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“A Simple Axiomatization of Neo-Additive Choquet Expected Utility Theory on a Finite State Space”

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A Simple Axiomatization of Neo-Additive Choquet Expected Utility Theory on a Finite State Space*

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Abstract

By assuming that a state space is finite, we provide an easy-to-understand axiomatization of neo-additive Choquet expected utility theory (CEU) following Chateauneuf et al. (2007). Our axiomatization based on Möbius inversions sheds new light on their usefulness for our understanding of the behaviors of decision makers and enables us to provide new interpretations for neo-additive CEU from uncertainty aversion (uncertainty lovingness) and certainty equivalent, respectively. Furthermore, our approach enables us to understand the relationship between existing contributions by Chateauneuf et al. (2007), Kajii et al. (2009), and Asano and Kojima (2022) in a clearer and more unified way.

JEL Classification Numbers: C71; D81; D90

Keywords: Neo-additive Choquet Expected Utility, Möbius Inversion

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1 Introduction

The importance of the distinction between *risk* and *uncertainty* or *ambiguity* has been recognized in the literature since Ellsberg (1961). Ellsberg (1961) experimentally presents evidence that decision-makers (DMs) are likely to dislike situations where they cannot assign a unique probability measure, which implies that the existing framework of expected utility theory cannot explain these behaviors. To overcome these shortcomings of expected utility theory, various representations have been duly proposed. As seminal work, Gilboa and Schmeidler (1989) propose maxmin expected utility theory (MEU), while Schmeidler (1989) proposes Choquet expected utility theory (CEU).[1]

Since MEU by Gilboa and Schmeidler (1989) and CEU by Schmeidler (1989), DMs' preferences under uncertainty or ambiguity have been axiomatized from various viewpoints. Neo-additive CEU by Chateauneuf et al. (2007) as a specific form of CEU, α-maxmin expected utility by Ghirardato et al. (2004) as a generalization of MEU, and smooth ambiguity models by Klibanoff et al. (2005) have been widely applied in economics and finance. For a survey of the decision-theoretic literature, see for example, Gilboa and Marinacci (2016). The purpose of this paper is to shed some light on neo-additive CEU by Chateauneuf et al. (2007) from the viewpoint of Möbius inversions and belief functions. Our axiomatization in this regard enables us to deepen our understanding of DMs' behavior based on neo-additive CEU.

For a state space Ω with a generic element ω, an act f, a utility function u, and a probability measure P, neo-additive CEU is represented as

$$V(f) \;=\; (1-\delta)\int u(f(\omega))dP(\omega) + \delta\alpha \max_{\omega\in\Omega} u(f(\omega)) + \delta(1-\alpha)\min_{\omega\in\Omega} u(f(\omega)),$$

where $\alpha, \delta \in [0,1]$. This representation is tractable compared with the standard CEU by Schmeidler (1989) and allows a distinction between ambiguity itself and the ambiguity attitudes of DMs. Furthermore, neo-additive CEU can capture the optimistic and pessimistic attitudes of DMs in a unified way. Therefore, neo-additive CEU has been widely analyzed in the literature. For example, Eichberger et al.

[1]For further details, including the role of risk and ambiguity, see Gilboa (2009) or Wakker (2010). Note that throughout this paper, we refer to ambiguity and uncertainty interchangeably.

(2009) analyze Bertrand and Cournot competition models by considering optimism and pessimism under neo-additive CEU. Eichberger and Kelsey (2011) experimentally show that many of the inconsistencies pointed out by Goeree and Halt (2001) can be explained by strategic ambiguity under neo-additive CEU.[2] Diamantaras and Gilles (2011) apply neo-additive CEU to a model of the tragedy of the commons. Based on Jaffray and Philippe (1997), in which DMs' beliefs are captured by a more general capacity than that in Chateauneuf et al. (2007), Eichberger and Kelsey (2014) analyze the effects of ambiguity and ambiguity attitudes (ambiguity averse and ambiguity loving agents) in games with strategic complements. In recent work, Kishishita and Sato (2021) analyze the effect of the monopolist's optimism on the level of preventive effort.

In the fields of economics and statistics, the notion of belief functions has been widely investigated. In statistics, Dempster (1967) and Shafer (1976) propose a *belief function* to model uncertain situations. A belief function is a function that assigns any event to a lower bound of the likelihood that cannot be necessarily evaluated by probability measures. The reason for this is that DMs do not have sufficient information about the likelihood of any event. Shafer (1976) shows that a set function is considered to be a belief function if and only if its Möbius inversion is nonnegative. Furthermore, Chateauneuf and Jaffray (1989) investigate the relationship between the inclusion–exclusion formula for a capacity v and its Möbius inversion. Chateauneuf and Jaffray's (1989) analysis paves the way for the application of belief functions to economics. In one example, Rigotti et al. (2011) shed light on the occupational choices of individuals and firm formation, while Rigotti et al. (2011) analyze individuals' choices of occupation and technology.[3]

Our contribution of this paper is threefold. Under the assumption that a state space is finite, by proposing Properties 1 and 2 in Section 4, we provide an easy-to-understand derivation of neo-additive CEU by Chateauneuf et al. (2007) based on Möbius inversions. Properties 1 and 2 are the applications of Asano and Kojima

[2]Goeree and Halt (2001, pp.1401–1402) experimentally demonstrate that "for each of these games, however, we show that a change in the payoff structure can produce a large inconsistency between theoretical prediction(s) and human behavior."

[3]See the axiomatizations of belief functions from the decision-theoretic viewpoint, for example, Jaffray and Wakker (1994), Mukerji (1997), Ghirardato (2001), and Asano and Kojima (2022).

(2022) that provide a characterization of a class of belief functions. The reason that we can derive neo-additive CEU in a relatively easy way is that we make the most of the properties of Möbius inversions. As shown by Shapley (1953), each game is uniquely represented with a basis of unanimity games (see Lemma 1), and similarly, uniquely represented with that of the conjugate of the unanimity games, defined as one minus the value of the unanimity game for complements. Moreover, as shown by Gilboa and Schmeidler (1994), Choquet integrals with respect to an unanimity game and its conjugate coincide with the minimum and maximum operators, respectively. Properties 1 and 2 make it possible to express capacity in the representation as the sum of unanimity games and its conjugate using Chateaunueuf and Jaffray (1998).

The second contribution of this paper is to axiomatize neo-additive CEU by imposing *uncertainty aversion* and *uncertainty lovingness* on comaximum functions (acts) and cominimum functions (acts), respectively, where these functions are defined by Kajii et al. (2007). Because the former *comaximum uncertainty aversion* and the latter *cominimum uncertainty lovingness* axioms correspond to the pessimistic and optimistic behaviors of DMs, respectively, they capture certain properties of neo-additive CEU. The third contribution is to axiomatize neo-additive CEU by proposing a *two-states partial certainty equivalent*, which is an extended notion of the certainty equivalent in CEU. This axiomatization indicates that neo-additive CEU can be axiomatized from the viewpoint of certainty equivalent.

Our axiomatization based on Möbius inversions thus sheds new light on the usefulness of Möbius inversions for understanding DMs' behaviors. Moreover, our approach based on the properties of Möbius inversions enables us to understand the relationship between Chateauneuf et al. (2007), Kajii et al. (2009), and Asano and Kojima (2022) in a clearer and more unified way.

The organization of this paper is as follows. Section 2 provides the preliminaries, Section 3 explains neo-additive CEU, and Section 4 details an alternative axiomatization of neo-additive CEU. Section 5 presents an example that explains the essence of our axiomatization, while Section 6 discusses the relationship between this paper and the related literature, particularly, Chateauneuf et al. (2007) and Kajii et al. (2009).

2 Preliminaries

In this section, we provide the definitions and well-known results concerning the modularity of a game and its Möbius inversion. Let Ω be a nonempty state space. This set may be finite or infinite. A generic element $\omega \in \Omega$ denotes a state of the world, and a generic element $E \in 2^{\Omega}$ denotes an event. Let $\mathcal{F}$ be the collection of all nonempty subsets of Ω. Let $\mathbb{R}^{\Omega} = \{x | \, x : \Omega \to \mathbb{R}\}$ denote the set of all real-valued functions on Ω.

A set function $v : 2^{\Omega} \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *game* or a *non-additive signed measure*, *monotone* if $E \subseteq F$ implies $v(E) \leq v(F)$ for all E, $F \in 2^{\Omega}$, *finitely additive* if $v(E \cup F) = v(E) + v(F)$ for all E, $F \in 2^{\Omega}$ with $E \cap F = \emptyset$, *convex* if $v(E \cup F) + v(E \cap F) \geq v(E) + v(F)$ for all E, $F \in 2^{\Omega}$, *concave* if $v(E \cup F) + v(E \cap F) \leq v(E) + v(F)$ for all E, $F \in 2^{\Omega}$, *normalized* if $v(\Omega) = 1$, and a *non-additive measure* if it is monotone. A monotone and normalized game v is called a *capacity*. For a capacity v, the *conjugate of* v, denoted v', is defined by $v'(E) = 1 - v(E^c)$ for all $E \in 2^{\Omega}$. Furthermore, a game v is *k-monotone* for $k \geq 2$ if $v\left(\cup_{i=1}^{k} A_i\right) \geq \sum_{\emptyset \neq L \subset \{1,\ldots,k\}} (-1)^{|L|+1} v\left(\cap_{i \in L} A_i\right)$ for all $A_1, \ldots, A_k \in 2^{\Omega}$, a game v is *totally monotone* if it is monotone and k-monotone for all $k \geq 2$, and a game v is a *belief function* if it is totally monotone and $v(\Omega) = 1$, where $|L|$ denotes the cardinality of L.

For $x \in \mathbb{R}^{\Omega}$ and a capacity v, the *Choquet integral* of x is defined as $\int_{\Omega} x dv = \int_0^{\infty} v(x \geq \alpha) d\alpha + \int_{-\infty}^{0} (v(x \geq \alpha) - 1) d\alpha$, where $v(x \geq \alpha) = v(\{\omega \in \Omega \,|\, x(\omega) \geq \alpha\})$.

To provide an alternative axiomatization of Chateauneuf et al. (2007), in the following analyses, we assume that a state space Ω is finite. To provide the notions of the unanimity game and the well-known results in the following, we follow the expositions in Kajii et al. (2009). For $T \in \mathcal{F}$, let a game μ_T be the *unanimity game* on T defined by the following rule: $\mu_T(S) = 1$ if $T \subseteq S$ and $\mu_T(S) = 0$ otherwise. Let w_T be the conjugate of μ_T. Then, it can be easily shown that $w_T(E) = 1$ if $T \cap E \neq \emptyset$ and $w_T(E) = 0$ otherwise. When $T = \{\omega\}$, i.e., T is a singleton set, $u_T = w_T$ and they are additive (see Kajii et al. (2009)). When T is not a singleton set, they are not necessarily additive.

Lemma 1 (Shapley (1953)). *Each game v is uniquely represented as a linear com-*

bination of unanimity games and its Möbius inversion: $v = \sum_{T \in \mathcal{F}} \beta_T \mu_T$*, or equivalently,* $v(E) = \sum_{T \subseteq E} \beta_T$ *for all* $E \in \mathcal{F}$*, where* $\beta_T = \sum_{E \subseteq T} (-1)^{|T|-|E|} v(E)$.

By convention, we omit the empty set in the summation indexed by subsets of Ω. The set of coefficients $\{\beta_T\}_{T \in \mathcal{F}}$ is referred to as the *Möbius inversion*. Shafer (1976) shows that for any game v, $v = \sum_{T \in \mathcal{F}} \beta_T \mu_T$ is totally monotone if and only if β_T is non-negative for all $T \in \mathcal{F}$. It holds that if $v = \sum_{T \in \mathcal{F}} \beta_T \mu_T$, the conjugate v' is given by $v' = \sum_{T \in \mathcal{F}} \beta_T w_T$.

Lemma 2 (Kajii et al. (2009))**.** *For each* $E \in \mathcal{F}$*, it holds that* $w_E = \sum_{T \subseteq E} (-1)^{|T|-1} \mu_T$.

Gilboa and Schmeidler (1994) clarify the relationship between Choquet integrals and Möbius inversions in a finite state space. Gilboa and Schmeidler (1994) show that the Choquet integral of x with respect to v can be represented by a weighted sum of all minima of x with respect to the Möbius inversions $\{\beta_T\}_{T \in \mathcal{F}}$.

Proposition 1 (Gilboa and Schmeidler (1994))**.** *For all* $x \in \mathbb{R}^\Omega$ *and a capacity* $v = \sum_{T \in \mathcal{F}} \beta_T \mu_T$*,*

$$\begin{aligned} \int_\Omega x dv &= \sum_{T \in \mathcal{F}} \beta_T \int_\Omega x d\mu_T = \sum_{T \in \mathcal{F}} \beta_T \min_{\omega \in T} x(\omega), \\ \int_\Omega x dv' &= \sum_{T \in \mathcal{F}} \beta_T \int_\Omega x dw_T = \sum_{T \in \mathcal{F}} \beta_T \max_{\omega \in T} x(\omega). \end{aligned}$$

Chateauneuf and Jaffray (1989) analyze the relationship between the inclusion–exclusion formula for a game v and its Möbius inversion.

Lemma 3 (Chateauneuf and Jaffray (1989))**.** *Let* $v = \sum_{T \in \mathcal{F}} \beta_T u_T$ *be a game, and let* k *be an integer satisfying* $k \geq 2$*. Then,*
$v(\bigcup_{1 \leq i \leq k} T_i) - \sum_{\emptyset \neq L \subseteq \{1,2,\ldots,k\}} (-1)^{|L|+1} v(\bigcap_{j \in L} T_j) = \sum_{T \subseteq \bigcup T_i, T \not\subseteq T_i (1 \leq i \leq k)} \beta_T$.

3 Neo-additive CEU

In this section, we explain the notion of a *neo-additive capacity* and its properties. Chateauneuf et al. (2007) propose a *neo-additive capacity*. A neo-additive capacity is defined by

$$v(A) = \begin{cases} 1 & \text{for} \quad A = \Omega, \\ \alpha\delta + (1-\delta)\pi(A) & \text{for} \quad \emptyset \subsetneq A \subsetneq \Omega \\ 0 & \text{for} \quad A = \emptyset, \end{cases}$$

where $\alpha, \delta \in [0,1]$, and π is a finitely additive probability.[4] Note that a neo-additive capacity is a special case of capacities introduced by Jaffray and Philippe (1997); we call this JP-capacities, as in Eichberger and Kelsey (2014, pp.456–487). A capacity v on Ω is a JP-capacity if there exist a convex capacity v_1 and $\alpha \in [0,1]$ such that $v = \alpha v_1' + (1-\alpha)v_1$, where v_1' denotes the conjugate of v_1. By assuming a finite or infinite state space in a Savage's (1954) framework, Chateauneuf et al. (2007) axiomatize neo-additive CEU as follows:[5]

$$V(f) \;=\; (1-\delta)\int u(f(\omega))dP(\omega) + \delta\alpha \max_{\omega\in\Omega} u(f(\omega)) + \delta(1-\alpha)\min_{\omega\in\Omega} u(f(\omega)), \quad (1)$$

where $\alpha, \delta \in [0,1]$. As pointed out in Chateauneuf et al. (2007, pp.541–542), $V(f)$ is reduced to expected utility for $\delta = 0$, to pure pessimism for $\delta = 1$ and $\alpha = 0$, to pure optimism for $\delta = 1$ and $\alpha = 1$, and to the Hurwicz criterion for $\delta = 1$ and $\alpha \in (0,1)$. Ambiguity about the true probability measure is reflected by the parameter δ. The highest level of ambiguity is captured by $\delta = 1$, and the lowest (no ambiguity) by $\delta = 0$. Conversely, the ambiguity attitude of DMs for optimism and pessimism are reflected by the parameter α. By letting $\gamma = 1-\delta$ and $\beta = \delta\alpha$ in Equation (1), it follows that

$$V(f) = \gamma\int u(f(\omega))dP(\omega) + \beta \max_{\omega\in\Omega} u(f(\omega)) + (1-\gamma-\beta)\min_{\omega\in\Omega} u(f(\omega)).$$

These values α and β capture the degrees of optimism and pessimism, respectively, for example, see Gilboa and Marinacci (2016). See also Eichberger and Kelsey (2011, pp.316–317) for the interpretation of these parameters.[6]

[4] Chateauneuf et al. (2007) refer to this capacity as a *neo-additive capacity* because it is additive on non-extreme outcomes.

[5] Strictly speaking, the representation (1) differs from that obtained by Chateauneuf et al. (2007). Because we consider the neo-additive CEU within the framework of Anscombe and Aumann (1963), we slightly modify the representation in a Savage's (1954) framework as derived by Chateauneuf et al. (2007).

[6] Jaffray and Philippe (1997) show that the CEU with respect to a JP-capacity $v = \alpha v_1' + (1-\alpha)v_1$ is represented as follows:

$$\begin{aligned} & \int u(f(\omega))dv(\omega) \\ = \;& \alpha\int u(f(\omega))dv_1'(\omega) + (1-\alpha)\int u(f(\omega))dv_1(\omega) \\ = \;& \alpha \max_{P\in\mathrm{core}(v_1)}\int u(f(\omega))dP(\omega) + (1-\alpha)\min_{P\in\mathrm{core}(v_1)}\int u(f(\omega))dP(\omega), \end{aligned}$$

4 Axiomatization for Neo-additive CEU

This section provides an alternative axiomatization for neo-additive CEU proposed by Chateauneuf et al. (2007). Whereas Chateauneuf et al. (2007) assumes a finite or infinite state space within the framework of Savage (1954), we assume a finite state space. By taking advantage of the properties of Möbius inversions, we provide an easy-to-understand axiomatization of neo-additive CEU in a finite state space.

Let X be the nonempty finite set of all deterministic outcomes, and let Y be the set of all distributions over X with finite supports, that is, $Y = \{y : X \to [0,1] \mid y(x) \neq 0 \text{ for finitely many } x \in X \text{ and } \sum_{x \in X} y(x) = 1\}$. We call an element of Y a *lottery*. For notational simplicity, we identify $x \in X$ with the Dirac measure $\delta_x \in Y$: δ_x is the probability measure that assigns probability one to $\{x\}$. The set of all Σ-measurable finite step functions from Ω to Y is denoted by L_0, and elements of L_0 are called *simple lottery acts* or *acts*. The set of all constant functions in L_0 is denoted by L_c, and elements of L_c are called *constant acts*. For all f, $g \in L_0$ and $\lambda \in [0,1]$, the compound lottery is defined by $(\lambda f + (1-\lambda)g)(\omega) \equiv \lambda f(\omega) + (1-\lambda)g(\omega)$ for all $\omega \in \Omega$. We assume that DMs' preferences are captured by a binary relation $\succeq$ on L_0. The asymmetric ($\succ$) and symmetric ($\sim$) parts of $\succeq$ are defined as usual. A binary relation $\succeq$ on Y is defined by restricting $\succeq$ on L_c and denoted by the same symbol $\succeq$. That is, for all y, $z \in Y$, $y \succeq z$ if and only if $y^\Omega \succeq z^\Omega$, where y^Ω and z^Ω denote constant functions on Ω. The notation $(f, A; g, A^c)$ denotes the act that equals to act f on event A and act g on event A^c, respectively.

We provide the following two properties for capacity v that play significant roles in our axiomatization.

Property 1. For any T with $3 \leq |T| \leq n-1$ and any $\omega_i, \omega_j \in T$,

$$\text{(P1)} \quad v(T) + v(T \backslash \{\omega_i, \omega_j\}) = v(T \backslash \{\omega_i\}) + v(T \backslash \{\omega_j\}).$$

Property 2. There exists a two-point set $T^0 = \{\omega^*, \omega^{**}\}$ such that

$$\text{(P2-a)} \quad v(T^0) \leq v(T^0 \backslash \{\omega^*\}) + v(T^0 \backslash \{\omega^{**}\})$$

where $\text{core}(v_1)$ is the set of probability measures dominating the capacity v_1. DMs' preferences correspond to the cases in which they are optimistic and pessimistic for $\alpha \in (0,1)$, purely optimistic for $\alpha = 1$, and purely pessimistic for $\alpha = 0$. This representation also enables us to separate ambiguity from ambiguity attitude.

$$\text{(P2-b)} \quad v(\Omega) + v(\Omega \backslash T^0) \geq v(\Omega \backslash \{\omega^*\}) + v(\Omega \backslash \{\omega^{**}\}).$$

Theorem 1. *Let $|\Omega| \geq 4$. Suppose that a binary relation $\succeq$ defined on L_0 is represented by CEU with a utility function u and a capacity v, that is, Schmeidler's (1989) five axioms*[7] *are satisfied. If this capacity v satisfies Properties 1 and 2, then there exist a unique set of nonnegative coefficients $\{\eta^0_{\{\omega_i\}}\}_{\omega_i \in \Omega}$, and a unique nonnegative coefficients $\tilde{\eta}$ and $\bar{\eta}$ with $\sum_{\omega_i \in \Omega} \eta^0_{\{\omega_i\}} + \tilde{\eta} + \bar{\eta} = 1$ such that for all f and g in L_0,*

$$f \succeq g \Leftrightarrow I(f) \geq I(g),$$

where

$$I(f) = \sum_{i=1}^{n} \eta^0_{\{\omega_i\}} u(f(\omega_i)) + \tilde{\eta} \min_{\omega_i \in \Omega} u(f(\omega_i)) + \bar{\eta} \max_{\omega_i \in \Omega} u(f(\omega_i)). \tag{2}$$

The converse also holds.

To prove Theorem 1, we provide the following three lemmas. Lemmas 4, 5 and 6 are shown by Chateauneuf and Jaffray (1989). In the following analyses, let $v = \sum_T \beta_T \mu_T$.

Lemma 4 (Comodularity)**.** *Let $T \in 2^\Omega$ with $|T| \geq 2$ and let $\omega_i, \omega_j \in T$ with $\omega_i \neq \omega_j$. Then, the following are equivalent:*
(i) $v(T) + v(T \backslash \{\omega_i, \omega_j\}) = v(T \backslash \{\omega_i\}) + v(T \backslash \{\omega_j\})$.
(ii) $\sum_{\{\omega_i, \omega_j\} \subseteq S \subseteq T} \beta_S = 0$.

Lemma 5 (Coconvexity)**.** *Let $T \in 2^\Omega$ with $|T| \geq 2$ and let $\omega_i, \omega_j \in T$ with $\omega_i \neq \omega_j$. Then, the following are equivalent.*
(i) $v(T) + v(T \backslash \{\omega_i, \omega_j\}) \geq v(T \backslash \{\omega_i\}) + v(T \backslash \{\omega_j\})$.
(ii) $\sum_{\{\omega_i, \omega_j\} \subseteq S \subseteq T} \beta_S \geq 0$.

Lemma 6 (Coconcavity)**.** *Let $T \in 2^\Omega$ with $|T| \geq 2$ and let $\omega_i, \omega_j \in T$ with $\omega_i \neq \omega_j$. Then, the following are equivalent.*
(i) $v(T) + v(T \backslash \{\omega_i, \omega_j\}) \leq v(T \backslash \{\omega_i\}) + v(T \backslash \{\omega_j\})$.
(ii) $\sum_{\{\omega_i, \omega_j\} \subseteq S \subseteq T} \beta_S \leq 0$.

[7] Schmeidler (1989) axiomatizes CEU by Weak Order, Comonotonic Independence, Continuity, Monotonicity, and Nondegeneracy.

5 Proof of Theorem 1 for $|\Omega| = 4$

In this section, we provide a proof of Theorem 1 for $|\Omega| = 4$. In the general case, see Appendix. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, and let $\omega^* = \omega_1$ and $\omega^{**} = \omega_2$. Let

$$v = \sum_{1 \le k \le 4} \eta_{\{\omega_k\}} \mu_{\{\omega_k\}} + \sum_{|T| \ge 2} \eta_T \mu_T.$$

It follows from (P2-a) in Property 2 and Lemma 6 that

$$\sum_{\{\omega_1,\omega_2\} \subseteq S \subseteq \{\omega_1,\omega_2\}} \eta_S = \eta_{\{\omega_1,\omega_2\}} \le 0.$$

In the following, let $\eta_{\{\omega_1,\omega_2\}} = -\bar{\eta}$ ($\bar{\eta} \ge 0$). By using Property 1 and Lemma 4 for $T = \{\omega_1, \omega_2, \omega_3\}$, it holds that

$$\sum_{\{\omega_1,\omega_2\} \subseteq S \subseteq \{\omega_1,\omega_2,\omega_3\}} \eta_S = \eta_{\{\omega_1,\omega_2\}} + \eta_{\{\omega_1,\omega_2,\omega_3\}} = 0,$$

which implies that $\eta_{\{\omega_1,\omega_2,\omega_3\}} = -\eta_{\{\omega_1,\omega_2\}} = \bar{\eta}$. Similarly, by using Property 1 and Lemma 4 for $T = \{\omega_1, \omega_2, \omega_4\}$, it holds that $\eta_{\{\omega_1,\omega_2\}} + \eta_{\{\omega_1,\omega_2,\omega_4\}} = 0$, which implies that

$$\eta_{\{\omega_1,\omega_2,\omega_4\}} = -\eta_{\{\omega_1,\omega_2\}} = \bar{\eta}. \tag{3}$$

Moreover, by using Property 1 and Lemma 4 for $T = \{\omega_1, \omega_2, \omega_4\}$, it holds that $\eta_{\{\omega_1,\omega_4\}} + \eta_{\{\omega_1,\omega_2,\omega_4\}} = 0$. Thus, by (3), it holds that $\eta_{\{\omega_1,\omega_4\}} = -\eta_{\{\omega_1,\omega_2,\omega_4\}} = -\bar{\eta}$. Then, it follows from successive calculations that

$$\eta_{\{\omega_1,\omega_2\}} = \eta_{\{\omega_1,\omega_3\}} = \eta_{\{\omega_1,\omega_4\}} = \eta_{\{\omega_2,\omega_3\}} = \eta_{\{\omega_2,\omega_4\}} = \eta_{\{\omega_3,\omega_4\}} = -\bar{\eta} \quad \text{and}$$
$$\eta_{\{\omega_1,\omega_2,\omega_3\}} = \eta_{\{\omega_1,\omega_2,\omega_4\}} = \eta_{\{\omega_1,\omega_3,\omega_4\}} = \eta_{\{\omega_2,\omega_3,\omega_4\}} = \bar{\eta}.$$

Finally, by using (P2-b) in Property 1 and Lemma 5 for $T = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, it holds that

$$\sum_{\{\omega_1,\omega_2\} \subseteq S \subseteq \{\omega_1,\omega_2,\omega_3,\omega_4\}} \eta_S \ge 0,$$

which implies that

$$0 \le \eta_{\{\omega_1,\omega_2\}} + \eta_{\{\omega_1,\omega_2,\omega_3\}} + \eta_{\{\omega_1,\omega_2,\omega_4\}} + \eta_\Omega$$

$$= \quad -\bar{\eta} + \bar{\eta} + \bar{\eta} + \eta_{\Omega} = \bar{\eta} + \eta_{\Omega}$$

By letting $\tilde{\eta} = \bar{\eta} + \eta_{\Omega}$, it holds that $\tilde{\eta} \geq 0$. Because v is a capacity, it holds that

$$\begin{aligned} & 0 \leq v(\{\omega_i, \omega_k\}) - v(\{\omega_i\}) \\ = \quad & \eta_{\{\omega_i\}} + \eta_{\{\omega_k\}} + \eta_{\{\omega_i,\omega_k\}} - \eta_{\{\omega_i\}} = \eta_{\{\omega_i\}} + \eta_{\{\omega_k\}} - \bar{\eta} - \eta_{\{\omega_i\}} = \eta_{\{\omega_k\}} - \bar{\eta}. \end{aligned}$$

Thus, by letting $\eta^0_{\{\omega_k\}} = \eta_{\{\omega_k\}} - \bar{\eta}$ for $k = 1, 2, 3, 4$, it holds that $\eta^0_{\{\omega_k\}} \geq 0$. The above argument shows that

$$\begin{aligned} v \quad &= \quad \sum_{1 \leq k \leq 4} \eta_{\{\omega_k\}} \mu_{\{\omega_k\}} + \sum_{|T| \geq 2} \eta_T \mu_T \\ &= \quad \sum_{1 \leq k \leq 4} \eta^0_{\{\omega_k\}} \mu_{\{\omega_k\}} + \bar{\eta} \left(\sum_{|T|=1} \mu_T + \sum_{|T|=2} (-\mu_T) + \sum_{|T|=3} \mu_T + (-\mu_{\Omega}) \right) + \tilde{\eta} \mu_{\Omega}, \end{aligned}$$

which implies that

$$v = \sum_{1 \leq k \leq 4} \eta^0_{\{\omega_k\}} \mu_{\{\omega_k\}} + \tilde{\eta} \mu_{\Omega} + \bar{\eta} w_{\Omega}$$

where $\eta^0_{\{\omega_k\}} \geq 0$, $\tilde{\eta} \geq 0$, and $\bar{\eta} \geq 0$. By using Proposition 1, the rest of this proof is completed.

6 Discussion and Related Literature

In this section, we discuss the relationship between our paper and the related literature. In Section 4, we proposed Properties 1 and 2 to axiomatize neo-additive CEU in a finite state space. We particularly discuss how Properties 1 and 2 play a role in deriving representations of neo-additive CEU.

6.1 The Linkage between Chateauneuf et al. (2007) and This Paper

Chateauneuf et al. (2007) provide an axiomatization of neo-additive CEU. To this purpose, Chateauneuf et al. (2007) prove the following proposition, which plays a crucial part in deriving a neo-additive CEU.

Proposition 2 (Chateauneuf et al. (2007), Proposition 3.1)**.** *Let Ω be a finite state space.*[8] *Let v be a capacity on $(\Omega, 2^\Omega)$. Then, the following statements are equivalent:*
(i) v is a neo-additive capacity.
(ii) the capacity v satisfies the following properties:
(a) for any non-empty events E, F, G with $E \neq \Omega$, $F \neq \Omega$ and $G \neq \Omega$ such that $E \cap F = \emptyset = E \cap G$, $E \cup F \neq \Omega \neq E \cup G$, $v(E \cup F) - v(F) = v(E \cup G) - v(G)$,
(b) for some E, F such that $E \cap F = \emptyset$, $v(E \cup F) \leq v(E) + v(F)$,
(c) for some E, F such that $E \cap F = \emptyset$, $v'(E \cup F) \leq v'(E) + v'(F)$,
(d) for any E, $v(E \cup \emptyset) = v(E)$,
where v' denotes the conjugate of v.

First, Chateauneuf et al. (2007) derive biseparable preferences by Ghirardato and Marinacci (2001) based on four axioms (Ordering, Continuity, Eventwise Monotonicity, and Binary Comonotonic Act Independence).[9] On top of that, Chateauneuf et al. (2007) derive Condition (ii) in Proposition 2. The key in axiomatizing neo-additive CEU is to derive Condition (ii) in Proposition 2. The relationship between Chateauneuf et al. (2007) and this paper is as follows. In our approach, by Properties 1 and 2, we derive the following:

$$\eta_S = (-1)^{|S|-1}\bar{\eta} \text{ for all } S \text{ with } 2 \leq |S| \leq n-1$$
$$(-1)^{n-2}\bar{\eta} + \eta_\Omega \geq 0,$$

which can derive (a), (b), and (c) in Condition (ii) of Proposition 2. Because we consider a finite state space, (d) in Condition (ii) of Proposition 2 holds. Therefore, Properties 1 and 2 can derive neo-additive CEU by way of Proposition 2. This paper provides an easy-to-understand axiomatization of neo-additive CEU by way of Möbius inversions under the assumption that a state space is finite.

Proof. See Appendix. □

[8] To analyze an infinite state space, more conditions are required for collections of events. See Chateauneuf et al. (2007) in detail.
[9] For the definition of biseparable preferences, see Appendix.

6.2 The Linkage between Kajii et al. (2007) and This Paper

In this subsection, we lead to neo-additive CEU by proposing *uncertainty lovingness axiom* and *uncertainty averse axiom* based on Chateauneuf et al. (2007). First, we discuss the relationship between Kajii et al. (2007) and this paper. Kajii et al. (2007) proposed the notions of $\mathcal{E}$-cominimum and $\mathcal{E}$-comaximum functions.[10]

Definition 1. Two functions $x, y \in \mathbb{R}^{\Omega}$ are *cominimum* if $\mathrm{argmin}_{\omega\in\Omega}\, x \cap \mathrm{argmin}_{\omega\in\Omega}\, y \neq \emptyset$. Two functions $x, y \in \mathbb{R}^{\Omega}$ are *comaximum* if $\mathrm{argmax}_{\omega\in\Omega}\, x \cap \mathrm{argmax}_{\omega\in\Omega}\, y \neq \emptyset$.

Based on this definition, Asano and Kojima (2015) provide the following definition.

Definition 2. Two acts $f,\ g \in L_0$ are said to be *cominimum* if $\{\omega \in \Omega \,|\, f(\omega') \succeq f(\omega)\text{ for all }\omega' \in \Omega\} \cap \{\omega \in \Omega \,|\, g(\omega') \succeq g(\omega)\text{ for all }\omega' \in \Omega\} \neq \emptyset$. Two acts $f,\ g \in L_0$ are said to be *comaximum* if $\{\omega \in \Omega \,|\, f(\omega) \succeq f(\omega')\text{ for all }\omega' \in \Omega\} \cap \{\omega \in \Omega \,|\, g(\omega) \succeq g(\omega')\text{ for all }\omega' \in \Omega\} \neq \emptyset$.

In other words, two acts f and g are cominimum if the set of states of act f with worst outcomes and that of g have a common element. Similarly, two acts f and g are comaximum if the set of states of act f with best outcomes and that of g have a common element. Under the assumption that a state space Ω is finite, if we adopt the notions of cominimum and comaximum acts, then we can rewrite the Extreme Events Sensitivity Axiom (see Axiom 7 in Appendix) as follows.

Axiom 1. For any $f, g, h \in L_0$ such that $f \sim g$ and g and h are cominimum and comaximum,
1. if f and h are cominimum, then $(1/2)g \oplus (1/2)h \succeq (1/2)f \oplus (1/2)h$,
2. if f and h are comaximum, then $(1/2)f \oplus (1/2)h \succeq (1/2)g \oplus (1/2)h$.

Note that $(1/2)f \oplus (1/2)g$ denotes a preference average (or a subjective mixture) of acts f and g (see Definition 6 in Appendix).

The following axiom captures the uncertainty loving behaviors of DMs that are reflected in neo-additive CEU as optimistic attitudes. Axiom 2 corresponds to Condition 1 in Extreme Sensitivity Axiom by Chateauneuf et al. (2007) (see Appendix).

[10] The notions of cominimum and comaximum functions depend on a collection $\mathcal{E}$ of Ω. However, we do not mention a collection $\mathcal{E}$ because we consider only $\mathcal{E} = \{\Omega\}$.

Axiom 2 (Cominimum Uncertainty Lovingness)**.** For all cominimum acts f, g, if $f \sim g$, then $(1/2)f + (1/2)g \preceq f$.

Alternatively, the following axiom captures the uncertainty averse behaviors of DMs that are reflected in neo-additive CEU as pessimistic attitudes. Axiom 3 corresponds to Condition 2 in the Extreme Sensitivity Axiom by Chateauneuf et al. (2007) (see Appendix).

Axiom 3 (Comaximum Uncertainty Aversion)**.** For all comaximum acts f, g, if $f \sim g$, then $(1/2)f + (1/2)g \succeq f$.

To avoid confusion, some caveats are worth mentioning. Axiom 2 capturing the optimistic attitudes of DMs leads to the nonnegative coefficient $\bar{\eta}$ of the maximum part in Representation (2), while Axiom 3 capturing the pessimistic attitudes of DMs leads to the nonnegative coefficient $\tilde{\eta}$ of the minimum part in Representation (2) (see the proof of Theorem 1 in detail). Therefore, Axioms 2 and 3 are natural behavioral axioms to impose.

Under the assumptions that a state space is finite and the preferences of DMs are represented by CEU, Properties 1 and 2 follow from Axioms 2 and 3, which means that these preferences are represented by the neo-additive CEU from Theorem 1. Note that we need to assume that DMs' preferences are represented by CEU.

Theorem 2. *If a binary relation $\succeq$ defined on L_0 is represented by CEU and it satisfies Axioms 2 and 3, then Properties 1 and 2 are satisfied. That is, DM's preferences are represented by neo-additive CEU. Conversely, if a binary relation $\succeq$ on L_0 is represented by neo-additive CEU, then it satisfies Axioms 2 and 3.*

Proof. See Appendix. □

6.3 Two-States Partial Certainty Equivalents

In this subsection, we derive Properties 1 and 2 by proposing a new notion, *two-states partial certainty equivalents*, which is like the certainty equivalent.

Definition 3. Let an act $f \in L_0$ and an ordered pair of two states (ω_i, ω_j) be fixed. A lottery $c \in Y$ is called a *two-states partial certainty equivalent of f with respect to ω_i and ω_j* (simply *TPCE*) if $c \in Y$ satisfies $(c, \{\omega_i, \omega_j\}; f, \Omega \backslash \{\omega_i, \omega_j\}) \sim f$.

We call such an act c TPCE because it is defined as a constant outcome (constant act) on ω_i and ω_j that makes the act $(c, \{\omega_i, \omega_j\}; f, \Omega \backslash \{\omega_i, \omega_j\})$ indifferent to a fixed act f. Note that $(\omega_i, \omega_j) \neq (\omega_j, \omega_i)$ because these are ordered pairs. Under the assumption that a binary relation $\succeq$ on L_0 satisfies Schmeidler's (1989) axioms, it satisfies the Monotonicity and Continuity axioms. Therefore, there exists $c \in Y$ with $(c, \{\omega_i, \omega_j\}; f, \Omega \backslash \{\omega_i, \omega_j\}) \sim f$ such that $f(\omega_i) \succeq c \succeq f(\omega_j)$ or $f(\omega_j) \succeq c \succeq f(\omega_i)$. If there is a TPCE, then there also exists a number $\alpha \in [0, 1]$ such that $c \sim \alpha f(\omega_i) + (1 - \alpha) f(\omega_j)$. We call this α *two-states partial certainty equivalent weight of f with respect to ω_i and ω_j* (simply *TPCE weight*). Note that TPCE weight α denotes the coefficient of the first component ω_i because (ω_i, ω_j) is an ordered pair. Also note that TPCE weight $\alpha \in [0, 1]$ is uniquely determined or equal to the interval itself $[0, 1]$.

Lemma 7. *Suppose that a binary relation $\succeq$ on L_0 is represented by the expected utility with an affine utility function u and a probability p. Then, there exists a common TPCE weight $\alpha \in [0, 1]$ for all f.*

Proof. See Appendix. $\square$

This lemma states that whether a common TPCE weight (that is, α) for all $f \in \mathcal{F}$ exists depends on the additivity of operator I. When operator I is the Choquet integrals, we impose the existence of a common weight for small classes of acts on our analyses.

Let y_1 and y_0 acts such that $u(y_1) = 1$ and $u(y_0) = 0$ (see the proof of Theorem 2 in Appendix or Schmeidler (1989)). For any T with $|T| \geq 2$ and any ordered pair (ω_i, ω_j) with $\omega_i, \omega_j \in T$, we define the following act f_{T,ω_i}:

$$f_{T,\omega_i}(\omega) = \begin{cases} y_1 & \text{if} \quad \omega \in T \backslash \{\omega_i\} \\ y_0 & \text{if} \quad \omega \in T^c \cup \{\omega_i\} \end{cases}$$

This act assigns a good outcome y_1 if state ω occurs in $T \backslash \{\omega_i\}$, and a bad outcome y_0 if state ω occurs in $T^c \cup \{\omega_i\}$. We define this act f_{T,ω_i} because we investigate certainty equivalents of f_{T,ω_i}.

Lemma 8. *Let T with $|T| \geq 2$ and ordered pair (ω_i, ω_j) with $\omega_i, \omega_j \in T$ be fixed. Then, the following are equivalent:*

(1) The number α_{ω_i} is the TPCE weight for act f_{T,ω_i} with ordered pair (ω_i, ω_j).
(2) For capacity v,

$$\alpha_{\omega_i} v(T\backslash\{\omega_i, \omega_j\}) + (1 - \alpha_{\omega_i}) v(T) = v(T\backslash\{\omega_i\}). \tag{4}$$

Proof. See Appendix. □

Lemma 9 states that the modularity of v (that is, (5)) restricted to some sets holds if there exists a common TPCE for f_{T,ω_i} and f_{T,ω_j}.[11]

Lemma 9. *Let T with $|T| \geq 2$ and ordered pair (ω_i, ω_j) with $\omega_i, \omega_j \in T$ be fixed. Then, if there exists a common number for the TPCE weight for act f_{T,ω_i} with ordered pair (ω_i, ω_j) and the TPCE weight for act f_{T,ω_j} with ordered pair (ω_i, ω_j), then it holds that*

$$v(T\backslash\{\omega_i, \omega_j\}) + v(T) = v(T\backslash\{\omega_j\}) + v(T\backslash\{\omega_i\}). \tag{5}$$

Proof. See Appendix. □

The modularity of v restricted to some sets represents the additivity of Choquet integrals. Therefore, by assuming the following axiom together with CEU, Property 1 can be obtained.

Axiom 4. For any T with $3 \leq |T| \leq n - 1$ and any $\omega_i, \omega_j \in T$, there exists a common TPCE weight for f_{T,ω_i} and f_{T,ω_j} with ordered pair (ω_i, ω_j).

In contrast to Lemma 9, the difference in TPCE leads to the convexity of v on some sets.

Lemma 10. *Let T with $|T| \geq 2$ and ordered pair $(\omega_i, \omega_j) \in T$ be fixed. Let α_{ω_i} be a TPCE weight for f_{T,ω_i} with ordered pair (ω_i, ω_j) and α_{ω_j} be a TPCE weight for f_{T,ω_j} with ordered pair (ω_i, ω_j). Then, if there exist α_{ω_i} and α_{ω_j} such that $\alpha_{\omega_i} \geq \alpha_{\omega_j}$, then it holds that*

$$v(T\backslash\{\omega_i, \omega_j\}) + v(T) \geq v(T\backslash\{\omega_j\}) + v(T\backslash\{\omega_i\}). \tag{6}$$

[11] We can impose the existence of a common TPCE wight for larger classes of acts than those analyzed in this paper. See Asano and Kojima (2022) in detail.

Proof. See Appendix. □

By Lemma 10, the following axiom together with CEU show (P2-b) in Property 2.

Axiom 5. There exists a two-point set $T^0 = \{\omega^*, \omega^{**}\}$ satisfying the following: there exist α_{ω^*} and $\alpha_{\omega^{**}}$ such that $\alpha_{\omega^*} \geq \alpha_{\omega^{**}}$, where α_{ω^*} is a TPCE weight for f_{Ω,ω^*} with ordered pair (ω^*, ω^{**}) and $\alpha_{\omega^{**}}$ is a TPCE weight for $f_{\Omega,\omega^{**}}$ with ordered pair (ω^*, ω^{**}).

Lemma 11. *Under Axiom 5, (P2-b) in Property 2 holds.*

Proof. See Appendix. □

Together, the following axiom and CEU show (P2-a) in Property 2.

Axiom 6. For $T^0 = \{\omega^*, \omega^{**}\}$ in Axiom 5, it holds that $\alpha_{\omega^{**}} \geq \alpha_{\omega^*}$, where α_{ω^*} is a TPCE weight for f_{T^0,ω^*} with ordered pair (ω^*, ω^{**}) and $\alpha_{\omega^{**}}$ is a TPCE weight for $f_{T^0,\omega^{**}}$ with ordered pair (ω^*, ω^{**}).

Lemma 12. *Under Axiom 6, (P2-a) in Property 2 holds.*

Proof. See Appendix. □

In summary, we obtain the following theorem.

Theorem 3. *If a binary relation $\succeq$ defined on L_0 is represented by CEU and satisfies Axioms 4–6, then Properties 1 and 2 are satisfied. That is, DM's preferences are represented by neo-additive CEU. Conversely, if a binary relation $\succeq$ defined on L_0 is represented by neo-additive CEU, then it satisfies Axioms 4–6.*

Proof. See Appendix. □

7 Conclusion

In this paper, by assuming that a state space is finite and Properties 1 and 2 hold, we provided an easy-to-understand derivation of neo-additive CEU by Chateauneuf et al. (2007) based on Möbius inversions. Properties 1 and 2 are the applications of

Asano and Kojima (2022) that provide a characterization of a class of belief functions. Our axiomatization based on Möbius inversions then shed new light on their usefulness for our understanding of the behaviors of decision makers. Furthermore, our approach based on the properties of Möbius inversions enabled us to understand the relationship between Chateauneuf et al. (2007), Kajii et al. (2009), and Asano and Kojima (2022) in a clearer and more unified way.

The key point of our methodology is to form the conjugate of unanimity games using the equality that the sum of a class of Möbius inversions is zero, which is mentioned in the proof of Theorem 1 in Section 5. We provided the axiomatization of JP-capacity for $v_1 = \mu_\Omega$. As future research, an axiomatization of JP-capacity for a general v_1 from a behavioral viewpoint would be worth investigating.

Appendix

Appendix A

Appendix A presents the definitions and the results provided by Ghirardato and Marinacci (2001), Ghirardato et al. (2003), and Chateauneuf et al. (2007).

Definition 4. Let Ω be a state space, and let $\mathcal{M}$ be a σ-algebra of subsets of Ω. The collection of *null events*, $\mathcal{N}$, satisfies (i) $\emptyset \in \mathcal{N}$, (ii) if $A \in \mathcal{N}$, then $B \in \mathcal{N}$ for all $B \subset A$, and (iii) if $A, B \in \mathcal{N}$, then $A \cup B \in \mathcal{N}$. The collection of *universal events* is defined by $\mathcal{U} = \{E \in \mathcal{M} | \Omega \backslash E \in \mathcal{N}\}$. The collection of *essential sets* is defined by $\mathcal{M}^* = \mathcal{M} \backslash (\mathcal{N} \cup \mathcal{U})$.

Ghirardato and Marinacci (2001) propose the notion of *biseparable preferences.* A functional form $J : L_0 \to \mathbb{R}$ such that $J(f) \geq J(g)$ if and only if $f \succeq g$ is a *representation* of $\succeq$. J is monotonic if $J(f) \geq J(g)$ whenever $f(\omega) \succeq g(\omega)$ for all $\omega \in \Omega$, and J is *nontrivial* if $J(f) \neq J(g)$ for some $f, g \in L_0$. Given a binary relation, a functional $J : L_0 \to \mathbb{R}$ is a *canonical representation* of $\succeq$ if it is a nontrivial and monotonic representation of $\succeq$ and letting $u(x) := J(x)$ for all $x \in X$, there exists a function $\rho : \mathcal{F} \to [0, 1]$ such that

$$J(xEy) = u(x)\rho(E) + u(y)(1 - \rho(E))$$

for all outcomes $x \succeq y$ and all $E \in \mathcal{F}$, where xEy denotes an act that equals x if $\omega \in E$ and equals y if $\omega \notin \Omega$.

Definition 5 (Ghirardato and Marinacci (2001))**.** A binary relation $\succeq$ on L_0 is a *biseparable preference* if (1) it has a canonical representation, (2) in the case that $\succeq$ has at least one essential event, then such representation is unique up to a positive affine transformation.

For any act $f \in L_0$, the certainty equivalent of f, denoted by c_f, is the set of outcomes indifferent to f, $\{x \in X : x \sim f\}$.

Definition 6 (Ghirardato et al. (2003))**.** Given $x, y \in X$ such that $x \succeq y$, an outcome $z \in X$ is a *preference average of x and y (given E)* if $x \succeq z \succeq y$ and $xEy \succeq c_{xEz}Ec_{zEy}$.

Proposition 3 (Ghirardato et al. (2003, Proposition 1))**.** *Let $\succeq$ be a biseparable preference. For each $x, y \in X$ and each essential event $E \in \mathcal{M}$, a outcome $z \in X$ is a preference average of x and y given E if and only if*

$$u(z) = \frac{1}{2}u(x) + \frac{1}{2}u(y).$$

Hence, preference averages of x and y given E exist for every essential event $E \in \mathcal{M}$, they do not depend on either on the choice of E or on the normalization of u, and they form an indifference class.

Based on this proposition, we can denote a preference average of x and y by $(1/2)x \oplus (1/2)y$.

Definition 7 (Ghirardato et al. (2003))**.** Given $f, g \in L_0$ and $\alpha \in [0,1]$, a *subjective mixture* of f and g with weight α is any act $h \in L_0$ such that $h(\omega) \sim \alpha f(\omega) \oplus (1-\alpha)g(\omega)$ for every $\omega \in \Omega$.

Definition 8 (Chateauneuf et al. (2007))**.** Fix $f \in L_0$. An outcome $z \in X$ is *in the indifference set of the infimum of* f, denoted by $z \in \inf_{\succeq}(f)$, if for $A := f^{-1}(x : z \succ x)$, $z_A f \sim f$ and if for every $y \succ z$ and $B := f^{-1}(x : y \succeq x)$, $y_B f \succ f$. Similarly, an outcome $z \in X$ is *in the indifference set of the supremum of* f, denoted by $z \in \sup_{\succeq}(f)$, if for $A := f^{-1}(x : x \succ z)$, $z_A f \sim f$ and if for every y such that $z \succ y$ and $B := f^{-1}(x : x \succeq y)$, $f \succ y_B f$.

Axiom 7 (Extreme Events Sensitivity (Chateauneuf et al. (2007)))**.** For any $f, g, h \in L_0$ such that $f \sim g$ and $h \in \underline{L_0}(g) \cap \overline{L_0}(g)$,
1. if $h \in \underline{L_0}(f)$, then $(1/2)g \oplus (1/2)h \succeq (1/2)f \oplus (1/2)h$,
2. if $h \in \overline{L_0}(f)$, then $(1/2)f \oplus (1/2)h \succeq (1/2)g \oplus (1/2)h$, where $\underline{L_0}(f)$ and $\overline{L_0}(f)$ are defined as follows:

$$\underline{L_0}(f) := \left\{ h \in L_0 \,\middle|\, \left\{ \omega \in \Omega : f(\omega) \succ \inf_{\succeq}(f) \right\} \cup \left\{ \omega \in \Omega : h(\omega) \succ \inf_{\succeq}(h) \right\} \notin \mathcal{U} \right\},$$
$$\overline{L_0}(f) := \left\{ h \in L_0 \,\middle|\, \left\{ \omega \in \Omega : f(\omega) \succeq \sup_{\succeq}(f) \right\} \cap \left\{ \omega \in \Omega : h(\omega) \succeq \sup_{\succeq}(h) \right\} \notin \mathcal{N} \right\}.$$

Appendix B: Proofs

Proof of Theorem 1. **(Step 1)** Because we assume that Schmeidler's (1989) five axioms are satisfied, there exist a unique capacity v and a unique affine function u such that

$$f \succeq g \Leftrightarrow I(f) \geq I(g),$$

where $I(f) = \int_\Omega u(f)dv$.
(Step 2) It follows from Lemma 1 that

$$v = \sum_{1 \leq k \leq n} \eta_{\{\omega_k\}} \mu_{\{\omega_k\}} + \sum_{|T| \geq 2} \eta_T \mu_T.$$

(Step 3) By (P2-a) in Property 2 and Lemma 6, it holds that

$$\begin{aligned} & v(T^0) + v(T^0 \backslash \{\omega_i, \omega_j\}) \leq v(T^0 \backslash \{\omega_i\}) + v(T^0 \backslash \{\omega_j\}) \\ \Leftrightarrow \quad & \sum_{\{\omega_i, \omega_j\} \subseteq T \subseteq T^0} \eta_T \leq 0 \\ \Leftrightarrow \quad & \eta_{T^0} \leq 0. \end{aligned}$$

Here, let $\eta_{T^0} = -\bar{\eta}$.
(Step 4) It follows from Property 1 and Lemma 4 that for any $T = \{\omega_i, \omega_j, \omega_k\}$,

$$\begin{aligned} & v(T) + v(T \backslash \{\omega_i, \omega_j\}) = v(T \backslash \{\omega_i\}) + v(T \backslash \{\omega_j\}) \\ \Leftrightarrow \quad & \sum_{\{\omega_i, \omega_j\} \subseteq T \subseteq \{\omega_i, \omega_j, \omega_k\}} \eta_T = 0 \\ \Leftrightarrow \quad & \eta_{\{\omega_i, \omega_j\}} + \eta_{\{\omega_i, \omega_j, \omega_k\}} = 0. \end{aligned}$$

Because ω_i, ω_j, and ω_k are arbitrarily chosen, it holds that

$$\eta_{\{\omega_i, \omega_j\}} = \eta_{\{\omega_j, \omega_k\}} = \eta_{\{\omega_i, \omega_k\}} = -\eta_{\{\omega_i, \omega_j, \omega_k\}}.$$

(Step 5) First, we show the following lemma.

Lemma 13. *For any S with $2 \leq |S| \leq n-1$, $\eta_S = (-1)^{|S|-1}\bar{\eta}$.*

Proof. We show this lemma by induction.
(i) The case of $|S| = 2$ holds by Step 4.
(ii) We assume that the statement holds for $|S| = m$ with $2 \leq m < n-2$. Consider

a set S with $|S| = m+1$, and pick $\omega_i, \omega_j \in S$ arbitrarily. By Property 1 and Lemma 4, it holds that

$$\sum_{\{\omega_i,\omega_j\}\subseteq T\subseteq S} \eta_T = 0.$$

Now, it follows from the binomial theorem that

$$\begin{aligned} & \sum_{\{\omega_i,\omega_j\}\subseteq T\subseteq S} \eta_T \\ = \ & \bar{\eta}\left((-1) + \binom{m-1}{1}(-1)^2 + \binom{m-1}{2}(-1)^3 + \cdots + \binom{m-1}{m-2}(-1)^{m-1}\right) + \eta_S. \end{aligned}$$

Here,

$$\begin{aligned} & (-1) + \binom{m-1}{1}(-1)^2 + \binom{m-1}{2}(-1)^3 + \cdots + \binom{m-1}{m-2}(-1)^{m-1} \\ = \ & (-1)\left((1+(-1))^{m-1} - (-1)^{m-1}\right) = (-1)^{m-1}. \end{aligned}$$

Therefore, by $(-1)^{m-1}\bar{\eta} + \eta_S = 0$, it holds that $\eta_S = (-1)^m\bar{\eta}$, which proves this lemma. □

(Step 6) By (P2-b) in Property 2 (b) and Lemma 5, it holds that for $T^0 = \{\omega_i, \omega_j\}$,

$$\sum_{\{\omega_i,\omega_j\}\subseteq T\subseteq \Omega} \eta_T \geq 0.$$

Now, it follows from the binomial theorem that

$$\begin{aligned} & \sum_{\{\omega_i,\omega_j\}\subseteq T\subseteq \Omega} \eta_T \\ = \ & \bar{\eta}\left((-1) + \binom{n-2}{1}(-1)^2 + \binom{n-2}{2}(-1)^3 + \cdots + \binom{n-2}{n-3}(-1)^{n-2}\right)\bar{\eta} + \eta_\Omega \\ = \ & (-1)\left((1+(-1))^{n-2} - (-1)^{n-2}\right)\bar{\eta} + \eta_\Omega = (-1)^{n-2}\bar{\eta} + \eta_\Omega. \end{aligned}$$

Thus, $(-1)^{n-2}\bar{\eta} + \eta_\Omega \geq 0$.

(Step 7) By Lemma 13 in Step 5, it holds that

$$v = \sum_{1\leq i\leq n} \eta_{\{\omega_i\}}\mu_{\{\omega_i\}} + \sum_{2\leq |S|<n} (-1)^{|S|-1}\bar{\eta}\mu_S + \eta_\Omega\mu_\Omega,$$

which turns out to be the following:

$$v = \sum_{1 \le i \le n} (\eta_{\{\omega_i\}} - \bar{\eta})\mu_{\{\omega_i\}} + \bar{\eta} \sum_{S} (-1)^{|S|-1} \mu_S + ((-1)^{n-2}\bar{\eta} + \eta_\Omega)\mu_\Omega. \tag{7}$$

Here, let $\tilde{\eta} = (-1)^{n-2}\bar{\eta} + \eta_\Omega$. Then, $\tilde{\eta} \ge 0$ by Step 6. Equation (7) can be rewritten as follows:

$$v = \sum_{1 \le i \le n} (\eta_{\{\omega_i\}} - \bar{\eta})\mu_{\{\omega_i\}} + \bar{\eta} \sum_{S} (-1)^{|S|-1} \mu_S + \tilde{\eta}\mu_\Omega.$$

Because $\sum_S (-1)^{|S|-1}\mu_S = w_\Omega$ by Lemma 13, it holds that

$$v = \sum_{1 \le i \le n} (\eta_{\{\omega_i\}} - \bar{\eta})\mu_{\{\omega_i\}} + \tilde{\eta}\mu_\Omega + \bar{\eta} w_\Omega.$$

By letting $\eta^0_{\{\omega_i\}} = \eta_{\{\omega_i\}} - \bar{\eta}$, it follows that

$$v = \sum_{1 \le i \le n} \eta^0_{\{\omega_i\}}\mu_{\{\omega_i\}} + \tilde{\eta}\mu_\Omega + \bar{\eta} w_\Omega.$$

Because v is a capacity, it holds for all i, k that

$$\begin{aligned} 0 &\le v(\{\omega_i, \omega_k\}) - v(\{\omega_i\}) \\ &= \eta_{\{\omega_i\}} + \eta_{\{\omega_k\}} + \eta_{\{\omega_i, \omega_k\}} - \eta_{\{\omega_i\}} \\ &= \eta_{\{\omega_k\}} - \bar{\eta} = \eta^0_{\{\omega_k\}}. \end{aligned}$$

Thus, it holds that

$$I(f) = \int_\Omega u(f)dv = \sum_{i=1}^{n} \eta^0_{\{\omega_i\}} u(f(\omega_i)) + \tilde{\eta} \min_{\omega_i \in \Omega} u(f(\omega_i)) + \bar{\eta} \max_{\omega_i \in \Omega} u(f(\omega_i)).$$

Next, we show the converse. Define a capacity by $v = \sum_{i=1}^n \eta^0_{\{\omega_i\}}\mu_{\{\omega_i\}} + \tilde{\eta}\mu_\Omega + \bar{\eta} w_\Omega$. Then, it follows from Proposition 1 that $\int_\Omega u(f)dv = I(f)$, which shows that Schmeidler's (1989) five axioms are satisfied.

Let $T \in \mathcal{F}$ be any event with $3 \le |T| \le n-1$, and let $\omega_i, \omega_j \in T$ be any distinct points. Then, it holds that

$$\begin{aligned} & v(T) + v(T \backslash \{\omega_i, \omega_j\}) - v(T \backslash \{\omega_i\}) - v(T \backslash \{\omega_j\}) \\ = & \sum_{1 \le k \le n} \eta^0_{\{\omega_k\}} + \bar{\eta} + \sum_{1 \le k \le n,\, k \ne i,\, k \ne j} \eta^0_{\{\omega_k\}} + \bar{\eta} - \left(\sum_{1 \le k \le n,\, k \ne i} \eta^0_{\{\omega_k\}} + \bar{\eta} + \sum_{1 \le k \le n,\, k \ne j} \eta^0_{\{\omega_k\}} + \bar{\eta} \right) \end{aligned}$$

$$= \quad 0,$$

which shows that Property 1 holds.

Let $\omega_i, \omega_j \in T$ be any distinct points, and let $T^0 = \{\omega_i, \omega_j\}$. Then, it holds that

$$\begin{aligned} & v(T^0) + v(T^0 \backslash \{\omega_i, \omega_j\}) - v(T^0 \backslash \{\omega_i\}) - v(T^0 \backslash \{\omega_j\}) \\ = \quad & (\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}} + \bar{\eta}) + 0 - (\eta^0_{\{\omega_j\}} + \bar{\eta}) - (\eta^0_{\{\omega_i\}} + \bar{\eta}) = -\bar{\eta} \leq 0, \end{aligned}$$

which shows that (P2-a) in Property 2 holds. Furthermore, it also holds that

$$\begin{aligned} & v(\Omega) + v(\Omega \backslash \{\omega_i, \omega_j\}) - v(\Omega \backslash \{\omega_i\}) - v(\Omega \backslash \{\omega_j\}) \\ = \quad & \sum_{1 \leq k \leq n} \eta^0_{\{\omega_k\}} + \tilde{\eta} + \bar{\eta} \\ & + \sum_{1 \leq k \leq n,\, k \neq i,\, k \neq j} \eta^0_{\{\omega_k\}} + \bar{\eta} - \left(\sum_{1 \leq k \leq n,\, k \neq i} \eta^0_{\{\omega_k\}} + \bar{\eta} + \sum_{1 \leq k \leq n,\, k \neq j} \eta^0_{\{\omega_k\}} + \bar{\eta} \right) \\ = \quad & \tilde{\eta} \geq 0, \end{aligned}$$

which shows that (P2-b) in Property 2 holds. $\square$

Proof of Proposition 2. First, we show (a) in Condition (ii) of Proposition 2. Fix a set E be fixed. It suffices to show that $v(E \cup F) - v(F) =$ constant for any F with $E \cap F = \emptyset$ and $E \cup F \neq \Omega$. Note that $v(E \cup F) - v(F) - v(E)$ is equal to the summation of η_T over T with $T \subseteq E \cup F$, $E \cap T \neq \emptyset \neq F \cap T$. Let $\mathcal{E}$ be the collection of such T. Then, it follows from Lemma 3 that

$$v(E \cup F) - v(F) - v(E) = \sum_{T \in \mathcal{E}} \eta_T.$$

Because $\eta_S = (-1)^{|S|-1} \bar{\eta}$ for all S with $2 \leq |S| \leq n - 1$, it holds that

$$v(E \cup F) - v(F) - v(E) = \bar{\eta} \sum_{T \in \mathcal{E}} (-1)^{|T|-1}. \tag{8}$$

Because $E \cap F = \emptyset$, $T \subseteq E \cup F$, and $E \cap F \neq \emptyset \neq F \cap T$, any $T \in \mathcal{E}$ is equal to $E' \cup F'$, where $\emptyset \neq E' \subseteq E$ and $\emptyset \neq F' \subseteq F$. Letting $|E'| = x$ and $|F'| = y$, $|T| = x + y$ holds, which implies that $(-1)^{|T|-1} = (-1)(-1)^x(-1)^y$. Thus, it holds that

$$\sum_{T \in \mathcal{E}} (-1)^{|T|-1} = (-1) \sum_{1 \leq x \leq |E|,\, 1 \leq y \leq |F|} \begin{pmatrix} |E| \\ x \end{pmatrix} \begin{pmatrix} |F| \\ y \end{pmatrix} (-1)^{x+y}$$

$$
\begin{aligned}
&= (-1)\left(\sum_{1\leq x\leq |E|}\binom{|E|}{x}(-1)^x\right)\left(\sum_{1\leq y\leq |F|}\binom{|F|}{y}(-1)^y\right)\\
&= (-1)\left((1-1)^{|E|}-1\right)\left((1-1)^{|F|}-1\right)=-1. \qquad (9)
\end{aligned}
$$

Because (8) and (9) imply that $v(E\cup F)-v(F)-v(E)=-\bar{\eta}$, we show that $v(E\cup F)-v(F)=\text{constant}$.

Next, we show (b) and (c) in Condition (ii) of Proposition 2. By setting $E=\{\omega^*\}$ and $F=\{\omega^{**}\}$, it holds that

$$
v(E\cup F)\leq v(E)+v(F)\Leftrightarrow v(\{\omega^*,\omega^{**}\})\leq v(\{\omega^*\})+v(\{\omega^{**}\}),
$$

which is the same as (P2-a) in Property 2. It also holds that

$$
\begin{aligned}
& v'(E\cup F)\leq v'(E)+v'(F)\Leftrightarrow v'(\{\omega^*,\omega^{**}\})\leq v'(\{\omega^*\})+v'(\{\omega^{**}\})\\
\Leftrightarrow\ & 1-v(\Omega\backslash\{\omega^*,\omega^{**}\})\leq 1-v(\Omega\backslash\{\omega^*\})+1-v(\Omega\backslash\{\omega^{**}\})\\
\Leftrightarrow\ & v(\Omega\backslash\{\omega^*\})+v(\Omega\backslash\{\omega^{**}\})\leq v(\Omega)+v(\Omega\backslash\{\omega^*,\omega^{**}\}),
\end{aligned}
$$

which is the same as (P2-b) in Property 2. □

Proof of Theorem 2

To prove Theorem 2, we first show the following lemma.

Lemma 14. *Suppose that a binary relation $\succeq$ defined on L_0 is represented by CEU and it satisfies Axioms 2 and 3. Let $a:\Omega\to\mathbb{R}$ be a function such that $a(\omega)=u(f(\omega))$ for an act $f\in L_0$, and define an operator $J(a)$ by $J(a)=\int_\Omega u(f)dv$. Then, (1) if a and b are cominimum, then $J(a+b)\leq J(a)+J(b)$, and (2) if a and b are comaximum, then $J(a+b)\geq J(a)+J(b)$.*

Proof of Lemma 14. We prove this lemma based on Gilboa and Schmeidler (1989, Lemma 3.3). Let f,g be any pair of cominimum acts. Let $p=u(f)$ and $q=u(g)$. Then, p and q are cominimum functions. First, if $J(p)=J(q)$, then $f\sim g$, and by Axiom 2, $(1/2)f+(1/2)g\preceq f$ for any cominimum acts f,g. Therefore, it follows that

$$
J\left(u\left(\frac{1}{2}f+\frac{1}{2}g\right)\right)\leq J(p)=\frac{1}{2}J(p)+\frac{1}{2}J(q).
$$

By affinity of u,

$$J\left(u\left(\frac{1}{2}f+\frac{1}{2}g\right)\right)=J\left(\frac{1}{2}u(f)+\frac{1}{2}u(g)\right)=J\left(\frac{1}{2}p+\frac{1}{2}q\right).$$

By these two formulas, it holds that $J(p+q)\leq J(p)+J(q)$.

Next, assume $J(p)>J(q)$. Let $\gamma=J(p)-J(q)$. Set $q'=q+\gamma 1_\Omega$, where 1_Ω denotes the indication function. Note that p and q' are also cominimum functions. Furthermore, it holds that $J(q')=J(q+\gamma 1_\Omega)=J(q)+\gamma=J(p)$, where the second equality holds by the property of Choquet integrals. Similar to the above argument, it holds that $J((1/2)p+(1/2)q')\leq(1/2)J(p)+(1/2)J(q')$. The left-hand side of the inequality is equal to $J((1/2)p+(1/2)q)+(1/2)\gamma$, and the right-hand side is equal to $(1/2)J(p)+(1/2)J(q)+(1/2)\gamma$, which means that $J((1/2)p+(1/2)q)\leq(1/2)J(p)+(1/2)J(q)$. Thus, the first half of the claim is shown. The second half of the claim can be similarly shown. □

Proof of Theorem 2. First, we suppose that a binary relation $\succeq$ on L_0 is represented by CEU and it satisfies Axioms 2 and 3. Note that by Schmeidler (1989), there exist acts y_1 and y_0 such that $u(y_1)=1$ and $u(y_0)=0$. For these acts y_1 and y_0, let $y'=(1/2)y_1+(1/2)y_0$. Then, $u(y')=1/2$. For any ω^*,ω^{**}, take two functions $a=(1/2)1_{\{\omega^*\}}$ and $b=(1/2)1_{\{\omega^{**}\}}$. Indeed, two functions a and b are well-defined because for $f=(y',\{\omega^*\};y_0,\{\Omega\backslash\{\omega^*\}\})$ and $g=(y',\{\omega^{**}\};y_0,\{\Omega\backslash\{\omega^{**}\}\})$, $a=u(f)$ and $b=u(g)$. Because $a=(1/2)1_{\{\omega^*\}}$ and $b=(1/2)1_{\{\omega^{**}\}}$ are cominimum, it holds for $J(a)$ defined in Lemma 14 that

$$J\left(\frac{1}{2}1_{\{\omega^*\}}+\frac{1}{2}1_{\{\omega^{**}\}}\right)\leq J\left(\frac{1}{2}1_{\{\omega^*\}}\right)+J\left(\frac{1}{2}1_{\{\omega^{**}\}}\right).$$

The left-hand side of the inequality is equal to $(1/2)v(\{\omega^*,\omega^{**}\})$, and the right-hand side is equal to $(1/2)v(\{\omega^*\})+(1/2)v(\{\omega^{**}\})$, which means that (P2-a) in Property 2 holds.

Similar to the above argument, we can set $a=(1/2)1_{\{\Omega\backslash\{\omega^*\}\}}$ and $b=(1/2)1_{\{\Omega\backslash\{\omega^{**}\}\}}$. Because a and b are comaximum, it holds by Lemma 14 that

$$J\left(\frac{1}{2}1_{\{\Omega\backslash\{\omega^*\}\}}+\frac{1}{2}1_{\{\Omega\backslash\{\omega^{**}\}\}}\right)\geq J\left(\frac{1}{2}1_{\{\Omega\backslash\{\omega^*\}\}}\right)+J\left(\frac{1}{2}1_{\{\Omega\backslash\{\omega^{**}\}\}}\right).$$

The left-hand side of the inequality is equal to $(1/2)v(\Omega) + (1/2)v(\Omega\backslash\{\omega^*, \omega^{**}\})$, and the right-hand side is equal to $(1/2)v(\Omega\backslash\{\omega^*\})+(1/2)v(\Omega\backslash\{\omega^{**}\})$, which means that (P2-b) in Property 2 holds.

Finally, we show Property 1. Take any T with $3 \leq |T| \leq n-1$ and any $\omega_i, \omega_j \in T$. Similarly, we can set $a = (1/2)1_{\{T\backslash\{\omega_i\}\}}$ and $b = (1/2)1_{\{T\backslash\{\omega_j\}\}}$. Because a and b are cominimum and comaximum, it holds by Lemma 14 that

$$J\left(\frac{1}{2}1_{\{T\backslash\{\omega^*\}\}} + \frac{1}{2}1_{\{T\backslash\{\omega^{**}\}\}}\right) = J\left(\frac{1}{2}1_{\{T\backslash\{\omega^*\}\}}\right) + J\left(\frac{1}{2}1_{\{T\backslash\{\omega^{**}\}\}}\right).$$

The left-hand side of the inequality is equal to $(1/2)v(T)+(1/2)v(T\backslash\{\omega^*, \omega^{**}\})$, and the right-hand side is equal to $(1/2)v(T\backslash\{\omega^*\})+(1/2)v(T\backslash\{\omega^{**}\})$, which means that Property 1 holds.

Now, we show the converse of Theorem 2. Suppose DMs' preferences are represented by (2). If $f \sim g$, then $I(f) = I(g)$. If f and g are cominimum, then it holds that

$$\min_{\omega_i \in \Omega}\left\{\frac{1}{2}u(f(\omega_i)) + \frac{1}{2}u(f(\omega_j))\right\} = \frac{1}{2}\min_{\omega_i \in \Omega} u(f(\omega_i)) + \frac{1}{2}\min_{\omega_i \in \Omega} u(f(\omega_i)).$$

Furthermore, for any acts (functions) f, g, it holds that

$$\max_{\omega_i \in \Omega}\left\{\frac{1}{2}u(f(\omega_i)) + \frac{1}{2}u(f(\omega_j))\right\} \leq \frac{1}{2}\max_{\omega_i \in \Omega} u(f(\omega_i)) + \frac{1}{2}\max_{\omega_i \in \Omega} u(f(\omega_i)).$$

Therefore, keeping in mind that the first term in (2) is additive, it follows that

$$I\left(\frac{1}{2}f + \frac{1}{2}g\right) \leq \frac{1}{2}I(f) + \frac{1}{2}I(g) = I(f),$$

which means that $(1/2)f + (1/2)g \preceq f$. Thus, Axiom 2 holds. A similar argument shows that Axiom 3 holds, which completes the proof of Theorem 2. □

Proof of Lemma 7. Because v is a probability (that is, additive), it holds that

$$\begin{aligned} & (c, \{\omega_i, \omega_j\}; f, \Omega\backslash\{\omega_i, \omega_j\}) \sim f \\ \Leftrightarrow \quad & u(c)(v(\{\omega_i\}) + v(\{\omega_j\})) = u(f(\omega_i))v(\{\omega_i\}) + u(f(\omega_j))v(\{\omega_j\}). \end{aligned}$$

If $v(\{\omega_i\})+v(\{\omega_j\}) = 0$, then the equality holds for any c. If $v(\{\omega_i\})+v(\{\omega_j\}) \neq 0$, then by letting $\alpha = v(\{\omega_i\})/(v(\{\omega_i\}) + v(\{\omega_j\}))$, the affinity of u shows that $c \sim \alpha f(\omega_i) + (1-\alpha)f(\omega_j)$ for all f. □

Proof of Lemma 8. Let $u(y_1) = 1$, $u(y_0) = 0$, and let I be a Choquet integral. Then, it holds that

$$\begin{aligned} & \alpha_{\omega_i} \in [0,1] \text{ is the TPCE weight for act } f_{T,\omega_i} \text{ with ordered pair } (\omega_i, \omega_j) \\ \Leftrightarrow\ & (\alpha_{\omega_i} y_0 + (1-\alpha_{\omega_i}) y_1, \{\omega_i, \omega_j\}; y_1, T \backslash \{\omega_i, \omega_j\}; y_0, T^c) \sim f_{T,\omega_i} \\ \Leftrightarrow\ & \alpha_{\omega_i} v(T \backslash \{\omega_i, \omega_j\}) + (1-\alpha_{\omega_i}) v(T) = v(T \backslash \{\omega_i\}). \end{aligned}$$

□

Proof of Lemma 9. Let α_{ω_i} be a common TPCE weight. Because (ω_i, ω_j) is an ordered pair, it holds that

$$\begin{aligned} & \alpha_{\omega_j} \in [0,1] \text{ is the TPCE weight for act } f_{T,\omega_j} \text{ with ordered pair } (\omega_i, \omega_j) \\ \Leftrightarrow\ & (\alpha_{\omega_j} y_1 + (1-\alpha_{\omega_j}) y_0, \{\omega_i, \omega_j\}; y_1, T \backslash \{\omega_i, \omega_j\}; y_0, T^c) \sim f_{T,\omega_j} \\ \Leftrightarrow\ & (1-\alpha_{\omega_j}) v(T \backslash \{\omega_i, \omega_j\}) + \alpha_{\omega_j} v(T) = v(T \backslash \{\omega_j\}). \end{aligned} \tag{10}$$

When $\alpha_{\omega_i} = \alpha_{\omega_j}$, adding both sides of (4) and (10) shows the claim. □

Proof of Lemma 10. By Lemma 8, it holds that

$$\begin{aligned} & \alpha_{\omega_i} \in [0,1] \text{ is the TPCE weight for act } f_{T,\omega_i} \text{ with ordered pair } (\omega_i, \omega_j) \\ \Leftrightarrow\ & \alpha_{\omega_i} v(T \backslash \{\omega_i, \omega_j\}) + (1-\alpha_{\omega_i}) v(T) = v(T \backslash \{\omega_i\}). \end{aligned} \tag{11}$$

On the other hand, it holds that

$$\begin{aligned} & \alpha_{\omega_j} \in [0,1] \text{ is the TPCE weight for act } f_{T,\omega_j} \text{ with ordered pair } (\omega_i, \omega_j) \\ \Leftrightarrow\ & (\alpha_{\omega_j} y_1 + (1-\alpha_{\omega_j}) y_0, \{\omega_i, \omega_j\}; y_1, T \backslash \{\omega_i, \omega_j\}; y_0, T^c) \sim f_{T,\omega_j}. \end{aligned}$$

If $\alpha_{\omega_i} \geq \alpha_{\omega_j}$, then the monotonicity of a binary relation $\succeq$ shows that

$$(\alpha_{\omega_i} y_1 + (1-\alpha_{\omega_i}) y_0, \{\omega_i, \omega_j\}; y_1, T \backslash \{\omega_i, \omega_j\}; y_0, T^c) \succeq f_{T,\omega_j}.$$

Similar to the argument in the proof of Lemma 9, it holds that

$$(1-\alpha_{\omega_i}) v(T \backslash \{\omega_i, \omega_j\}) + \alpha_{\omega_i} v(T) \geq v(T \backslash \{\omega_j\}). \tag{12}$$

Adding both sides of (11) and (12) shows (6) . □

Proof of Lemma 11. Letting $T = \Omega$, $\omega_i = \omega^*$, and $\omega_j = \omega^{**}$ in Lemma 10 proves this lemma. $\square$

Proof of Lemma 12. Note that Axioms 5 and 6 are the axioms for converse preference relations because (ω^*, ω^{**}) is an ordered pair. Thus,

$$\begin{aligned} & \alpha_{\omega^*} \in [0,1] \text{ is the TPCE weight for act } f_{T^0,\omega^*} \text{ with ordered pair } (\omega^*, \omega^{**}) \\ \Leftrightarrow\ & (\alpha_{\omega^*} y_0 + (1-\alpha_{\omega^*}) y_1, \{\omega^*, \omega^{**}\}; y_0, \Omega \backslash \{\omega^*, \omega^{**}\}) \sim f_{T^0,\omega^*}. \end{aligned}$$

$\alpha_{\omega^{**}} \geq \alpha_{\omega^*}$ and the monotonicity of a binary relation $\succeq$ show that

$$(\alpha_{\omega^{**}} y_0 + (1-\alpha_{\omega^{**}}) y_1, \{\omega^*, \omega^{**}\}; y_0, \Omega \backslash \{\omega^*, \omega^{**}\}) \preceq f_{T^0,\omega^*}.$$

An argument similar to Lemma 10 shows the claim. $\square$

Proof of the Converse of Theorem 3. Let $I(f)$ be the neo-additive CEU represented by (2). Note that $v(T) = I(1_T)$.

First, we show that Axiom 4 holds. Let T be an event with $3 \leq |T| \leq n-1$, and let $S(T) = \sum_{\omega \in \Omega} \eta^0_{\{\omega\}}$. Note that it holds that $\min\{1_T\} = \min\{1_{T \backslash \{\omega_i\}}\} = \min\{1_{T \backslash \{\omega_j\}}\} = \min\{1_{T \backslash \{\omega_i, \omega_j\}}\} = 0$ because $3 \leq |T| \leq n-1$. Then, it holds that

$$\begin{aligned} (4) \quad \Leftrightarrow\ & \alpha_{\omega_i}(S(T) - \eta^0_{\{\omega_i\}} - \eta^0_{\{\omega_j\}} + \bar{\eta}) + (1-\alpha_{\omega_i})(S(T) + \bar{\eta}) = S(T) - \eta^0_{\{\omega_i\}} + \bar{\eta} \\ \Leftrightarrow\ & \alpha_{\omega_i}(\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}}) = \eta^0_{\{\omega_i\}}, \text{ and} \\ (10) \quad \Leftrightarrow\ & (1-\alpha_{\omega_j})(S(T) - \eta^0_{\{\omega_i\}} - \eta^0_{\{\omega_j\}} + \bar{\eta}) + \alpha_{\omega_j}(S(T) + \bar{\eta}) = S(T) - \eta^0_{\{\omega_j\}} + \bar{\eta} \\ \Leftrightarrow\ & \alpha_{\omega_j}(\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}}) = \eta^0_{\{\omega_i\}}. \end{aligned}$$

Thus, there exist α_{ω_i} and α_{ω_j} such that $\alpha_{\omega_i} = \alpha_{\omega_j}$, and Axiom 4 holds.

Next, we show that Axiom 5 holds. Let ω^*, ω^{**} be any states. Let $T = \Omega$, $\omega_i = \omega^*$, and $\omega_j = \omega^{**}$. Note that $\min\{1_{T \backslash \{\omega_i\}}\} = \min\{1_{T \backslash \{\omega_j\}}\} = \min\{1_{T \backslash \{\omega_i, \omega_j\}}\} = 0$. Then, it holds that

$$\begin{aligned} (4) \quad \Leftrightarrow\ & \alpha_{\omega_i}(S(T) - \eta^0_{\{\omega_i\}} - \eta^0_{\{\omega_j\}} + \bar{\eta}) + (1-\alpha_{\omega_i})(S(T) + \tilde{\eta} + \bar{\eta}) = S(T) - \eta^0_{\{\omega_i\}} + \bar{\eta} \\ \Leftrightarrow\ & \alpha_{\omega_i}(\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}}) = \eta^0_{\{\omega_i\}} + (1-\alpha_{\omega_i})\tilde{\eta}, \text{ and} \\ (10) \quad \Leftrightarrow\ & (1-\alpha_{\omega_j})(S(T) - \eta^0_{\{\omega_i\}} - \eta^0_{\{\omega_j\}} + \bar{\eta}) + \alpha_{\omega_j}(S(T) + \tilde{\eta} + \bar{\eta}) = S(T) - \eta^0_{\{\omega_j\}} + \bar{\eta} \\ \Leftrightarrow\ & \alpha_{\omega_j}(\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}}) = \eta^0_{\{\omega_i\}} - \alpha_{\omega_j}\tilde{\eta}, \end{aligned}$$

which implies that $\alpha_{\omega_i}(\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}}) \geq \alpha_{\omega_j}(\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}})$. Thus, there exist α_{ω_i} and α_{ω_j} such that $\alpha_{\omega_i} \geq \alpha_{\omega_j}$, and Axiom 5 holds.

Finally, we show that Axiom 6 holds. Let $T = T^0 = \{\omega^*, \omega^{**}\}$, $\omega_i = \omega^*$, and $\omega_j = \omega^{**}$. Then, it holds that

$$\begin{aligned}
(4) \quad &\Leftrightarrow \quad (1 - \alpha_{\omega_i})(\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}} + \bar{\eta}) = \eta^0_{\{\omega_j\}} + \bar{\eta} \\
&\Leftrightarrow \quad \alpha_{\omega_i}(\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}}) = \eta^0_{\{\omega_i\}} - \alpha_{\omega_i}\bar{\eta}, \text{ and} \\
(10) \quad &\Leftrightarrow \quad \alpha_{\omega_j}(\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}} + \bar{\eta}) = \eta^0_{\{\omega_i\}} + \bar{\eta} \\
&\Leftrightarrow \quad \alpha_{\omega_j}(\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}}) = \eta^0_{\{\omega_i\}} + (1 - \alpha_{\omega_j})\bar{\eta},
\end{aligned}$$

which implies that $\alpha_{\omega_j}(\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}}) \geq \alpha_{\omega_i}(\eta^0_{\{\omega_i\}} + \eta^0_{\{\omega_j\}})$. Thus, there exist α_{ω_i} and α_{ω_j} such that $\alpha_{\omega_j} \geq \alpha_{\omega_i}$, and Axiom 6 holds. $\square$

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