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"Preference Aggregation with a Robust Pareto Criterion"

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Abstract

We present a new Pareto criterion to provide a minimal guidance to a social planner, who is concerned by the robustness of social decision with respect to imprecise beliefs of the true probability distribution over the state space. This new criterion, the obvious belief-free Pareto criterion, implies that the social planner is necessarily ambiguity averse. We show that given the set of reasonable beliefs and the set of individual risk preferences, the obvious belief-free Pareto criterion is the only axiom needed to characterize Maxmin Expected Utility social preferences. This result further brings us a preference aggregation theorem for Subjective Expected Utility individuals: A Maxmin Expected Utility social planner can linearly aggregate individual tastes and beliefs simultaneously if and only if s/he respects the obvious belief-free Pareto criterion.

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1 Introduction

This study proposes a robust Pareto criterion for social decision in the presence of imprecise beliefs. The decision maker is a social planner, who has complete knowledge of the state space, but only has imprecise knowledge of the probability distribution over it. Suppose the imprecise knowledge of probability distribution can be represented by an exogenously given possible scenario set, called the reasonable belief set. Furthermore, suppose the social planner completely understands the risk preferences of all individuals in the society. Equipped with these two sets of data, how should a social planner evaluate a social alternative?

The new Pareto criterion, which we dub the obvious belief-free Pareto criterion, is intended to provide a minimal guidance for social decision. The obvious belief-free Pareto criterion tells us that the attractiveness of a social alternative depends on the underlying scenarios: Given two social alternatives f and g, a social planner respecting the obvious belief-free Pareto criterion would conclude that f is better than g if some possible scenario after choosing g is outperformed by all possible scenarios, hence by the worst scenario, after choosing f, with the performance of a social alternative evaluated by all individual risk preferences. In this sense, a social planner who respects the obvious belief-free Pareto criterion behaves similar to an ambiguity averse decision maker à la Gilboa and Schmeidler (1989)[11]. We formally show that a social planner respecting the obvious belief-free Pareto criterion is necessarily ambiguity averse in the sense of Epstein (1999)[6].

The obvious belief-free Pareto criterion reduces to the standard Pareto principle in von Neumann-Morgenstern (vNM)'s Expected Utility setting when only constant acts are under consideration, thus is an alternative way to strengthen the standard Pareto principle.

The virtue of the obvious belief-free Pareto criterion can be clearly illustrated if one persues a functional representation of social preferences. The main contribution of this paper is to establish a social preference representation result in the Maxmin Expected Utility (MEU) environment, with the obvious belief-free Pareto criterion being the only axiom.

To be exact, in order to capture a cautious social planner when probability distribution over the state space is imprecise, we further assume that the social planner's preference relation can be represented by an MEU functional. Given the set of reasonable beliefs and the set of individual risk preferences, we show that the obvious belief-free Pareto criterion is equivalent to the following two conditions: (i) The social planner's risk preference is a linear aggregation of individual risk preferences; (ii) The social planner's subjective belief set and the exogenously given reasonable belief set coincide. Our representation result makes it possible to treat the social planner's MEU functional as a social welfare function and to compare two social alternatives by direct evaluation.

If we restrict our attention to preference aggregation problems, the above characterization of social preferences would bring us a new preference aggregation theorem. Assume that all individuals are Subjective Expected Utility (SEU) maximizers and the reasonable belief set is the convex hull of all individual beliefs, then our main result implies that one can aggregate these individual preferences into an MEU social preference: An MEU social planner respecting the obvious belief-free Pareto criterion is able to linearly aggregate both individual tastes and individual beliefs.

We shall discuss the related literature for the rest of this section. This study is mainly related to two strands of literature: studies on preference aggregation and studies on characterizing alternative Pareto criteria.

In the literature on preference aggregation, it is known that Harsanyi's aggregation theorem cannot be extended directly to SEU setting. (See, for example, Mongin 2016[14].) Positive aggregation results are established in many studies by modifying the Pareto principle. Our SEU-MEU aggregation result can be treated as one of them and we briefly review the related literature along this line in Section 6.1.

Among preference aggregation studies, Alon and Gayer (2016)[1] is closely related to our paper. They modify the standard Pareto principle into two weaker conditions, Lottery Pareto and Likelihood Pareto. Alon and Gayer's (2016)[1] main result tells us that these two conditions are necessary and sufficient conditions for aggregation of SEU individual preferences, with individual beliefs partially aggregated: While an MEU social planner respecting Lottery Pareto and Likelihood Pareto can aggregate individual vNM utility functions in a utilitarian style, her/his subjective belief set only has to be a subset of the convex hull of individual beliefs.

Given this result, can one further establish exactly the same aggregation result as ours, that is, can one ensure that the social planner's subjective belief set and the convex hull of individual beliefs coincide? Alon and Gayer (2016)[1] show that to do this, one needs an additional axiom, Social Ambiguity Avoidance, which has the same ambiguity averse flavor as our obvious belief-free Pareto criterion. Hence, focusing on this SEU-MEU preference aggregation problem, one can treat the obvious belief-free Pareto criterion as a new and simple re-interpretation of Alon and Gayer's (2016)[1] whole axiom set. However, we also argue that our social planner's attitude towards individual subjective beliefs is distinct from theirs: While their social planner has to identify the owner of each subjective belief, ours treats individual beliefs anonymously. Detailed discussions on Alon and Gayer (2016)[1] can be found in Section 5 and Section 6.1, where we also show that their axiom set is not equivalent to the obvious belief-free Pareto criterion in general.

In the literature on characterizing alternative Pareto criteria, several criteria have been suggested to capture various concerns about social decision making. Gilboa, Samuelson and Schmeidler (2014)[10] offer a refinement of the standard Pareto principle, which they call the no-betting Pareto dominance. Their main message is that a social planner respecting the no-betting Pareto dominance criterion will not encourage voluntary trade if it can not be rationalized by any common belief. In Section 6.2, we show that the obvious belief-free Pareto criterion has a similar implication as the no-betting Pareto dominance criterion. When MEU social planner assumption is imposed, a "risky" trade will never be strictly preferred by a social planner who respects the obvious belief-free Pareto criterion. As a result, both a social planner respecting the no-betting Pareto dominance criterion and a social planner respecting the obvious belief-free Pareto criterion would not encourage a voluntary trade from a fully insured allocation to some other one, if the incentive to trade only comes from difference in subjective beliefs.

Brunnermeier, Simsek and Xiong (2014)[3] ask what social welfare criterion should be adopted if the social planner knows that individual beliefs are distorted, but does not know the objective probability distribution over the state space. Their belief-neutral Pareto efficiency criterion tells us that a social planner should then take the convex hull of all individual beliefs as the reasonable belief set, and conclude that one social alternative is better than another if the former Pareto dominates the latter under every reasonable belief. In Section 6.2, we show that although the implicit presumptions under the belief-neutral Pareto efficiency criterion and the obvious belief-free Pareto criterion are conceptually different, these two criteria still have several features in common.

The unanimity Pareto dominance criterion of Gayer et al. (2014)[7] essentially requires the social planner to conclude that a social alternative is better than another if the former Pareto dominates the latter under every individual's belief. Some comparison of the obvious belief-free Pareto criterion with the no-betting Pareto dominance criterion and the belief-neutral Pareto efficiency criterion applies to the unanimity Pareto dominance criterion. We also discuss some possible relations between the obvious belief-free Pareto criterion and the unanimity Pareto dominance criterion in Section 6.2.

We start by collecting the preliminaries of the model in Section 2. The definition of the obvious belief-free Pareto criterion is proposed in Section 3, where we also show that this criterion implies that the social planner is ambiguity averse in the sense of Epstein (1999)[6]. In Section 4, we provide the main representation theorem. A leading application of our new Pareto criterion, a preference aggregation result, is established in Section 5. Related literature is discussed in detail in Section 6.

2 Preliminaries

There are I individuals in the society we consider. Denote the set of all individuals by $\mathcal{I} = \{1, ..., I\}$ and let our desicion maker (DM), the social planner, be individual 0.

Let Ω be a set of states of the world endowed with a σ-field Σ and let $\mathcal{P}$ be the set of all probability measures over Ω. Let X be a non-empty set of social outcomes and let $\Delta(X)$ stand for the set of all simple (i.e., finitely supported) lotteries on X. For $x \in X$, denote $\delta_x \in \Delta(X)$ to be the degenerated lottery that gives probability 1 to outcome x. We sometimes use x to stand for degenerated lottery δ_x.

Call a finitely valued, Σ-measurable mapping $f : \Omega \to \Delta(X)$ a simple act and let $\mathcal{F}$ be the set of all simple acts, which is the set of social alternatives. The DM is endowed with a preference relation $\succsim_0$ over $\mathcal{F}$. Let $\succ_0$ and $\sim_0$ be the asymmetric and symmetric part of $\succsim_0$ respectively.

In the natural example where X stands for the set of all possible consumption plans for individuals in the society, i.e., $X = \mathbb{R}^I$, $f : \Omega \to \Delta\left(\mathbb{R}^I\right)$ is interpreted as a state contingent randomized consumption plan, and the DM's preference relation is defined over all such plans.

The DM can access two sets of data. First, s/he completely understands individual risk preferences. That is, for any $i \in \mathcal{I}$, there is a corresponding vNM utility function $u_i : X \to \mathbb{R}$ and the DM knows it. Note that each vNM utility function u has a natural extension to $\Delta(X)$, that is, $u(p) = \mathbf{E}_p(u) = \int_X u dp$ for all $p \in \Delta(X)$. We apply this extension to all u_i's whenever needed.

Second, the DM is exogenously given a set of "reasonable" beliefs over the state space, denoted by $\Pi \subset \mathcal{P}$. At this stage, we do not impose any restriction on Π, and we do not ask how this set is derived. In interpretation, the DM is confident that the true probability distribution belongs to Π.

Remark 1. There are many candidates for the reasonable belief set Π. For instance, Π can be a collection of beliefs which do not contradict hard evidence. This may happen when Π is the set of reasonable scenarios of the world provided by government consultants who have expertise in the issue of the social decision problem under consideration. It may also be the case that Π comes directly from individual subjective beliefs when they are known to the DM: If each individual holds a unique belief over the state space, which is the classical Bayesian case, a DM who listens to all individual opinions would take the set of all individual beliefs as Π. If each individual holds a set of beliefs, a DM who values the consensus of opinions would take the intersection of all individual belief sets as Π, while on the contrary, a DM who cares about collecting all possible probability distributions would take the union of all individual belief sets as Π.

By construction, the outcomes of a simple act $f \in \mathcal{F}$ consists of finitely many simple lotteries over X. Whenever $\pi \in \mathcal{P}$ is given, π and f constitute a two-stage lottery and we can reduce them to a one-stage lottery over X. Denote $L^{\pi}(f)$ the reduced lottery of π and f, with outcomes given by the union of outcomes of all the simple lotteries in f's support and probability of each outcome given by $\Pr(x) = \sum_{\omega \in \Omega} \pi(\omega) f(\omega)(x)$.

Denote $\mathbf{L}^{\Pi} = \{L^{\pi}(f) \mid f \in \mathcal{F},\ \pi \in \Pi\}$ to be the set of all reduced lotteries derived from Π and $\mathcal{F}$, which is a subset of $\Delta(X)$. Note that by the linearity of expectation operation, $u_i(L^{\pi}(f)) = \mathbf{E}_{\pi}(u_i(f))$ for all $i \in \mathcal{I}$. Moreover, since our DM can access individual risk preference data, s/he is able to rank any pair of reduced lotteries in $\mathbf{L}^{\Pi}$ from any individual's perspective.

3 The Obvious Belief-Free Pareto Criterion

We aim to provide a new Pareto criterion to guide our DM's decision making, with the environment given in Section 2 being the primitive. For this purpose, we first recall the standard Pareto dominance on the set of reduced lotteries.

Definition 1. (Dominance of Reduced Lotteries) For any $f, g \in \mathcal{F}$ and any $\pi, \pi' \in \mathcal{P}$, say $L^{\pi}(f)$ weakly dominates $L^{\pi'}(g)$ if $u_i(L^{\pi}(f)) \geq u_i\left(L^{\pi'}(g)\right)$ for all $i \in \mathcal{I}$. Say $L^{\pi}(f)$ strictly dominates $L^{\pi'}(g)$ if $u_i(L^{\pi}(f)) > u_i\left(L^{\pi'}(g)\right)$ for all $i \in \mathcal{I}$.

Since the DM thinks reasonable beliefs are contained in Π, if s/he seeks benefits of the whole society, s/he would prefer an act f to another act g if

f dominates g no matter which reasonable belief turns out to be true. Thus we propose the next new Pareto criterion.

Definition 2. (The Obvious Belief-Free Pareto Criterion) The DM's preference relation $\succsim_0$ respects the obvious belief-free Pareto criterion if the following two conditions hold: (i) For any $f, g \in \mathcal{F}$, if there exists $\pi' \in \Pi$ such that $L^{\pi}(f)$ weakly dominates $L^{\pi'}(g)$ for all $\pi \in \Pi$, then $f \succsim_0 g$; (ii) For any $f, g \in \mathcal{F}$, if there exists $\pi' \in \Pi$ such that $L^{\pi}(f)$ strictly dominates $L^{\pi'}(g)$ for all $\pi \in \Pi$, then $f \succ_0 g$.

Given $f, g \in \mathcal{F}$, if the DM can find a $\pi' \in \Pi$ such that $L^{\pi}(f)$ dominates $L^{\pi'}(g)$ for all $\pi \in \Pi$, then s/he is sure that one scenario (π') after choosing g is outperformed by all scenarios, hence by the worst scenario, in Π after choosing f. Therefore, one possible interpretation is that a DM respecting the obvious belief-free Pareto criterion is concerned by the worst case scenario thus exhibits a kind of ambiguity aversion described by Gilboa and Schmeidler (1989)[11].

However, we do not impose any explicit structure on the DM's preference relation. In particular, the DM's preference relation might not conform with the Gilboa and Schmeidler (1989)[11] model. Therefore, it is a non-trivial question to ask whether a DM respecting the obvious belief-free Pareto criterion is indeed ambiguity averse in some sense. We shall show that the answer to this question is yes.

Following the classical way to define ambiguity aversion proposed by Epstein (1999)[6] and Ghirardato and Marinacci (2002)[8], in order to define an absolute version of ambiguity aversion, we first define comparative ambiguity aversion and then provide a class of benchmark preferences which are intuitively ambiguity neutral. To do this, assume the following minimal agreement condition.

Assumption 1 (Minimal Agreement)**.** There exist $x^*, x_* \in X$ such that $u_i(x^*) > u_i(x_*)$ for all $i \in \mathcal{I}$.

We assume that Assumption 1 holds throughout the rest of this paper. Since individual vNM utility function u_i's are unique up to positive linear transformations, we will from now on normalize them so that $u_i(x^*) = 1$ and $u_i(x_*) = 0$ for all $i \in \mathcal{I}$.

For an event $E \in \Sigma$ and two lotteries $p, q \in \Delta(X)$, denote pEq to be the act that gives p in states in E and gives q otherwise. The acts under consideration here are binary acts which have the form x^*Ex_* for $E \in \Sigma$, where x^* and x_* are outcomes specified by the minimal agreement condition. The next observation would be useful: For all $\pi \in \Pi$ and all $E \in \Sigma$, the reduced lottery of π and x^*Ex_* is a lottery that gives x^* with probability $\pi(E)$ and gives x_* with probability $(1 - \pi(E))$. Hence for all $i \in \mathcal{I}$, $u_i(L^\pi(x^*Ex_*)) = \mathbf{E}_\pi(u_i(x^*Ex_*)) = \pi(E)$.

Let $\mathcal{E} := \{E \in \Sigma : \pi(E) = \pi'(E)\ \forall \pi, \pi' \in \Pi\}$. This is the set of events whose probabilities are agreed on by all reasonable beliefs, hence it is natural to call $\mathcal{E}$ the set of *unambiguous events*.[1] Call x^*Ex_* an *unambiguous act* if $E \in \mathcal{E}$.

Given any pair of preference relations $\succsim$ and $\succsim'$, say $\succsim'$ is *more ambiguity averse than* $\succsim$ if for any unambiguous event $E \in \mathcal{E}$ and any event $F \in \Sigma$

$$x^*Ex_* \succsim (\text{resp. } \succ) x^*Fx_* \Rightarrow x^*Ex_* \succsim' (\text{resp. } \succ') x^*Fx_* \tag{1}$$

The next step is to provide a class of ambiguity neutral benchmark preferences. Following Epstein (1999)[6], we choose the set of probabilistically sophisticated preferences as the benchmark. Let $\succsim^{(\mu,W)}$ be the preferene relation of some probabilistically sophisticated individual, where $\mu \in \Delta(\Omega)$ is a probability measure over the state space and $W : \Delta(X) \to \mathbb{R}$ is a real function of the set of simple lotteries. When evaluating an act $f \in \mathcal{F}$, this individual first calculates the reduced lottery $L^\mu(f)$, then concludes that her/his utility is $W(L^\mu(f))$. As usual, we assume that W satisfies monotonicity: W is strictly increasing with respect to the first-order stochastic

[1]Epstein's (1999)[6] unambiguous event set is exogenously given thus is not defined by probabilities. However, when considering multiple-priors (MEU) preferences, where there is a set of probability measures that the DM believes, he argues that a natural way to model the nature of unambiguous events is to require that all measures in the DM's subjective belief set agree when restricted to the unambiguous event set. (See Section 3.2 of Epstein (1999)[6].) In our model, the DM starts from the reasonable belief set, thus we define the set of unambiguous events by the reasonable beliefs in the same spirit.

dominance.

Say a preference relation $\succsim$ is *ambiguity averse* if there exists a probabilistically sophisticated preferene relation $\succsim^{(\mu,W)}$ such that $\succsim$ is more ambiguity averse than $\succsim^{(\mu,W)}$. Similarly, say a preference relation $\succsim$ is *ambiguity loving* if there exists a probabilistically sophisticated preferene relation $\succsim^{(\nu,V)}$ such that for any unambiguous event $E \in \mathcal{E}$ and any event $F \in \Sigma$

$$x^* E x_* \precsim^{(\nu,V)} (\text{resp. } \prec^{(\nu,V)}) x^* F x_* \Rightarrow x^* E x_* \precsim (\text{resp. } \prec) x^* F x_* \qquad (2)$$

Finally, define a preference relation $\succsim$ to be *ambiguity neutral* if it is both ambiguity averse and ambiguity loving.

The next Lemma 1 shows that this definition of ambiguity neutrality is internal consistent. To state the lemma, say the set of unambiguous events $\mathcal{E}$ is rich under preference relation $\succsim$ if $x^* \succ x_*$ and if for every event $\bar{F} \subset F \in \Sigma$ and $E \in \mathcal{E}$ satisfying $x^* E x_* \sim x^* F x_*$, there exists a set $\bar{E} \in \mathcal{E}$, $\bar{E} \subset E$ such that $x^* \bar{E} x_* \sim x^* \bar{F} x_*$.

In Epstein (1999)[6], an unambiguous act is an act that is measurable to a λ-system of unambiguous events which are given exogenously, and $\succsim'$ is more ambiguity averse than $\succsim$ if $\succsim'$ prefers an unambiguous act whenever $\succsim$ does. We define the set of unambiguous events $\mathcal{E}$ using probabilities and one can readily confirm that $\mathcal{E}$ is a λ-system. It follows that Epstein's (1999)[6] proof also works in our setting, so we put the proof of Lemma 1 in Appendix A only for completeness.

Lemma 1. If a preference relation $\succsim$ is probabilistically sophisticated, then it is ambiguity neutral. If the set of unambiguous events $\mathcal{E}$ is rich under preference relation $\succsim$, then ambiguity neutrality of $\succsim$ implies that it is probabilistically sophisticated.

Proof. See Appendix A. □

We shall prove that a DM respecting the obvious belief-free Pareto criterion is ambiguity averse in the above sense. In fact, we can find some probabilistically sophisticated preference relation $\succsim^{(\mu,W)}$ that supports $\succsim_0$ in the sense of (1).

Proposition 1. The DM's preference relation $\succsim_0$ respecting the obvious belief-free Pareto criterion is ambiguity averse.

Proof. Consider a probabilistically sophisticated individual who ranks x^* higher than x_* and whose belief is in the set Π. Let $\succsim^{(\pi,W)}$ be her/his preference relation.

We shall show that $\succsim_0$ is more ambiguity averse than $\succsim^{(\pi,W)}$. For this purpose, pick any $E \in \mathcal{E}$ and any $F \in \Sigma$ and suppose that $x^*Ex_* \succsim^{(\pi,W)} x^*Fx_*$. By the monotonicity property of W, we have $\pi(E) \geq \pi(F)$. Since $E \in \mathcal{E}$, for any $\pi' \in \Pi$, we have $\pi'(E) = \pi(E) \geq \pi(F)$, which implies that for all $i \in \mathcal{I}$, $u_i\left(L^{\pi'}(x^*Ex_*)\right) \geq u_i\left(L^{\pi}(x^*Fx_*)\right)$ holds for any $\pi' \in \Pi$. Thus we can conclude that for any $\pi' \in \Pi$, $L^{\pi'}(x^*Ex_*)$ weakly dominates $L^{\pi}(x^*Fx_*)$. By the obvious belief-free Pareto criterion, $x^*Ex_* \succsim_0 x^*Fx_*$.

The strict part, $x^*Ex_* \succ^{(\pi,W)} x^*Fx_*$ implies $x^*Ex_* \succ_0 x^*Fx_*$, follows from an analogous argument. $\square$

We next state an implication of the obvious belief-free Pareto criterion on the set of constant acts. Call $f_c \in \mathcal{F}$ a constant act if f_c maps every state to one particular lottery. To state formally, $f_c \in \mathcal{F}$ is a constant act if there exists $p \in \Delta(X)$ such that for all $\omega \in \Omega$, $f_c(\omega) = p$. Denote the set of all constant acts by $\mathcal{F}_c$. Given such a constant act $f_c \in \mathcal{F}_c$, the reduced lottery of any $\pi \in \Pi$ and f_c is p, which implies that for all $i \in \mathcal{I}$, $u_i(L^{\pi}(f_c)) = u_i(p)$ for all $\pi \in \Pi$. Thus the knowledge of Π does not affect the evaluation of constant acts.

With the observation above, the restriction of Definition 2 to constant acts reduces to the standard Pareto principle in the vNM setting[2] : For any $f_c \in \mathcal{F}_c$ and $g_c \in \mathcal{F}_c$ such that f_c always gives $p \in \Delta(X)$ and g_c always gives $q \in \Delta(X)$, if $u_i(p) \geq u_i(q)$ (resp. $u_i(p) > u_i(q)$) for all $i \in \mathcal{I}$, then $f_c \succsim_0 g_c$ (resp. $f_c \succ_0 g_c$).

We close this section by a remark on the set-up of this paper. Our DM starts from two sets of data: individual risk preferences and reasonable beliefs. These two sets of data are given separately, but this separation is

[2]The environment where each individual's preference domain is the set of all (simple) objective lotteries, i.e., $\Delta(X)$.

inessential for the results reported so far.[3] However, this separation is valuable since under this assumption, the obvious belief-free Pareto criterion makes a utilitarian type aggregation result possible if the DM's preference relation $\succsim_0$ can be represented by an MEU functional, as we shall show in the next section.

4 Main Result

In this section, we establish a representation theorem for the DM's preference relation when it can be represented by an MEU functional.[4] The only axiom we use to characterize this social welfare function is the obvious belief-free Pareto criterion.

Now, suppose $\succsim_0$ is represented by an MEU functional. That is, we assume that there exist a utility function $u_0 : \Delta(X) \to \mathbb{R}$ and a unique non-empty, convex, closed[5] set of countably additive, non-atomic subjective probabilities over Ω, denoted by P_0, such that $\succsim_0$ is represented by

$$U_0(f) = \min_{p \in P_0} \mathbf{E}_p(u_0(f)), \quad \forall f \in \mathcal{F}$$

Remark 2. We are interested in the situations where the true probability

[3]For example, one can assume that probabilistically sophisticated individuals constitute the society. Redefining Definition 1 using these probabilistically sophisticated preferences and Definition 2 using the probability measures that represent these probabilistically sophisticated preferences, one may find that the argument in this section still works, since in the proof for this section, only the monotonicity property of probabilistically sophisticated preferences is used.

[4]Of course there are social preferences which respect the obvious belief-free Pareto criterion but are not MEU. For example, let $\Omega = \{\omega_1, \omega_2\}$, $X = \{1, 2\}$, $\Pi = \{\pi_1 = (1/4, 3/4), \pi_2 = (2/3, 1/3)\}$. Suppose there are two individuals, 1 and 2, and let $u_i(x) = 1 + \ln(x)/\ln(2)$ for $i = 0, 1, 2$. Consider $f_1 = (2, 2)$, $f_2 = (1, 2)$, $f_3 = (2, 1)$, $f_4 = (1, 1)$. One can confirm that f_k obvious belief-free Pareto dominates f_j for $j \geq k$ and that this ranking is respected by an ambiguity averse social planner with utility function $U_0(f) = (1/2)\phi_0(\mathbf{E}_{\pi_1}(u_0(f))) + (1/2)\phi_0(\mathbf{E}_{\pi_2}(u_0(f)))$, where $\phi_0(y) = (1 - \exp(-y))/(1 - \exp(-1))$. This is a smooth ambiguity utility function (see Klibanoff, Marinacci and Mukerji 2005 [13]), thus this social planner's preference exhibits smooth ambiguity aversion. However, the purpose of this paper is to illustrate the virtue of the obvious belief-free Pareto criterion, not to specify the whole class of social preferences that respect this criterion.

[5]In weak* topology.

distribution over the state space is imprecise, that is, the social planner faces ambiguity when s/he has to choose a social alternative. We use MEU social preferences here to characterize a cautious DM, which is justified by Proposition 1. As also noted in Alon and Gayer (2016)[1], caution plays a role when the DM is making influential, especially irreversible social decisions. While we take MEU social preference as a primitive, a similar characterization can be found in Danan et al. (2016)[4], where they assume Bewley incomplete social preference (Bewley 2002[2]). See Alon and Gayer (2016)[1], Danan et al. (2016)[4] for further discussions on the ambiguity averse DM assumption.

We will focus on the case where the reasonable belief set is the convex hull of a finite number of countably additive, non-atomic probability measures. So let $\Pi = co\left(\{\pi_1, \cdots, \pi_K\}\right)$, where $\pi_k \in \mathcal{P}$ for $k = 1, ..., K$.

The following representation theorem shows that in this environment, the obvious belief-free Pareto criterion is the only axiom we need to characterize an MEU social preference.

Theorem 1. The DM's preference relation $\succsim_0$ respects the obvious belief-free Pareto criterion if and only if there exists a non-zero social weight vector $\{\lambda_i\}_{i=1}^I \geq 0$ such that $u_0 = \sum_{i\in\mathcal{I}} \lambda_i u_i$ and $P_0 = \Pi$.

Proof. We only prove the if part in the main text. The proof of the only if part is technically involved thus is left for Appendix B.

If part: Suppose there exists a non-zero social weight vector $\{\lambda_i\}_{i=1}^I \geq 0$ such that $u_0 = \sum_{i\in\mathcal{I}} \lambda_i u_i$ and $P_0 = \Pi$. Given any $f \in \mathcal{F}$, the DM's MEU evaluation of f is $U_0(f) = \min_{\pi\in\Pi} \mathbf{E}_\pi\left(u_0(f)\right)$. Note that for each $\pi \in \Pi$, the expectation $\mathbf{E}_\pi\left(u_0(f)\right)$ only depends on the reduced lottery of π and f, thus $\mathbf{E}_\pi\left(u_0(f)\right) = u_0\left(L^\pi(f)\right)$.

To show that the DM's preference relation respects the obvious belief-free Pareto criterion, we only have to show that condition (i) and condition (ii) in Definition 2 hold. To show condition (i), pick $f, g \in \mathcal{F}$ and suppose there exists $\pi' \in \Pi$ such that $L^\pi(f)$ weakly dominates $L^{\pi'}(g)$ for all $\pi \in \Pi$. That is, for all $\pi \in \Pi$, $u_i(L^\pi(f)) \geq u_i\left(L^{\pi'}(g)\right)$ for all $i \in \mathcal{I}$. Then we have

$$\begin{aligned} U_0(f) &= \min_{\pi\in\Pi} \mathbf{E}_\pi(u_0(f)) = \min_{\pi\in\Pi} u_0(L^\pi(f)) = \min_{\pi\in\Pi} \sum_{i\in\mathcal{I}} \lambda_i u_i(L^\pi(f)) \\ &\geq \sum_{i\in\mathcal{I}} \lambda_i u_i\left(L^{\pi'}(g)\right) \geq \min_{\pi\in\Pi} \sum_{i\in\mathcal{I}} \lambda_i u_i(L^\pi(g)) = \min_{\pi\in\Pi} u_0(L^\pi(g)) = U_0(g). \end{aligned}$$

Thus $f \succsim_0 g$.

To show condition (ii), pick $f, g \in \mathcal{F}$ and suppose there exists $\pi' \in \Pi$ such that $L^\pi(f)$ strictly dominates $L^{\pi'}(g)$ for all $\pi \in \Pi$. That is, for all $\pi \in \Pi$, $u_i(L^\pi(f)) > u_i\left(L^{\pi'}(g)\right)$ for all $i \in \mathcal{I}$. Let $\mathcal{I}_> := \{i \in \mathcal{I} : \lambda_i > 0\}$. That is, $\mathcal{I}_>$ is the set of individuals with positive social weights. Note that i's risk preference affects the DM's utility only when $i \in \mathcal{I}_>$. Then, similar to the proof of condition (i), we have

$$\begin{aligned} U_0(f) &= \min_{\pi\in\Pi} \sum_{i\in\mathcal{I}} \lambda_i u_i(L^\pi(f)) = \min_{\pi\in\Pi} \sum_{i\in\mathcal{I}_>} \lambda_i u_i(L^\pi(f)) > \sum_{i\in\mathcal{I}_>} \lambda_i u_i\left(L^{\pi'}(g)\right) \\ &\geq \min_{\pi\in\Pi} \sum_{i\in\mathcal{I}_>} \lambda_i u_i(L^\pi(g)) = \min_{\pi\in\Pi} \sum_{i\in\mathcal{I}} \lambda_i u_i(L^\pi(g)) = U_0(g). \end{aligned}$$

Thus $f \succ_0 g$. □

The idea for the proof of the only if part of Theorem 1 can be described as following. To show that the DM's preference relation respecting the obvious belief-free Pareto criterion implies linear aggregation of individual risk preferences, it is enough to recall that the obvious belief-free Pareto criterion reduces to the standard Pareto principle when considering constant acts. Then, Harsanyi's theorem (Harsanyi 1955[12], De Meyer and Mongin 1995[5]) applies. The obvious belief-free Pareto criterion implying $P_0 = \Pi$ is proved by seeking a contradiction: Assuming that either direction of the two inclusions does not hold will make it possible to construct two acts contradicting the obvious belief-free Pareto criterion by applying a separation result of convex sets.

By Theorem 1, our DM uses a utilitarian way to aggregate individual risk preferences by assigning social weight λ_i to individual i. Also, if the DM believes that the obvious belief-free Pareto criterion must be respected, then her/his subjective belief set must coincide with the set of reasonable beliefs.

5 Application: Aggregating SEU Individual Preferences

In this section, we restrict attention to preference aggregation problems, assuming that all the individuals in the society are SEU maximizers. In fact, a preference aggregation result can be established as an immediate implication of Theorem 1.

To state it formally, assume that each individual i is now equipped with a subjective belief π_i in addition to her/his vNM utility function u_i. Thus individual i can rank simple acts directly by evaluating its expected utility, where the expectation is taken with respect to her/his own belief π_i. Let the set of reasonable beliefs, Π, be the convex hull of all individual subjective beliefs, i.e., $\Pi = co\left(\{\pi_1, \cdots, \pi_I\}\right) \subset \mathcal{P}$.

The next corollay of Theorem 1 tells us that an MEU DM respecting the obvious belief-free Pareto criterion is able to linearly aggregate both individual tastes and individual beliefs.

Proposition 2. (Aggregation of SEU Individual Preferences) The DM's preference relation $\succsim_0$ respects the obvious belief-free Pareto criterion if and only if there exists a non-zero social weight vector $\{\lambda_i\}_{i=1}^{I} \geq 0$ such that $u_0 = \sum_{i\in\mathcal{I}} \lambda_i u_i$ and $P_0 = co(\{\pi_1, \cdots, \pi_I\})$.

Alon and Gayer (2016)[1] also provide a model to aggregate SEU individual preferences into an MEU social preference. In order to linearly aggregate both beliefs and tastes for SEU preferences, they modify the standard Pareto principle in two directions. We briefly review their Pareto conditions and compare them with the obvious belief-free Pareto criterion. For ease of comparison, the terminology mentioned below follows Alon and Gayer (2016)[1].[6]

[6] They use Savage acts while we do not. Definition 3 and Definition 4 are restatements

Call a partition $\{E_1, \cdots, E_M\}$ of Ω a *socially unambiguous partition* if, for all $i \in \mathcal{I}$, $\mathbf{E}_{\pi_i}(u_i(x^* E_m x_*)) = \mathbf{E}_{\pi_i}(u_i(x^* E_n x_*))$ for all $1 \leq m, n \leq M$. A *socially unambiguous act* is an act that is measurable with respect to a socially unambiguous partition.

Definition 3. (Lottery Pareto) The DM's preference relation $\succsim_0$ satisfies the Lottery Pareto condition if, for two socially unambiguous acts f and g, $\mathbf{E}_{\pi_i}(u_i(f)) > \mathbf{E}_{\pi_i}(u_i(g))$ for all $i \in \mathcal{I}$ implies $f \succ_0 g$.

Definition 4. (Likelihood Pareto) The DM's preference relation $\succsim_0$ satisfies the Likelihood Pareto condition if, for two events E and F, $\mathbf{E}_{\pi_i}(u_i(x^* E x_*)) \geq \mathbf{E}_{\pi_i}(u_i(x^* F x_*))$ for all $i \in \mathcal{I}$ implies $x^* E x_* \succsim_0 x^* F x_*$.

The above two Pareto conditions stem from the idea that Pareto principle in the SEU setting is more convincing when individual beliefs or tastes agree. The Lottery Pareto condition weaken the standard Pareto principle so that it only adopts to acts for which individual beliefs agree. The Likelihood Pareto condition, which adopts to bettings on different events, is a natural analogy of the Lottery Pareto condition when individual tastes agree.

Although the obvious belief-free Pareto criterion does not capture the idea that Pareto principle should be devided into common belief part and common taste part, it turns out that the obvious belief-free Pareto criterion implies both the Lottery Pareto condition and the Likelihood Pareto condition.

Proposition 3. If the DM's preference relation $\succsim_0$ respects the obvious belief-free Pareto criterion, then it satisfies the Lottery Pareto condition and the Likelihood Pareto condition.

Proof. First recall that for all $i \in \mathcal{I}$ and all $\pi \in \Pi$, $\mathbf{E}_{\pi}(u_i(x^* E x_*)) = \pi(E)\, u_i(x^*) + (1 - \pi(E))\, u_i(x_*) = \pi(E)$.

The obvious belief-free Pareto criterion implies the Lottery Pareto condition: Fix two socially unambiguous acts $f, g \in \mathcal{F}$ such that $\mathbf{E}_{\pi_i}(u_i(f)) > \mathbf{E}_{\pi_i}(u_i(g))$ for all $i \in \mathcal{I}$. Suppose f is measurable with respect to a socially

in our setting, but these restatements do not change the idea since outcomes of Savage act can be treated as degenerated lotteries.

unambiguous partition $\{E_1, \cdots, E_M\}$. By the definition of socially unambiguous partition, for all $i \in \mathcal{I}$, $\pi_i(E_m) = \pi_i(E_n)$ for all $1 \leq m, n \leq M$. Since every $\pi \in \Pi$ is a convex combination of $\pi_1, \cdots, \pi_I$, $\pi(E_m) = \pi(E_n)$ for all $1 \leq m, n \leq M$, all $\pi \in \Pi$. The measurability of f with respect to $\{E_1, \cdots, E_M\}$ thus implies π and f always give the same reduced lottery $L^{\pi}(f)$ for all $\pi \in \Pi$. Similarly, π and g always give the same reduced lottery $L^{\pi}(g)$ for all $\pi \in \Pi$.

By the assumption of the Lottery Pareto condition, $u_i(L^{\pi_i}(f)) = \mathbf{E}_{\pi_i}(u_i(f)) > \mathbf{E}_{\pi_i}(u_i(g)) = u_i(L^{\pi_i}(g))$ for all $i \in \mathcal{I}$. Since $L^{\pi_i}(f) = L^{\pi}(f)$, $L^{\pi_i}(g) = L^{\pi}(g)$ for all $\pi \in \Pi$, we have $u_i(L^{\pi}(f)) > u_i\left(L^{\pi'}(g)\right)$ for all $\pi, \pi' \in \Pi$ and all $i \in \mathcal{I}$. Then there exists a $\pi' \in \Pi$ such that $L^{\pi}(f)$ strictly dominates $L^{\pi'}(g)$ for all $\pi \in \Pi$. By the obvious belief-free Pareto criterion, we can conclude $f \succ_0 g$.

The obvious belief-free Pareto criterion implies the Likelihood Pareto condition: By the assumption of the Likelihood Pareto condition, for events $E, F \in \Sigma$, $\pi_i(E) = \mathbf{E}_{\pi_i}(u_i(x^* E x_*)) \geq \mathbf{E}_{\pi_i}(u_i(x^* F x_*)) = \pi_i(F)$ for all $i \in \mathcal{I}$. Pick up one individual who gives the lowest belief to event E. Without loss of generality, suppose individual 1 does so, i.e., $\pi_1(E) \in \min\{\pi_1(E), \cdots, \pi_I(E)\}$. We then have $\pi_i(E) \geq \pi_1(E) \geq \pi_1(F)$ for all $i \in \mathcal{I}$. Since each $\pi \in \Pi$ is a convex combination of $\pi_1, \cdots, \pi_I$, $\pi(E) \geq \pi_1(F)$ for all $\pi \in \Pi$. Hence for all $i \in \mathcal{I}$, $u_i(L^{\pi}(x^* E x_*)) = \mathbf{E}_{\pi}(u_i(x^* E x_*)) = \pi(E) \geq \pi_1(F) = \mathbf{E}_{\pi_1}(u_i(x^* F x_*)) = u_i(L^{\pi_1}(x^* F x_*))$ for all $\pi \in \Pi$. That is, we find $\pi_1 \in \Pi$ such that $L^{\pi}(x^* E x_*)$ weakly dominates $L^{\pi_1}(x^* F x_*)$ for all $\pi \in \Pi$. By the obvious belief-free Pareto criterion, $x^* E x_* \succsim_0 x^* F x_*$. □

A more detailed literature review on aggregating SEU individual preferences and more discussions on Alon and Gayer (2016)[1] can be found in Section 6.1.

6 Related Literature

More detailed discussions on related literature are provided in this section. Here we extensively use the MEU social preference representation given by

Theorem 1 (Proposition 2).

6.1 Aggregating SEU Individual Preferences

In this subsection, we compare our preference aggregation result (Proposition 2) with the existing literature on aggregating SEU preferences. Again, assume that each individual preference is represented by an SEU functional and that $\Pi = co(\{\pi_1, \cdots, \pi_I\})$.

Harsanyi (1955)[12] studies the role of standard Pareto principle in the vNM setting. Harsanyi's aggregation theorem states that standard Pareto principle is equivalent to the DM's vNM utility function being a linear aggregation of all those of individuals. However, it is known that Harsanyi's aggregation theorem can not be extended directly to other settings. Especially, when individual preferences are represented by SEU functionals, the following analogy of standard Pareto principle will make it impossible to linearly aggregate both individual vNM utility functions and beliefs, if the existence of a dictator is not allowed.

Definition 5. (Pareto Principle for SEU Individuals) If $\mathbf{E}_{\pi_i}(u_i(f)) \geq \mathbf{E}_{\pi_i}(u_i(g))$ for all $i \in \mathcal{I}$, then $f \succsim_0 g$.

Gilboa, Samet and Schmeidler (2004)[9] point out that, the above axiom used to derive the impossibility result for aggregation is itself counterintuitive. In the vNM setting, Pareto principle only describes the difference in tastes, while in the SEU setting, Pareto principle describes the mixture of differences in both tastes and beliefs. As a result, even if two SEU individuals have opposite beliefs and opposite tastes, it is still possible that these effects cancel out when taking expectations, so that these two individuals give the same ranking when comparing two acts, which leads to spurious unanimity.

Gilboa, Samet and Schmeidler (2004)[9] give a duel example to show how Pareto principle for SEU individuals leads to uncompelling social decisions and suggest that Pareto principle is more convincing when individual beliefs or tastes agree. Indeed, they show that when the DM also maximizes some SEU functional, it becomes possible to linearly aggregate both beliefs and tastes of SEU individuals if a condition similar to the Lottery Pareto condition, which they call "the restricted Pareto condition", holds.[7]

As also mentioned in Section 5, following Gilboa, Samet and Schmeidler's (2004)[9] idea, Alon and Gayer (2016)[1] successfully aggregate SEU individual preferences into an MEU social preference. Theorem 1 of Alon and Gayer (2016)[1] shows that the DM's utility function linearly aggregates individual ones and $P_0 \subset \Pi$ if and only if the Lottery Pareto condition and the Likelihood Pareto condition hold. The difference between our result and theirs is that in our aggregation theorem, not only $P_0 \subset \Pi$ but also the other inclusion hold. This is because the Lottery Pareto condition and the Likelihood Pareto condition are weaker than the obvious belief-free Pareto criterion, as we have shown in Proposition 3.

Alon and Gayer (2016)[1] also provide a stronger aggregation result. They show that we can further impose the following axiom which states that the social preference relation is more ambiguity averse than any individual, to ensure $P_0 = \Pi$.

Definition 6 (Social Ambiguity Avoidance)**.** Let E be a socially unambiguous event and F be any event. If $\mathbf{E}_{\pi_i}(u_i(x^*Ex_*)) > \mathbf{E}_{\pi_i}(u_i(x^*Fx_*))$ for some individual $i \in \mathcal{I}$, then $x^*Ex_* \succ_0 x^*Fx_*$.

Alon and Gayer's (2016)[1] socially unambiguous events are defined as the unions of events in a socially unambiguous partition (see Section 5), hence belong to our unambiguous event set $\mathcal{E}$. It is straightforward to verify that Social Ambiguity Avoidance condition is implied by the obvious belief-free Pareto criterion.

An interesting question is that whether Alon and Gayer's (2016)[1] axiom set (Lottery Pareto, Likelihood Pareto and Social Ambiguity Avoidance) is equivalent to the obvious belief-free Pareto criterion. This is true only in a limited sense. If we consider exactly Alon and Gayer's (2016)[1] Savage set-up or our set-up, including all the technical details, the equivalence is

[7] The restricted Pareto condition weakens the standard Pareto principle so that it only applies to acts which are measurable with respect to the set of events whose probabilities are agreed on by all individuals.

straightforward because their axiom set and the obvious belief-free Pareto criterion both give us the same aggregation result. Hence, when focusing on SEU-MEU preference aggregation problem, the obvious belief-free Pareto criterion can be treated as a simple way to reinterpret Alon and Gayer's (2016)[1] whole axiom set.

However, they are not equivalent in general. In the next example, we consider Savage acts as Alon and Gayer (2016)[1] do, assume MEU social preferences, but consider a finite state space where an additional assumption on the DM's preference[8] used in Alon and Gayer (2016)[1] does not hold. In this example, their axiom set does not imply the obvious belief-free Pareto criterion. Thus in general, there is a gap between their axiom set and the obvious belief-free Pareto criterion, and one should be careful when understanding or applying these axioms.

Example 1. There are three states of the world, represented by the state space $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Two individuals, 1 and 2, whose utility functions are u_1 and u_2, respectively. Individual 1 and 2 hold different subjective beliefs over Ω: $\pi_1(\omega_1, \omega_2, \omega_3) = (1/3, 1/3, 1/3)$, $\pi_2(\omega_1, \omega_2, \omega_3) = (0, 1/2, 1/2)$. Let the DM be an MEU maximizer with utility function u_0 and let the reasonable belief set, which is supposed to be identical to the DM's subjective belief set, be $\Pi = \{\pi_1, \pi_2\}$. Note that the only socially unambiguous event in this example is Ω. Let $X = \{1, 2, 3, 4\}$ and suppose that $u_1(4) > u_1(3) > u_1(2) > u_1(1)$, $u_2(1) > u_2(3) > u_2(2) > u_2(4)$, $u_0(4) > u_0(1) > u_0(3) > u_0(2)$.

Note that x^*, x_* in the minimal agreement condition can only be 3 and 2. We first check that the Lottery Pareto condition holds. In this example, any socially unambiguous act is a constant act. Thus there is only one pair of acts that satisfy the assumption of Lottery Pareto: 3 and 2. The rankings are respected also by the DM, so the Lottery Pareto condition holds.

Next we show that the Likelihood Pareto condition holds. Since the ranking of x^* and x_* is agreed on by all individuals and the DM, we only have to check the probabilities. For each pair of events E and F such that $\pi_i(E) \geq$

[8] The existence of some socially agreed "fair" event E such that $U_0(x^* E x_*) = 0.5u_0(x^*) + 0.5u_0(x_*)$.

$\pi_i(F)$ for $i = 1,2$, our MEU DM with subjective belief set $\Pi = \{\pi_1, \pi_2\}$ concludes that $x^*Ex_* \succsim_0 x^*Fx_*$. Thus the Likelihood Pareto condition holds.

To show that the Social Ambiguity Avoidance condition holds, note that the socially unambiguous event E in the assumption can only be Ω. For any event F, if $x^*\Omega x_* \succ_i x^*Fx_*$ for some i, we have $\pi_i(F) < 1$. Hence our MEU DM with subjective belief set $\Pi = \{\pi_1, \pi_2\}$ also concludes that $x^*\Omega x_* \succ_0 x^*Fx_*$.

Now, assume further that $u_0(x) = (x - 9/4)^2$, $u_1(x) = \ln(x)/\ln 2$, $u_2(x) = (x - 9/4)^2$ if $x \leq 3$ and $u_2(x) = -1$ otherwise. It is straightforward to check that the rankings mentioned above hold for these utility functions. Consider act $f = (f(\omega_1), f(\omega_2), f(\omega_3)) = (3,1,4)$ and $g = (g(\omega_1), g(\omega_2), g(\omega_3)) = (3,3,3)$. It can be readily confirmed that $L^{\pi_1}(g)$ strictly dominates $L^{\pi_2}(f)$, $L^{\pi_2}(g)$ strictly dominates $L^{\pi_2}(f)$. Thus if the obvious belief-free Pareto criterion holds, we have $g \succ_0 f$. However, $U_0(f) > U_0(g)$, which implies $f \succ_0 g$.

Furthermore, Alon and Gayer (2016)[1] maintain the convention that i's belief must be attached to i's risk preference. In the SEU model of individual decision making, i's belief is simply i's subjective view of the possible scenario. Thus, whether i's belief is "right" or not, it has nothing to do with j's evaluation of any alternative. To maintain this convention in a social decision making model, Alon and Gayer's (2016)[1] social planner has to identify the owner of each possible scenario.

On the other hand, similar to Brunnermeier, Simsek and Xiong (2014)[3] and Gayer et al. (2014)[7], we allow the DM to pool all the individual opinions into one reasonable belief set to capture the idea that all individual beliefs of possible scenarios should be treated anonymously. When the DM cannot pin down a single scenario, what only matters is the set of reasonable scenarios, not the owner of each one. Even in the same SEU setting, whether the DM preserves the anonymity of individual beliefs draws a distinction between our model and that of Alon and Gayer (2016)[1].

6.2 Alternative Pareto Criteria

This paper directly relates to studies on characterizing compelling alternative Pareto criteria when individuals hold heterogeneous beliefs. In this subsection, we explore the connection between the obvious belief-free Pareto criterion and other existing ones.

No-Betting Pareto Dominance: Gilboa, Samuelson and Schmeidler (2014)

Gilboa, Samuelson and Schmeidler (2014)[10] offer a refinement of the standard Pareto principle in the SEU setting. Given two acts f and g, say individual $i \in \mathcal{I}$ is involved in a trade from g to f if there exists a state $\omega \in \Omega$ where i is not indifferent between the outcomes of f and g, i.e., $u_i(f(\omega)) \neq u_i(g(\omega))$. Denote $\mathcal{I}(f,g) \subset \mathcal{I}$ the set of individuals involved in the trade from g to f. Say f is an improvement upon g if $\mathcal{I}(f,g) \neq \emptyset$ and $\mathbf{E}_{\pi_i}(u_i(f)) > \mathbf{E}_{\pi_i}(u_i(g))$ for all $i \in \mathcal{I}(f,g)$.

Now we can state the definition of no-betting Pareto dominance. The following condition (i) requires that all individuals involved in a trade strictly prefer the act after trade, according to their subjective belief. Condition (ii) requires that the voluntary trade, possibly based on heterogeneous beliefs, has to be rationalized by some common belief.

Definition 7. (No-Betting Pareto Dominance) An act f no-betting Pareto dominates g if: (i) f is an improvement upon g; (ii) there exists a probablity measure π_0 over the state space such that $\mathbf{E}_{\pi_0}(u_i(f)) > \mathbf{E}_{\pi_0}(u_i(g))$ for all $i \in \mathcal{I}(f,g)$.

Note that condition (i) in the definition of no-betting Pareto dominance is not comparable to the obvious belief-free Pareto criterion unless Π is singleton set, which corresponds to the common belief case.[9]

Although it seems difficult to compare our criterion with the no-betting Pareto dominance criterion clearly, the obvious belief-free Pareto criterion

[9] Also notice the fact that condition (i) in no-betting Pareto dominance always implies standard Pareto principle, while the obvious belief-free Pareto criterion implies standard Pareto principle only when Π is singleton set (or, when only $\mathcal{F}_c$ is under consideration).

still provides a similar insight into the social attitude towards "bet", the key concept in Gilboa, Samuelson and Schmeidler (2014)[10].

Gilboa, Samuelson and Schmeidler (2014)[10] call a feasible trade from some act f to g a bet if it satisfies two conditions: (i) Each individual involved in the trade strictly prefers g to f; (ii) For each individual who strictly prefers to trade, the initial act f fully insures her/him. Condition (i) in their definition captures the idea that unless regulated, individuals in the market would voluntarily trade from f to g.

However, shifting to the risky act g from the riskless act f is often thought of as socially undesirable, as long as the incentive to shift comes from different subjective beliefs only. Gilboa, Samuelson and Schmeidler (2014)[10] argue that such a trade should be regulated if there is no common belief that supports it, that is, when g does not dominate f under the no-betting Pareto dominance criterion.

We use the next example, which is a leading example (Example 2) of Gilboa, Samuelson and Schmeidler (2014)[10], to illustrate that a DM respecting the obvious belief-free Pareto criterion would not prefer such a trade neither.

Example 2. There are two states of the world, represented by the state space $\Omega = \{\omega_1, \omega_2\}$. Two risk neutral SEU individuals, A and B, each has one dollar whatever the state is. That is, the initial endowment is an act $f = (f_A, f_B) \in \mathcal{F}$ such that $f(\omega_1) = f(\omega_2) = (1, 1)$. However, A and B hold different subjective beliefs over Ω: $\pi_A(\omega_1, \omega_2) = (2/3, 1/3)$, $\pi_B(\omega_1, \omega_2) = (1/3, 2/3)$. Consider another act $g = (g_A, g_B) \in \mathcal{F}$ such that $g(\omega_1) = (2, 0)$ and $g(\omega_2) = (0, 2)$. Under the SEU assumption, both individuals prefer g to f. Suppose all reasonable beliefs lie between π_A and π_B, i.e., $\Pi = co(\{\pi_A, \pi_B\})$. Then it can be readily confirmed that a DM respecting the obvious belief-free Pareto criterion will never rank the act g strictly higher than f, by taking $\pi'(\omega_1, \omega_2) = (1/2, 1/2) \in \Pi$ in the definition of the obvious belief-free Pareto criterion.

In general, a similar result can be established if we further assume that the DM's preference relation can be represented by an MEU functional, i.e., social welfare function is that in Proposition 2.

To illustrate this, consider a one-good exchange economy studied in Section 3.2 of Gilboa, Samuelson and Schmeidler (2014)[10]. Let $X \subset \mathbb{R}^I$, where $x \in X$ specifies an allocation, x_i, for each individual $i \in \mathcal{I}$. Further, consider the acts that are state contingent non-randomized consumption plans, i.e., whose outcomes are degenerated lotteries over X. Individual vNM utility functions are assumed to be standard: strictly monotone and (weakly) concave. Given an act g, we say a trade from g to another act f is feasible if for each state, the aggregate consumption of f does not exceed the aggregate consumption of g. That is, we say a trade from g to f is feasible if $\sum_i f(\omega)_i \leq \sum_i g(\omega)_i$ for each $\omega \in \Omega$.

The following definition says that trading from a fully insured act to some other one is "not safe".

Definition 8 (Risky Trade)**.** A feasible trade from g to f is risky if $g(\omega)_i$ does not depend on ω for all $i \in \mathcal{I}$.

A risky trade is a feasible trade to a risky allocation from a fully insured riskless one. Unlike the way to define a bet in Gilboa, Samuelson and Schmeidler (2014)[10], in order to define a trade to be risky, it is not necessary to require that f is an improvement upon g. But these two definitions both imply that if a trade is proposed by the individuals in the market, the motivation behind it can never be risk sharing.

The next proposition shows that an MEU DM respecting the obvious belief-free Pareto criterion would never strictly prefer a risky trade, thus would never encourage such voluntary trades.

Proposition 4. Suppose that the DM is an MEU maximizer respecting the obvious belief-free Pareto criterion. If a feasible trade from g to f is risky, then $f \succ_0 g$ does not hold.

Proof. By Proposition 2, if the DM is an MEU maximizer and respects the obvious belief-free Pareto criterion, then her/his utility of any act $h \in \mathcal{F}$ is $\min_{\pi \in \Pi} \mathbf{E}_\pi \left(\sum_{i \in \mathcal{I}} \lambda_i u_i(h) \right)$.

Now, suppose $f \succ_0 g$ holds, then

$$\min_{\pi\in\Pi} \mathbf{E}_\pi \left(\sum_{i\in\mathcal{I}} \lambda_i u_i (f)\right) > \min_{\pi\in\Pi} \mathbf{E}_\pi \left(\sum_{i\in\mathcal{I}} \lambda_i u_i (g)\right) \tag{3}$$

Take any π^f that achieves the minimum of LHS and let $g(\omega) = \bar{g}$ for all $\omega \in \Omega$, we have

$$\mathbf{E}_{\pi^f} \left(\sum_{i\in\mathcal{I}} \lambda_i u_i (f)\right) > \sum_{i\in\mathcal{I}} \lambda_i u_i (\bar{g}) \tag{4}$$

since the RHS of equation (3) is unaffected by taking expectations. Note that due to the concavity of individual vNM utility functions,

$$\mathbf{E}_{\pi^f} \left(\sum_{i\in\mathcal{I}} \lambda_i u_i (f)\right) = \sum_{i\in\mathcal{I}} \lambda_i \mathbf{E}_{\pi^f} (u_i (f)) \leq \sum_{i\in\mathcal{I}} \lambda_i u_i (\mathbf{E}_{\pi^f} (f)) \tag{5}$$

Combining equation (4) and (5), we have

$$\sum_{i\in\mathcal{I}} \lambda_i u_i (\mathbf{E}_{\pi^f} (f)) > \sum_{i\in\mathcal{I}} \lambda_i u_i (\bar{g})$$

By strict monotonicity, $\mathbf{E}_{\pi^f} (f) > \bar{g}$, which further implies $\mathbf{E}_{\pi^f} (f_i) > \bar{g}_i$. Summation over i yields

$$\sum_{i\in\mathcal{I}} \mathbf{E}_{\pi^f} (f_i) = \mathbf{E}_{\pi^f} \left(\sum_{i\in\mathcal{I}} f_i\right) > \sum_{i\in\mathcal{I}} \bar{g}_i$$

which leads to a contradiction because feasibility of trade from g to f implies that for each $\omega \in \Omega$, $\sum_{i\in\mathcal{I}} f(\omega)_i \leq \sum_{i\in\mathcal{I}} \bar{g}_i$. □

Belief-Neutral Pareto Efficiency: Brunnermeier, Simsek and Xiong (2014)

Brunnermeier, Simsek and Xiong (2014)[3] provide a social welfare criterion under distorted beliefs, called belief-neutral Pareto efficiency (inefficiency). Assume that the set of social reasonable beliefs is the convex hull of beliefs of SEU individuals, that is, $\Pi = co(\{\pi_1, \cdots, \pi_I\})$, where π_i is i's subjective belief. When judging whether an act is belief-neutral Pareto efficient or not, it is sufficient to see whether the act is Pareto efficient if we first fix any belief in Π as the common belief for all individuals.

Definition 9. (Belief-Neutral Pareto Efficiency/Inefficiency) An act f is belief-neutral Pareto inefficient if for every reasonable belief $\pi \in \Pi$, there exists another act g_π such that $\mathbf{E}_\pi (u_i (f)) \leq \mathbf{E}_\pi (u_i (g_\pi))$ for all $i \in \mathcal{I}$, with strict inequality holding for at least one $i \in \mathcal{I}$. An act f is belief-neutral Pareto efficient if for every reasonable belief $\pi \in \Pi$, there does not exist such g_π.

The analysis of Brunnermeier, Simsek and Xiong (2014)[3] is based on the presumption that distorted beliefs come from distortions in updating. Thus, irrationality plays a crucial role in their study. In contrast, we require the elements of reasonable belief set Π to be derived rationally. Hence, we assert that our stance on heterogeneity of beliefs is similar to that of Morris (1995)[15].

Although our study and Brunnermeier, Simsek and Xiong (2014)[3] are conceptually different, these two studies still share some common features. First, Brunnermeier, Simsek and Xiong (2014)[3] propose belief-neutral Pareto efficiency criterion to characterize a DM who knows that individual beliefs are distorted but does not know the objective probability distribution. Thus, similar to our DM, a DM who adopts belief-neutral Pareto efficiency criterion is also concerned by the robustness of social decision with respect to imprecise probability distribution over the state space. Furthermore, under both criteria, when evaluating the utility of some act to individual i, not only i's belief but also the others' beliefs are taken into account. Since the DM has no knowledge of the "correct" belief, the way to deal with the robustness problem is to treat individual beliefs anonymously.

Secondly, both belief-neutral Pareto efficiency criterion and the obvious belief-free Pareto criterion can be interpreted as a kind of ambiguity aversion in the DM's decision process. Our obvious belief-free Pareto criterion has the following intuitive explanation. Facing a set of reasonable beliefs Π, a cautious DM can ask her/himself the following question: Suppose nature will choose a distribution from Π against me *after* I choose an act to implement, in what circumstances can I guarantee that an act f does better than an act g, if I also would like to take care of individual risk attitudes? The obvious belief-free Pareto criterion suggests that if a bad case when our DM chooses g is worse than every case, thus the worst case, when the DM chooses f, then our DM should conclude that g really does worse than f. As also pointed out in Section 3, our DM concerns the worst case, with a flavor of ambiguity aversion as the Gilboa and Schmeidler (1989)[11] model. On the other hand, a DM adopts belief-neutral Pareto efficiency criterion is ambiguity averse in a Bewley-type way (Bewley 2002[2]). That is, s/he only concludes that an act f is better than g if f does better than g under every belief in Π. This can be understood as another kind of caution when the DM's view of nature's moving time is different. A belief-neutral DM believes nature will choose a distribution against her/him *before* s/he chooses an act to implement, which makes it reasonable to say an act is better if it is better whatever nature's choice is.

Finally, an immediate implication of Harsanyi's theorem tells us that a belief-neutral DM's preference can be represented by a set of utilitarian social welfare functions, one for each reasonable belief in Π, with social weights depending on the belief chosen. This is Proposition 1 provided by Brunnermeier, Simsek and Xiong (2014)[3]. Although their social welfare function set only provides an incomplete ranking over social alternatives, this approach, as our Proposition 2, offers a direct way to compare two social alternatives.

Unanimity Pareto Dominance: Gayer et al. (2014)

Gayer et al. (2014)[7] suggest another Pareto criterion called unanimity Pareto dominance, requiring that the social planner ranks an act f above g if for every individual belief, f is a Pareto improvement over g.

Definition 10. (Unanimity Pareto Dominance) An act f unanimity Pareto dominates g if for every π_j, $j \in \mathcal{I}$, $\mathbf{E}_{\pi_j}(u_i(f)) > \mathbf{E}_{\pi_j}(u_i(g))$ for all $i \in \mathcal{I}(f,g)$.

The unanimity Pareto dominance criterion is similar to the belief-neutral

Pareto efficiency criterion because they both require Pareto dominance for every belief in a fixed belief set. Hence, the comparison in the previous subsection applies.

Also, one can immediately check that unanimity Pareto dominance implies no-betting Pareto dominance. Thus a DM respecting this criterion and a DM respecting the obvious belief-free Pareto criterion have similar attitude towards risky trades.

A direct comparison of the obvious belief-free Pareto criterion with the unanimity Pareto dominance criterion seems to be difficult. Specifically, the following facts hold when comparing these two criteria:

1. An act g may be unanimity Pareto dominated by an act f, but may not be obvious belief-free Pareto dominated by f. The unanimity Pareto dominance requires Pareto dominance of f over g for any fixed belief, which tells us nothing about the individual rankings when f and g can be attached to different worst case beliefs respectively.

2. An act g may be obvious belief-free Pareto dominated by an act f, but may not be unanimity Pareto dominated by f. The obvious belief-free Pareto criterion ensures that there exists a π' such that f constitutes a Pareto improvement over g under π', but is silent about the Pareto dominance relation between f and g under other individual beliefs.

3. There may be no act that unanimity Pareto dominates g, while g may still be obvious belief-free Pareto dominated by some f. This is because the unanimity undomination of g only tells us for any f, there is some individual belief π under which f is not a Pareto improvement over g, thus provides no clue when comparing the worst case individual expected utilities.

4. There may be no act that obvious belief-free Pareto dominates g, while g may still be unanimity Pareto dominated by some f. This is because while the obvious belief-free Pareto undomination of g implies g is no worse than any f in the worst case, it is still possible that f improves over g if we first fix a common belief.

7 Conclusion

This paper proposes a new Pareto criterion for social decision making. This new Pareto criterion, the obvious belief-free Pareto criterion, suggests that a social planner should conclude that an act f is better than another act g if the worst case performance of f is better than the performance of g under some case, thus is better than the performance of g under the worst case, with the performance of an act evaluated in view of all individuals in the society. The obvious belief-free Pareto criterion provides a minimal guidance when the social planner persues the robustness of social decision with respect to imprecise probability distribution over the state space. We formally show that this criterion implies that the social planner is ambiguity averse in the sense of Epstein (1999)[6].

We further show that the obvious belief-free Pareto criterion is the only axiom needed to characterize an MEU social preference: Given the set of individual risk preferences and the set of reasonable beliefs, an MEU social planner respecting the obvious belief-free Pareto criterion will aggregate individual risk preferences linearly and conclude that her/his subjective belief set is in fact the same as the reasonable belief set. A direct implication of this characterization is a preference aggregation result: Suppose individuals are SEU maximizers and the reasonable belief set is the convex hull of all individual subjective beliefs, then an MEU social planner respecting the obvious belief-free Pareto criterion is able to linearly aggregate both individual tastes and individual beliefs.

Appendices

A Proof for Section 3

Proof of Lemma 1. The first part follows directly from the definition of ambiguity neutrality.

To prove the second part, suppose that $\succsim$ is ambiguity neutral. Then there are probabilistically sophisticated preference relations $\succsim^{(\mu,W)}$ and $\succsim^{(\nu,V)}$ such that (1) and (2) hold. By the definition of richness of $\mathcal{E}$, $x^* \succ x_*$.

From (1), we can show that $\succsim^{(\mu,W)}$ agrees with $\succsim$ on the set of unambiguous acts. It suffices to show that the converse of (1) holds for all $E, F \in \mathcal{E}$. For any $E, F \in \mathcal{E}$, suppose that $x^*Ex_* \prec^{(\mu,W)}$ (resp. $\precsim^{(\mu,W)}$)x^*Fx_*, then by (1), we have $x^*Ex_* \prec$ (resp. $\precsim$)x^*Fx_*. Similarly, $\succsim^{(\nu,V)}$ and $\succsim$ agree on the set of unambiguous acts by (2). Thus, we know that $x^* \succ^{(\mu,W)} x_*$ and $x^* \succ^{(\nu,V)} x_*$.

By the monotonicity assumption of probabilistically sophisticated preferences, we have

$$x^*Ex_* \succsim^{(\mu,W)} (\text{resp. } \succ^{(\mu,W)}) x^*Fx_* \Leftrightarrow \mu(E) \geq (\text{resp. } >) \mu(F)$$

So we can restate the definition of ambiguity aversion of $\succsim$ as

$$\mu(E) \geq (\text{resp. } >) \mu(F) \Rightarrow x^*Ex_* \succsim (\text{resp. } \succ) x^*Fx_* \tag{6}$$

for all $E \in \mathcal{E}$ and all $F \in \Sigma$.

Similarly, the definition of ambiguity lovingness of $\succsim$ can be restated as

$$\nu(E) \leq (\text{resp. } <) \nu(F) \Rightarrow x^*Ex_* \precsim (\text{resp. } \prec) x^*Fx_* \tag{7}$$

for all $E \in \mathcal{E}$ and all $F \in \Sigma$.

Note that for all $E \in \mathcal{E}$, $F \subset E$, $G \subset E^c$, $\mu(G) \leq \mu(F) \Rightarrow \nu(G) \leq \nu(F)$.

Indeed, we have

$$\begin{aligned}\mu(G) \leq \mu(F) &\Leftrightarrow \mu(G) + \mu(E) - \mu(F) \leq \mu(F) + \mu(E) - \mu(F) \\ &\Leftrightarrow \mu(E + G - F) \leq \mu(E) \\ &\Rightarrow x^*(E + G - F)\, x_* \precsim x^* E x_* \\ &\Rightarrow \nu(E + G - F) \leq \nu(E) \\ &\Leftrightarrow \nu(G) \leq \nu(F)\end{aligned}$$

where the third line follows from (6) and the fourth line follows from (7). Apply the same argument to E^c instead of E, we have $\mu(G) \geq \mu(F) \Rightarrow \nu(G) \geq \nu(F)$. Thus, for all $E \in \mathcal{E}$, $F \subset E$, $G \subset E^c$,

$$\mu(G) \leq \mu(F) \Leftrightarrow \nu(G) \leq \nu(F) \tag{8}$$

Let U represents $\succsim$ and define $u(F) := U(x^* F x_*)$ for all $F \in \Sigma$. We first show that u is ordinally equivalent to μ and ν.

From (6), (7) and (8), we have

$$\begin{aligned}\mu(G) \leq \mu(F) &\Leftrightarrow \mu(E + G - F) \leq \mu(E) \\ &\Rightarrow u(E + G - F) \leq u(E) \\ &\Rightarrow \nu(E + G - F) \leq \nu(E) \\ &\Leftrightarrow \nu(G) \leq \nu(F) \\ &\Leftrightarrow \mu(G) \leq \mu(F)\end{aligned}$$

where the fifth line follows from (8). Hence for all $E \in \mathcal{E}$, $F \subset E$, $G \subset E^c$

$$\mu(G) \leq \mu(F) \Leftrightarrow u(E + G - F) \leq u(E) \tag{9}$$

For any $F \in \Sigma$, $F - E \cap F \subset E^c$, $E - E \cap F \subset E$. Apply (9) to get

$$\begin{aligned}u(F) \leq u(E) &\Leftrightarrow u(E + F - E \cap F - (E - E \cap F)) \leq u(E) \\ &\Leftrightarrow \mu(F - E \cap F) \leq \mu(E - E \cap F) \\ &\Leftrightarrow \mu(F) \leq \mu(E)\end{aligned}$$

for all $E \in \mathcal{E}$ and all $F \in \Sigma$.

Thus, every lower contour set (hence, every indifference curve) for u containing some unambiguous event corresponds to a lower contour set (hence, an indifference curve) for μ containing the same unambiguous event. By the richness assumption of $\mathcal{E}$, every indifference curve for u contains some unambiguous event. Therefore, $\succsim$ is ordinally equivalent to probability measure μ (and ν by (8)).

We next show that $\succsim$ and $\succsim^{(\mu,W)}$ coincide.

For each $F \in \Sigma$, there exists $E \in \mathcal{E}$ such that $x^*Ex_* \sim^{(\mu,W)} x^*Fx_*$ and $x^*Ex_* \sim^{(\nu,V)} x^*Fx_*$. To see this, use the richness of $\mathcal{E}$ to find $E \in \mathcal{E}$ such that $u(E) = u(F)$. Since u is ordinally equivalent to μ and ν, $\mu(E) = \mu(F)$ and $\nu(E) = \nu(F)$. Thus $L^\mu(x^*Ex_*) = L^\mu(x^*Fx_*)$, $L^\nu(x^*Ex_*) = L^\nu(x^*Fx_*)$, which imply that $x^*Ex_* \sim^{(\mu,W)} x^*Fx_*$ and $x^*Ex_* \sim^{(\nu,V)} x^*Fx_*$.

At the beginning of the proof of Lemma 1, we showed that $\succsim^{(\mu,W)}$ and $\succsim^{(\nu,V)}$ agree with $\succsim$ on all unambiguous acts. Thus $\succsim^{(\mu,W)}$ and $\succsim^{(\nu,V)}$ agree on the set of all unambiguous acts. By the argument of the above paragraph, $\succsim^{(\mu,W)}$ and $\succsim^{(\nu,V)}$ agree on the set of all binary acts.

Then, (1) and (2) further imply that for all $E \in \mathcal{E}$ and all $F \in \Sigma$,

$$x^*Ex_* \succsim^{(\mu,W)} x^*Fx_* \Leftrightarrow x^*Ex_* \succsim x^*Fx_*$$

From which we can conclude that every indifference curve of $\succsim^{(\mu,W)}$ containing some unambiguous act is also an indifference curve for $\succsim$. But we know that every indifference curve of $\succsim^{(\mu,W)}$ contains some unambiguous act, thus $\succsim$ and $\succsim^{(\mu,W)}$ coincide. That is, $\succsim$ is probabilistically sophisticated. □

B Proof for Section 4

Proof of Theorem 1. We show the remaining only if part here.

Recall that the reduced lottery of any $\pi \in \Pi$ and a constant act is the outcome of that act, which is a vNM (objective) lottery. Focusing on constant acts and extending u_i's domain to $\Delta(X)$ for $i \in \mathcal{I}$, we go back to the vNM setting. As confirmed in the main text, the obvious belief-free Pareto criterion reduces to the standard Pareto principle if we restrict attention to constant acts. Hence the DM's problem reduces to preference aggregation with the presumption that all the individuals and the DM respect vNM axioms, which is exactly a version of Harsanyi's aggregation problem (Harsanyi 1955[12]).

Under the vNM assumption, the range of u_i is always convex for all $i \in \{0\} \cup \mathcal{I}$. Thus we can apply the aggregation result of De Meyer and Mongin (1995)[5]:[10] If the obvious belief-free Pareto criterion (applied to constant acts) and the minimal agreement assumption hold, then there are non-negative ξ_i for $i \in \mathcal{I}$, not all of them zero, and there is a real number μ, such that $u_0(p) = \sum_{i \in \mathcal{I}} \xi_i u_i(p) + \mu$ for all $p \in \Delta(X)$. The term μ does not affect the DM's ranking since u_0 is cardinal. Also, since at least one ξ_i is strictly positive, we can normalize $\{\xi_i\}_{i=1}^I$ to a set of social weights $\{\lambda_i\}_{i=1}^I$ such that $\sum_{i=1}^I \lambda_i = 1$.

It remains to show $P_0 = \Pi$. To show this, we adopt a separation theorem of probabilities used in Alon and Gayer (2016)[1]. To explore the details, we need some further notation. Denote the space of all bounded and finitely additive real functions over Σ by $ba(\Sigma)$. Let $B(\Omega, \Sigma)$ stand for the set of all bounded, Σ-measurable real funtions over Ω. Let $B_0(\Omega, \Sigma) \subset B(\Omega, \Sigma)$ be the set of all finite-valued, bounded, Σ-measurable real funtions over Ω. Weak* topology of $ba(\Sigma)$, denoted by $\mathscr{T}$, is the topology of $ba(\Sigma)$ generated by linear functionals $f(p) = \int_\Omega f dp$ for each $f \in B(\Omega, \Sigma)$ for $p \in ba(\Sigma)$. Denote $\mathscr{T}_0$ the topology of $ba(\Sigma)$ generated by linear functionals $f(p) = \int_\Omega f dp$ for each $f \in B_0(\Omega, \Sigma)$ for $p \in ba(\Sigma)$. It is known that $\mathscr{T}$ and $\mathscr{T}_0$ coincide when restricted to probability measures in $ba(\Sigma)$. Hence we know both P_0 and Π are closed in $\mathscr{T}_0$ as well.

By definition, if a function α is an element of $B_0(\Omega, \Sigma)$, there exist finitely many real numbers $\alpha_1, \cdots, \alpha_M$ and a partition $\{E_1, \cdots, E_M\}$ of Ω such that one can write $\alpha = \sum_{m=1}^M \alpha_m \mathbf{1}_{E_m}$, where $\mathbf{1}_E$ is the indicator function of $E \in \Sigma$. With this observation, we can state the following two existing results used to prove $P_0 = \Pi$.

Lemma 2 (Alon and Gayer 2016, Claim 5)**.** Let $P \subset ba(\Sigma)$ be a convex set of

[10] Their aggregation result only requires that the vector $(u_0, u_1, \cdots, u_I)$ is convex ranged.

probability measures, closed in the $\mathscr{T}_0$ topology, and $p' \in ba(\Sigma)$ a probability measure such that $p' \notin P$. Then there exsit $\alpha = \sum_{m=1}^{M} \alpha_m \mathbf{1}_{E_m} \in B_0(\Omega, \Sigma)$ with $0 \leq \alpha_m \leq 1$, and a scalar $0 < c \leq 1$ such that $\alpha(q) \geq c > \alpha(p')$ for all $q \in P$.

Lemma 3 (Alon and Gayer 2016, Claim 6)**.** Let $q_1, \cdots, q_R$ be non-atomic, countably additive probabilities over Σ and suppose there is $\alpha = \sum_{m=1}^{M} \alpha_m \mathbf{1}_{E_m} \in B_0(\Omega, \Sigma)$ such that $0 \leq \alpha_m \leq 1$. Then there exists an event $G \in \Sigma$ such that $\alpha(q_r) = q_r(G)$ for all $r = 1, \cdots, R$.

In Lemma 4 and Lemma 5, we show the two inclusions. Both results are proved by seeking a contradiction: We first assume that the inclusion does not hold, then construct two acts that contradict the obvious belief-free Pareto criterion. The non-atomic assumption of elements of P_0 and Π is used when we apply Lemma 3 to find a key event to construct the acts we need.

Lemma 4. $P_0 \subset \Pi = co(\{\pi_1, \cdots, \pi_K\})$.

Proof. Suppose there is some $p_0 \in P_0$ such that $p_0 \notin \Pi$. Since Π is closed under $\mathscr{T}_0$ topology, by Lemma 2, there exsit $\alpha = \sum_{m=1}^{M} \alpha_m \mathbf{1}_{E_m} \in B_0(\Omega, \Sigma)$ with $0 \leq \alpha_m \leq 1$, and a scalar $0 < c \leq 1$ such that

$$\alpha(\pi) \geq c > \alpha(p_0) \quad \forall \pi \in \Pi \tag{10}$$

Since probabilities $\pi_1, \cdots, \pi_K, p_0$ are non-atomic and countably additive over Σ by assumption, Lemma 3 implies that there exists an event $G \in \Sigma$ such that $\alpha(\pi_k) = \pi_k(G)$ for all $k = 1, \cdots, K$ and $\alpha(p_0) = p_0(G)$.

Let $f = x^* G x_*$, that is, let f be the act that gives δ_{x^*} in states in G and δ_{x_*} otherwise. Then $\alpha(\pi_k) = \pi_k(G) = \mathbf{E}_{\pi_k}(u_i(f))$ for all $i \in \mathcal{I}$, $\alpha(p_0) = p_0(G) = \mathbf{E}_{p_0}(u_0(f))$. Note that $\alpha(\pi_k) = \mathbf{E}_{\pi_k}(u_i(f)) = u_i(L^{\pi_k}(f))$ and $\alpha(p_0) = \mathbf{E}_{p_0}(u_0(f)) \geq \min_{p \in P_0} \mathbf{E}_p(u_0(f)) = U_0(f)$.

Let $p = cx^* + (1-c)x_*$, that is, let $p \in \Delta(X)$ be a lottery that gives x^* with probability c and gives x_* with probability $(1-c)$. Define g to be the constant act that always gives lottery p whatever the state is. Then for all π_k, $k = 1, \cdots, K$, the reduced lottery of π_k and g is p. We then have for all $i \in \mathcal{I}$, $u_i(L^{\pi_k}(g)) = u_i(p) = cu_i(x^*) + (1-c)u_i(x_*) = c = u_0(p) = U_0(g)$.

Since for each $\pi \in \Pi$, $\pi = \sum_{k=1}^{K} \beta_k \pi_k$ for some $0 \leq \beta_k \leq 1$ with $\sum_{k=1}^{K} \beta_k = 1$, $u_i\left(L^{\pi}\left(f\right)\right) = \pi\left(G\right) = \sum_{k=1}^{K} \beta_k \pi_k\left(G\right) = \sum_{k=1}^{K} \beta_k \alpha\left(\pi_k\right) = \alpha\left(\pi\right)$, where the last equality holds because α is linear. Since g is a constant act, we also have $u_i\left(L^{\pi}\left(g\right)\right) = c$ for all $i \in \mathcal{I}$, all $\pi \in \Pi$. Hence by equation (10), for any $\pi, \pi' \in \Pi$, $u_i\left(L^{\pi}\left(f\right)\right) = \alpha\left(\pi\right) \geq c = u_i\left(L^{\pi'}\left(g\right)\right)$, that is, $L^{\pi}\left(f\right)$ weakly dominates $L^{\pi'}\left(g\right)$ for all $\pi, \pi' \in \Pi$. By the obvious belief-free Pareto criterion, this implies $f \succsim_0 g$. However, $U_0\left(g\right) = c > \alpha\left(p_0\right) \geq U_0\left(f\right)$, a contradiction. □

Lemma 5. $\Pi = co\left(\{\pi_1, \cdots, \pi_K\}\right) \subset P_0$.

Proof. Suppose there is some $\pi_l \in \Pi$ such that $\pi_l \notin P_0$. Since P_0 is closed under $\mathscr{T}_0$ topology, by Lemma 2, there exsit $\alpha = \sum_{m=1}^{M} \alpha_m \mathbf{1}_{E_m} \in B_0\left(\Omega, \Sigma\right)$ with $0 \leq \alpha_m \leq 1$, and a scalar $0 < c \leq 1$ such that

$$\alpha\left(p\right) \geq c > \alpha\left(\pi_l\right) \quad \forall p \in P_0 \tag{11}$$

Since probabilities $\pi_1, \cdots, \pi_K$ are non-atomic and countably additive over Σ by assumption, Lemma 3 implies that there exists an event $G \in \Sigma$ such that $\alpha\left(\pi_k\right) = \pi_k\left(G\right)$ for all $k = 1, \cdots, K$. As shown in Lemma 4, each $p \in P_0$ is a convex combination of $\pi_1, \cdots, \pi_K$, we then have $\alpha\left(p\right) = p\left(G\right)$ for all $p \in P_0$.

Let $f = x^* G x_*$. Then $\alpha\left(\pi_l\right) = \pi_l\left(G\right) = \mathbf{E}_{\pi_l}\left(u_i\left(f\right)\right)$ for all $i \in \mathcal{I}$, $\alpha\left(p\right) = p\left(G\right) = \mathbf{E}_p\left(u_0\left(f\right)\right)$ for all $p \in P_0$. Note that $\alpha\left(\pi_l\right) = \mathbf{E}_{\pi_l}\left(u_i\left(f\right)\right) = u_i\left(L^{\pi_l}\left(f\right)\right)$.

Pick any $d < c$ such that $\alpha\left(p\right) \geq c > d > \alpha\left(\pi_l\right)$ holds for all $p \in P_0$. Let $q = dx^* + \left(1-d\right)x_*$, that is, let $q \in \Delta\left(X\right)$ be a lottery that gives x^* with probability d and gives x_* with probability $\left(1-d\right)$. Define h to be the constant act that always gives lottery q whatever the state is. By a similar argument as Lemma 4, for all $i \in \mathcal{I}$, all $\pi \in \Pi$, $u_i\left(L^{\pi}\left(h\right)\right) = d = U_0\left(h\right)$.

Hence for any $\pi \in \Pi$, $u_i\left(L^{\pi}\left(h\right)\right) = d > \alpha\left(\pi_l\right) = u_i\left(L^{\pi_l}\left(f\right)\right)$, that is, $L^{\pi}\left(h\right)$ strictly dominates $L^{\pi_l}\left(f\right)$ for all $\pi \in \Pi$. By the obvious belief-free Pareto criterion, this implies $h \succ_0 f$. However, $U_0\left(f\right) = \min_{p \in P_0} \mathbf{E}_p\left(u_0\left(f\right)\right) = \min_{p \in P_0} \alpha\left(p\right) \geq c > d = U_0\left(h\right)$, which implies $f \succ_0 h$, a contradiction. □

Combining Lemma 4 and Lemma 5 above, we have $P_0 = \Pi$. □

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