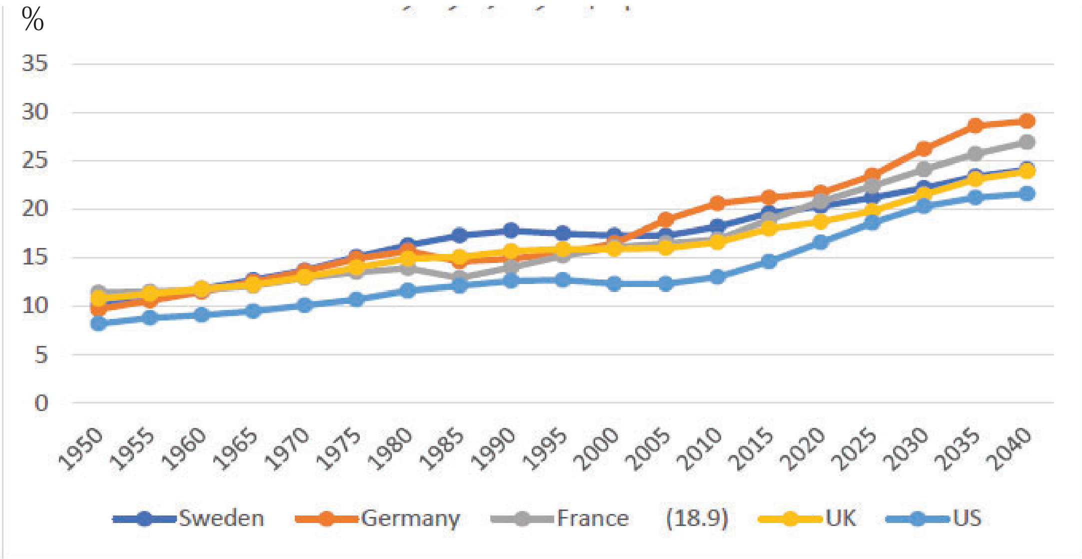


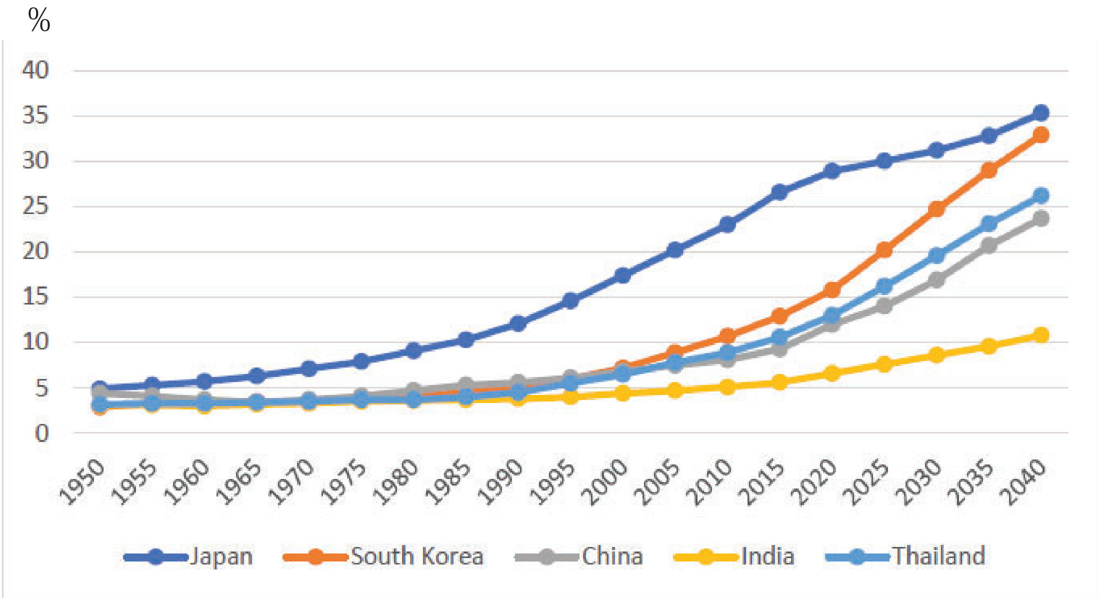


Figure 1: Changes in the population share in European and Asian countries and the United States

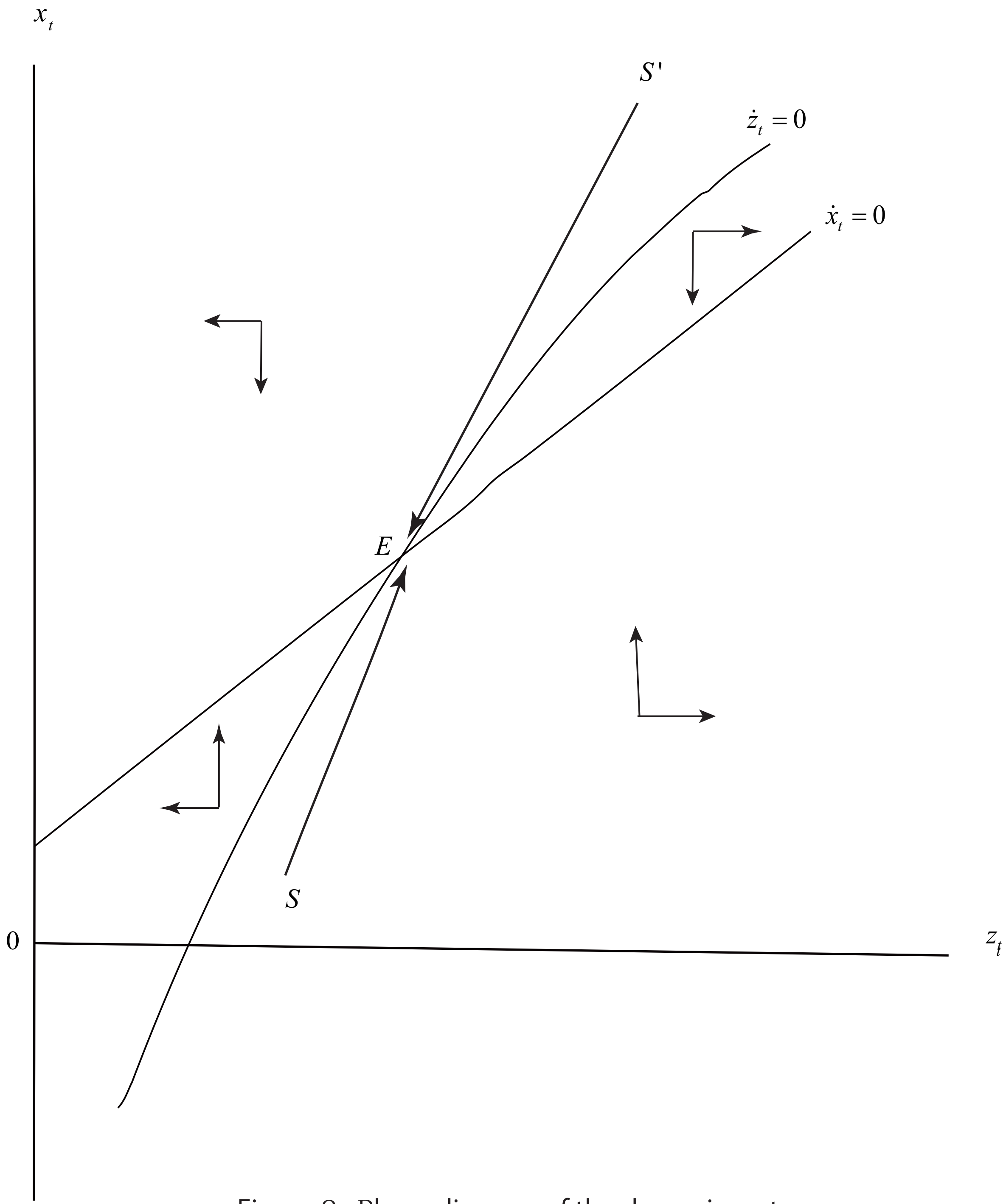


Figure 2: Phase diagram of the dynamic system

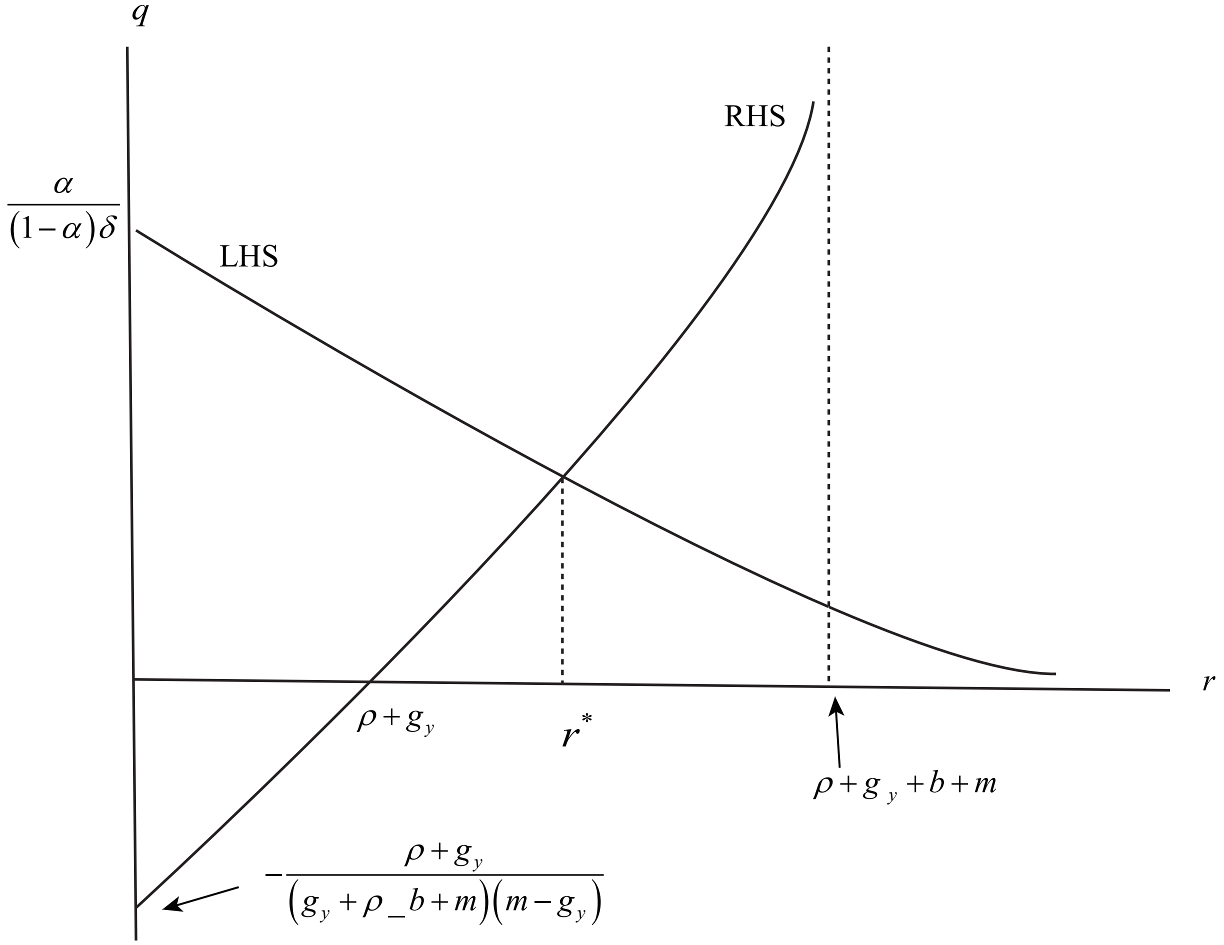


Figure 3: Dtermination of the steady-state rate of rerurn

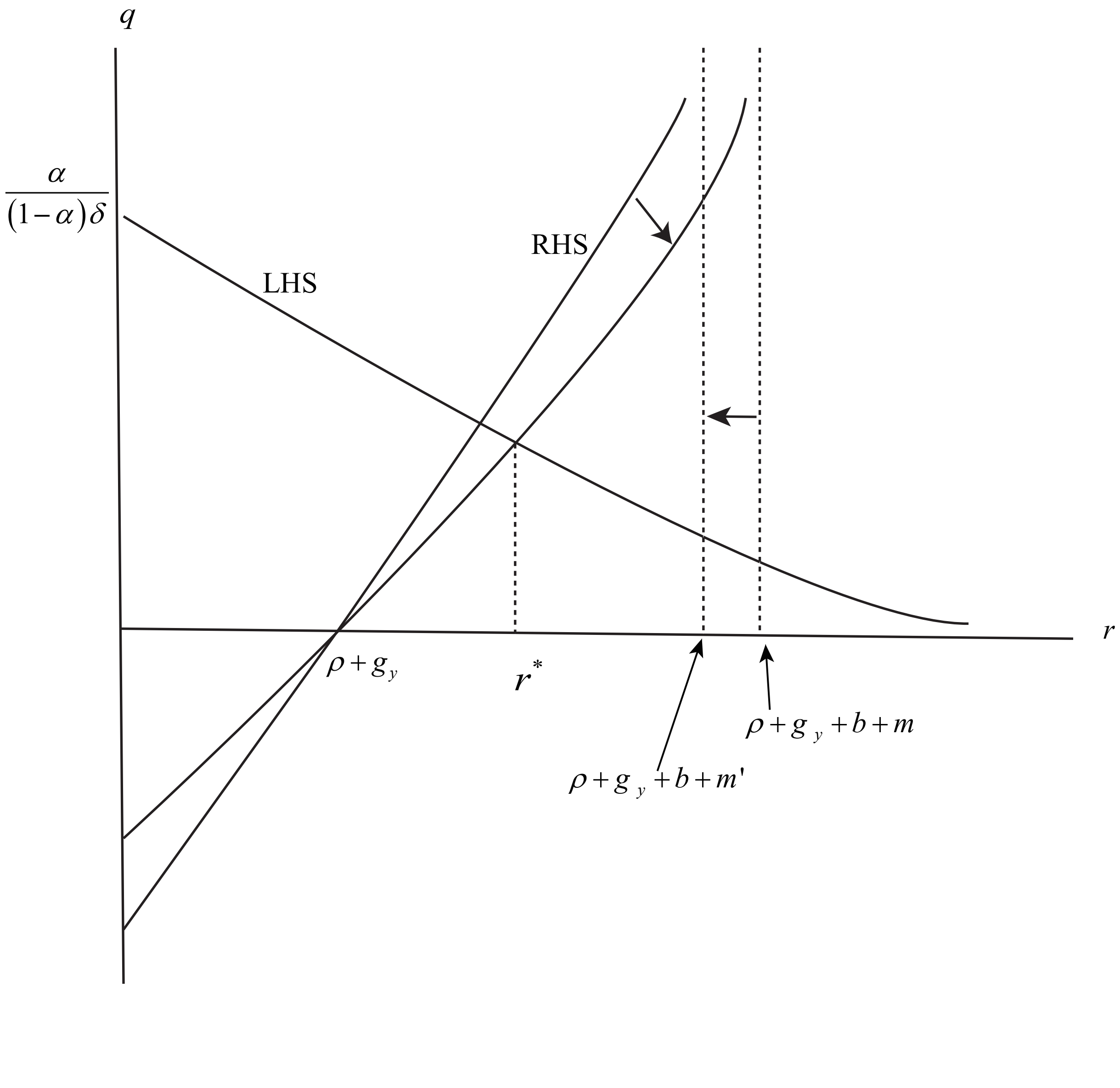


Figure 4 : The effect of a fall in m

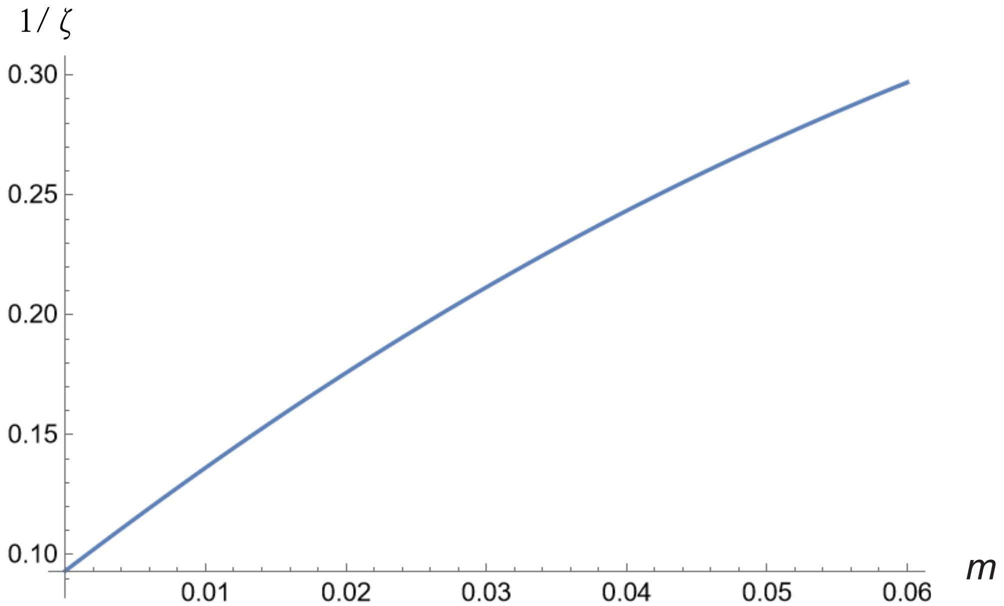


Figure 5 : Relationship between the fertility rate and the degree of inequality

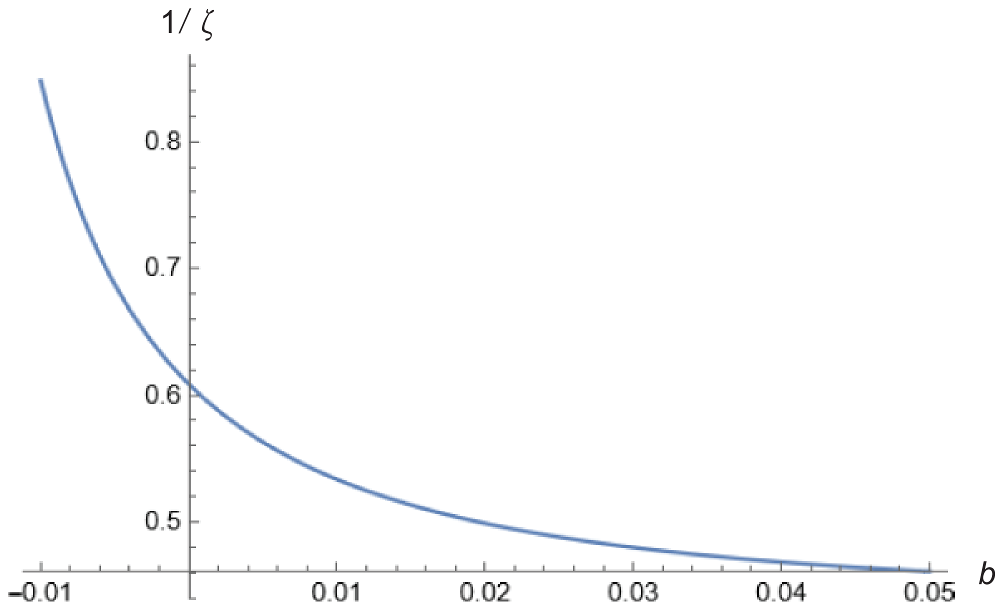


Figure 6: Relashionship between the birth rate and the degee inequality

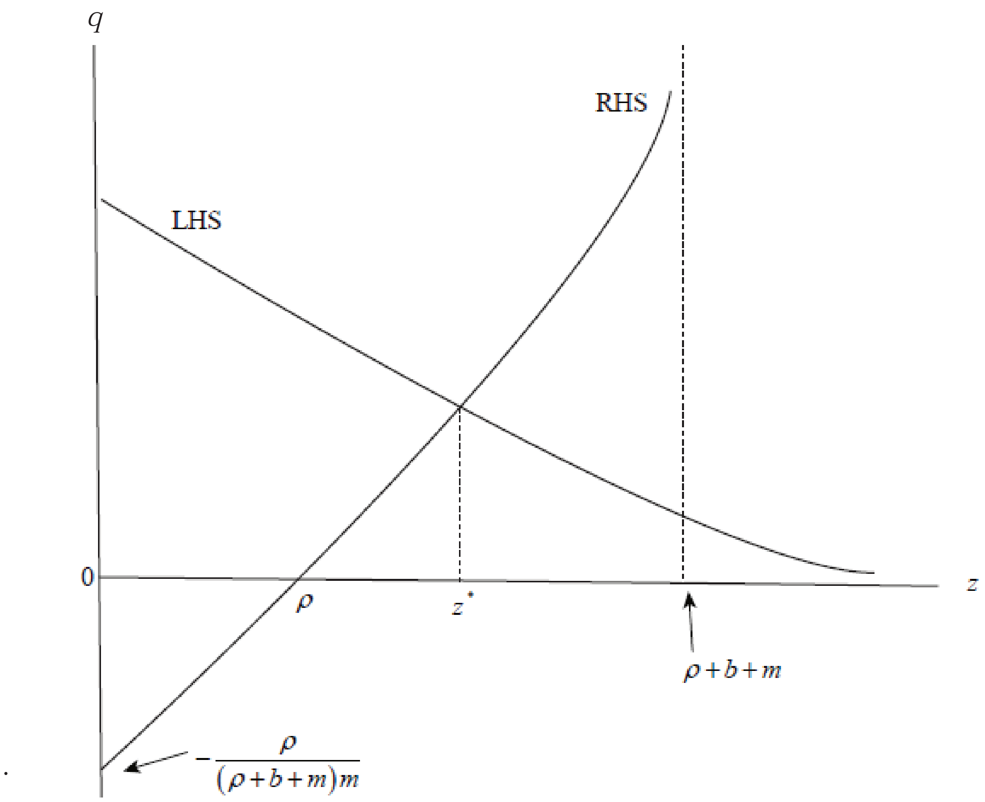


Figure 7: Determination of the steady-state level of z

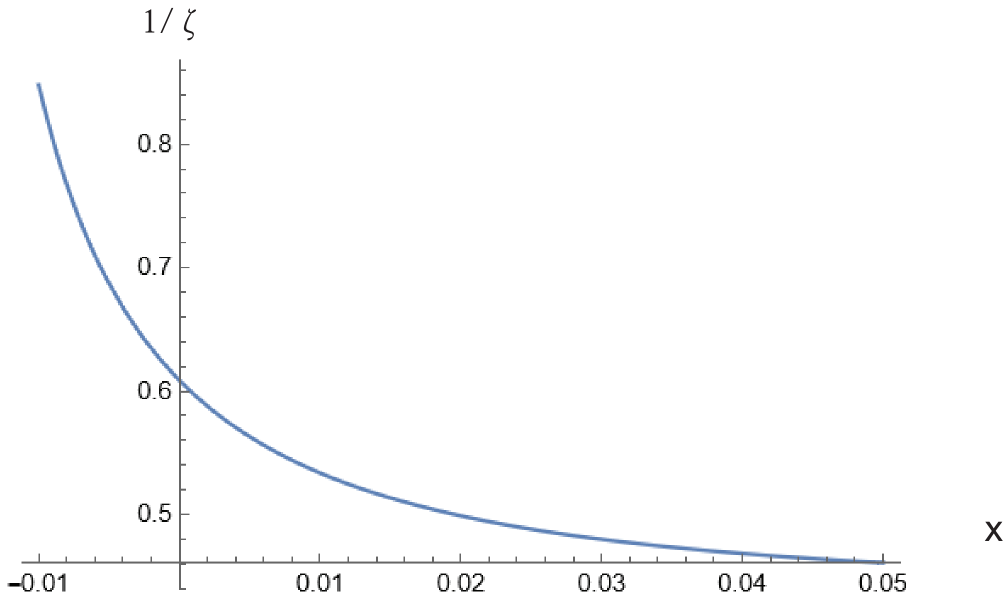


Figure 8: Relationship between the productivity growth rate and the degee of inequality

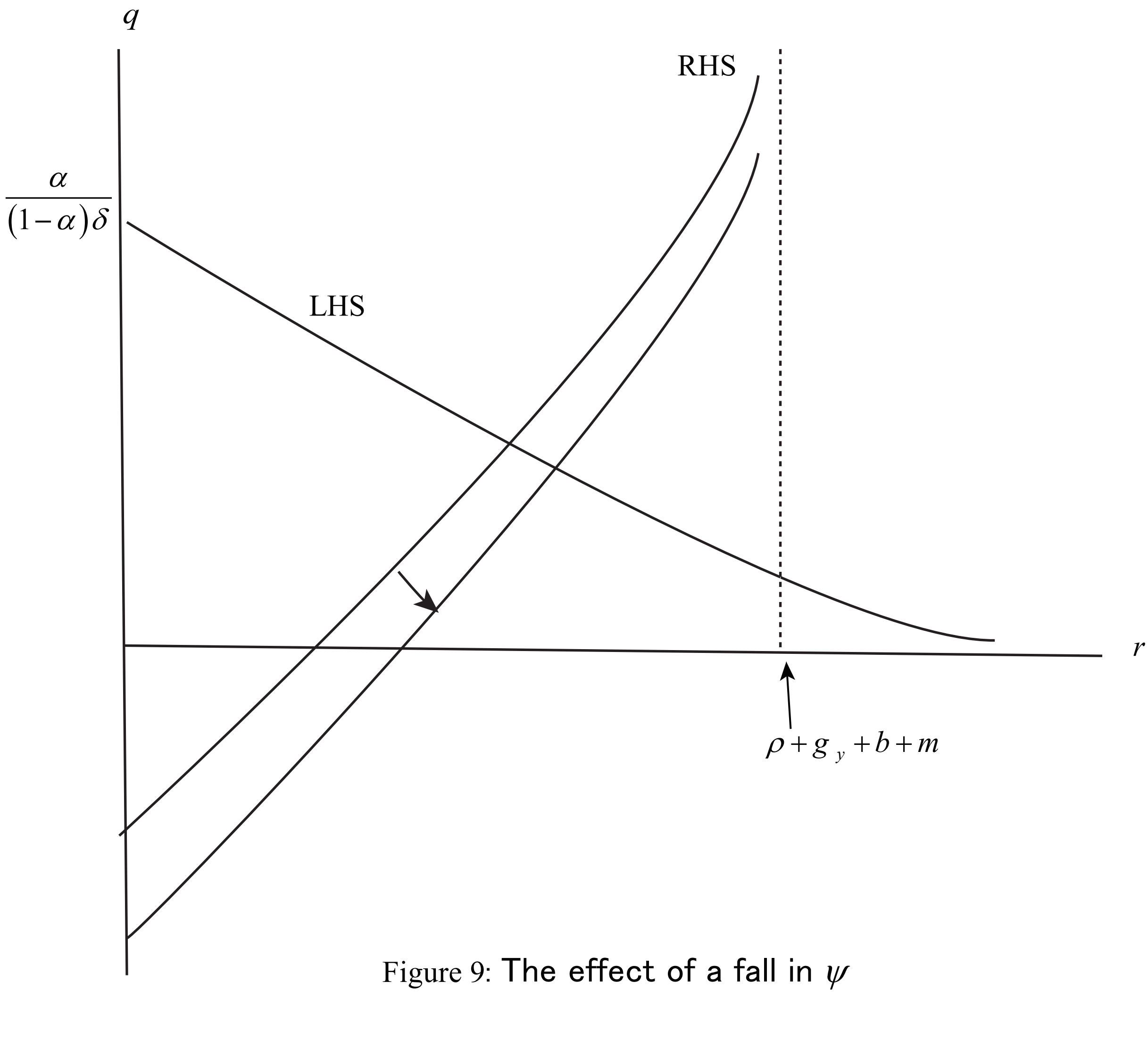
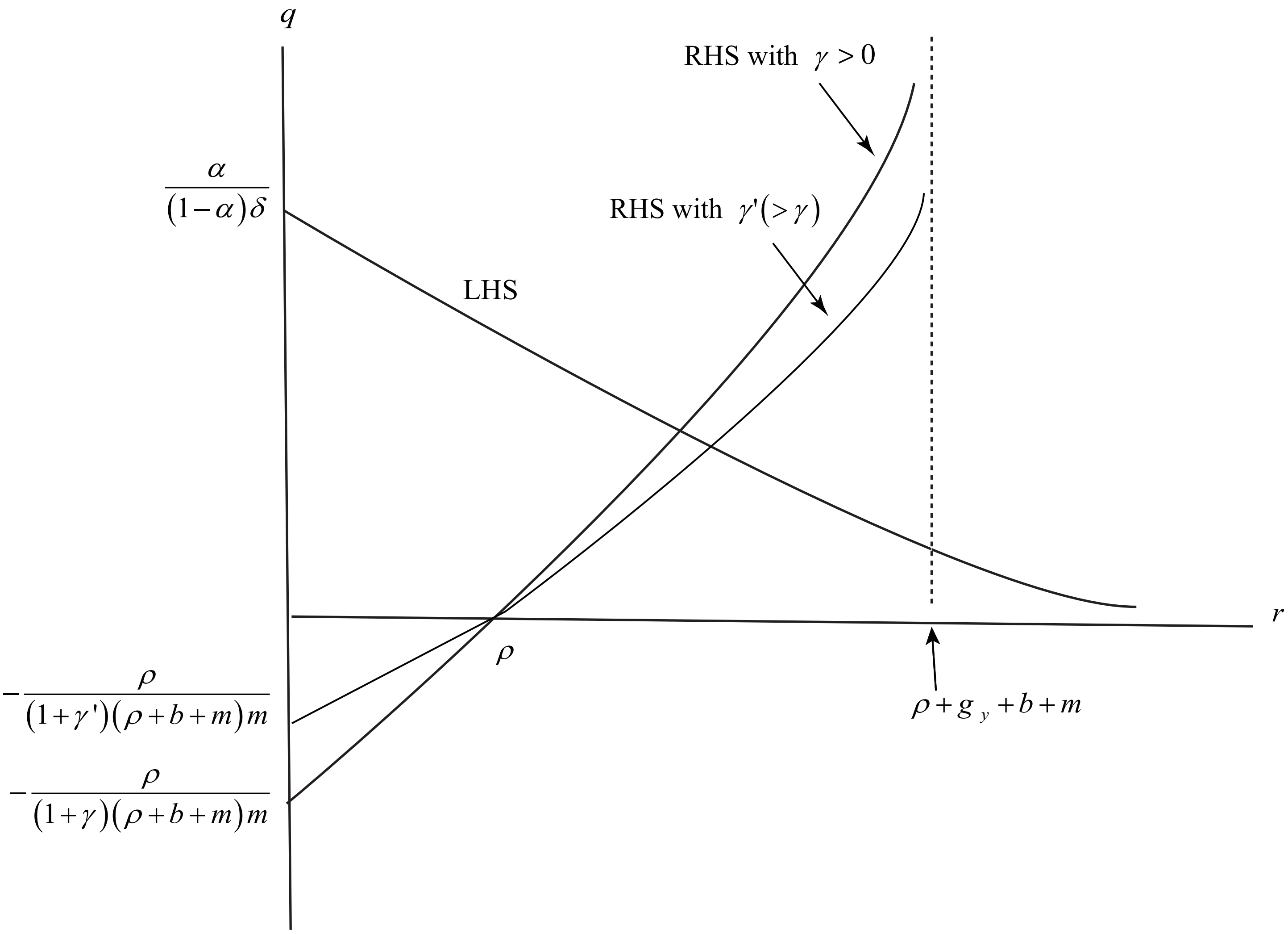


Figure 9: The effect of a fall in ψ



Fiigure 10: Determination of the steady-state level of rate of return under alternative values of γ