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"Learning and Strategic Delay in a Dynamic Coordination Game"

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Abstract

Heterogeneously informed agents decide their optimal action timings while observing past activities over time. We construct such a dynamic global coordination game to investigate the impact of learning and delay options on coordination behaviors and outcomes. A unique monotone equilibrium is characterized, which is analytically convenient for all ranges of learning efficiencies, and we demonstrate that learning improves coordination success, while the delay options alone have no impact, relative to the one-shot game. Dynamics of agents' behaviors and welfare implications are then presented. In addition, we show that full learning about the state achieves in the limit, and find the condition on which observing actions reveals more accurate information about the state than directly observing it.

Keywords: Learning; Strategic delay; Global games; Dynamics

JEL classification: D82; D83

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1. Introduction

Coordination games of incomplete information like currency crises or investment crashes impact the economy massively and draw much attention from economists. One prominent approach to analyze such problems is the global games model pioneered by Carlsson and Van Damme (1993) and Morris and Shin (1998). It introduces asymmetric information into the traditional coordination game framework and remarkably obtains a unique and analytically convenient equilibrium, shedding testable insights on policy guidance and welfare implications. However, the existing studies are mostly in static contexts, despite the economic activities are inherently dynamic. Budget-constrained agents delay their investment decisions to learn from their predecessors' behaviors, for example. That said, learning and delaying behaviors of agents are prevalent in practice and worth exploring, but the static models cannot provide predictions or analysis for those dynamic aspects.

Particularly in coordination games, learning and delay behaviors become more notable because agents face not only the payoff uncertainty about the economic fundamentals, but also the strategic uncertainty about their opponents' (past, current, and future) beliefs. Consequently, it is almost inevitable to extend the static models into multi-periods and consider learning and delaying behaviors of agents, to capture their intrinsic motivations to mitigate both sorts of uncertainties. And the investigation into dynamic environments is not a simple extension of the static model because of the strategic delay consideration of agents to try to select the optimal action timing. That is, delay provides informational gains through agents' observation of past activities, but is also costly due to discounting and shrinking opportunities, so agents must constantly trade off the benefit and the cost of delay to determine when to act, and this crucial trade-off cannot be captured in static frameworks.

Therefore in this paper, we construct an $N \in \mathbb{N}$ period model in an investment context to investigate the impact of learning and delay options on agents' behaviors, based on the static global game of Morris and Shin (2000). The prospect of an investment project, or the state, is deterministic but ex ante unknown, and a continuum of heterogeneously informed agents can undertake a fixed-size investment once. They independently select the investment timing (if at all) out of N periods, while observing a stream of noisy signals about past activities over time, which represents the learning behavior and is the informational gain of delay. To capture the coordination motive and the opportunity cost of delay, we let the payoff of the investment to an agent, paid at the end of the game, be positively correlated with the aggregate investment, while negatively with her investment timing, if ever invested. Hence agents with one-time investment opportunity need to trade off the informational gain of delay versus its opportunity cost to decide their optimal investment timing.

After constructing the model, we solve for its equilibrium and demonstrate the existence and the uniqueness of a monotone equilibrium, in which agents take a symmetric threshold strategy profile (i.e., an agent invests in one period if and only if her belief about the state exceeds some threshold prescribed for that period.) This monotone form of strategy is documented in almost all relevant literature and is as well intuitively appealing in this dynamic environment. To see it, agents select their investment timing by trading off the informational gain of delay against its opportunity cost, so if an agent believes the state is good enough in one period, she expects the investment is profitable and consequently, the expected opportunity cost to her is huge while the informational gain is little; thereby she invests immediately. Otherwise she delays to the next period, in which she updates her information by observing what others have done, and then make decisions by the same trade-off logic as before, and so on.

Noteworthy in equilibrium, the investment decisions of an agent only depend on her beliefs about the state, even though the payoff involves her fellow agents' behaviors. This is so because (i) the payoff depends on the aggregate investment and (ii) the aggregate investment is (shown to be) deterministic given the state. There two properties are standard in global games literature and essentially stem from the Law of Large Numbers. Recall that agents form a continuum, which allows us to characterize a one-to-one relation between the aggregate action and the state. Hence a belief about the state suffices to evaluate the corresponding aggregate investment. Also with this deterministic relation, the information learned from past actions is shown to be summarized in a closed-form statistic centered around the state, for all learning precision levels. This is one of the novelty of this paper because the past literature in dynamic environments only allows analysis in limit accurate learning situations (cf., Dasgupta (2007)). The comparison to the existing literature will be elaborated in the literature review section soon.

Our following analysis is thus focused on this unique monotone equilibrium and addresses two questions. The first probes the dynamics of agents' behaviors in equilibrium, and the second investigates comparative statics, particularly the impact of learning and delay options on coordination success and welfare.

First, we summarize agents' equilibrium behaviors. In period 1, the optimistic agents who observe favorable signals (that exceed the equilibrium threshold of period 1) invest immediately, since they believe the investment's prospect is already good and thus outweigh the opportunity cost of delay over its informational gain. In the subsequent intermediate periods, the remaining agents constantly revise their expectations about the investment through cumulative learning, and depending on learning efficiency, a large or small fraction of agents will switch into investing. Noteworthy, if the learning efficiency is modest (i.e., the accuracy of endogenous signals is low), in every period will a few agents newly invest, so the relatively inertia phenomenon documented in the literature (Angeletos et al. (2007)) is expected. Also by implication, had no learning effect existed, agents would only act in the first period and stay inactive till the last period. We indeed verify this conjecture and show that the mere delay option without learning opportunities has no impact on the game, relative to the one-shot game. In the last period, there is no stage to delay to, so another positive fraction of remaining agents will choose to invest.

Next we discuss comparative statics. To begin with, we contrast agents' behaviors with that in the one-shot game. Results show that agents are less aggressive (i.e., less likely to invest) in the intermediate periods than in the static game. Intuitively, agents are tempted by the information learned from delaying and hence choose to wait. On the other hand, agents behave more aggressively in the last period of the dynamic game, due to a higher expected total investment and the coordination motive.

We next investigate the values of learning opportunities and the consequent welfare implications. It is demonstrated that learning opportunities increase agents' expected continuation payoffs and thus improve coordination success and social welfare. Intuitively, coordination fails because agents, facing the uncertainty about whether others will cooperate, may choose not to invest, even if it is their collective interest to do so. Learning alleviates this problem by reducing the strategic uncertainty among agents, since it makes agents' signals more accurate and thereby better aligned. Also we show that agents more accurately infer the state in the presence of learning, indicating the payoff uncertainty is also mitigated.

Note that agents learn the state through observing past activities, and we are interested in how efficient such a learning mechanism is, relative to learning from directly observing the state. We find that as long as agents' initial information is precise enough, observing actions reveals more accurate information than directly observing the state. Intuitively, learning efficiency of observing actions depends on (i) how accurately agents' private information is about the state and (ii) how accurately endogenous signals reflect their actions (and hence their private information). The two channels are shown to be mutually reinforced and therefore, if one of them is accurate enough, it is possible that indirect learning delivers more accurate information about the state than direct learning.

Lastly, we discuss the equilibrium selection in the dynamic model. One of the remarkable results that static global game models provide is the uniqueness in equilibrium when information among agents is sufficiently diffused (see Morris and Shin (2003)), resolving the indeterminacy of equilibria problem in complete information coordination games. And we indeed obtain a unique monotone equilibrium in this dynamic environment. However, other forms of strategies than a threshold strategy cannot be excluded to constitute an equilibrium. This is because the dynamic environment provides other dimensions for coordination and thus multiplicity. For instance, if all agents believe their opponents will take some specific strategy form, so may they, and this mutual effect in turn justifies the usage of that strategy form. Aside from this, as Angeletos and Werning (2006) demonstrate, when learning is through public observation of actions, multiple equilibria can arise even when agents are endowed with limit accurate private information. The feature is also present in our model when we consider that learning is through public observation of actions in Section 4.

Also in section 4, we extend the game to infinite periods and show that the properties of the N-period game are still valid; furthermore, we find that agents completely learn the true state in the limit, avoiding the usual information cascade when learning is through observing past activities (Bikhchandani et al. (1992) and Banerjee (1992)). Indeed, in our model, pooled information of agents reveals the true state, so it is at least plausible for agents to fully learn the state. And the signal structures we consider are continuous due to normal noise; as Lee (1993) demonstrate, this continuity prevents information cascade, because any tiny variation in agents' behaviors will be, at least noisily, reflected by signals.

1.1 Related literature

This paper is most related to Angeletos et al. (2007) and Dasgupta (2007). Angeletos et al. (2007) investigate a dynamic regime change game in which short-lived agents (in the sense that agents are new and given a unit of perishable endowment every period) repeatedly decide whether to attack a regime, while observing the outcomes of the past attacks. By contrast, agents in our model are long-lived and have budget constraints in the sense that they can only act at most once, and thus face an active timing problem. Moreover, we consider that all past activities cumulatively affect the payoff of the investment, while they assume only the action of the present period affects agents' payoffs. In addition, a continuous payoff structure is assumed in our paper, as opposed to the discrete payoff structures of the regime change game (which pays either a lump sum or nothing, depending on whether the regime switches), so our result about the dynamics of agents' behaviors complements that of Angeletos et al. (2007): agents in our model respond continuously to information variations, while their agents have complete inertia unless receiving a large change of information.

It is worth stressing that though payoffs are continuous in parameters in our model, agents' strategies are not because of the feature of the threshold strategy. That said, agents' actions can change discontinuously and dramatically with a small perturbation of information (even given the state of the world); to see it, consider those with signals around the threshold. Consequently, volatile non-fundamental variations of actions exist in our model, which is one of the highlights of the global games approach to explain sudden changes of behaviors in crises phenomena; see Morris and Shin (2003).

Dasgupta (2007) considers a regime change game in a two-period span, with agents endowed with limit accurate private information as well as learning is of limit accurate, so learning is almost immaterial there. Our analysis instead spans N, and further infinite, periods and applies to all learning efficiencies. Furthermore, the almost fully informed agents in Dasgupta's work always benefit from the delay option, while we find that, when agents are not fully informed, what helps improve coordination success is the learning effects and that the delay option alone does not affect the outcomes, relative to the static game.

Some works focus exclusively on learning effects, especially the effects of public signals on equilibrium selection in global games. The pioneers are Angeletos and Werning (2006), who show the rise of multiple equilibria when learning is through public observation. Most distinctively, our paper differs from theirs because in that their game is essentially static, in the sense that one group of agents act in the first period in the financial market of Grossman and Stiglitz (1976), and then another group, observing price or activity in the market, act in a static global game; the two groups share no payoff transfers. Also connecting to the rational learning literature, our learning mechanism, particularly the Gaussian signal structure, has the similar updating rule as in Vives (1993).

There are works on global coordination games that study different aspects than this paper. For example, Hellwig et al. (2006) consider endogenous interest rates, Angeletos et al. (2006) analyze the signaling effects, and Szkup and Trevino (2015) study costly information acquisition. See also Morris and Shin (1998) for currency crises, Goldstein and Pauzner (2005) for bank runs, and Edmond (2013) for sociopolitical revolutions.

In very different setups, the option value of delay has been examined by Chamley and Gale (1994) in a noncooperation environment with perfect observation of past activities. Gale (1995) studies strategic delay in a complete information coordination game.

The rest of the paper is structured as follows. Section 2 investigates the game comprising two periods and captures our core results. Section 3 considers multiple periods and confirms the validity of the results in the two-period model. Section 4 discusses the extension concerning infinite periods and public learning.

2. The Two-Period Model

In this section, we examine a two-period game with a linear payoff structure; it captures our core results. The stage game is based on Morris and Shin (2000). The general model that comprises $N \in \mathbb{N}$ periods and a general payoff will be explored in Section 3.

2.1 Setup

A measure-one continuum of agents, denoted i or j, independently decide whether to invest in a risky project at time $t = 1$ or $t = 2$, if at all. An agent can invest at most once irreversibly. Let $a_{ti} \in \{0, 1\}$ denote agent i's action at time t, where 1 (or 0) refers to investing (or not investing); it is then required that $a_{1i} + a_{2i} \in \{0, 1\}$. Moreover, let $a_t = \int_i a_{ti} di$ be the aggregate investment at time t, and $\hat{a}_t = \sum_1^t a_k$ the cumulative investment till t. The return of the project is determined after all investment decisions are completed, so payoffs are realized at the end of time 2. The payoff of an agent who does not invest is normalized to 0, and that to investing is the sum of two factors. The first is the total investment $\hat{a}_2$, and the second is the exogenous investment environment, which is driven by other economic fundamentals. We summarize the second factor by a single parameter $r \in \mathbb{R}$. In sum, the return to an agent who chooses $a_{ti} \in \{0,1\}$ equals

$$a_{ti}(r + \hat{a}_2). \tag{2.1}$$

In each $t = 1, 2$, agent i chooses $a_{ti} \in \{0,1\}$ to maximize her aggregate expected payoff, $E[\sum_{t=1}^{2} \delta^{t-1} a_{ti}(r + \hat{a}_2)]$, given her available information at that time, where $\delta \in (0,1)$ is the timing cost on investment. Note that δ acts similarly as a discount factor, but since agents only receive payments at the end of the game, δ is interpreted as shrinking opportunities.

The state parameter r is deterministic but ex ante unknown, and is uniformly distributed over the entire real line, so agents hold an improper prior about it: $r \sim \text{Unif}(\mathbb{R})$. In period 1, agent i observes a private signal x_{1i} about the realization of r:

$$x_{1i} = r + \frac{1}{\sqrt{\tau_1}} \varepsilon_{1i}, \tag{2.2}$$

and in period 2, agent i additionally receives x_{2i} about the past activity a_1:

$$x_{2i} = \Phi^{-1}(a_1) + \frac{1}{\sqrt{\tau_2}} \varepsilon_{2i}, \tag{2.3}$$

where Φ is the CDF of the standard normal and $\tau_t > 0$, $t = 1, 2$, measures the information quality, and ε_{ti} is a standard normal variable, independent across time and agents and of r (i.e., $\varepsilon_{ti}|_r = \varepsilon_{ti} \sim \mathcal{N}(0,1)$, i.i.d. for any t and i). Here we follow the literature (Dasgupta (2007) and Angeletos and Werning (2006)) to choose the analytically convenient information aggregation technology Φ^{-1}, but as we will see soon, the qualitative results of the paper are valid for other learning technologies. Furthermore, we impose the Law of Large Numbers (LLN) convention through out the paper, namely, the proportion of agents who receive signals higher than some real number is equal to the probability of an individual agent receiving such signals. Consequently, no aggregate uncertainty about r exists since the idiosyncratic noise cancels out: $\int_i \varepsilon_{1i} di = 0$.

In summary, the game proceeds as follows.

0. Nature randomly draws r from $\mathbb{R}$.

1. In period 1, agent i privately observes x_{1i} about r and then makes an investment decision. The total investment a_1 is thus determined.

2. Subsequently in period 2, agent i privately observes x_{2i} about a_1 and then takes a feasible action. The aggregate investment $\hat{a}_2(= a_1 + a_2)$ of the game is thus determined.

3. The payoffs to investment depending on r and $\hat{a}_2$ are realized at the end of period 2.

Recall that in period 2, the only feasible action to agents who have invested is action 0.

2.2 Threshold Strategies and Monotone Equilibria

In line with the literature, we consider that agents play a symmetric threshold strategy in each period - an agent invests iff her expectation of r at that period exceeds some threshold number. Specifically, a *threshold strategy* σ_1 *in period 1* for agent i who observes x_{1i} takes the form

$$\sigma_1(x_{1i}) = \begin{cases} 1, & \text{if } x_{1i} > x_1 \\ 0, & \text{otherwise,} \end{cases}$$

for some $x_1 \in \mathbb{R}$ (we differentiate signals and thresholds by subscript i). By construction, the agent selects not investing at a tie when $x_{1i} = x_1$. The expression of a threshold strategy in period 2 requires closer inspection because of endogenous learning. To see it, note that $a_1(r) = P(x_{1i} > x_1 \mid r) = \Phi(\sqrt{\tau_1}(r - x_1))$ for any realization of r, when all agents follow a threshold strategy with threshold x_1 in period 1. Hence endogenous signal x_{2i} becomes

$$x_{2i} = \sqrt{\tau_1}(r - x_1) + \frac{1}{\sqrt{\tau_2}}\varepsilon_{2i},$$

rearranging which we obtain

$$\frac{x_{2i}}{\sqrt{\tau_1}} + x_1 = r + \frac{1}{\sqrt{\tau_2 \tau_1}}\varepsilon_{2i}.$$

Therefore, if we define

$$x'_{2i} \equiv \frac{x_{2i}}{\sqrt{\tau_1}} + x_1,$$

then

$$x'_{2i} = r + \frac{1}{\sqrt{\tau'_2}}\varepsilon_{2i},$$

where $\tau'_2 \equiv \tau_1\tau_2$. Note that x'_{2i} is informationally equivalent to x_{2i} with respect to r. Therefore by Bayes' rule, agent i's updated belief about r in period 2 can be summarized by a sufficient statistic $\hat{x}_{2i}(x_{1i}, x_{2i})$ with

$$\hat{x}_{2i}(x_{1i}, x_{2i}) = \hat{x}_{2i}(x_{1i}, x'_{2i}) = \frac{\tau_1 x_{1i} + \tau'_2 x'_{2i}}{\tau_1 + \tau'_2} = r + \frac{1}{\sqrt{\hat{\tau}_2}}\varepsilon_{2i}, \tag{2.4}$$

where $\hat{\tau}_2 \equiv \tau_1 + \tau'_2$.[1] A *threshold strategy* σ_2 *in period 2* is defined by the rule

$$\sigma_2(x_{1i}, x_{2i}) = \begin{cases} 1 - \sigma_1(x_{1i}), & \text{if } \hat{x}_{2i}(x_{1i}, x_{2i}) > x_2 \\ 0, & \text{otherwise,} \end{cases}$$

for some $x_2 \in \mathbb{R}$, given agents follow σ_1 in period 1. This completes the definition of a threshold strategy profile in the dynamic game. [2] When no confusion might occur, we write $\hat{x}_{2i} \equiv \hat{x}_{2i}(x_{1i}, x_{2i})$ and $\hat{x}_{1i} \equiv x_{1i}$ (and hence define $\hat{\tau}_1 = \tau'_1 \equiv \tau_1$), and denote a threshold strategy profile by its thresholds, say, (x_1, x_2).

It should be noted that when all the agents follow (x_1, x_2), the size of investment a_t at time $t = 1, 2$ is a deterministic function of r, such that

$$a_1(r) = \int_i P(\hat{x}_{1i} > x_1 \mid r)di = P(\hat{x}_{1i} > x_t \mid r), \quad a_2(r) = P(x_{1i} < x_1, \hat{x}_{2i} > x_2 | r),$$

by LLN.[3] That said, when all agents play a threshold strategy, the payoff to investment

[1] Recall ε_{1i} and ε_{2i} are both standard normals, and by abusing notation, we let ε_{2i} in (2.4) denote a normal noise in agent i's belief towards r at $t = 2$.

[2] Note that σ_2 is only well defined when agents take a threshold strategy in period 1; it suffices for our purpose since we restrict to agents playing a threshold strategy profile.

[3] Note that $a_2(r) = P(x_{1i} < x_1, \hat{x}_{2i} > x_2|r) = P(\hat{x}_{2i} > x_2|r, x_{1i} < x_1)P(x_{1i} < x_1|r) = P(\hat{x}_{2i} > x_2|r)P(x_{1i} < x_1|r)$, since r suffices to estimate $\hat{x}_{2i}$ by (2.4).

depends on $(r, \hat{a}_2(r))$, so that the estimation about r suffices to evaluate decisions even $\hat{a}_2$ enters the payoff function. In this paper, we consider symmetric perfect Bayesian equilibria in which all agents follow a threshold strategy profile, and call such equilibria monotone equilibria. In what follows, we refer to monotone equilibria as equilibria unless otherwise stated.

It is worth stressing that the key in obtaining an analytical form of posterior belief $\hat{x}_{2i}(x_{1i}, x_{2i})$ in period 2 is the transformation from x_{2i} centered around $\Phi^{-1}(a_1)$ to x'_{2i} centered around r. The transformation is plausible because no aggregate uncertainty exists in the model ($\int_i x_{1i} di = r$ indicates pooling the continuum's information reveals r), so that when all agents follow a threshold strategy in period 1, the aggregate activity a_1 is deterministic given r, and thus the observation of a monotone function of it (i.e., $\Phi^{-1}(a_1)$) leads to an estimation of r. This line of reasoning implies that the specific aggregation rule Φ^{-1} of x_{2i} is not qualitatively restrictive: any one-to-one aggregation rule results in an estimation of r from observing a_1; we choose Φ^{-1} to obtain the well-behaved transformed signal.

Moreover, since the estimation of r is derived from x_{2i}, the quality of the estimation depends on how precise (i.e., τ_2) x_{2i} reflects a_1 and how precise (i.e., τ_1) a_1 reflects r. Indeed, the induced precision level $\tau'_2 = \tau_1 \tau_2$ of x'_{2i} verifies this. It also highlights that the endogenous information is generated by social learning, or from individuals' private information, so the more accurate information agents initially hold, the more accurate information their actions convey. Noteworthy, the precision level τ'_2 is the same as that of endogenous signals obtained from rational expectations equilibrium price (Grossman and Stiglitz (1976)), underscoring that the specific learning rule Φ^{-1} provides results consistent with the literature.

We close this section by characterizing agent i's cross-period beliefs about one another, and show that a higher expectation of r leads to a higher expectation of $\hat{a}_2$. The results are useful in the equilibrium characterization later.

Lemma 1. *When agents follow a threshold strategy profile* (x_1, x_2), *for time* $t \neq k \in$

$\{1, 2\}$ *and any signal realization* $\hat{x}_{ki}, \hat{x}_{tj}$, *we have, for* $i \neq j$,

$$\hat{x}_{tj}|_{\hat{x}_{ki}} \sim \mathcal{N}\left(\hat{x}_{ki}, \frac{\hat{\tau}_k + \hat{\tau}_t}{\hat{\tau}_k \hat{\tau}_t}\right),$$

and for $i = j$

$$\hat{x}_{2i}|_{x_{1i}} \sim \mathcal{N}\left(x_{1i}, \frac{\tau_2'}{\hat{\tau}_2 \tau_1}\right). \tag{2.5}$$

Moreover, $E[\hat{a}_2|\hat{x}_{ki}]$ *strictly increases in* $\hat{x}_{ki}$.

Proof. For $i \neq j$, since $\hat{x}_{ki}(= r + \varepsilon_{ki}/\sqrt{\hat{\tau}_k})$, we have

$$\hat{x}_{tj} = r + \frac{1}{\sqrt{\hat{\tau}_t}}\varepsilon_{tj} = \hat{x}_{ki} - \frac{1}{\sqrt{\hat{\tau}_k}}\varepsilon_{ki} + \frac{1}{\sqrt{\hat{\tau}_t}}\varepsilon_{tj}.$$

For $i = j$, since $x_{2i}' = x_{1i} - \varepsilon_{1i}/\sqrt{\tau_1} + \varepsilon_{2i}/\sqrt{\tau_2'}$, we have

$$\hat{x}_{2i} = \frac{\tau_1 x_{1i} + \tau_2' x_{2i}'}{\tau_1 + \tau_2'} = x_{1i} + \frac{\tau_2'}{\tau_1 + \tau_2'}\left(-\frac{1}{\sqrt{\tau_1}}\varepsilon_{1i} + \frac{1}{\sqrt{\tau_2'}}\varepsilon_{2i}\right),$$

so (2.5) holds.

For the second part, note that $(E[a_1|\hat{x}_{ki}])' = (P(x_{1j} > x_1|\hat{x}_{ki}))' > 0$, and that $E[a_2|\hat{x}_{ki}] = P(x_{1j} < x_1, \hat{x}_{2j} > x_2|\hat{x}_{ki}) = (1 - E[a_1|\hat{x}_{ki}])P(\hat{x}_{2j} > x_2|\hat{x}_{ki})$, so

$$\begin{aligned}\frac{d}{d\hat{x}_{ki}}E[\hat{a}_2|\hat{x}_{ki}] &= \frac{d}{d\hat{x}_{ki}}E[(a_1 + a_2) \mid \hat{x}_{ki}] \\ &= (E[a_1 \mid \hat{x}_{ki}])'(1 - P(\hat{x}_{2j} > x_2|\hat{x}_{ki})) + (1 - E[a_1 \mid \hat{x}_{ki}])(\underbrace{P(\hat{x}_{2j} > x_2|\hat{x}_{ki})}_{=\Phi(\sqrt{\cdot}(\hat{x}_{ki} - x_2))})' > 0.\end{aligned}$$

Q.E.D.

Note that, complying with our intuition, the point (iii) states that i makes more accurate inferences about her own belief than about others'. In what follows, we abbreviate $E[a_t|\hat{x}_{ki}]$ and $E[\hat{a}_t|\hat{x}_{ki}]$ to $a_t(\hat{x}_{ki})$ and $\hat{a}_t(\hat{x}_{ki})$, respectively.

2.3 Equilibrium Characterization

2.3.1 The Static Game

We first consider when the game only consists of the first period, namely a static game, to illustrate how to solve for the unique monotone equilibrium; it also serves as a benchmark for later comparative statics analysis. Since the one-shot game is a standard static global game, the equilibrium can be easily characterized as in the following proposition.

Proposition 1. *In the static game with signal structure $x_{1i} = r + \varepsilon_{1i}/\sqrt{\tau_1}$ and payoff structure $r + a_1$, there exists a unique equilibrium which is a monotone equilibrium characterized by a threshold strategy x^*_{st} with $x^*_{st} = -1/2$.*

A detailed proof can be found in Morris and Shin (2000), and here we sketch it. Think of a marginal agent with signal x_{1i} such that she is indifferent between investing or not, namely, $E[r + a_1|x_{1i}] = 0$. It is straightforward to verify it has a unique solution, $x_{1i} = -1/2$, and we claim it is x^*_{st}.[4] Indeed, following symmetric strategy $-1/2$ (investing iff $x_{1i} > -1/2$) is optimal, because $E[r + a_1|x_{1i}] > 0$ iff $x_{1i} > -1/2$. The global uniqueness is obtained by the standard iterated dominance argument so we omit its proof.

Note that in verifying the threshold strategy x^*_{st} constitutes an equilibrium, it suffices to consider the marginal agent who is indifferent between the two actions. This is due to payoff's monotonicity in r and a_1: a higher signal realization indicates higher r and a_1, resulting in higher expected payoffs from investing, so an agent with signals higher than the cutoff signal expects the payoff to investing exceeds 0 and thus invests. Since the monotonicity of the payoff holds in the dynamic game, one can expect that an equilibrium shall be readily found by identifying such a marginal agent. However, her role is more subtle there, since instead of balancing whether to invest or not, the agent trades off between acting now versus delaying and acting optimally later. A careful analysis is therefore required and conducted below.

[4] Formally, $a_1(x_{1i}) = \Phi(\sqrt{\tau_1/2}(x_{1i} - x_1))$ given other agents follow some threshold strategy x_1. Thus the only symmetric solution to the equation is $x_1 = -1/2$.

2.3.2 The Two-Period Game

We now solve the two-period model. Let $R_1(x_{1i};(x_1,x_2))$ denote the expected continuation payoff for agent i who observes x_{1i} and delays in period 1, given all other agents follow some threshold strategy (x_1,x_2) in the game. To compute R_1, agent i infers her to be received signal $\hat{x}_{2i}$ from x_{1i}, since $\hat{x}_{2i}$ determines whether she will invest later and if so, her expected payoff. We claim

$$\begin{aligned} R_1(x_{1i};(x_1,x_2)) &= \delta E\big[E\left[r+\hat{a}_2|\hat{x}_{2i}>x_2\right]\big|x_{1i}\big] \\ &= \delta\int_{x_2}^{\infty} E[r+\hat{a}_2\mid\hat{x}_{2i}]f(\hat{x}_{2i}\mid x_{1i})d\hat{x}_{2i}, \end{aligned}$$

where $f(\hat{x}_{2i}|x_{1i})$ is the density of $\hat{x}_{2i}$ given x_{1i} and by (2.5) in Lemma 1, equals $(P(\cdot\leqslant \hat{x}_{2i}|x_{1i}))' = (\Phi(\sqrt{\tau_1\hat{\tau}_2/\tau_2'}(\hat{x}_{2i}-x_{1i})))'$; also see Lemma 1 for the formula of $E[\hat{a}_2|\hat{x}_{2i}]$ given thresholds (x_1,x_2). Let $R_2(\hat{x}_{2i};(x_1,x_2))\equiv 0$ for any $\hat{x}_{2i}$ and (x_1,x_2), meaning the continuation payoff at the last stage is 0.

To better understand the formula of R_1, suppose the agent who observes $\hat{x}_{2i}$ has reached period 2; then given others follow (x_1,x_2) in the game, her expected payoff to following x_2, denoted $\tilde{R}_1(\hat{x}_{2i};(x_1,x_2))$, is

$$\tilde{R}_1(\hat{x}_{2i};(x_1,x_2)) = \begin{cases} E[r+\hat{a}_2\mid\hat{x}_{2i}], & \text{if } \hat{x}_{2i}>x_2 \\ 0, & \text{otherwise.} \end{cases}$$

In period 1, the agent forms an expectation of this value through her current signal x_{1i}, which is what she expects to obtain by delaying and thus is R_1, so

$$R_1(x_{1i};(x_1,x_2)) = \delta E[\tilde{R}_1(\hat{x}_{2i};(x_1,x_2))\mid x_{1i}] = \delta\int_{x_2}^{\infty} E[r+\hat{a}_2\mid\hat{x}_{2i}]f(\hat{x}_{2i}\mid x_{1i})d\hat{x}_{2i}.$$

With this result, the following proposition establishes the existence and the uniqueness of a monotone equilibrium.

Proposition 2. *A unique monotone equilibrium characterized by a threshold strategy* *profile* (x_1^*, x_2^*) *exists in the two-period game, where* x_t^* *uniquely solves*

$$E[r + \hat{a}_2 | x_t^*] = R_t(x_t^*; (x_1^*, x_2^*)), \quad \textit{for } t = 1, 2.$$

Proof. Let (x_1, x_2) denote an arbitrary threshold strategy. In period 2, given the others follow (x_1, x_2) in the game, agent i with belief $\hat{x}_{2i}$ expects her investment payoff to be

$$G_2(\hat{x}_{2i}; (x_1, x_2)) \equiv E[r + \hat{a}_2 \mid \hat{x}_{2i}; (x_1, x_2)], \tag{2.6}$$

where we write (x_1, x_2) to emphasize it is used to compute $E[\hat{a}_2|\hat{x}_{2i}]$, which by Lemma 1 increases in $\hat{x}_{2i}$, so (2.6) increases in $\hat{x}_{2i}$. If a threshold strategy $x_2' \in \mathbb{R}$ is an equilibrium strategy in period 2, agent i should invest (i.e., $G_2(\hat{x}_{2i}; (x_1, x_2')) > 0$) if $\hat{x}_{2i} > x_2'$ and should not if $\hat{x}_{2i} < x_2'$; therefore by the continuity of (2.6) in $\hat{x}_{2i}$, i observing $\hat{x}_{2i} = x_2'$ must be indifferent between investing or not, namely,

$$G_2(x_2'; (x_1, x_2')) = 0. \tag{2.7}$$

We show in Appendix that $G_2(x_2'; (x_1, x_2'))$ is continuous, strictly increasing in x_2' and converges to $-\infty$ (*resp.* ∞) as $x_2' \to -\infty$ (*resp.* ∞). Hence a unique solution, given any x_1, to (2.7) exists, and we call it $x_2^*(x_1)$. Note that $x_2^*(x_1)$ is the only candidate for an equilibrium threshold in period 2, given x_1. The increasing monotonicity of (2.6) thus verifies $x_2^*(x_1)$ constitutes an equilibrium in period 2, since $G_2(\hat{x}_{2i}; (x_1, x_2^*(x_1))) > 0$ if $\hat{x}_{2i} > x_2^*(x_1)$.

With the above result, we proceed to period 1. When all agents except i follow some threshold strategy x_1 in period 1 and all agents follow $x_2^*(x_1)$ in period 2 (note that the deviation of a measure-zero agent does not affect the optimal strategy in period 2), the investment payoff of agent i with x_{1i} equals

$$G_1(x_{1i}; (x_1, x_2^*(x_1))) \equiv E[r + \hat{a}_2 \mid x_{1i}; (x_1, x_2^*(x_1))],$$

which increases in x_{1i} by Lemma 1. Let $\Delta(x_{1i};(x_1, x_2^*(x_1)))$ denote the payoff difference of agent i between investing and delaying in period 1, namely,

$$\Delta(x_{1i};(x_1, x_2^*(x_1))) = G_1(x_{1i};(x_1, x_2^*(x_1))) - R_1(x_{1i};(x_1, x_2^*(x_1))).$$

A threshold strategy x_1' constitutes an equilibrium in period 1 only if agent i observing $x_{1i} = x_1'$ is indifferent between investing and delaying, that is, the payoff difference is zero:

$$\Delta(x_1';(x_1', x_2^*(x_1'))) = 0. \tag{2.8}$$

Likewise, we show in Appendix that $\Delta(x_1';(x_1', x_2^*(x_1')))$ is continuous, strictly increasing in x_1' and converges to $-\infty$ (*resp.* ∞) as x_1' converges to $-\infty$ (*resp.* ∞), so that a unique solution, denoted by x_1^*, to (2.8) exists such that $\Delta(x_1^*;(x_1^*, x_2^*(x_1^*))) = 0$. And x_1^* is the only candidate for equilibrium threshold strategies in period 1. Indeed, it is optimal because

$$\frac{\partial}{\partial x_{1i}}\Delta(x_{1i};(x_1^*, x_2^*(x_1^*))) > 0, \tag{2.9}$$

namely, investing in period 1 is optimal ($\Delta(x_{1i};(x_1^*, x_2^*(x_1^*))) > 0$) if $x_{1i} > x_1^*$. And (2.9) can be obtained by the same way in which we compute $\partial\Delta(x_1';(x_1', x_2^*(x_1')))/\partial x_1' > 0$ in Appendix. Setting $x_2^* = x_2^*(x_1^*)$, then (x_1^*, x_2^*) is the unique threshold equilibrium stated in the proposition.

Q.E.D.

We have focused on symmetric strategies and this is without loss of generality, as Remark 1 shows; the essence is that every agent is infinitesimally small and faces the same decision problem.

Remark 1 (Exclusion of Asymmetric Strategies)**.** *There can only be symmetric threshold strategies in equilibrium. Suppose by contradiction that the unit agents are divided into groups and each group in equilibrium follow threshold* $(x_{11}^*, x_{12}^*, \cdots, x_{1N}^*)$ *respectively in period 1 and* $(x_{21}^*, x_{22}^*, \cdots, x_{2M}^*)$ *respectively in period 2, where* $N, M \in \mathbb{N}$*. Then in period 2, an agent in group* $i \in \{1, \cdots, M\}$ *who observes* x_{2i}^* *must be indifferent between* *investing or not:*

$$E[r + \hat{a}_2 | x_{2i}^*; (x_{11}^*, \cdots, x_{1N}^*, x_{21}^*, \cdots, x_{2i}^*, \cdots, x_{2M}^*)] = 0. \tag{2.10}$$

Since agents are infinitesimally small, the value of $\hat{a}_2$ *only depends on the thresholds of the population and is invariant of individuals' actions. Hence we have by* (2.10) *that* x_{2i}^* *equals the negative* $\hat{a}_2$*. Similarly for a group* $j \neq i$ *agent, she solves*

$$E[r + \hat{a}_2 | x_{2j}^*; (x_{11}^*, \cdots, x_{1N}^*, x_{21}^*, \cdots, x_{2M}^*)] = 0,$$

and thus x_{2j}^* *also equals the negative* $\hat{a}_2$ *and thus equals* x_{2i}^**. The similar argument applies to period 1.*

Remark 2 (Investment of Variable Size)**.** *If the action space is replaced by an interval* $[0, 1]$ *with* $\sum_{t=1}^{2} a_{ti} \in [0, 1]$*, the monotone equilibrium stays unchanged. That said, no agent will split their endowment even if they can. This is so because the monotone equilibrium* (x_1^*, x_2^*) *holds due to* $\{0, 1\} \subset [0, 1]$*, and its uniqueness gives the result. Intuitively, when agents expect positive returns and face delay costs, it is not wise for them to keep endowment unused, whereas when they expect negative returns, being infinitesimal means that investing has neither payoff nor information values.*

The proposition establishes the uniqueness in a monotone equilibrium, consistent with the static global games literature. Yet the general uniqueness, namely the exclusion of other strategy forms, does not obtain. Even though taking a threshold strategy is intuitively appealing because of payoff's increasing monotonicity in the state. However, due to coordination motives, if agents believe their opponents take some other specific strategy, they may follow that form of the strategy. More technically speaking, when iteratively eliminating strictly dominated strategies (which is the key to establish general uniqueness; see Proposition 1), we will encounter an open interval of signal realizations in which all strategy forms are plausible to constitute an equilibrium.

What is more severe here is that, with endogenous learning from past activity, since arbitrary strategy forms in period 1 need to be taken into account in solving for general equilibria, agents face arbitrary information structures about r from observing a_1, thereby leaving the room for other forms of equilibria.

Remark 3 (Complementarities in Action Timings)**.** *In the model, agents contemplate their own action timing but not those of others, because the payoff depends on the activities $\hat{a}_2$ throughout the game. If, instead, the payoff to investing at time t depends only on the current investment size a_t, multiplicity also can occur. This is so because agents now have coordination motives in action timing, and if an agent believes all others will act in one particular period, so will her; see Dasgupta et al. (2012) for further discussion on this line of reasoning.*

2.4 Equilibrium Analysis

In this section, we contrast agents' behaviors between the static and the dynamic games, and investigate the values of learning and the delay option. The consequent welfare implications are also discussed. All the conclusions apply to the general N-period model.

2.4.1 Changes in Behavior

The two-period game can be perceived as (i) adding a stage before the static game or (ii) adding a stage afterwards. We first consider case (i) and compare agents' behavior in period 2 of the dynamic game to that in the static game. Results show that agents in period 2 tend to invest more frequently than in the static game ($x_2^* < x_{st}^*$). Intuitively, in period 2, there is no delay option, which is the same as in the static game; meanwhile, agents know if the game were static, the same fraction of agents would invest, and adding an additional previous stage means weakly more agents invest. So the conclusion follows by the strategic complementarity. For case (ii), agents in period 1 of the dynamic game are tempted by the delay option and thus invest less frequently than in the static game (i.e., $x_{st}^* < x_1^*$), reflecting the informational value from learning. The following proposition establishes these results.

Proposition 3. *Comparing to in the static game, agents are more aggressive in period 2, and less in period 1:* $x_2^* < x_{st}^* < x_1^*$.

Proof. Note that $\hat{a}_2(x_2^*) = a_1(x_2^*) + (1 - a_1(x_2^*))P(\hat{x}_{2j} > x_2^*|x_2^*) > 1/2 = a_1(x_{st}^*)$, so if $x_2^* \geqslant x_{st}^*$, then $E[r + \hat{a}_2|x_2^*] > E[r + a_1|x_{st}^*] = 0$, contradicting the equilibrium condition of period 2 in the dynamic game.

For the second half, note that if $\delta = 0$, then $R_1(x_{1i}; (x_1^*, x_2^*)) = 0$ and $a_2(x_{1i}) = 0$ (recall that agents in period 2 being indifferent to invest or not choose action 0), for any x_{1i}. Hence x_1^* solves $E[r + a_1|x_1^*] = 0$ and thus equals x_{st}^*. As δ increase, R_1 also increases, so x_1^* must increase to balance the equilibrium condition (2.8) of period 1. Therefore $x_1^* > x_{st}^*$.

Q.E.D.

Noteworthy, within the two-period game, agents behave more aggressively in period 2 than in period 1 since $x_2^* < x_1^*$. Contrasting this phenomenon with case (i) earlier, in which that agents in period 2 invest more frequently than in the static game ($x_2^* < x_{st}^*$) is due to coordination motives. Here for the dynamic game, coordination motives do not play a role since agents enjoy the same payment $\hat{a}_2$ whichever period they invest. Instead, here is because of the decreasing continuation payoff that changes from a strictly positive value R_1 to 0 at the last stage. Following this line of logic, we obtain the effect of continuation payoffs on agents' behavior: the lower continuation payoffs to delaying to the next period, the more aggressively agents behave in the current period. Its proof follows the proof of the second half in Proposition 3.

2.4.2 The Value of Information and Welfare Analysis

Note that in equilibrium,

$$\delta E[r + \hat{a}_2 \mid x_{1i}] = R_1(x_{1i}) + \underbrace{\delta \int_{-\infty}^{x_2^*} E[r + \hat{a}_2 \mid \hat{x}_{2i}] f(\hat{x}_{2i} \mid x_{1i}) d\hat{x}_{2i}}_{<0} < R_1(x_{1i}),$$

where the negativity of the second term is by the definition of x_2^*. The term $\delta E[r + \hat{a}_2 \mid x_{1i}]$ is the expected payoff to delaying without learning (i.e., when agent i holds a constant signal x_{1i}), which is shown strictly lower than the continuation payoff with learning existed; hence the value of information is positive. To see the intuition of why learning improves agents' expected payoffs, note that learning makes agents' signals better aligned, alleviating their strategic uncertainty and thereby making them better coordinate. In addition, learning mitigates the payoff uncertainty, as is reflected in $\hat{\tau}_2 > \tau_1$, namely, agents better infer the state in the presence of learning.

We next compare the interim welfare of agents between the dynamic and the static games, after agents' signals are realized yet before the state is revealed. Results show that agent i expects a higher payoff in the dynamic game when (i) she invests in the first stage in the dynamic game ($x_{1i} > x_1^*(> x_{st}^*)$), or (ii) she invests in period 2 ($x_{1i} \leqslant x_1^*$ and $\hat{x}_{2i} > x_2^*$) and her belief is driven upward after learning ($\hat{x}_{2i} > x_{1i}$). The increased welfare in case (i) originates from the higher expected total investment in the dynamic game, and that in case (ii) is due to, by $\hat{x}_{2i} > x_{1i}$, both higher state and higher investment size.

Some computation gives the conclusion. For example, in period 1, the expected payoff for agent i with x_{1i} is

$$\begin{cases} x_{1i} + a_1(x_{1i}), & \text{if } x_{1i} > x_{st}^* \\ 0, & \text{otherwise,} \end{cases}$$

in the static game, and

$$\begin{cases} x_{1i} + a_1(x_{1i}) + a_2(x_{1i}), & \text{if } x_{1i} > x_1^* \\ R_1(x_{1i}; (x_1^*, x_2^*)), & \text{if } x_{1i} < x_1^*, \end{cases}$$

in the dynamic game. Therefore, i's welfare increases if $x_{1i} > x_1^*(> x_{st}^*)$. Otherwise if $x_{1i} < x_1^*$, the agent proceeds to period 2 and similar comparison can be made. Note that there are inconclusive situations in which the direction of the welfare change depends on cost parameter δ versus information precision τ_1 and τ_2. For example, when i invests in both games but learning drives her belief down ($x_2^* < \hat{x}_{2i} < x_{st}^* < x_{1i} < x_1^*$), then welfare comparison depends on $x_{1i} + a_1(x_{1i})$ versus $\hat{x}_{2i} + \hat{a}_2(\hat{x}_{2i})$.

2.4.3 The Value of Delay

This subsection explores the option value of delay in isolation from the learning effects. To this end, we consider the game in which agent i cannot observe x_{2i}; one can think of it as $\tau_2 \to 0$, so that x_{2i} is completely noisy and ignored.

Proposition 4. *When x_{2i} is not observable, the dynamic game is essentially static: $x_2^* \geqslant x_1^* = x_{st}^*$.*

Proof. Suppose that agent i holds a constant belief x_{1i}. Her expected payoff to investing in period 2 is $\delta E[r + \hat{a}_2 \mid x_{1i}]$, so that if she will invest, she will only invest in period 1 due to $\delta < 1$. That said, the agent in period 2 stays inactive for sure, so x_2^* can be any number larger than x_1^*. Since agents will not invest in period 2, the payoff to delaying to period 2 is 0 and also $\hat{a}_2 = a_1$, so x_1^* is such that $E[r + a_1 | x_1^*] = 0$ and thus equals x_{st}^*. *Q.E.D.*

Intuitively, with no learning benefit but only cost from delaying, agents act (if at all) as soon as possible, as is indicated by that the continuation payoff to delaying to period 2 is at most $\delta E[r + \hat{a}_2 \mid x_{1i}]$, a mere discounted current payoff. Consequently $\hat{a}_2 = a_1$, so that threshold $x_1^* = x_{st}^*$ and the strategic stage ends there.

2.5 Learning Efficiency

In this subsection, we pay attention to the learning mechanism in our paper, and contrast it with learning through directly observing the state r. That is, instead of observing an endogenous signal about past activity as in (2.3), if agent i is directly endowed with an exogenous signal x_{2i} such that

$$x_{2i} = r + \frac{1}{\sqrt{\tau_2}}\varepsilon_{2i}, \tag{2.11}$$

will she infer the state r more accurately at $t = 2$? Surprisingly, the agent estimates r more accurately when she learns though observing action, as long as her initially information precision $\tau_1 > 1$. To see it, by Bayes' rule, there exists a sufficient signal $\hat{\hat{x}}_{2i}$ for agent i that summarizes her information about r contained in x_{1i} and x_{2i}, such that

$$\hat{\hat{x}}_{2i} = r + \frac{1}{\sqrt{\tau_1 + \tau_2}} \varepsilon_{2i},$$

when learning is through directly observing r as in (2.11). Recall that the precision level of agent i's information by indirect learning is $\hat{\tau}_2 = \tau_1 + \tau_1\tau_2$ at $t = 2$. Therefore, as long as $\tau_1 > 1$, learning through observation of actions reveals more accurate information about r.

Intuitively, learning efficiency of direct observation on r is fixed $(=\tau_1 + \tau_2)$, while its precision level depends on how accurate agents know about r (measured by τ_1) and how accurate the endogenous signal reflects their private information (measured by τ_2), when learning is through observing the past activity. The two channels are mutually reinforced, as is reflected by that indirect learning precision in period 2 is $\tau_2' = \tau_1\tau_2$. Therefore, indirect learning can be more accurate when one of the channels is accurate enough, and we confirm the condition is $\tau_1 > 1$.

3. The N-Period Model

We now augment the game to $N \in \mathbb{N}$ periods and consider a general payoff structure. In the game, the unit of agents decide the optimal timing of investment (if at all) between $t = 1, 2, \cdots, N$. The first two periods run identically as before, and notations a_{ti}, a_t, and $\hat{a}_t$ bear similar meanings. Agent i's payoff in period t to action $a_{ti} = 0$ is 0, and her payoff to $a_{ti} = 1$ is now summarized by an increasing and continuously differentiable function $U(r, \hat{a}_N)$, namely, the return of investment increases in state r and aggregate investment $\hat{a}_N$, indicating the coordination feature of the game. We assume that $U(r, \hat{a}_N)$ is concave in each component and that $\lim_{r\to\infty} U(r, \hat{a}_N) = \infty$ and $\lim_{r\to-\infty} U(r, \hat{a}_N) = -\infty$, for any $\hat{a}_N \in [0, 1]$. By implication, when the state is extremely good (or bad), investing strictly dominates (or is strictly dominated) regardless of others' actions.

We now describe the endogenous signals that agents receive in periods $t = 3, \cdots, N$. To maintain analyticity and similarity to Section 2, we let agent i observe, in $t = 3, \cdots, N$,

$$x_{ti} = \Phi^{-1}(\bar{a}_{t-1}) + \frac{1}{\sqrt{\tau_t}}\varepsilon_{ti}, \quad \tau_t > 0, \tag{3.1}$$

where $\bar{a}_{t-1} = (\hat{a}_{t-1} - \hat{a}_{t-2})/(1 - \hat{a}_{t-2})$ is average action in $t-1$ (note that $\hat{a}_{t-1} - \hat{a}_{t-2}$ denotes the new investment at $t-1$ and that $1 - \hat{a}_{t-2}$ the fraction of agents who reach $t-1$), and $\varepsilon_{ti} \sim \mathcal{N}(0,1)$ is independent of all other variables. By LLN, the average action equals the likelihood of investment for an individual agent; therefore, when agents follow a threshold strategy profile denoted by $(x_1, x_2, \cdots, x_N)$ in period $1, 2, \cdots, N$, $\bar{a}_t(r) = P(\hat{x}_{ti} > x_t|r)$, where $\hat{x}_{ti}$ is i's expectation of r at time t. Note that the structure is consistent with x_{2i} defined in Section 2 since the average action $\bar{a}_1 = a_1$.

In each period t, agent i still chooses a_{ti} to maximize her conditional expected total payoff $E[\sum_{t=1}^{N} a_{ti}\delta^{t-1}U(r, \hat{a}_N)|x_{1i}, \cdots, x_{ti}]$, where $\delta \in (0,1)$.

3.1 Learning Under a Threshold Strategy

As in the two-period setup, for $t = 3, \cdots, N$, x_{ti} can be transformed into an informationally equivalent (with respect to r) signal x'_{ti} centered around r, when agents follow a threshold strategy profile before time t. The definition of a threshold strategy profile is similar to that in Section 2 and thus omitted. Agent i's updated belief about r in period t can be summarized by a unidimensional statistic $\hat{x}_{ti}(x_{1i}, x_{2i}, \cdots, x_{ti})$ that is normally distributed given r. We still let $\hat{x}_{1i} = x'_{1i} = x_{1i}$ and $\hat{\tau}_1 = \tau'_1 = \tau_1$. It turns out that the precisions of x'_{ti} and $\hat{x}_{ti}$, denoted by τ'_t and $\hat{\tau}_t$ respectively, are such that $\tau'_t = \tau_t\hat{\tau}_{t-1}$ and $\hat{\tau}_t = \sum_{k=1}^{t} \tau'_k$, for all $t \geqslant 2$. Lemma 2 summarizes the results.

Lemma 2. *Suppose that agents follow a threshold strategy profile with respective thresholds* $\{x_1, x_2, \cdots, x_N\}$.

(i) Let $x'_{ti} \equiv x_{ti}/\sqrt{\hat{\tau}_{t-1}} + x_{t-1}$ *for any i and* $t \geqslant 2$*; then* x'_{ti} *is sufficient for* x_{ti} *with*

respect to r and

$$x'_{ti} = r + \frac{1}{\sqrt{\tau'_t}}\varepsilon_{ti}.$$

(ii) $\hat{x}_{ti}$ *can be expressed by* $\hat{x}_{ti}(x'_{1i}, \cdots, x'_{ti}) = (\sum_{k=1}^{t} \tau'_k)^{-1}(\sum_{k=1}^{t} \tau'_k x'_{ki})$ *and particularly,*

$$\hat{x}_{ti} = r + \frac{1}{\sqrt{\hat{\tau}_t}}\varepsilon_{ti}.$$

(iii) For any $t, k \in \{1, 2, \cdots, N\}$, $\hat{x}_{tj}$ *is normally distributed given* $\hat{x}_{ki}$ *(when* $i = j$*, let* $t > k$*). And moreover,* $E[\hat{a}_N|\hat{x}_{ki}]$ *increases in* $\hat{x}_{ki}$*.*

Proof. The proofs are by indication on t. For (i), it holds at $t = 2$ by Section 2. Assume inductively that it holds until $t = N - 1$. Then $\bar{a}_{N-1}(r) = \Phi(\sqrt{\hat{\tau}_{N-1}}(r - x_{N-1}))$. So at $t = N$,

$$x_{Ni} = \sqrt{\hat{\tau}_{N-1}}(r - x_{N-1}) + \frac{1}{\sqrt{\tau_N}}\varepsilon_{ti}.$$

Rearranging and comparing it with x'_{Ni} and τ'_N give the conclusion. Then (ii) follows by Bayes' rule.

For (iii), when $i \neq j$, the first part is similar to Lemma 1 (iii). When $i = j$ and let $t > k$, it follows

$$\hat{x}_{ti}|_{\hat{x}_{ki}} = \frac{\hat{\tau}_k \hat{x}_{ki} + \tau'_{k+1} x'_{(k+1)i} + \cdots \tau'_t x'_{ti}}{\hat{\tau}_k + \tau'_{k+1} + \cdots \tau'_t}|_{\hat{x}_{ki}},$$

and note that for $n \in \{k+1, \cdots, t\}$, $x'_{ni}|_{\hat{x}_{ki}} = r + \varepsilon_{ni}/\sqrt{\tau'_n} = \hat{x}_{ki} - \varepsilon_{ki}/\sqrt{\hat{\tau}_k} + \varepsilon_{ni}/\sqrt{\tau'_n}$ is normally distributed given $\hat{x}_{ki}$.

The monotonicity of $E[\hat{a}_2|\hat{x}_{ki}]$ holds by Lemma 1. Assume inductively that $\hat{a}'_{t-1}(\hat{x}_{ki}) > 0$ till $t = N - 1$. Then at $t = N$,

$$\frac{d}{d\hat{x}_{ki}}\hat{a}_N(\hat{x}_{ki}) = \frac{d}{d\hat{x}_{ki}}[\hat{a}_{N-1} + (1 - \hat{a}_{N-1})\bar{a}_N](\hat{x}_{ki}) = (1 - \bar{a}_N)\hat{a}'_{N-1} + (1 - \hat{a}_{N-1})\bar{a}'_N > 0.$$

Q.E.D.

These results are extensions to those in Section 2 and follow the discussions there.

3.2 Equilibrium Characterization

We still restrict to a monotone equilibrium and now solve for it. Provided that agents follow a threshold strategy profile denoted $(x_1, x_2, \cdots, x_N)$, the expected continuation payoff $R_t(\hat{x}_{ti}; \{x_t\}_{t=1}^N)$ of agent i with $\hat{x}_{ti}$ at time t is

$$\begin{aligned} R_t(\hat{x}_{ti}; \{x_t\}_{t=1}^N) = \delta \int_{x_{t+1}}^{\infty} & E[U(r, \hat{a}_N) \mid \hat{x}_{(t+1)i}] f(\hat{x}_{(t+1)i} \mid \hat{x}_{ti}) d\hat{x}_{(t+1)i} \\ + \delta \int_{-\infty}^{x_{t+1}} & R_{t+1}(\hat{x}_{(t+1)i}; \{x_t\}_{t=1}^N) f(\hat{x}_{(t+1)i} \mid \hat{x}_{ti}) d\hat{x}_{(t+1)i}, \end{aligned} \tag{3.2}$$

where $f(\hat{x}_{(t+1)i} \mid \hat{x}_{ti})$ is the conditional density of $\hat{x}_{(t+1)i}$ on $\hat{x}_{ti}$, whose value can be deduced by Lemma 2 (iii). Let $R_N(\hat{x}_{Ni}; \{x_t\}_{t=1}^N) \equiv 0$ for any $\hat{x}_{Ni} \in \mathbb{R}$. Note that we have let $R_t(\cdot)$ represent the face value at time t, instead of being discounted to time 1. The following proposition characterizes the unique monotone equilibrium.

Proposition 5. *There exists a unique monotone equilibrium characterized by* $(x_1^*, x_2^*, \cdots, x_N^*)$ *in the N-period game, where* x_t^* *is the unique solution to*

$$E[U(r, \hat{a}_N)|x_t^*] = R_t(x_t^*; \{x_t^*\}_{t=1}^N), \quad t = 1, 2, \cdots, N. \tag{3.3}$$

Similar to that in the two-period model, the proof starts from the last period N and takes as given that all agents in all previous periods play some threshold strategy profile, so as to characterize x_N^*. Next proceeding the argument backward and in each period $1 \leqslant t \leqslant N-1$, it is taken as given that agents play some threshold strategy profile before t and act optimally after t, and sequentially obtains $x_{N-1}^*, x_{N-2}^*, \cdots, x_1^*$. Recall that in checking that no agent wants to deviate at time t, the key is that $x_{t+1}^*, \cdots, x_N^*$ will not be disturbed by an infinitesimally small agent's deviation.

Proof. Fix an arbitrary threshold strategy profile $(x_1, x_2, \cdots, x_N)$ and an agent i. At $t = N$, given that all agents expect i follow $(x_1, x_2, \cdots, x_N)$ in the game, the payoff of agent i with belief $\hat{x}_{Ni}$ to investing is

$$E[U(r, \hat{a}_N) \mid \hat{x}_{Ni}; (x_1, \cdots, x_N)], \tag{3.4}$$

which increases in $\hat{x}_{Ni}$. An threshold strategy x'_N constitutes an equilibrium threshold at $t = N$ only if observing it makes agent i indifferent between investing or not, that is, it is such that

$$E[U(r, \hat{a}_N) \mid x'_N; (x_1, \cdots, x_{N-1}, x'_N)] = 0.$$

Similarly as in Proposition 2, the LHS is continuous, converges to infinity as x'_N converges to infinity, and strictly increases in x'_N, so there exists a unique such x'_N that solves the above equation. And the increasing monotonicity of (3.4) verifies that the solution indeed constitutes an equilibrium threshold at $t = N$. We denote it by $x^*_N(x_1, x_2, \cdots, x_{N-1})$ and shorthand it by x^*_N.

Proceeding to $t = N - 1$, taken as given that all agents expect some agent i follow $(x_1, x_2, \cdots, x_{N-1}, x^*_N)$, the payoff of investing immediately to agent i with $\hat{x}_{(N-1)i}$ is

$$E[U(r, \hat{a}_N) \mid \hat{x}_{(N-1)i}; (x_1, \cdots, x_{N-1}, x^*_N)],$$

while delaying to the next period has an expectation value given by

$$R_{N-1}(\hat{x}_{(N-1)i}; (x_1, \cdots, x_{N-1}, x^*_N)).$$

Let $\Delta_{N-1}(\hat{x}_{(N-1)i}; (x_1, \cdots, x_{N-1}, x^*_N))$ denote the payoff difference for i investing at $N - 1$ or delaying, that is,

$$\Delta_{N-1}(\hat{x}_{(N-1)i}; (x_1, \cdots, x_{N-1}, x_N*)) \equiv$$
$$E[U(r, \hat{a}_N) \mid \hat{x}_{(N-1)i}; (x_1, \cdots, x_{N-1}, x^*_N)] - R_{N-1}(\hat{x}_{(N-1)i}; (x_1, \cdots, x_{N-1}, x^*_N)).$$

An threshold strategy x'_{N-1} that constitutes an equilibrium threshold at $t = N-1$ must be such that

$$\Delta_{N-1}(x'_{N-1}; (x_1, \cdots, x_{N-2}, x'_{N-1}, x^*_N)) = 0. \tag{3.5}$$

We demonstrate in Appendix the unique existence of such x'_{N-1} that solves (3.5), by showing the LHS is strictly increasing in x'_{N-1} and converges to infinity as x'_{N-1} goes infinity. Also, we confirm that the solution indeed constitutes an equilibrium at $t = N-1$ by showing in Lemma 3 in Appendix that $\Delta_{N-1}(\hat{x}_{(N-1)i}; (x_1, \cdots, x_{N-1}, x_N*))$ increases in $\hat{x}_{(N-1)i}$. Let the solution be denoted by $x^*_{N-1}(x_1, x_2, \cdots, x_{N-2})$, or for notational simplicity, by x^*_{N-1}. By backward induction and similarly, we can characterize $x^*_{N-2}, x^*_{N-3}, \cdots, x^*_1$. *Q.E.D.*

It is noteworthy that $\Delta_t \neq \Delta_k$ so that $x^*_t \neq x^*_k$ for $t \neq k \in \{1, \cdots, N\}$, indicating agents respond to information changes continuously and that a positive fraction of them move from not investing to investing every period. This observation is in contrast to the dynamic regime change games (*cf.* Angeletos et al. (2007)) in whose model agents stay inertia for a series of periods. The difference occurs because the payoff structure in this paper is continuous in r and $\hat{a}_N$, while it is discrete in their regime change game.

However, if learning precisions $\{\tau_t\}_{t \geqslant 2}$ are moderate (so $\hat{x}_{ti} \approx \hat{x}_{(t+1)i}$) and the delaying cost is not too severe (e.g., $\delta \to 1$ so $R_t \approx R_{t+1}$), the number of new active agents between periods should be small, since the differences between the continuations payoffs evaluated at t and $t+1$ are small. So x^*_t and x^*_{t+1} are near. In this situation, agents' behaviors experience relative inertia in intermediate periods (also documented in Angeletos et al. (2007)), and the dynamics of the game are now such that an active first stage followed by a relative tranquil phase, till the last stage at which less optimistic agents also invest, because the continuation value of delay in the last period drops discontinuously to 0 from some positive number R_{N-1}.

3.3 Equilibrium Analysis

In this section, we confirm our two-period results. The positive information value is easy to obtain, since

$$\begin{aligned}\delta E[U(r, \hat{a}_N) \mid \hat{x}_{ti}] = \delta \int_{x^*_{t+1}}^{\infty} & E[U(r, \hat{a}_N) \mid \hat{x}_{(t+1)i})] f(\hat{x}_{(t+1)i} \mid \hat{x}_{ti}) d\hat{x}_{(t+1)i} \\ & + \delta \int_{-\infty}^{x^*_{t+1}} \underbrace{E[U(r, \hat{a}_N) \mid \hat{x}_{(t+1)i})]}_{< R_{t+1}(\hat{x}_{(t+1)i})} f(\hat{x}_{(t+1)i} \mid \hat{x}_{ti}) d\hat{x}_{(t+1)i} < R_t(\hat{x}_{ti}),\end{aligned}$$

by the definition of $R_t(\hat{x}_{ti})$. Perceiving the first term $\delta E[U(r, \hat{a}_N) \mid \hat{x}_{ti}]$ as the expected payoff at time $t+1$ in the absence of learning, the strict inequality then shows the information is of positive value. Next, we verify (i) comparing to in the static game, agents are more aggressive in the last stage ($x_N^* < x_{st}^*$) and less ($x_{st}^* < x_t$) in earlier periods $t < N$, and (ii) the mere delay option has zero impact. The proofs are relegated to Appendix.

Proposition 6. *(i)* $x_N^* < x_{st}^* < x_t^*$*, for* $t = 1, 2, \cdots, N-1$*.*
(ii) When learning does not exist such that x_{ti} *for any* $t \geqslant 2$ *is unobservable, the game is essentially static:* $x_1^* = x_{st}^*$ *and agents stay inactive after period 1.*

4. Discussions

4.1 Learning Efficiency

We now investigate the learning efficiency of observing actions, by comparing it with learning through directly observing the state r. That is, if the signal structures of x_{ti}, for $t = 2, 3, \cdots, N$, are such that

$$x_{ti} = r + \frac{1}{\sqrt{\tau_t}} \varepsilon_{ti}, \tag{4.1}$$

will it improve the accuracy with which agents infer the state r, relative to the signal structures (3.1) in the paper?

Proposition 7. *If the initial information is precise* $\tau_1 > 1$*, observing the actions as in* (3.1) *reveals more accurate information about the state* r *than directly observing* r *as in* (4.1)*, for all periods* $t \geqslant 2$*.*

Recall that the information precision through observing actions is $\hat{\tau}_t = \sum \tau_k'$. Let $\hat{\hat{\tau}}_t$ denote the precision level of learning through observing r. We conclude by comparing them.

Proof. In period $t \geqslant 2$, when agents directly observe r as in (4.1), there exists a sufficient statistic, denoted $\hat{\hat{x}}_{ti}$, of $x_{1i}, x_{2i}, \cdots, x_{ti}$ with respect to r; by Bayes' rule,

$$\hat{\hat{x}}_{ti} = r + \varepsilon_{ti}/\sqrt{\hat{\hat{\tau}}_t}, \quad \text{with } \hat{\hat{\tau}}_t = \tau_1 + \tau_2 + \cdots + \tau_N.$$

Therefore, $\hat{\tau}_t > \hat{\hat{\tau}}_t$ for all $t \geqslant 2$, whenever $\tau_1 > 1$. *Q.E.D.*

4.2 Infinite Periods

Now we augment the game into infinite periods by setting $N \to \infty$, and demonstrate that the equilibrium properties are similar as when N is finite. Also, we find that agents fully learn the true state in the limit. Defined analogously, a threshold strategy is denoted by $\{x_t\}_{t=1}^{\infty}$ and continuation payoffs by $\{R_t\}_{t=1}^{\infty}$. We restrict to that agents follow a symmetric threshold strategy $\{x_t\}_{t=1}^{\infty}$ in the game, so we obtain the similar transformed endogenous signals x'_{ti} and cumulative signals $\hat{x}_{ti}$ as in Section 3, since the learning processes only depend on that agents play a threshold strategy.

The equilibrium concept we consider now is, however, an ε-symmetric monotone equilibrium which consists of a symmetric threshold strategy, such that no agent can expect to gain more than $\varepsilon > 0$ by deviating from the strategy, given others also follow it. This enables us to implement the previous backward induction argument in characterizing the equilibrium. In detail, for any $\varepsilon > 0$, due to $\delta \in (0, 1)$, there exists $N^*_\varepsilon \in \mathbb{N}$ such that

$$R_t(\hat{x}_{ti}; \{x_t\}_t) < \varepsilon,$$

for every $t \geqslant N^*_\varepsilon$, signal $\hat{x}_{ti}$, and threshold strategy $\{x_t\}$. Henceforth fix a random $\varepsilon > 0$ and consequently an N^*_ε. We claim there exists an ε-monotone equilibrium with an identical equilibrium threshold after period N^*_ε. In what follows, we assume that agents follow some identical threshold after N^*_ε to solve for an equilibrium, and then verify it is indeed optimal for agents to follow such a constant threshold strategy after period N^*_ε.

For any $t \geqslant N^*_\varepsilon$, a threshold strategy $x^*_{N^*_\varepsilon}$ constitutes an equilibrium strategy in period t only if it solves

$$E[U(r, \hat{a}_\infty) \mid x^*_{N^*_\varepsilon}; (x_1, \cdots, x^*_{N^*_\varepsilon}, x^*_{N^*_\varepsilon}, \cdots)] = 0,$$

where $\hat{a}_\infty = \sum_t^\infty a_t \in (0,1)$. Such $x^*_{N^*_\varepsilon}$ exists; to see its monotonicity, as $x^*_{N^*_\varepsilon}$ increases, the state r increases and the expected fractions of investors in periods other than t increase while the expected fraction of investors in period t remains constant (which is 1/2). To check $x^*_{N^*_\varepsilon}$ indeed is an ε-equilibrium strategy in period t, note that when observing a signal higher than $x^*_{N^*_\varepsilon}$, deviating from investing (which gives a positive payoff) to delaying (which gives $R_t < \varepsilon$) increases the expected payoff by at most ε; when observing a signal lower than $x^*_{N^*_\varepsilon}$, deviating from not investing to investing clearly lowers the expected payoff. Since t is arbitrary as long as larger than N^*_ε, we have shown that for periods $t = N^*_\varepsilon, N^*_\varepsilon+1, \cdots$, it is optimal for agents to follow a constant threshold strategy $x^*_{N^*_\varepsilon}$. Next, proceed to period $N^*_\varepsilon - 1$ and take as given that all agents in periods $t \geqslant N^*_\varepsilon$ follow $x^*_{N^*_\varepsilon}$; the argument goes exactly the same as in Proposition 5, so we obtain its unique equilibrium threshold $x^*_{N^*_\varepsilon-1}$, and proceeding backward to obtain $x^*_{N^*_\varepsilon-2}$,$x^*_{N^*_\varepsilon-3}$ till x^*_1.

Noteworthy, agents fully learn the actual state in the limit, because their information precision $\sum_{t=1}^\infty \tau_t \to \infty$. Such a property holds even when the equilibrium thresholds are now constant after some certain periods. To see the reason, recall that equilibrium thresholds start to be constantly $x^*_{N^*_\varepsilon}$ from period N^*_ε; then at $t = N^*_\varepsilon$, a positive fraction of agents will move to invest because $x^*_{N^*_\varepsilon-1} \neq x^*_{N^*_\varepsilon}$. This movement changes the total investment size in period N^*_ε (from that in $N^*_\varepsilon - 1$) and thereby makes agents in period $t = N^*_\varepsilon + 1$ learn new information and consequently, a further fraction of agents will move to invest in period $N^*_\varepsilon + 1$, and so on. Essentially, the fully learning of the state stems from that (i) there is no public learning and hence no crowding out effect as in the herding literature, and (ii) pooling everyone's information reveals the true state. Of course, that observational precisions τ_t being exogenously given and bounded away from zero is also a reason.

4.3 Proper Priors and Public Learning

The analysis till now is conducted with agents holding an improper prior, and we claim it is almost without loss of generality. Now we mention how to extend the model to a proper prior game. Let agents hold a common prior as follows:

$$r \sim \mathcal{N}(\alpha, 1/\beta),$$

where $\alpha \in \mathbb{R}$ and $\beta > 0$. Still restrict to agents taking a threshold strategy profile; agent i's belief about r at each stage is summarized by a unidimensional statistic $\hat{x}_{ti}$, by the same Gaussian updating process as in Section 3.1. Consequently, the equilibrium characterization is analogous, so is the analysis part *when* there exists a unique monotone equilibrium.

Moreover, consider the case where learning is from public observation of actions, so the signal structure of period $t \in \{2, 3, \cdots, N\}$ becomes

$$x_{ti} = \Phi^{-1}(\bar{a}_{t-1}) + \frac{1}{\sqrt{\tau_t}}\varepsilon_t,$$

where $\varepsilon_t \sim \mathcal{N}(0, 1)$ represents the market-wise noise, independent of all other variables. Let x_{1i} still be private.

We now elaborate on the potential arise of multiple equilibria in the presence of public learning. To this end, it suffices to consider period 2 and show there exist multiple optimal strategies. Results from Section 2 state that the informativeness of the public signal about r (assuming an improper prior) is $\tau_2' = \tau_1\tau_2$, which converges to infinity as $\tau_1 \to \infty$. Therefore, the ratio of the precision of the public information to the square root of that of private information, namely $\tau_2'/\sqrt{\tau_1}$, diverges to ∞ as $\tau_1 \to \infty$. Hence with public learning, multiplicity in monotone equilibria arises even if private information is infinitely diffused; see Morris and Shin (2004) and Angeletos and Werning (2006) for proofs on why the ratio determines multiplicity. Noteworthy, the proof shows that there are multiple optimal symmetric threshold strategies, to say nothing of optimal strategies in other forms. Intuitively, it is known that complete information coordination games admit multiple equilibria; when the ratio is large, indicating public information dominates private information, global games exhibit similarity to the complete information environment and thus have multiple equilibria. Note that adding a common prior only increases the ratio and thereby only contributes to the rise of multiplicity.

How about the learning property in the limit when learning is public? We have shown that full learning of the true state obtains in the limit with only private observation, and attributed it to the absence of the crowding-out effect from the public information. However, even with public learning as in this subsection, full learning is plausible when agents interact long enough, as long as they play a threshold strategy. To see it, the learning mechanism stays the same as Section 3.1 when agents follow a threshold strategy, so the information precision about the state is always increasing and due to observational precisions τ_t are exogenous, agents in the limit learn the true state. This result crucially depends on the continuous signal structures in this paper, which, shown generally by Lee (1993), avoids the information cascade.

5. Conclusions

This paper constructs a dynamic coordination game with learning and delay opportunities factored in. It tractably analyzes agents' optimal action timings, which are determined though constantly trading off the information gain of delay against its opportunity costs. A unique monotone equilibrium is characterized and in it, learning is shown to improve agents' expected payoff, while the mere delay option impose no impact on agents' behaviors, relative to the one-shot game. Additionally, the dynamics of agents' behaviors are characterized and depending on the learning efficiency, the tranquil intermediate periods documented in the literate obtain. Conditions of welfare enhancement, and the contrast to learning by directly observing the state, are also given. The analysis applies for all ranges of learning efficiencies, generalizing the existing studies that usually focus on the limit accurate signals. We illustrate the paper in an investment context; the applicability to other coordination scenarios including currency crises or bank runs is straightforward.

6. Appendix

Computations in Proposition 2 The monotonicity of $G_2(x_2'; (x_1, x_2'))$ in x_2' follows from that, given x_1,

$$\begin{aligned}\frac{\partial}{\partial x_2'} G_2(x_2'; (x_1, x_2')) &= \frac{\partial}{\partial x_2'} \{E[r|x_2'] + P(x_{1j} > x_1|x_2') + (1 - P(x_{1j} > x_1|x_2')) \underbrace{P(\hat{x}_{2j} > x_2'|x_2')}_{=1/2}\} \\ &= \frac{\partial}{\partial x_2'} \left\{ x_2' + \frac{1}{2}\Phi(\sqrt{\cdot}(x_2' - x_1)) + \frac{1}{2} \right\} > 0.\end{aligned}$$

For the boundary value of $G_2(x_2'; (x_1, x_2'))$, since $E[r|\hat{x}_{2i}] = \hat{x}_{2i}$ and $\hat{a}_2 \in [0, 1]$, when $x_2' \to \infty$,

$$G_2(x_2'; (x_1, x_2')) = E[r + \hat{a}_2 \mid x_2'; (x_1, x_2')] \to \infty.$$

Now we prove the monotonicity of $\Delta(x_1'; (x_1', x_2^*(x_1')))$ in x_1'. Note that

$$\begin{aligned}\frac{\partial}{\partial x_1'} G_1(x_1'; (x_1', x_2^*(x_1'))) &= \frac{\partial}{\partial x_1'} \left\{ E[r \mid x_1'] + \underbrace{P(x_{1j} > x_1'|x_1')}_{=1/2} + (1 - P(x_{1j} > x_1'|x_1'))P(\hat{x}_{2j} > x_2^*(x_1')|x_1') \right\} \\ &= \frac{\partial}{\partial x_1'} \left\{ x_1' + \frac{1}{2} + \frac{1}{2}\Phi(\sqrt{\cdot}(x_1' - x_2^*(x_1'))) \right\} > 0,\end{aligned}$$

since $dx_2^*(x_1')/dx_1' \in [0, 1]$ by taking the total derivative of (2.7) with respect to x_1'. Also,

$$\begin{aligned}\frac{\partial}{\partial x_1'} R_1(x_1'; (x_1', x_2^*(x_1'))) =& \delta \frac{d}{dx_1'} \int_{-\infty}^{\infty} E[r + \hat{a}_2 \mid \hat{x}_{2i}] f(\hat{x}_{2i} \mid x_1') \mathbf{1}_{\hat{x}_{2i} > x_2^*(x_1')} d\hat{x}_{2i} \\ \leqslant& \delta \frac{d}{dx_1'} \int_{-\infty}^{\infty} \underbrace{E[r + \hat{a}_2 \mid \hat{x}_{2i}]}_{=E[r + \hat{a}_2 | \hat{x}_{2i}, x_1']} f(\hat{x}_{2i} \mid x_1') d\hat{x}_{2i} \\ =& \delta \frac{d}{dx_1'} E[r + \hat{a}_2 \mid x_1'],\end{aligned} \tag{6.1}$$

where the first equation follows from $R_1's$ definition with $\mathbf{1}$ being the indicator function, the inequality is due to $\mathbf{1} \in [0,1]$. Therefore,

$$\frac{\partial}{\partial x_1'}\Delta(x_1';(x_1',x_2^*(x_1'))) \geqslant (1-\delta)\frac{\partial}{\partial x_1'}G_1(x_1';(x_1',x_2^*(x_1'))) > 0.$$

Next for the boundary value, since $R_1 \geqslant 0$, we have $\Delta(x_1';(x_1',x_2^*(x_1'))) \to -\infty$ as $x_1' \to -\infty$. On the other hand, as $x_1' \to \infty$, it is similar as in (6.1)) to obtain $R_1(x_1';(x_1',x_2^*(x_1'))) \leqslant \delta E[r+\hat{a}_2 \mid x_1';(x_1';x_2^*(x_1'))]$, so we also have

$$\Delta(x_1';(x_1',x_2^*(x_1'))) \geqslant (1-\delta)E[r+\hat{a}_2 \mid x_1';(x_1';x_2^*(x_1'))] \to \infty.$$

Existence of a Unique Solution to (3.5) For a pedagogical purpose, we verify the general case by showing

$$\begin{aligned}&\Delta_t(x_t';(x_1,\cdots,x_t',x_{t+1}^*,\cdots,x_N^*)) \equiv \\ &E[U(r,\hat{a}_N) \mid x_t';(x_1,\cdots,x_t',\cdots,x_N^*)] - R_t(x_t';(x_1,\cdots,x_t',\cdots,x_N^*)) = 0\end{aligned}$$

admits a unique solution x_t'. First recall by Lemma 2 we have $E[U(r,\hat{a}_N)|\hat{x}_{ti}]$ strictly increasing in $\hat{x}_{ti}$, given any threshold strategy profile the population plays. Hence in checking the monotonicity of $E[U(r,\hat{a}_N)|x_t';(x_1,\cdots,x_t',\cdots,x_N^*)]$ in x_t', if we can show that the investment at time t will not decrease, then by the result of Lemma 2, the aggregate investment increases in x_t'. Indeed, at time t, the fraction of agents who invest (in the eyes of the agent observing x_t') equals $P(\hat{x}_{tj} > x_t'|x_t') = 1/2$, namely, it is invariant. So we have $E[U(r,\hat{a}_N) \mid x_t']$ is strictly increasing in x_t'. Next we show that $\partial R_t(x_t')/\partial x_t' < \delta E[U(r,\hat{a}_N)|x_t']$ by backward induction.

For $t = N-1$, it is the same as in the two-period model to obtain that $\partial R_{N-1}(x_{N-1}')/\partial x_{N-1}' < \delta E[r+\hat{a}_{N-1}|x_{N-1}']$. Assume backward inductively that $\partial R_k(x_k')/\partial x_k' < \delta E[U(r,\hat{a}_N)|x_k']$ for all $k = N-2, N-3, \cdots, t+1$, so at time t,

$$\begin{aligned}&\frac{\partial}{\partial x_t'}\int_{-\infty}^{x_{t+1}^*} R_{t+1}(\hat{x}_{(t+1)i};x_1,\cdots,x_t',x_{t+1}^*,\cdots,x_N^*)f(\hat{x}_{(t+1)i} \mid x_t')d\hat{x}_{(t+1)i} \\ \leqslant&\delta\frac{\partial}{\partial x_t'}\int_{-\infty}^{x_{t+1}^*} E[r+\hat{a}_t \mid \hat{x}_{(t+1)i}]f(\hat{x}_{(t+1)i} \mid x_t')d\hat{x}_{(t+1)i},\end{aligned}$$

by noting that x^*_{t+1} increases in x'_t due to strategic complementarity (a higher signal to an individual does not affect x^*_{t+1}, but a higher threshold means fewer agents invest, which causes agents less willing to invest and thus x^*_{t+1} decreases). Therefore,

$$\begin{aligned}\frac{\partial}{\partial x'_1}R_t(x'_t;x_1,\cdots,x'_t,x^*_{t+1},\cdots,x^*_N) \leqslant & \delta\frac{d}{dx'_t}\int_{x^*_{t+1}}^{\infty}E[r+\hat{a}_t \mid \hat{x}_{(t+1)i}]f(\hat{x}_{(t+1)i}\mid x'_t)d\hat{x}_{(t+1)i} \\ & +\delta\frac{\partial}{\partial x'_t}\int_{-\infty}^{x^*_{t+1}}E[r+\hat{a}_t \mid \hat{x}_{(t+1)i}]f(\hat{x}_{(t+1)i}\mid x'_t)d\hat{x}_{(t+1)i} \\ = & \delta\frac{\partial}{\partial x'_t}\int_{-\infty}^{\infty}E[r+\hat{a}_t \mid \hat{x}_{(t+1)i}]f(\hat{x}_{(t+1)i}\mid x'_t)d\hat{x}_{(t+1)i} \\ = & \delta\frac{\partial}{\partial x'_t}E[r+\hat{a}_t \mid x'_t].\end{aligned}$$

Hence we conclude that

$$\frac{\partial}{\partial x'_t}\Delta_t(x'_t;(x_1,\cdots,x'_t,x^*_{t+1},\cdots,x^*_N)) \geqslant (1-\delta)\frac{\partial}{\partial x'_t}E[U(r,\hat{a}_N)|x'_t] > 0.$$

Computation in Proposition 5 The following lemma establishes the monotonicity of $\Delta_t(\hat{x}_{ti};(x_1,\cdots,x_N))$ in agent i's current belief $\hat{x}_{ti}$, given any $(x_1,\cdots,x_N)$ and t.

Lemma 3. *Let $(x_1,x_2,\cdots,x_N)$ be an arbitrary threshold strategy profile. The expected continuation payoff $R_t(\hat{x}_{ti};x_1,x_2,\cdots,x_N)$ for agent i with $\hat{x}_{ti}$ at time t satisfies*

$$\frac{d}{d\hat{x}_{ti}}R_t(\hat{x}_{ti};x_1,x_2,\cdots,x_N) < \delta_t\frac{d}{d\hat{x}_{ti}}E[U(r,\hat{a}_N)\mid \hat{x}_{ti}],$$

for $t=1,2,\cdots,N-1$ and any $\hat{x}_{ti}$.

Proof. Note that agents can at most invest once, so it suffices to show that the derivative of each integrand in $R_t(\hat{x}_{ti};\{x_t\}_t)$ satisfies the property stated in the Lemma. For example for its first term concerning $t+1$,

$$
\begin{aligned}
&\frac{d}{d\hat{x}_{ti}}\delta_{t+1}\int_{-\infty}^{\infty} E[U(r,\hat{a}_N) \mid \hat{x}_{(t+1)i}] f(\hat{x}_{(t+1)i} \mid \hat{x}_{ti}) \mathbf{1}_{\hat{x}_{(t+1)i}\geqslant x_{t+1}} d\hat{x}_{(t+1)i}\\
\leqslant&\frac{d}{d\hat{x}_{ti}}\delta_{t+1}\int_{-\infty}^{\infty} E[U(r,\hat{a}_N) \mid \hat{x}_{(t+1)i}] f(\hat{x}_{(t+1)i} \mid \hat{x}_{ti}) d\hat{x}_{(t+1)i}\\
=&\frac{d}{d\hat{x}_{ti}}\delta_{t+1}E[U(r,\hat{a}_N) \mid \hat{x}_{ti}].
\end{aligned}
$$

Therefore, $R_t'(\hat{x}_{ti}; x_1, \cdots, x_N) \leqslant \max\{\delta_{t+1}(E[U(r,\hat{a}_N) \mid \hat{x}_{ti}])', \cdots, \delta_N(E[U(r,\hat{a}_N) \mid \hat{x}_{ti}])'\} < \delta_t(E[U(r,\hat{a}_N) \mid \hat{x}_{ti}])'$. *Q.E.D.*

Proof of Proposition 6 For (i), recall

$$
E[U(r,\hat{a}_N)|x_N^*] = 0,
$$
$$
E[U(r,a_1)|x_{st}^*] = 0.
$$

Since $\hat{a}_N(x_N^*) > a_1(x_N^*)$ and $U(x,\hat{a}_N)$ increase in both elements, by contradiction it can be proved that $x_N^* < x_{st}^*$ and $\hat{a}_N(\hat{x}_N^*) > a_1(x_{st}^*)$. The second half is as in the two-period game. That is, if $\delta = 0$, then $R_t = 0$ and thus $x_t^* = x_{st}^*$ for all $t = 1, 2, \cdots, N$. Since R_t increases in δ, when $\delta > 0$, we must have x_t^* also increase to satisfy the equilibrium condition of period t.

For (ii), it suffices to confirm that $R_t(x_t^*; \{x_t^*\}_t) = 0$ at every $t = 1, 2, \cdots, N$ when learning lacks. Fix an arbitrary t and x_{1i}. Recall that

$$
\begin{aligned}
R_t(x_{1i}; \{x_t^*\}_t) = &\delta\int_{x_{t+1}^*}^{\infty} E[U(r,\hat{a}_N) \mid \hat{x}_{(t+1)i}] f(\hat{x}_{(t+1)i} \mid x_{1i}) d\hat{x}_{(t+1)i}\\
&+\delta^2\int_{x_{t+2}^*}^{\infty}\int_{-\infty}^{x_{t+1}^*} E[U(r,\hat{a}_N) \mid \hat{x}_{(t+2)i}] f(\hat{x}_{(t+2)i}, \hat{x}_{(t+1)i} \mid x_{1i}) d\hat{x}_{(t+1)i} d\hat{x}_{(t+2)i}\\
&+\cdots\\
&+\delta^{N-t}\int_{x_N^*}^{\infty}\int_{-\infty}^{x_{N-1}^*}\cdots\int_{-\infty}^{x_{t+1}^*} E[U(r,\hat{a}_N) \mid \hat{x}_{Ni}] f(\hat{x}_{Ni}, \cdots, \hat{x}_{(t+1)i} \mid x_{1i}) d\hat{x}_{(t+1)i}\cdots d\hat{x}_{Ni}.
\end{aligned}
$$

Since agent i holds constant belief x_{1i}, each integrant is mutually exclusive; therefore, only one integrant remains, so

$$R_t(x_{1i}; \{x_t^*\}_t) = \delta^k E[U(r, \hat{a}_N) \mid x_{1i}],$$

for some $k \in \{1, 2, \cdots, N-t\}$. To pin down x_t^*, it is required that

$$E[U(r, \hat{a}_N)|x_t^*] = R_t(x_t^*; \{x_t^*\}_t) = \delta^k E[U(r, \hat{a}_N) \mid x_t^*].$$

If $E[U(r, \hat{a}_N)|x_t^*] \neq 0$, the two sides can never be equal, so $E[U(r, \hat{a}_N)|x_t^*] = 0$.

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