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Sequential unit root test for first-order autoregressive processes with initial values*

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Abstract

This paper examines the effect of initial values and small-sample properties in sequential unit root tests of the first-order autoregressive (AR(1)) process with a coefficient expressed by a local parameter. Adopting a stopping rule based on observed Fisher information defined by Lai and Siegmund (1983), we use the sequential least squares estimator (LSE) of the local parameter as the test statistic. The sequential LSE is represented as a time-changed Brownian motion with drift. The stopping time is written as the integral of the reciprocal of twice of a Bessel process with drift generated by the time-changed Brownian motion. The time change is applied to the joint density and joint Laplace transform derived from the Bessel bridge of the squared Bessel process by Pitman and Yor (1982), by which we derive the limiting joint density and joint Laplace transform for the sequential LSE and stopping time. The joint Laplace transform is needed to calculate joint moments because the joint density oscillates wildly as the value of the stopping time approaches zero. Moreover, this paper also earns the exact distribution of stopping time by Imhof's formula for both normally distributed and fixed initial values. When the autoregressive coefficient is less than 1, the question arises as to whether the local-to-unity or the strong stationary model should be used. We make the decision by comparing joint moments for respective models with those calculated from the exact distribution or simulations.

Keywords: Stopping time, observed Fisher information, DDS Brownian motion, local asymptotic normality, Bessel process, initial values, exact distributions

JEL Classification: C12, C22, C46

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1 Introduction

The unit root test is a basic problem in economics, and a quicker test procedure will save the cost of doing the test. Anscombe (1953) first proposed the concept of sequential analysis. This is an approach in which statistical inferences are made while data are acquired sequentially, and significant decisions are made when sufficient information has been collected. After his study, some researchers considered the sequential estimation of autoregressive time series. Lai and Siegmund (1983) introduced a stopping rule based on the observed Fisher information, and they also showed the limiting approximation of the stopping time. Nagai, Nishiyama, and Hitomi (2018) considered a unit root test against a local-to-unity hypothesis. Using a 3/2 dimensional Bessel process, they obtained the joint approximation of the sequential autoregressive coefficient estimator and stopping time. For stationary AR(1) processes, Hitomi et al. (2021) demonstrated the asymptotic joint normality of the stopping time and autoregressive coefficient estimator.

We consider the sequential unit root test for AR(1) process with an initial value x_0 and error terms ε_n;

$$x_n = \beta x_{n-1} + \varepsilon_n \quad (n = 1, 2, ...)$$

In the large sample theory developed in this paper, $\varepsilon_1, \varepsilon_2, \cdots$ are assumed to be strict stationary and ergodic martingale differences with variance $\sigma^2 < \infty$. On the other hand, in the small-sample strict distribution theory, we make a stronger assumption that the error terms consist of independent normal random variables. For both theories, the influence of initial values is investigated. The limiting approximation and exact distribution of stopping time are compared by numerical computation.

2 Asymptotic Property of Test Statistic and Stopping Time

Based on the observed Fisher information $I_N = \sum_{n=1}^{N} x_{n-1}^2/\sigma^2$, Lai and Siegmund(1983) defined a stopping time by $\tau_c = \inf\{N : I_N \geq c\}$. They also proved $\tau_c/\sqrt{c}$ converges in distribution to U_1 under $\beta = 1$;

$$\tau_c/\sqrt{c} \Rightarrow U_1 \equiv \inf\left\{t \geq 0 : \int_0^t W^2 ds = 1\right\}.$$

where $\Rightarrow$ stands for weak convergence and W is a Brownian motion.

While β is close to 1, one can localize the regression coefficient with a local parameter δ and a localizing number c;

$$\beta = 1 + \frac{\delta}{\sqrt{c}}.$$

We consider a unit root test with respect to the null and alternative hypotheses;

$$H_0 : \delta = 0 \text{ vs } H_1 : \delta < 0.$$

We make the following asymptotic assumption to investigate the effect of the initial value x_0 in the AR(1) process (1). Letting X_0 be an L_2 random variable, we assume that as $c \to \infty$,

$$x_0/c^{1/4} \xrightarrow{p} X_0$$

where $\xrightarrow{p}$ represents convergence in probability. Of course, when considering small-sample theory and simulations, we set $X_0 = x_0/c^{1/4}$ since c is fixed at a constant value.

Following the argument of Nagai, Nishiyama, and Hitomi (2018) with this assumption, $x_{\lfloor\sqrt{c}t\rfloor}/c^{1/4}$ converges in distribution to an Ornstein-Uhlenbeck process X_t with coefficient δ (OU(δ)) and initial value X_0; as $c \to \infty$,

$$x_{\lfloor\sqrt{c}t\rfloor}/c^{1/4} \Rightarrow X_t = X_0 + \delta \int_0^t X_s ds + \sigma W_t \tag{1}$$

where $\lfloor a \rfloor$ represents the integer part of real and W is a Brownian motion. In most cases, including this one, weak convergences will be carried out on $D[0, \infty)$, the space of right continuous functions with left limits (càdlàg).

Nagai, Nishiyama, and Hitomi (2018) proved

$$\tau_c/\sqrt{c} \Rightarrow U_1 \equiv \inf\left\{t \geq 0 : \int_0^t X_s^2/\sigma^2 ds = 1\right\}.$$

They also gave the limiting approximation of the sequential LSE for the local parameter

$$\begin{aligned}\hat{\delta}_{\tau_c} = \sqrt{c}\left(\hat{\beta}_{\tau_c} - 1\right) &= \frac{\frac{1}{\sqrt{c}}\sum_{n=1}^{\tau_c} x_{n-1}\Delta x_n}{\frac{1}{c}\sum_{n=1}^{\tau_c} x_{n-1}^2} \\ \Rightarrow \frac{\int_0^{U_1} X_s dX_s}{\int_0^{U_1} X_s^2 ds} &= \delta + \int_0^{U_1} X_s/\sigma dW_s \equiv \hat{\delta}_{U_1}\end{aligned}$$

Let $M_t \equiv \int_0^t X_s/\sigma dW_s$, then its quadratic variation is $\langle M\rangle_t = \int_0^t X_s^2/\sigma^2 ds$. Extend the random variable U_1 into a stochastic process U_v by defining

$$U_v \equiv \inf\left\{t \geq 0 : \langle M\rangle_t = v\right\}.$$

Since $\langle M\rangle_t$ is a continuous increasing process, the definition of U_v implies $v = \int_0^{U_v} X_s^2/\sigma^2 ds$. By the inverse function theorem, $\frac{dU_v}{dv} = \frac{1}{X_{U_v}^2}$. Denote $\rho_v \equiv \frac{X_{U_v}^2}{2\sigma^2}$, then $U_v = \int_0^v \frac{1}{2\rho_s} ds$. According to the Dambis-Dubins-Schwarz (DDS) theorem, $B_v = M_{U_v}$ is called a time-changed Brownian motion or DDS Brownian motion. See Revuz and Yor (1999) for time change. By applying the Ito's lemma to the OU process, we have

$$X_t^2 = X_0^2 + 2\sigma\int_0^t X_s dW_s + 2\delta\int_0^t X_s^2 ds + \sigma^2 t. \tag{2}$$

Then, we can see that $\rho_v = X_{U_v}^2/2\sigma^2$ is a 3/2-dimensional Bessel process with drift δ. In general, a k-dimensional Bessel process with constant drift for $k \geq 0$ is defined as a process ρ_v satisfying the following stochastic integral equation (see Linetsky (2004)).

$$\rho_v = \rho_0 + B_v + \delta v + \frac{k-1}{2}\int_0^v \frac{1}{\rho_s} ds \tag{3}$$

where B_v is a Brownian motion. Dividing (2) by $2\sigma^2$ and substituting t with U_v yields

$$\begin{aligned}\rho_v = \frac{X_{U_v}^2}{2\sigma^2} &= \frac{X_0^2}{2\sigma^2} + \int_0^{U_v} \frac{X_s}{\sigma} dW_s + \int_0^{U_v} \frac{X_s^2}{\sigma^2} ds + \frac{U_v}{2}. \\ &= \rho_0 + B_v + \delta v + \frac{1}{4}\int_0^v \frac{1}{\rho_s} ds\end{aligned} \tag{4}$$

In conclusion, $\left(\hat{\delta}_{\tau_c}, \frac{\tau_c}{\sqrt{c}}\right) \Rightarrow (\delta + B_1, U_1)$. The joint probability density function (PDF) of B_1 and U_1 can be computed through Bessel process in the following subsections. Then the marginal distribution of U_1 can also be derived.

2.1 Joint Density of Bessel Process and Stopping Time with zero initial value under H_0

Let P^0 and E^0 be the probability and the expectation under H_0, and q_t be a k-dimensional ($k > 0$) squared Bessel process under P^0 defined as the following stochastic equation with initial value x.

$$q_t = x + 2\int_0^t \sqrt{q_s} d\mathcal{W}_s + kt \tag{5}$$

where $\mathcal{W}$ is a Brownian motion. The transition density of q_t is

$$f_q(x, y|t) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\nu/2} \exp\left(-\frac{x+y}{2t}\right) I_\nu\left(\frac{\sqrt{xy}}{t}\right) \tag{6}$$

where $\nu = k/2 - 1$ is the index of q_t and I_ν is the modified Bessel function for $\nu \geq -1$ and $z > 0$

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!\Gamma(\nu+k+1)}.$$

See Revuz and Yor (1999) for details on squared Bessel processes.

For a k-dimensional squared Bessel process q_t with initial value x, Pitman and Yor (1982) derived a conditional Laplace transform of $\int_0^t q_s ds$ given q_t in the following form, named Bessel bridge.

$$\begin{aligned} &E^0\left[\exp\left(-\gamma\int_0^t q_s ds\right)|q_t = y\right] \\ &= \frac{t\sqrt{2\gamma}}{\sinh(t\sqrt{2\gamma})} \exp\left\{\frac{x+y}{2t}(1 - t\sqrt{2\gamma}\coth(t\sqrt{2\gamma}))\right\} \frac{I_\nu\left(\frac{\sqrt{xy}\sqrt{2\gamma}}{\sinh(t\sqrt{2\gamma})}\right)}{I_\nu\left(\frac{\sqrt{xy}}{t}\right)}. \end{aligned} \tag{7}$$

We obtain the joint PDF of q_t and $\int_0^t q_s ds$ from the above Bessel bridge. See Borodin & Selminen (2002) for functions is_v,es_v, s_v and $D_\mu(x)$.

Lemma 1. *Let $f_{q_t,\int_0^t q_s ds}(y, v)$ be the joint PDF of q_t and $\int_0^t q_s ds$ for a k-dimensional squared Bessel process with initial value x defined in (5). Then,*

$$f_{q_t,\int_0^t q_s ds}(y, v) = \mathrm{is}_v\left(\nu, t, 0, \frac{x+y}{2}, \frac{\sqrt{xy}}{2}\right)\frac{1}{2}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} \tag{8}$$

where $\nu = k/2 - 1$ is the index of q_t ,and for $\nu \geq -1$, $t + \nu t + r + z > 0$, $t > 0$

$$\begin{aligned} \mathrm{is}_v(\nu, t, r, z, w) :&= \mathcal{L}_\gamma^{-1}\left(\frac{\sqrt{2\gamma}}{\sinh(t\sqrt{2\gamma})}\exp\left(-r\sqrt{2\gamma} - z\sqrt{2\gamma}\coth(t\sqrt{2\gamma})\right) I_\nu\left(\frac{2w\sqrt{2\gamma}}{\sinh(t\sqrt{2\gamma})}\right)\right) \\ &= \sum_{l=0}^{\infty}\frac{w^{\nu+2l}}{\Gamma(\nu+l+1)l!}\mathrm{es}_v(1+\nu+2l, 1+\nu+2l, t, r, z) \end{aligned}$$

$$\begin{aligned} \mathrm{es}_v(\mu, \nu, t, r, z) :&= \mathcal{L}_\gamma^{-1}\left(\frac{(2\gamma)^{\frac{\mu}{2}}}{\sinh^\nu(t\sqrt{2\gamma})}\exp\left(-r\sqrt{2\gamma} - z\sqrt{2\gamma}\coth(t\sqrt{2\gamma})\right)\right) \\ &= \sum_{k=0}^{\infty}\frac{(-z)^k}{k!}s_v(\mu+k, \nu+k, t, r+z+kt) \end{aligned}$$

$$\begin{aligned} \mathrm{s}_v(\mu, \nu, t, z) :&= \mathcal{L}_\gamma^{-1}\left(\frac{(2\gamma)^{\mu/2}}{\sinh^\nu(t\sqrt{2\gamma})}e^{-z\sqrt{2\gamma}}\right) \\ &= 2^\nu\sum_{k=0}^{\infty}\frac{\Gamma(\nu+k)e^{-(\nu t+z+2kt)^2/4y}}{\sqrt{2\pi}v^{1+\mu/2}\Gamma(\nu)k!}D_{\mu+1}\left(\frac{\nu t+z+2kt}{\sqrt{v}}\right), \quad \nu \geq 0, \quad \nu t + z > 0. \end{aligned}$$

$D_\mu(x)$ is the Parabolic cylinder function.

Proof. Multiplying the Bessel bridge in (7) by the right-side of (6), one obtains

$$\int_0^\infty \exp(-\gamma v) f_{q_t, \int_0^t q_s ds}(y,v)\, dv$$
$$= \frac{\sqrt{2\gamma}}{\sinh(t\sqrt{2\gamma})} \exp\left\{-\frac{x+y}{2}\sqrt{2\gamma}\coth(t\sqrt{2\gamma}))\right\} I_\nu\left(\frac{\sqrt{xy}\sqrt{2\gamma}}{\sinh(t\sqrt{2\gamma})}\right)\frac{1}{2}\left(\frac{y}{x}\right)^{\frac{\nu}{2}}.$$

An inverse Laplace transform yields the expression of $f_{q_t, \int_0^t q_s ds}(y,v)$ as in (8). □

The OU process X_t under $H_0 : \delta = 0$ can be expressed as $X_t = X_0 + \sigma W_t$, and it is well-known that $q_t = X_t^2/\sigma^2$ can be identified with one-dimensional squared Bessel process (see Revuz and Yor (1999)). In this case, as shown in (4), $\rho_v = X_{U_v}^2/2\sigma^2$ is a 3/2-dimensional Bessel process and $U_v = \int_0^v \frac{1}{2\rho_s} ds$.

Theorem 2. *The joint densities of $(2\rho_v, U_v)$ and $\left(q_t, \int_0^t q_s ds\right)$ have the following relationship.*

$$f_{2\rho_v, U_v}(y,t) = f_{q_t, \int_0^t q_s ds}(y,v) \cdot y \tag{9}$$

Proof. Let $u = U_v$ then $v = \int_0^u q_s ds$ and $dv = q_u du$. In the Laplace transform of the joint CDF for $2\rho_v$ and U_v, we change the integral variable from v to u.

$$\begin{aligned}\int_0^\infty e^{-\gamma v} P^0(2\rho_v \leqq y, U_v \leqq t)\, dv &= E^0\left[\int_0^\infty e^{-\gamma v} 1_{\{2\rho_v \leqq y, U_v \leqq t\}} dv\right] \\ &= E^0\left[\int_0^\infty e^{-\gamma \int_0^u q_s ds} 1_{\{q_s \leqq y, u \leqq t\}} q_u du\right] \\ &= \int_0^t E^0\left[e^{-\gamma \int_0^u q_s ds} 1_{\{q_u \leqq y\}} q_u\right] du\end{aligned}$$

Taking the derivative with respect to t, and expressing the expectation in the integral form,

$$\begin{aligned}\int_0^\infty e^{-\gamma v}\frac{\partial}{\partial t} P^0(2\rho_v \leqq y, U_v \leqq t)\, dv &= E^0\left[e^{-\gamma \int_0^t q_s ds} 1_{\{q_t \leqq y\}} q_t\right] \\ &= \int_0^y \int_0^\infty e^{-\gamma v} f_{q_t, \int_0^t q_s ds}(z,v)\, z dv dz\end{aligned}$$

Next, taking the derivative with respect to y ,

$$\int_0^\infty e^{-\gamma v}\frac{\partial^2}{\partial t \partial y} P^0(2\rho_v \leqq y, U_v \leqq t)\, dv = \int_0^\infty e^{-\gamma v} f_{q_t, \int_0^t q_s ds}(y,v)\, y dv$$

The uniqueness of the inverse Laplace transform gives (9) □

Combining Lemma 1 and Theorem 2 together, the joint PDF of $2\rho_v$ and U_v under $H_0 : \delta = 0$ can be written as

$$\begin{aligned}f_{2\rho_v, U_v}(y,t) &= f_{q_t, \int_0^t q_s ds}(y,v)\, y \\ &= \mathrm{is}_v\left(-\frac{1}{2}, t, 0, \frac{x+y}{2}, \frac{\sqrt{xy}}{2}\right)\frac{1}{2}\left(\frac{y}{x}\right)^{-1/4} y\end{aligned}$$

where x is the initial value of $q_t = X_t^2/\sigma^2$.

Since $2\rho_v = X_0^2/\sigma^2 + 2B_v + U_v$, we can obtain the joint PDF $f_{B_v, U_v}(z,t)$ of B_v and U_v by variable transformation. For $(z,t) \in R \times [0,\infty)$ satisfying $z \geq -t/2 - X_0^2/2\sigma^2$,

$$f_{B_v, U_v}(z,t) = 2 f_{2\rho_v, U_v}\left(X_0^2/\sigma^2 + 2z + t, t\right) \cdot \left(X_0^2/\sigma^2 + 2z + t\right).$$

2.2 Joint Density under Alternative via Girsanov Transformation

The joint PDF of B_v and U_v under the alternative hypothesis can be obtained by a Girsanov transformation. Denoting the probability measure of the null hypothesis as P and that of the alternative hypothesis as P^δ, the Bessel process ρ_v differs in drift for these two measures.

$$P^\delta : \rho_1 = \rho_0 + B_1 + \delta + \frac{1}{4}\int_0^1 \frac{1}{\rho_s} ds = \rho_0 + \hat{\delta}_{U_1} + U_1/2$$

Under the Girsanov's transformation, the Radon–Nikodym derivative is

$$\frac{dP^\delta}{dP^0}\,|_{\mathcal{G}_1=\mathcal{F}_{U_1}} = \exp\left(\int_0^1 \delta dB_s - \frac{1}{2}\int_0^1 \delta^2 ds\right) = \exp\left(\delta\hat{\delta}_{U_1} - \frac{\delta^2}{2}\right)$$

Therefore, under the alternative hypothesis, i.e. $\delta \neq 0$, the joint PDF of B_v and U_v is

$$f^\delta_{\hat{\delta}_{U_1},U_1}(z,t) = e^{\delta z - \frac{\delta^2}{2}} f_{B_1,U_1}(z,t)$$

2.3 Joint Density with Zero Initial Value under H_0

Since the OU process X_t with $X_0 = 0$ under H_0 can be written simply as $X_t = \sigma W$, letting $M_t \equiv \int_0^t W_s dW_s$, its quadratic variation is $\langle M\rangle_t = \int_0^t W_s^2 ds$. Define $U_v \equiv \inf\{t \geq 0 : \langle M\rangle_t = v\}$, then $\tau_c/\sqrt{c}$ converges in distribution to U_1. Since $\langle M\rangle_t$ is a continuous increasing process, the definition of U_v implies $v = \int_0^{U_v} W_s^2 ds$. By the inverse function theorem, $\frac{dU_v}{dv} = \frac{1}{2\rho_v}$ where $\rho_v \equiv \frac{W^2_{U_v}}{2}$, then $U_v = \int_0^v \frac{1}{2\rho_s} ds$. According to the Dambis-Dubins-Schwarz theorem, $B_v = M_{U_v}$ is a time-changed Brownian motion, which makes ρ_v to be a 3/2-dimensional Bessel process.

The distribution of W_t^2 and $\int_0^t W_s^2 ds$ is

$$f_{W_t^2,\int_0^t W_s^2 ds}(y,v) = \frac{y^{-\frac{1}{2}}}{\sqrt{2\pi}}\mathrm{es}_v\left(\frac{1}{2},\frac{1}{2},t,0,\frac{y}{2}\right)$$

By the theorem 2, we get

$$\begin{aligned} f_{2\rho_v,U_v}(y,t) &= y \times f_{W_t^2,\int_0^t W_s^2 ds}(y,v) \\ &= \frac{y^{\frac{1}{2}}}{\sqrt{2\pi}}\mathrm{es}_v\left(\frac{1}{2},\frac{1}{2},t,0,\frac{y}{2}\right) \end{aligned}$$

Since $\rho_v = B_v + \frac{U_v}{2}$ while $\rho_0 = 0$, the joint PDF of (B_v, U_v) is

$$\begin{aligned} f_{B_v,U_v}(z,t) &= 2 f_{2\rho_v,U_v}(2z+t,t) \\ &= 2\sqrt{\frac{2z+t}{2\pi}}\mathrm{es}_v\left(\frac{1}{2},\frac{1}{2},t,0,\frac{2z+t}{2}\right) \end{aligned}$$

Let $v = 1$, then by a Girsanov transformation, under the alternative hypothesis, i.e. $\delta \neq 0$,

$$f^\delta_{\hat{\delta}_{U_1},U_1}(z,t) = e^{\delta z - \frac{\delta^2}{2}} f_{B_1,U_1}(z,t)$$

Hence, the distribution of U_1^δ is its marginal distribution, which can be computed by integrating $f^\delta_{\hat{\delta}_{U_1},U_1}$ with respect to z.

$$f_{U_1^\delta}(z,t) = \int_{-\infty}^{\infty} f_{B_1^\delta,U_1^\delta}(z,t)\,dz$$

3 Exact Distribution of Stopping Time

3.1 Normally Distributed Initial Value

For a stationary AR(1) process, i.e. $\beta < 1$, we assume $\varepsilon_1, \varepsilon_2, \cdots$ are normal random variables with mean 0 and variance σ^2, and $x_0 \sim N\left(0, \sigma^2/\left(1-\beta^2\right)\right)$. Put $\boldsymbol{x} = (x_0, x_1, \cdots, x_N)'$ and $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_N)'$ with $\varepsilon_0 = \sqrt{1-\beta^2}x_0$. Using the recursion formula, we have the relation $\boldsymbol{x} = A\boldsymbol{\varepsilon}$ where

$$A = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & & & & \\ \frac{\beta}{\sqrt{1-\beta^2}} & 1 & & O & \\ \frac{\beta^2}{\sqrt{1-\beta^2}} & \beta & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{\beta^N}{\sqrt{1-\beta^2}} & \beta^{N-1} & \beta^{N-2} & \cdots & 1 \end{pmatrix}$$

Substituting N with $\lfloor\sqrt{c}t\rfloor - 1$, we express the distribution of $\tau_c/\sqrt{c}$ in a matrix form.

$$\begin{aligned} P\left(\frac{\tau_c}{\sqrt{c}} \le t\right) &= P\left(\frac{1}{\sigma^2}\sum_{n=1}^{\lfloor\sqrt{c}t\rfloor} x_{n-1}^2 \ge c\right) \\ &= P\left(\frac{x'x}{\sigma^2} \ge c\right) \\ &= P\left(\frac{\varepsilon' A' A \varepsilon}{\sigma^2} \ge c\right) \end{aligned}$$

Following Imhof (1961), we can compute the exact distribution of stopping time, which is

$$P\left(\frac{\tau_c}{\sqrt{c}} \le t\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin\left\{\frac{1}{2}\sum_{r=1}^{\lfloor\sqrt{c}t\rfloor}\left[\arctan\left(\lambda_r u\right)\right] - \frac{1}{2}cu\right\}}{u\prod_{r=1}^{\lfloor\sqrt{c}t\rfloor}\left(1+\lambda_r^2 u^2\right)^{\frac{1}{4}}} du$$

where λ_r are the eigenvalues of $A'A$.

3.2 Constant Initial Value

Put $x = (x_1, \cdots, x_{N-1})'$, $\mu = \left(\beta x_0, \beta^2 x_0, \cdots, \beta^{N-1}x_0\right)'$and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_{N-1})'$. Using the recursion formula, we have the relation $x = A\varepsilon + \mu$ where

$$A = \begin{pmatrix} 1 & & & & \\ \beta & 1 & & O & \\ \beta^2 & \beta & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \beta^{N-2} & \beta^{N-3} & \beta^{N-4} & \cdots & 1 \end{pmatrix}$$

The real symmetric matrix $A'A$ can be decomposed into $A'A = P'\Lambda P$, where $\Lambda = \text{diag}\left(\lambda_1, \lambda_2, \cdots, \lambda_{N-1}\right)$ is composed by the eigenvalues of $A'A$ and P is composed of their respective orthogonal eigenvectors, then we have

$$
\begin{aligned}
\frac{\sum_{n=1}^{N-1} x_n^2}{\sigma^2} &= \frac{x'x}{\sigma^2} \\
&= \frac{(A\varepsilon+\mu)'(A\varepsilon+\mu)}{\sigma^2} \\
&= \frac{\left(\varepsilon+A^{-1}\mu\right)' A'A\left(\varepsilon+A^{-1}\mu\right)}{\sigma^2} \\
&= \left(\frac{P\varepsilon}{\sigma}+\frac{PA^{-1}\mu}{\sigma}\right)' \Lambda \left(\frac{P\varepsilon}{\sigma}+\frac{PA^{-1}\mu}{\sigma}\right)
\end{aligned}
$$

Substituting N with $\lfloor\sqrt{c}t\rfloor$, we can express the exact distribution of $\tau_c/\sqrt{c}$ through matrices and then apply Imhof's formula.

$$
\begin{aligned}
P\left(\frac{\tau_c}{\sqrt{c}} \le t\right) &= P\left(\frac{\sum_{n=1}^{\lfloor\sqrt{c}t\rfloor} x_{n-1}^2}{\sigma^2} \ge c\right) \\
&= P\left(\frac{x'x}{\sigma^2} \ge c - \frac{x_0^2}{\sigma^2}\right) \\
&= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \frac{\sin\{\theta(u)\}}{u\rho(u)}du
\end{aligned}
$$

where

$$
\begin{aligned}
\theta(u) &= \frac{1}{2}\sum_{r=1}^{\lfloor\sqrt{c}t\rfloor-1}\left[\arctan(\lambda_r u)+\frac{\delta_r^2\lambda_r u}{1+\lambda_r^2u^2}\right]-\frac{1}{2}\left(c-\frac{x_0^2}{\sigma^2}\right)u \\
\rho(u) &= \prod_{r=1}^{\lfloor\sqrt{c}t\rfloor-1}\left(1+\lambda_r^2u^2\right)^{\frac{1}{4}}\exp\left(\sum_{r=1}^{\lfloor\sqrt{c}t\rfloor-1}\frac{(\delta_r\lambda_r u)^2}{2\left(1+\lambda_r^2u^2\right)}\right)
\end{aligned}
$$

and δ_r is the $r$ht element of $\frac{P'A^{-1}\mu}{\sigma}$.

4 Comparison of Local and Stationary Parameter in Sequential Analysis

This part compares the local and stationary parameters for the AR(1) process in sequential analysis. For the autoregressive coefficient β, the local and stationary estimators are both normally distributed. Thus, our primary targets are joint moments of the stopping time and autoregressive coefficient.

4.1 Asymptotic Property of Stationary Parameter and Stopping Time

Hitomi, et al.(2021) proved the asymptotic joint property of the sequential estimator and the stopping time for AR(1) process. Their main conclusions are

$$
\sqrt{c}\begin{pmatrix} \hat{\beta}_{\tau_{1c}}-\beta \\ \frac{\tau_{1c}}{c}-\left(1-\beta^2\right) \end{pmatrix} \Rightarrow N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -2\beta \\ -2\beta & 4\beta^2+\left(1-\beta^2\right)\omega^2 \end{pmatrix}\right)
$$

$$
\sqrt{c}\begin{pmatrix} \hat{\beta}_{\tau_{2c}}-\beta \\ \frac{\tau_{2c}}{c}-\left(1-\beta^2\right) \end{pmatrix} \Rightarrow N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -2\beta \\ -2\beta & 4\beta^2 \end{pmatrix}\right)
$$

Here τ_{1c} stands for the case that the variance of error term σ^2 is known, and τ_{2c} stands for the case of unknown σ^2, which is also estimated sequentially.

4.2 Joint Moments of Local Parameter δ and Stopping Time

The local parameter $\hat{\delta}_{\tau_c} = \sqrt{c}\left(\hat{\beta}_{\tau_c} - 1\right)$ is used for the sequential analysis. Nagai et al.(2018) showed

$$\left(\hat{\delta}_{\tau_c}, \frac{\tau_c}{\sqrt{c}}\right) \Rightarrow (\delta + B_1, U_1)$$

Although the joint PDF of B_1 and U_1 has already been obtained, it is highly oscillatory in the neighborhood of $t = 0$. Thus the joint moments can not be computed through the joint PDF directly. Therefore, this section computes the modified joint moment generating function of $2\rho_v$ and U_v, which is derived through the Bessel bridge and Taylor expansion.

While $q_t \sim BESQ_x^d$, section 2.1 has shown that the multiplication of Bessel bridge and the PDF of q_t is a Laplace transform of $f_{q_t, \int_0^t q_s ds}(y, v)$ w.r.t v. Recall that its explicit expression is

$$\mathcal{L}_\gamma\left\{f_{q_t, \int_0^t q_s ds}(y, v)\right\} = \frac{\sqrt{2\gamma}}{2\sinh(\sqrt{2\gamma}t)} \exp\left(-\frac{x+y}{2}\sqrt{2\gamma}\coth(t\sqrt{2\gamma})\right) I_\nu\left(\frac{\sqrt{xy}\sqrt{2\gamma}}{\sinh(t\sqrt{2\gamma})}\right)\left(\frac{y}{x}\right)^{\frac{\nu}{2}} \tag{10}$$

By taking the Laplace transform of 10 w.r.t y, we can compute the joint moment generating function of q_t and $\int_0^t q_s ds$.

$$\begin{aligned}
E_x^0\left[e^{-\alpha q_t - \gamma\int_0^t q_s ds}\right] &= \int_0^\infty \int_0^\infty e^{-\alpha y - \gamma v} f_{q_t, \int_0^t q_s ds}(y, v)\, dv dy \\
&= \int_0^\infty e^{-\alpha y}\mathcal{L}_\gamma\left\{f_{q_t, \int_0^t q_s ds}(y, v)\right\} dy \\
&= 2^{\frac{\nu+1}{2}}\gamma^{\frac{\nu+1}{2}} \exp\left(-\frac{\sqrt{2\gamma}\alpha x\coth\left(\sqrt{2\gamma}t\right) + \gamma x}{2\alpha + \sqrt{2\gamma}\coth\left(\sqrt{2\gamma}t\right)}\right) \times \\
&\quad \left(2\alpha\sinh\left(\sqrt{2\gamma}t\right) + \sqrt{2\gamma}\cosh\left(\sqrt{2\gamma}t\right)\right)^{-\nu-1}
\end{aligned}$$

Set $u(v) = U_v = \int_0^v \frac{1}{2\rho_s} ds$, then $\frac{du}{dv} = \frac{1}{2\rho_v}$. Since $\rho_v = q_u/2$ and $u(0) = 0$, by the inverse function theorem we have $v = \int_0^u q_s ds$. In the Laplace transform of the modified joint moment generating function of $2\rho_v$ and U_v, changing the integral variable from v to u via the relation $du = 1/(2\rho_v)\, dv$, then substituting $2\rho_v$ and v with q_u and $\int_0^u q_s ds$ respectively, we have

$$\begin{aligned}
\int_0^\infty e^{-\gamma v} E_x^0\left[\frac{e^{-2\alpha\rho_v - \beta U_v}}{2\rho_v}\right] dv &= E_x^0\left[\int_0^\infty e^{-\beta U_v} e^{-2\alpha\rho_v - \gamma v}\frac{1}{2\rho_v} dv\right] \\
&= E_x^0\left[\int_0^\infty e^{-\beta u} e^{-\alpha q_u - \gamma\int_0^u q_s ds} du\right] \\
&= \int_0^\infty e^{-\beta u} E_x^0\left[e^{-\alpha q_u - \gamma\int_0^u q_s ds}\right] du
\end{aligned} \tag{11}$$

Changing the integral variable from u to s by the relation $s = \exp\left(-2\sqrt{2\gamma}u\right)$ and taking its Taylor series expansion w.r.t. x at 0, we have

$$\begin{aligned}
&\int_0^\infty e^{-\gamma v} E_x^0\left[\frac{e^{-2\alpha\rho_v - \beta U_v}}{2\rho_v}\right] dv \\
&= \sum_{n=0}^\infty \frac{x^n}{n!}\int_0^1 \frac{2^{\frac{3\nu}{2}}(-1)^n\gamma^{\nu/2} s^{\frac{\beta}{2\sqrt{2}\sqrt{\gamma}} + \frac{\nu-1}{2}}\left(\gamma + \sqrt{2\gamma}\alpha(s+1) - s\gamma\right)^n}{\left(\sqrt{2\gamma}(s+1) - 2\alpha(s-1)\right)^{\nu+n+1}} ds
\end{aligned} \tag{12}$$

The last equation converts the trigonometric functions to complex exponential functions in the Taylor series. Applying the Taylor's theorem for the multivariate function[1], the equation (12) can be written as

$$\int_0^\infty e^{-\gamma v} E_x^0 \left[\frac{e^{-2\alpha\rho_v - \beta U_v}}{2\rho_v} \right] dv = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{x^n \alpha^j \beta^{m-j}}{n! m!} \left(\begin{array}{c} m \\ j \end{array} \right) \int_0^1 K(\gamma, s, \nu, n, j, m-j) ds$$

where[2,3]

$$\begin{aligned} K(\gamma, s, \nu, n, j, l) =& (-1)^n (1+s)^{-1-j-\nu-n} (1-s)^{j+n} s^{\frac{\nu-1}{2}} \log^l(s) (-n-\nu-1)^{(j)} \times \\ & 2^{\frac{1}{2}(j-3l-n-1)+\nu} {}_2F_1 \left(-j, -n; -j-n-\nu; \frac{(s+1)^2}{(s-1)^2} \right) \gamma^{\frac{1}{2}(-l-1-j+n)} \end{aligned}$$

Finally, by inverting this Laplace transform w.r.t. γ, we obtain the explicit expression of the modified moment generating function, which is

$$E_x^0 \left[\frac{e^{-2\alpha\rho_v - \beta U_v}}{2\rho_v} \right] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{x^n \alpha^j \beta^{m-j}}{n! m!} \left(\begin{array}{c} m \\ j \end{array} \right) \int_0^1 J(v, s, \nu, n, j, m-j) ds$$

where

$$\begin{aligned} J(v, s, \nu, n, j, l) =& (-1)^n (1+s)^{-1-j-\nu-n} (1-s)^{j+n} s^{\frac{\nu-1}{2}} \log^l(s) (-n-\nu-1)^{(j)} \times \\ & 2^{\frac{1}{2}(j-3l-n-1)+\nu} {}_2F_1 \left(-j, -n; -j-n-\nu; \frac{(s+1)^2}{(s-1)^2} \right) \frac{v^{\frac{1}{2}(j+l-n-1)}}{\Gamma\left(\frac{1}{2}(j+l-n+1)\right)} \end{aligned}$$

For the alternative hypothesis, the joint modified Laplace transform of $(2\rho_v, U_v)$ can be computed by a Girsanov transformation. Denote P to be the probability measure of the null hypothesis and P^δ to be the probability measure under the alternative hypothesis. Under measures P and P^δ, the Bessel processes ρ_v are different in the drift δ.

$$\begin{aligned} P &: \rho_v = \rho_0 + B_v + \frac{1}{4} \int_0^v \frac{1}{\rho_s} ds \\ P^\delta &: \rho_v = \rho_0 + B_v + \delta v + \frac{1}{4} \int_0^v \frac{1}{\rho_s} ds \end{aligned} \tag{13}$$

In the Girsanov's transformation, the Radon–Nikodym derivative is

$$\begin{aligned} \frac{dP^\delta}{dP} |_{\mathcal{G}_v = \mathcal{F}_{U_v}} &= \exp\left(\int_0^v \delta dB_s - \frac{1}{2} \int_0^v \delta^2 ds \right) \\ &= \exp\left(\delta \left(\rho_v - \rho_0 - \frac{1}{2} U_v \right) - \frac{\delta^2}{2} v \right) \end{aligned}$$

---

[1]Taylor expansion for function of two variables: $f(\alpha, \beta) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array} \right) \frac{\alpha^j \beta^{m-j}}{m!} \frac{\partial^m}{\partial\alpha^p \partial\beta^p} f(0,0)$

[2]Hypergeometric function: ${}_2F_1(a, b, c, z) = \sum_{k=0}^{\infty} \frac{a^{(k)} b^{(k)}}{c^{(k)}} \frac{z^k}{k!}$

[3]Factorial power: $a^{(k)} = a(a-1) \cdots (a-(k-1))$

Thus, the modified joint moment generating function of $(2\rho_v, U_v)$ under the alternative hypothesis is

$$
\begin{aligned}
E^{\delta}\left[\frac{e^{(-2\alpha\rho_v-\beta U_v)}}{2\rho_v}\right] &= \int_{\Omega} e^{-2\alpha\rho_v-\beta U_v}\frac{1}{2\rho_v}\frac{dP^{\delta}}{dP^{0}}dP^{0} \\
&= e^{-\delta\rho_0-\frac{\delta^2}{2}v}\int_{\Omega} e^{-(2\alpha-\delta)\rho_v-\left(\beta+\frac{\delta}{2}\right)U_v}\frac{1}{2\rho_v}dP^{0} \\
&= e^{-\delta\rho_0-\frac{\delta^2}{2}v}E_x^{0}\left[\frac{e^{-(2\alpha-\delta)\rho_v-\left(\beta+\frac{\delta}{2}\right)U_v}}{2\rho_v}\right] \\
&= e^{-\delta\rho_0-\frac{\delta^2}{2}v}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{m}\frac{x^n\left(\alpha-\frac{\delta}{2}\right)^j\left(\beta+\frac{\delta}{2}\right)^{m-j}}{n!j!\,(m-j)!}\int_0^1 J(v,s,\nu,n,j,m-j)ds
\end{aligned}
$$

This gives the joint moment of ρ_v and U_v, which is computed by

$$
\begin{aligned}
E^{\delta}\left[(2\rho_v)^{p-1}U_v^{q}\right] &= (-1)^{p+q}\frac{\partial^{p+q}}{\partial\alpha^{p}\partial\beta^{q}}E^{\delta}\left[\frac{e^{(-2\alpha\rho_v-\beta U_v)}}{2\rho_v}\right]|_{\alpha=\beta=0} \\
&= e^{-\delta\rho_0-\frac{\delta^2}{2}v}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{m}\frac{x^n(\delta/2)^m(-1)^{p+q+j}}{n!j!\,(m-j)!}\int_0^1 J(v,s,\nu,n,j+p,m-j+q)ds
\end{aligned}
$$

Further, the covariance of local parameter and stopping time has following asymptotic limit.

$$
\begin{aligned}
Cov\left(\hat{\beta}_{\tau_c},\tau_c\right) &= Cov\left(\sqrt{c}\left(\hat{\beta}_{\tau_c}-1\right),\frac{\tau_c}{\sqrt{c}}\right) \\
&= Cov\left(\hat{\delta}_{\tau_c},\frac{\tau_c}{\sqrt{c}}\right) \\
&\to Cov\left(B_1,U_1\right) \\
&= E\left(B_1U_1\right)
\end{aligned}
$$

In order to compute this covariance, multiplying (13) by U_1 and taking the expect value on both sides, we have

$$
E\left(\rho_1U_1\right) = \left(\rho_0+\delta\right)E\left(U_1\right)+E\left(B_1U_1\right)+\frac{1}{2}E\left(U_1^2\right). \tag{14}
$$

This equation enables us to compute $E\left(B_1U_1\right)$ through joint moments of ρ_v and U_v.

5 Simulation and Numerical Computation

5.1 Limiting and Exact Distribution of Stopping time

Simulations are conducted to examine the limiting and exact distribution of stopping time. We use Mathematica for all the computation reports. First, both distributions perform well if the initial value is normally distributed. Then, for the constant initial value, even thorough c is small, the limiting distribution is close to the simulation and exact distribution. Further, as c grows, the fitting degree of limiting distribution becomes even better.

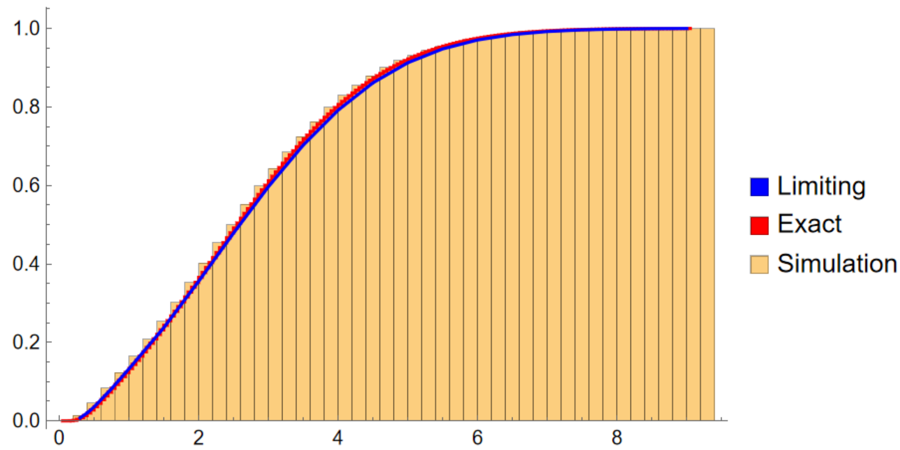

Fig 3: $\beta = 0.95$, $x_0 \sim N\left(0, \sigma^2/\left(1-\beta^2\right)\right)$, $c = 400$

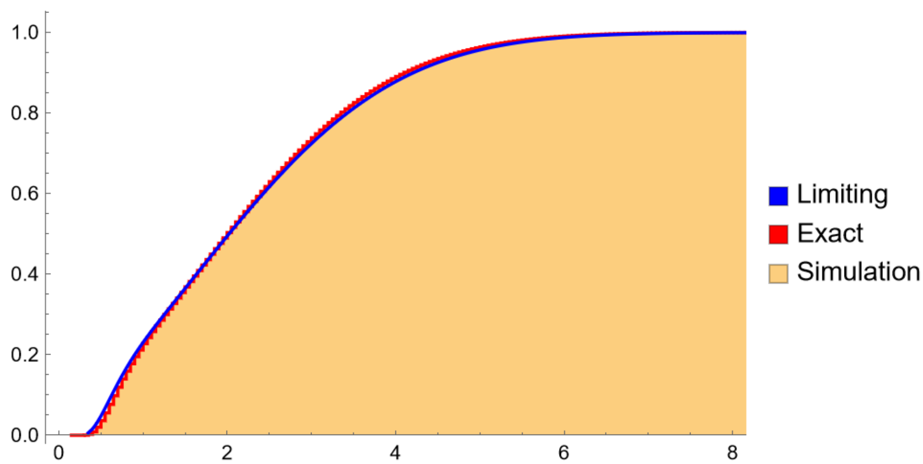

Fig 4: $\beta = 0.95$, $X_0 = 1$, $c = 400$

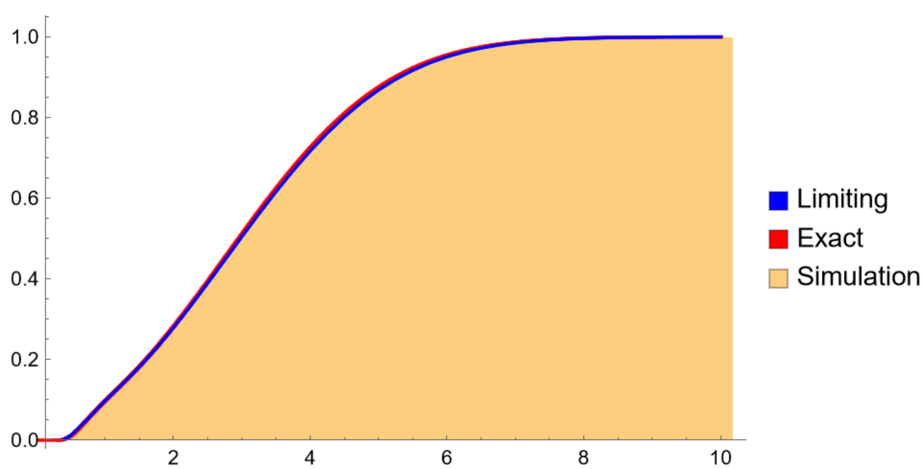

Fig 5: $\beta = 0.98$, $X_0 = 1$, $c = 6400$

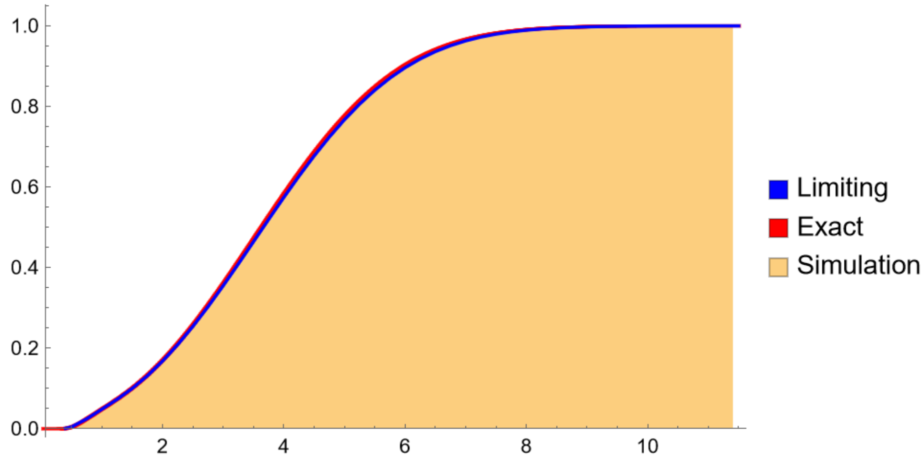

Fig 6: $\beta = 0.98$, $X_0 = 1$, $c = 10000$

5.2 Use the result in section 8.2, we can compute $E^{\delta}\left[(2B_v)^p U_v^q\right]$ with different δ.

For local parameter:

1. $\delta = \sqrt{c}\,(\beta - 1)$, $\left(\hat{\delta}_{\tau_c}, \frac{\tau_c}{\sqrt{c}}\right) \Rightarrow (\delta + B_1, U_1)$
2. $\rho_v = \frac{X_{U_v}^2}{2}$ starts at $\rho_0 = \frac{x_0^2}{2\sqrt{c}\sigma^2}$. $q_t = X_t^2$ starts at $x = \frac{x_0^2}{\sqrt{c}\sigma^2}$.
3. $Cov\left(\hat{\beta}_{\tau_c}, \tau_c\right) \rightarrow E\left(\rho_1 U_1\right) - \left(\rho_0 + \delta\right) E\left(U_1\right) - \frac{1}{2} E\left(U_1^2\right)$
4. $E\left(\tau_c\right) \rightarrow \sqrt{c} E\left(U_1\right)$, $SE\left(\tau_c\right) \rightarrow \sqrt{c Var\left(U_1\right)} = \sqrt{c\left[E\left(U_1^2\right) - \left[E\left(U_1\right)\right]^2\right]}$

Recall that for stationary parameter:

1. $\sqrt{c}\begin{pmatrix} \hat{\beta}_{\tau_{1c}} - \beta \\ \frac{\tau_{1c}}{c} - \left(1 - \beta^2\right) \end{pmatrix} \Rightarrow N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -2\beta \\ -2\beta & 4\beta^2 + \left(1 - \beta^2\right)\omega^2 \end{pmatrix}\right)$
2. $\sqrt{c}\begin{pmatrix} \hat{\beta}_{\tau_{2c}} - \beta \\ \frac{\tau_{2c}}{c} - \left(1 - \beta^2\right) \end{pmatrix} \Rightarrow N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -2\beta \\ -2\beta & 4\beta^2 \end{pmatrix}\right)$
3. $Cov\left(\hat{\beta}_{\tau_{ic}}, \tau_{ic}\right) = -2\beta$, $E\left[\tau_{ic}\right] = \left(1 - \beta^2\right) c$ for $i = 1, 2$
4. $SE\left(\tau_{1c}\right) = \sqrt{c\left(4\beta^2 + \left(1 - \beta^2\right)\omega^2\right)}$, $SE\left(\tau_{2c}\right) = 2\beta\sqrt{c}$

As β decreases from 0.99 to 0.8, comparison of local and stationary parameter for different c is conducted.

While the σ^2 is known. The comparison is shown in Figure 7-9. It is not surprising that the stationary parameter separates from the simulation or exact distribution while the initial value grows because the stationary theory assumes the affection of the initial value vanishes as c tends to ∞.

β	$Cov\left(\hat{\beta}_{\tau_c}, \tau_c\right)$			$E\left(\tau_c\right)$			$SE\left(\tau_c\right)$		
	Station	Local	Simu	Station	Local	Simu	Station	Local	Simu
0.99	-1.98	-1.15	-1.07	49.75	129.75	129.94	99.50	64.47	64.11
0.95	-1.90	-1.83	-1.65	243.75	279.85	273.70	97.53	91.05	88.66
0.90	-1.80	-2.04	-1.75	475.00	515.00	490.03	95.13	97.72	93.27
0.85	-1.70	-1.73	-1.67	693.75	755.36	703.77	92.80	106.45	91.71
0.80	-1.60	10.95	-1.58	900.00	436.07	907.92	90.55	435.40	90.00

Tab 9: $c = 2500$, $\sigma^2 = 1$, $x_0 = 0$

β	$Cov\left(\hat{\beta}_{\tau_c}, \tau_c\right)$			$E(\tau_c)$			$SE(\tau_c)$		
	Station	Local	Simu	Station	Local	Simu	Station	Local	Simu
0.99	-1.98	-1.12	-1.07	49.75	128.76	129.08	99.50	64.47	64.20
0.95	-1.90	-1.77	-1.67	243.75	278.85	272.91	97.53	91.05	89.07
0.90	-1.80	-1.94	-1.73	475.00	514.00	489.19	95.13	97.72	92.97
0.85	-1.70	-1.58	-1.67	693.75	754.44	703.26	92.80	106.31	91.92
0.80	-1.60	10.11	-1.58	900.00	434.54	906.70	90.55	439.01	90.14

Tab 10: $c = 2500$, $\sigma^2 = 1$, $x_0 = 1$

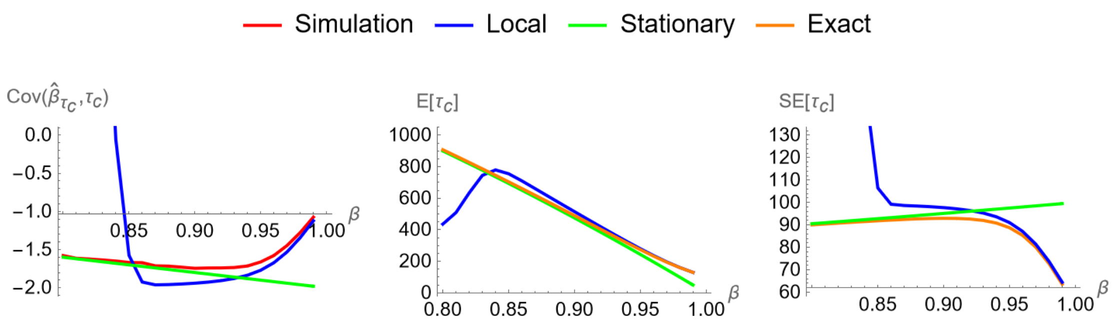

Fig 7: $c = 2500$, $x_0 = 0$

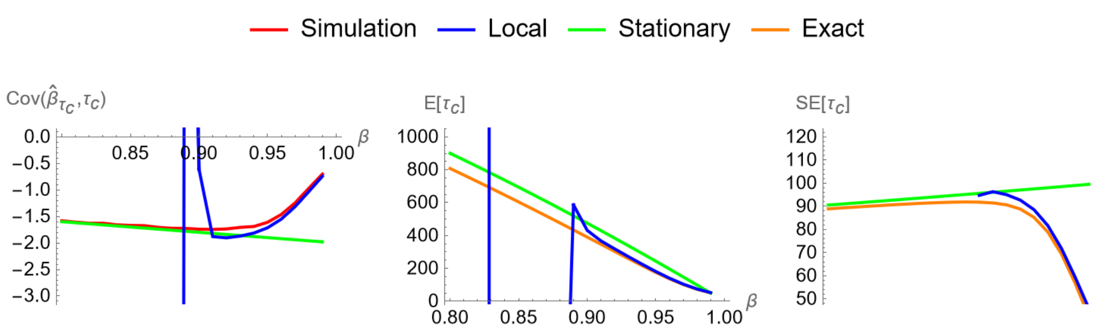

Fig 8: $c = 2500$, $x_0 = 10$

While σ^2 is unknown, the initial value of the Bessel process is unknown either. So only the zero initial value is compared in this case. We take 100000 times iterations in this simulation, using its estimator $s^2 = \frac{1}{N-1}\sum_{n=1}^{N}(x_n - x_{n-1})^2$. The result shows that the local parameter is closer to the simulation result when β is greater than around 0.93.

β	$Cov\left(\hat{\beta}_{\tau_c}, \tau_c\right)$			$E(\tau_c)$			$SE(\tau_c)$		
	Station	Local	Simu	Station	Local	Simu	Station	Local	Simu
0.99	-1.98	-1.15	-1.08	49.75	129.75	128.64	99	64.47	62.81
0.95	-1.90	-1.83	-1.67	243.75	279.85	271.16	95	91.05	86.34
0.9	-1.80	-2.04	-1.75	475.00	515.00	488.17	90	97.72	88.22
0.85	-1.70	-1.73	-1.69	693.75	755.36	702.75	85	106.45	84.49
0.8	-1.60	10.95	-1.60	900.00	436.07	906.34	80	435.40	79.81

Tab 11: $c = 2500$, unknown σ^2, $x_0 = 0$

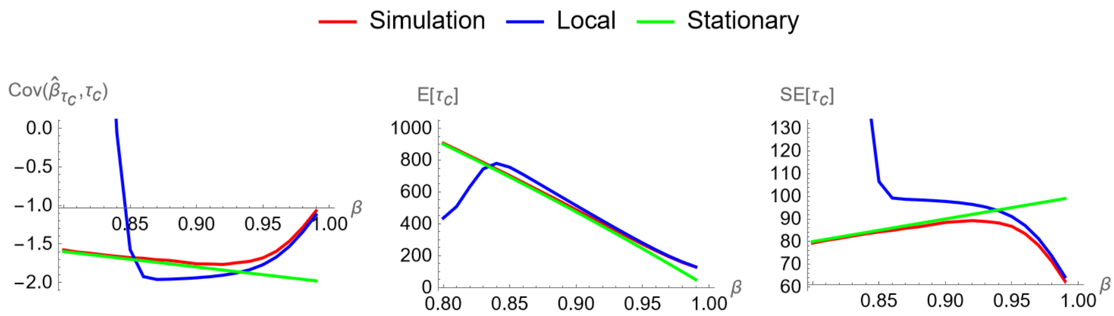

Fig 9: $c = 2500$, $x_0 = 0$

6 Conclusion

For an AR(1) process with a root near unity, we consider the effect of initial values in sequential unit root tests using stopping times based on the observed Fisher information. We use a time-change method deducing DDS Brownian motion, and we derive the theoretical joint density of the stopping time and sequential test statistic from the representation via the 3/2-dimensional Bessel process. Numerical studies show that the joint density performs well compared to simulation results, even when the level of observed Fisher information is small. Three types of marginal distributions of the stopping time are proposed: a limiting distribution calculated from the joint density above, an exact distribution derived from Imhof's method, and a cumulative relative frequency distribution computed from simulations. They agree well even when the expected sample size is small. Since we can not compute the joint moment directly due to the oscillatory property of the joint density, the joint Laplace transform is obtained. There is a question as to whether the local-to-unity or the strong stationary model should be used. We make the decision by comparing the joint moments for respective models with those calculated from the exact distribution or simulations. Of course, model selection for inference depends on the level of the observed Fisher information. For level $c = 2500$, we conclude as follows. When the autoregressive coefficient is greater than about 0.93, the joint moments obtained from the local-to-unity model give a better approximation. In contrast, when it is less than 0.93, those obtained from the stationary theory by Hitomi et al.(2021) provide a better approximation.

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