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"Choquet Integrals and Belief Functions"

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Choquet Integrals and Belief Functions*

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Abstract

This paper investigates characterizations of a class of belief functions. Our main contributions are threefold. First, using the notion of *invariant weights*, we characterize the Choquet integral with respect to belief functions along the lines of Schmeidler (1986). Second, we directly derive a class of belief functions on a state space and a collection of events that determines whether Möbius inversions are strictly positive or zero. Third, we show that the derived collection is *simple-complete*.

JEL Classification Numbers: C71; D81; D90
Key Words: Möbius inversions, Invariant weights, Simple-completeness

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1. Introduction

It is difficult to foresee what will happen in the future. This is important because this unpredictability or uncertainty affects individuals' decision-making in the real world. For example, in stock markets, uncertainty about firms' returns in the future affects investors' portfolio choices. Similarly, it is possible that even a sophisticated individual (or agent) cannot identify all relevant contingencies affecting outcomes in the future. Such an agent knows that he or she has only limited knowledge about detailed contingencies and this unawareness affects his or her decision-making. Therefore, uncertainty and unforeseen contingencies (or unawareness) should be taken into consideration. How should we analyze individuals who face uncertainty and unforeseen contingencies?

To consider decision-making under uncertainty and unforeseen contingencies, the notion of *belief functions* has been investigated in the literature. In the fields of economics and statistics, the beliefs of a decision-maker (DM) are usually captured by a probability measure when the DM faces "uncertain situations." However, researchers in these fields have expressed doubts about the validity of capturing a DM's beliefs in this manner.[1] In statistics, Dempster (1967) and Shafer (1976) proposed a *belief function* to model uncertain situations. First, we explain the definition of *belief functions* proposed by Shafer (1976). Let Ω be a finite set and let 2^Ω be the set of all subsets of Ω. Shafer (1976) defined the function $\mathrm{Bel} : 2^\Omega \to [0,1]$ as follows: (B1) $\mathrm{Bel}(\emptyset) = 0$, (B2) $\mathrm{Bel}(\Omega) = 1$, and (B3) for every positive integer n and every collection $A_1, A_2, \dots A_n$ of subsets of Ω, $\mathrm{Bel}(A_1 \cup \cdots \cup A_n) \geq \sum_{i=1}^n \mathrm{Bel}(A_i) - \sum_{i<j}^n \mathrm{Bel}(A_i \cap A_j) + - \cdots + (-1)^{n+1}\mathrm{Bel}(A_1 \cap \cdots \cap A_n)$, where $\emptyset$ denotes the empty set.[2] For example, if $\Omega = \{\omega_1, \omega_2\}$, then, from (B3), it holds that $1 = \mathrm{Bel}(\Omega) \geq \mathrm{Bel}(\{\omega_1\}) + \mathrm{Bel}(\{\omega_2\})$. Keeping this example in mind, consider a situation in which we decide whether a vase is a genuine product of the Ming dynasty or counterfeit (Shafer (1976, Example 1.1)). Let ω_1 denote the possibility that the vase is genuine, and let ω_2 denote the possibility that it is counterfeit. Let $\mathrm{Bel}(\{\omega_1\})$ and $\mathrm{Bel}(\{\omega_2\})$ denote the degrees of belief that the vase is genuine and counterfeit, respectively. For belief functions, because $1 > \mathrm{Bel}(\{\omega_1\}) + \mathrm{Bel}(\{\omega_2\})$

[1] For example, see Gilboa (2009) and Wakker (2010) for textbook presentations of risk and uncertainty in economics and see Yager and Liu (2008) in statistics. See also Grabisch (2016).

[2] Equivalently, Condition (B3) can be written as follows: for every positive integer n and every collection $A_1, A_2, \dots A_n$ of subsets of Ω, $\mathrm{Bel}(A_1 \cup \cdots \cup A_n) \geq \sum_{S \subset \{1,\cdots,n\}, S \neq \emptyset} (-1)^{|S|+1} \mathrm{Bel}(\cap_{i \in S} A_i)$, where $|E|$ denotes the cardinality of a set E.

can hold, it is possible that $\mathrm{Bel}(\{\omega_1\}) = 0.2$ and $\mathrm{Bel}(\{\omega_2\}) = 0.2$. Therefore, belief functions describe situations in which the DM does not have enough information about the likelihood of events.

To express the measure of the belief that is committed to each event A, Shafer (1976) proposed a *basic probability assignment*. A function $\beta : 2^\Omega \to [0, 1]$ is a basic probability assignment if (1) $\beta(\emptyset) = 0$ and (2) $\sum_{A\subseteq\Omega} \beta(A) = 1$. This value $\beta(A)$ measures the belief that the DM commits to A and just only to A, not the total belief that is assigned to A including the subsets of A. To calculate the total belief assigned to A in relation to the belief functions, Shafer (1976) defined the belief function $\mathrm{Bel}(A)$ as equal to the summation of basic probability assignments over all proper subsets B of A, that is, $\mathrm{Bel}(A) = \sum_{B\subseteq A} \beta(B)$.[3] Shafer (1976) showed that $\beta(S) = \sum_{E\subseteq S}(-1)^{|S|-|E|}\mathrm{Bel}(E)$ based on $\mathrm{Bel}(A) = \sum_{B\subseteq A} \beta(B)$. This function β is called the *Möbius inversion* in mathematics. In addition, Shafer (1976) demonstrated that $\beta(S) \geq 0$ by Condition (B3). Thus, belief functions can be defined by the non-negativity of Möbius inversions. In the literature on decision theory, to analyze DM's behaviors under uncertainty, researchers have adopted the notion of capacities v on Ω such that (1) $v(\emptyset) = 0$, (2) $v(\Omega) = 1$, and (3) $v(A) \leq v(B)$ for $A \subseteq B$. Because belief functions satisfy the conditions of capacities, our results in this paper can be applied to decision theory. In the following analyses, we simply denote $\sum_{E\subseteq S}(-1)^{|S|-|E|}v(E)$ by β_S.

Dempster (1967) considered a correspondence Γ from Ω to 2^X, where Ω is a state space and X is a set of outcomes. Because of imprecise or incomplete information, the outcome $\Gamma(\omega)$ is assumed to be multivalued, not single-valued. Although a probability is defined on Ω, it is not defined on X. Then, to approximate a probability on X, Dempster (1967) introduced upper and lower probabilities based on the correspondence Γ, and defined a belief function based on lower probabilities.[4] In economics, particularly in decision theory, axiomatizations of belief functions based

[3] In the example of the Ming dynasty vase, let $\beta(\{\omega_1\}) = \beta(\{\omega_2\}) = 0.2$. Then, $\beta(\{\omega_1, \omega_2\}) = 0.6$, $\mathrm{Bel}(\{\omega_1\}) = \mathrm{Bel}(\{\omega_2\}) = 0.2$, and $\mathrm{Bel}(\{\omega_1, \omega_2\}) = \beta(\{\omega_1\}) + \beta(\{\omega_2\}) + \beta(\{\omega_1, \omega_2\}) = 1$. This value $\beta(\{\omega_1, \omega_2\})$ captures the degree of "ignorance." Suppose that the DM has little to no knowledge of ancient chinaware. Then, he or she has no reason to decide whether the vase is genuine or counterfeit. Therefore, it is natural to assume that $\beta(\{\omega_1\}) = \beta(\{\omega_2\}) = 0$, and $\beta(\{\omega_1, \omega_2\}) = 1$, which corresponds to the case of "total ignorance." Conversely, if the DM is an expert in ancient chinaware, the basic probability assignments are $\beta(\{\omega_1\}) = \beta(\{\omega_2\}) = 0.5$, and $\beta(\{\omega_1, \omega_2\}) = 0$, which corresponds to a case of "no ignorance." For example, see Grabisch (2016, pp.382–383).

[4] Dempster (1967) also defined a plausibility function based on upper probabilities.

on a correspondence on a probability space have been investigated extensively (for example, see Jaffray and Wakker (1994), Mukerji (1997), Wakker (2000), and Ghirardato (2001)). However, to the best of our knowledge, there are few studies that have axiomatized or characterized belief functions from the perspective of Shafer (1967). One exception is the study by Asano and Kojima (2015). Within the framework of Schmeidler (1989), Asano and Kojima (2015) axiomatized a class of belief functions based on the cominimum additivity of Choquet integrals proposed by Kajii et al. (2007).

Our paper also characterizes belief functions from the perspective of Shafer (1976), but our study differs from Asano and Kojima (2015) in a number of ways. First, whereas Asano and Kojima (2015) adopted the properties of cominimum additivity of Choquet integrals, we use the notion of a *two-point condition* (TPC). Second, Asano and Kojima (2015) assumed that the collection on which Möbius inversions are positive is given, and derived belief functions. Conversely, from operator I, we directly derive the collection on which Möbius inversions are positive, and the belief functions. Third, we show that the collection coincides with the collection of all *simple-complete* collections. Fourth, whereas Asano and Kojima (2015) derived $\beta_T = 0$ and $\beta_T \geq 0$ based on the two axioms (the Cominimum Independence Axiom and the Uncertainty Aversion Axiom), in this paper, we derive $\beta_T = 0$ and $\beta_T \geq 0$ only based on TPC with respect to invariant weights.

To characterize the Choquet integral with respect to belief functions, we propose *I-comodularity* and *I-coconvexity* for an operator I. The notions of I-comodularity and I-coconvexity depend on the notion of *invariant weights*. For a real-valued function f on a state space Ω, consider the function $f^{\alpha}_{\omega_i,\omega_j}$ for any $\alpha \in [0,1]$.[5] For a Choquet integral I, an invariant weight is a value α that makes $I(f^{\alpha}_{\omega_i,\omega_j})$ equal to $I(f)$. The notions of I-comodularity and I-coconvexity can be obtained by invariant weights for a class of functions. The notion of I-comodularity plays a part in clarifying the relationship between v's modularity and the Möbius inversions. The notion of I-coconvexity clarifies the relationship between v's convexity and the Möbius inversions.

Next, we discuss a collection of events $\mathcal{E}$ that plays a central role in what follows.

[5] Let $\Omega = \{\omega_1, \ldots, \omega_n\}$, and let f be a real-valued function on Ω. For two distinct points $\omega_i, \omega_j \in \Omega$ and any $\alpha \in [0,1]$, the function $f^{\alpha}_{\omega_i,\omega_j}$ is defined by $f^{\alpha}_{\omega_i,\omega_j} := \alpha f(\omega_i) + (1-\alpha) f(\omega_j)$ if $\omega \in \{\omega_i, \omega_j\}$, and $f(\omega)$ otherwise.

This collection of events determines whether Möbius inversions are strictly positive or zero. Kajii et al. (2007) proposed the concept of an *$\mathcal{E}$-cominimum additive operator* (see Section 4 for details), which is stronger than the comonotonic additive operator of Schmeidler (1986) but weaker than an additive operator. In Kajii et al. (2007) and Asano and Kojima (2015), two problems remain to be solved. First, the structure of $\mathcal{E}$ is given. Second, the signs of the Möbius inversions are not determined. It is plausible that a collection $\mathcal{E}$ is derived and that the signs of the Möbius inversions are determined. The reason is that this collection $\mathcal{E}$ plays a crucial part in determining whether the Möbius inversions are strictly positive or zero. Furthermore, the DM's lack of information is captured by strictly positive Möbius inversions of a belief function.

The remainder of the paper is organized as follows. Section 2 provides mathematical results. Section 3 provides definitions for invariant weight, and for I-comodularity and I-coconvexity, and characterizes these two notions. Section 4 discusses the relationship between the previous literature and this paper. Section 5 concludes the paper. Proofs are relegated to Appendix.

2. Preliminaries

This section provides the mathematical preliminaries that play significant roles in this paper.

Let $\Omega = \{\omega_1, \ldots, \omega_n\}$ be a nonempty finite state space. A generic element $\omega \in \Omega$ denotes a state of the world and a generic element $E \in 2^\Omega$ denotes an event. Let $\mathcal{F}$ be the collection of all nonempty subsets of Ω. Let $\mathcal{F}_k$ be the collection of subsets with k elements. For example, $\mathcal{F}_1$ denotes the set of all singleton subsets of Ω, that is, $\mathcal{F}_1 = \{\{\omega\} \,|\, \omega \in \Omega\}$. Let $\mathbb{R}^\Omega = \{x | \, x : \Omega \to \mathbb{R}\}$ denote the set of all real-valued functions on Ω. Let $1_A \in \mathbb{R}^\Omega$ be the indicator function of an event $A \in 2^\Omega$.

A set function $v : 2^\Omega \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a game. A set function $v : 2^\Omega \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *capacity* if (i) $0 \leq v(A) \leq 1$ for all $A \in 2^\Omega$, and (ii) (monotonicity) $E \subseteq F$ implies that $v(E) \leq v(F)$ for all $E, F \in 2^\Omega$. A capacity v is *convex* if $v(E \cup F) + v(E \cap F) \geq v(E) + v(F)$ for any $E, F \in 2^\Omega$. A set function $v : 2^\Omega \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *finitely additive measure or probability measure* if $v(E) + v(F) = v(E \cup F)$ for any $E, F \in 2^\Omega$ with $E \cap F = \emptyset$. Note that throughout this paper, we refer to a finitely additive measure and a probability measure interchangeably.

Let v be a given capacity. Then, we define β_S for each S by

$$\beta_S = \sum_{E \subseteq S} (-1)^{|S|-|E|} v(E),$$

where $\sum_{E \subseteq A}$ denotes the summation with respect to all subsets E of A. The set of coefficients $\{\beta_S\}_{S \in \mathcal{F}}$ is referred to as the *Möbius inversion*. This is because it holds that for any event A, $v(A) = \sum_{S \subseteq A} \beta_S$. Note that by convention, we omit the empty set in the summation indexed by subsets of Ω. For $S \in \mathcal{F}$, let a capacity u_S be the *unanimity game* on S defined by the following rule: $u_S(A) = 1$ if $S \subseteq A$ and $u_S(A) = 0$ otherwise. The following result states that any capacity v can be uniquely represented by a linear combination of unanimity games and Möbius inversions.

Lemma 1 (Shapley (1953)). *Each game v is uniquely represented as a linear combination of unanimity games and its Möbius inversion: $v = \sum_{T \in \mathcal{F}} \beta_T u_T$.*

Definition 1 (Shafer (1976)). A capacity $v : 2^\Omega \to [0, 1]$ is a *belief function* if its Möbius inversions β_S for all S are nonnegative.

Schmeidler (1986) investigated the properties of Choquet integrals and showed the following representation theorem that plays important roles in decision theory. For $x \in \mathbb{R}^\Omega$ and a capacity v, the *Choquet integral* of x is defined as $\int_\Omega x dv = \int_0^\infty v(x \geq \alpha) d\alpha + \int_{-\infty}^0 (v(x \geq \alpha) - 1) d\alpha$, where $v(x \geq \alpha) = v(\{\omega \in \Omega \,|\, x(\omega) \geq \alpha\})$. Let $\langle E_i \rangle_{i=1}^n$ be a partition of Ω and let $f = \sum_{i=1}^n x_i 1_{E_i}$ with $x_1 \geq x_2 \geq \cdots \geq x_n$ be a step function, where 1_E denotes the indicator function of an event $E \in 2^\Omega$. Then, the Choquet integral can be written as follows: $\int_\Omega f dv = \sum_{j=1}^n (x_j - x_{j+1}) v(\cup_{i=1}^j E_i)$, where $x_{n+1} \coloneqq 0$. Two functions x, $y \in \mathbb{R}^\Omega$ are *comonotonic* if $(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) \geq 0$ for all $\omega, \omega' \in \Omega$. An operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ is *comonotonic additive* if $I(x + y) = I(x) + I(y)$ for any comonotonic functions $x, y \in \mathbb{R}^\Omega$, and I is *monotonic* if $I(x) \geq I(y)$ for any $x, y \in \mathbb{R}^\Omega$ with $x \geq y$. Schmeidler (1986) showed that for an operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ with $I(1_\Omega) = 1$, I satisfies comonotonic additivity and monotonicity if and only if I can be represented by the Choquet integral with respect to the capacity v defined by $v(E) = I(1_E)$ for any $E \in 2^\Omega$.

Gilboa and Schmeidler (1994) clarified the relationship between Choquet integrals and Möbius inversions. The following proposition states that the Choquet integral of x with respect to v can be represented by a weighted sum of all minima of x with respect to the Möbius inversions $\{\beta_T\}_{T \in \mathcal{F}}$.

Proposition 1 (Gilboa and Schmeidler (1994)). *For all* $x \in \mathbb{R}^\Omega$ *and a capacity* $v = \sum_{T \in \mathcal{F}} \beta_T u_T$*:*

$$\int_\Omega x dv = \sum_{T \in \mathcal{F}} \beta_T \int_\Omega x du_T = \sum_{T \in \mathcal{F}} \beta_T \min_{\omega \in T} x(\omega).$$

From the uniqueness of the Möbius inversions, we can show that a capacity v is additive if and only if its Möbius inversions $\beta_T = 0$ for all $|T| \geq 2$. Thus, $\beta_T \neq 0$ for $|T| \geq 2$ captures some kind of deviation from additivity.

3. Characterizations of I-Comodularity and I-Coconvexity

In this section, we provide characterizations for Choquet integrals with respect to belief functions. Therefore, in the following analyses, suppose that an operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ is a Choquet integral. First, we provide the following definition.

Definition 2. For $f \in \mathbb{R}^\Omega$, two distinct points $\omega_i, \omega_j \in \Omega$, and any $\alpha \in [0, 1]$, we define the following function $f^\alpha_{\omega_i,\omega_j}$:

$$f^\alpha_{\omega_i,\omega_j} := \begin{cases} \alpha f(\omega_i) + (1-\alpha) f(\omega_j) & \text{if} \quad \omega \in \{\omega_i, \omega_j\} \\ f(\omega) & \text{otherwise.} \end{cases}$$

The function $f^\alpha_{\omega_i,\omega_j}$ means that the values for ω_i, $f(\omega_i)$, and for ω_j, $f(\omega_j)$, are replaced with the weighted average, $\alpha f(\omega_i) + (1-\alpha) f(\omega_j)$ for any $\alpha \in [0, 1]$.

If $f(\omega_i) \geq f(\omega_j)$, then it holds by the monotonicity of I that $I(f^1_{\omega_i,\omega_j}) \geq I(f^0_{\omega_i,\omega_j})$. Moreover, the continuity of I with respect to α yields the following lemma (see Lemma 6 in Appendix).

Lemma 2. *Let* $I : \mathbb{R}^\Omega \to \mathbb{R}$ *be a Choquet integral. Then, for* $f \in \mathbb{R}^\Omega$*, there exists an* $\alpha \in [0, 1]$ *such that:*

$$I(f^\alpha_{\omega_i,\omega_j}) = I(f). \tag{1}$$

Proof. See Appendix. □

Remark 1. The number α satisfying (1) is not necessarily uniquely determined. For example, if $f(\omega_i) = f(\omega_j)$, then $f^\alpha_{\omega_i,\omega_j} = f$ for all $\alpha \in [0, 1]$, which implies that any $\alpha \in [0, 1]$ satisfies (1).

Remark 2. Let $E^p(f)$ denote the expectation of f with respect to a probability measure p. If the operator $I(f)$ is equal to $E^p(f)$ with $p(\omega_i)+p(\omega_j) \neq 0$, then there exists a unique $\alpha \in [0,1]$ in (1) that is determined only by p, and does not depend on f.

Proof. See Appendix. □

Now that the existence of α satisfying (1) has been shown, we provide the following definition.

Definition 3. For $f \in \mathbb{R}^\Omega$, and two distinct points $\omega_i, \omega_j \in \Omega$, the value $\alpha \in [0,1]$ satisfying (1) is called the *invariant weight with respect to* ω_i *and* ω_j, and the set of those α is denoted by $\{\alpha_f^{\omega_i,\omega_j}\}$.

Remark 3. The invariant weight $\alpha_f^{\omega_i,\omega_j}$ depends on the order of ω_i and ω_j. Precisely, $\alpha \in \{\alpha_f^{\omega_j,\omega_i}\}$ implies $1-\alpha \in \{\alpha_f^{\omega_i,\omega_j}\}$.

Note that from Remark 2, there is a unique element in $\{\alpha_f^{\omega_i,\omega_j}\}$ for all f if I is an expectation. In the case that I is a Choquet integral, the existence of such an α is not necessarily guaranteed. By assuming that there exists a common element in $\{\alpha_f^{\omega_i,\omega_j}\}$ for restricted classes of functions, we can characterize the Choquet integrals with respect to belief functions. For that purpose, we define a class of functions as follows.

Definition 4. Let $T \in 2^\Omega$ with $|T| \geq 2$, and $\{\omega_i, \omega_j\} \subseteq T$. The set of functions $f \in \mathbb{R}^\Omega$ with the following property is denoted by $B^T_{\omega_i,\omega_j}$: for all $\omega' \in T \backslash \{\omega_i, \omega_j\}$ and all $\omega'' \in T^c$:

$$f(\omega') \geq f(\omega_i) \geq f(\omega''), \text{ and}$$
$$f(\omega') \geq f(\omega_j) \geq f(\omega'').$$

This definition is closely related to the definition proposed by Sarin and Wakker (1998). Sarin and Wakker (1998, pp.233–234) stated that event D is a *dominating event* for event A if the states in D are rank-ordered higher than those of A, and the remaining states are rank-ordered lower than A. In other words, given a function $f \in \mathbb{R}^\Omega$, event D is dominating for event A if $A \cap D = \emptyset$ and $f(t) \geq f(s) \geq f(t')$ for all $t \in D$, $s \in A$, and $t' \in (A \cup D)^c$. Based on Sarin and Wakker's (1998) notion, Definition 4 is rewritten as follows: Event $T\backslash\{\omega_i, \omega_j\}$ is dominating for event $\{\omega_i, \omega_j\}$.

Next, we provide a technical lemma that plays an important role in proving Propositions 2 and 3.

Lemma 3. *Let I be a Choquet integral. Let $f \in B^T_{\omega_i,\omega_j}$. If $f(\omega_i) \geq f(\omega_j)$, then*

$$\begin{aligned} & I(f^\alpha_{\omega_i,\omega_j}) - I(f) \\ = \;& (f(\omega_i) - f(\omega_j))\left\{(1-\alpha)v(T\backslash\{\omega_i,\omega_j\}) + \alpha v(T) - v(T\backslash\{\omega_j\})\right\}. \end{aligned}$$

If $f(\omega_j) \geq f(\omega_i)$, then

$$\begin{aligned} & I(f^\alpha_{\omega_i,\omega_j}) - I(f) \\ = \;& (f(\omega_j) - f(\omega_i))\left\{\alpha v(T\backslash\{\omega_i,\omega_j\}) + (1-\alpha) v(T) - v(T\backslash\{\omega_i\})\right\}. \end{aligned}$$

Proof. See Appendix. □

Next, we provide the following definition if there exists a common element in $\{\alpha_f^{\omega_i,\omega_j}\}$ for all $f \in B^T_{\omega_i,\omega_j}$.

Definition 5. Fix an event $T \in 2^\Omega$ with $|T| \geq 2$, and fix two distinct points $\omega_i, \omega_j \in T$. If $\cap_{f \in B^T_{\omega_i,\omega_j}} \{\alpha_f^{\omega_i,\omega_j}\} \neq \emptyset$, then we say that ω_i and ω_j are *I-comodular on T*.

Note that "co" in "comodularity" in Definition 5 and "coconvexity" in Definition 6 below is an analogy to "co" in the definition of "codimension" in linear algebra. Chateauneuf and Jaffray (1989) analyzed the relationship between the inclusion–exclusion formula for a game v and its Möbius inversion.

Lemma 4 (Chateauneuf and Jaffray (1989))**.** *Let $v = \sum_{T \in \mathcal{F}} \beta_T u_T$ be a game, and let k be an integer satisfying $k \geq 2$. Then,*
$v(\bigcup_{1\leq i\leq k} T_i) - \sum_{\emptyset \neq S \subseteq \{1,2,\ldots,k\}} (-1)^{|S|+1} v(\bigcap_{j\in S} T_j) = \sum_{T \subseteq \bigcup T_i, T \not\subseteq T_i (1\leq i\leq k)} \beta_T$.

To characterize the Choquet integral with respect to belief functions, we provide the following proposition. This proposition clarifies the relationship between I-comodularity, v's modularity, and the Möbius inversions.

Proposition 2. *Let $T \in 2^\Omega$ with $|T| \geq 2$ and let $\omega_i, \omega_j \in T$ with $\omega_i \neq \omega_j$. Then, the following are equivalent:*

(i) ω_i *and* ω_j *are* I*-comodular on* T*.*
(ii) $v(T) + v(T\backslash\{\omega_i, \omega_j\}) = v(T\backslash\{\omega_i\}) + v(T\backslash\{\omega_j\})$*.*
(iii) $\sum_{\{\omega_i,\omega_j\}\subseteq S\subseteq T} \beta_S = 0$*.*

Note that v is *modular* if $v(E \cup F) = v(E) + v(F) - v(E \cap F)$ for all $E, F \in 2^\Omega$. Thus, Condition (ii) indicates modularity for two sets excluding the one-point set from T, that is, $T\backslash\{\omega_i\}$ and $T\backslash\{\omega_j\}$.

Proof. To show the equivalence of (ii) and (iii), we use Lemma 4. See Appendix. $\square$

Conditions (ii) and (iii) in Proposition 2 hold by equality. We define one more property for $\{\alpha_f^{\omega_i,\omega_j}\}$.

Definition 6. Fix an event $T \in 2^\Omega$ with $|T| \geq 2$, and fix two distinct points $\omega_i, \omega_j \in T$. When, for any $f, g \in B^T_{\omega_i,\omega_j}$, there exists a real number $\bar{\alpha} \in \{\alpha_f^{\omega_i,\omega_j}\}$ such that $I(g^{\bar{\alpha}}_{\omega_i,\omega_j}) \geq I(g)$, then we say that the two distinct points ω_i and ω_j are I*-coconvex on* T.

Similar to Proposition 2, the following proposition clarifies the relationship between I-coconvexity, v's convexity, and the Möbius inversions.

Proposition 3. *Let* $T \in 2^\Omega$ *with* $|T| \geq 2$ *and let* $\omega_i, \omega_j \in T$ *with* $\omega_i \neq \omega_j$*. Then, the following are equivalent.*
(i) ω_i *and* ω_j *are* I*-coconvex on* T*.*
(ii) $v(T) + v(T\backslash\{\omega_i, \omega_j\}) \geq v(T\backslash\{\omega_i\}) + v(T\backslash\{\omega_j\})$*.*
(iii) $\sum_{\{\omega_i,\omega_j\}\subseteq S\subseteq T} \beta_S \geq 0$*.*

Proof. To show the equivalence of (ii) and (iii), we use Lemma 4. See Appendix. $\square$

Condition (ii) indicates some kind of convexity for two sets, excluding the one-point set from T, that is, $T\backslash\{\omega_i\}$ and $T\backslash\{\omega_j\}$.

Now, for an operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ and fixed two points ω_i, ω_j, we consider the following condition.

Two-Point Condition (TPC): For any $T \in 2^\Omega$ with $|T| \geq 2$, there exist two distinct points such that:
(a) ω_i and ω_j are I-coconvex on T, and
(b) for all S with $\{\omega_i, \omega_j\} \subseteq S \subsetneq T$, ω_i and ω_j are I-comodular on S.

The following two propositions (Propositions 4 and 5) provide characterizations of the Choquet integral I with respect to the belief function $v = \sum_{T\in\mathcal{F}} \beta_T u_T$, where $\beta_T \geq 0$ for all $T \in \mathcal{F}$. Recall that u_E denotes the unanimity game on E. Furthermore, Proposition 4 pins down the collection $\mathcal{E}$ where the Möbius inversion $\{\beta_E\}_{E\in\mathcal{E}}$ is positive, and Proposition 5 shows that the collection of events derived by operator I on which the signs of Möbius inversions are strictly positive is *simple-complete* (see Definition 8).

Proposition 4. *Let I be a Choquet integral satisfying TPC. Then, there exist a unique collection $\mathcal{E} \subseteq 2^{\Omega}\backslash\mathcal{F}_1$, a unique set of* **positive** *coefficients $\{\beta_E\}_{E\in\mathcal{E}}$, and a unique set of nonnegative coefficients $\{\beta_{\{\omega\}}\}_{\omega\in\Omega}$ such that*

$$I(f) = \sum_{i=1}^{n} \beta_{\{\omega_i\}} f(\omega_i) + \sum_{E\in\mathcal{E}} \beta_E \min_{\omega\in E} f(\omega).$$

Proof. See Appendix. □

Note that this derived collection $\mathcal{E}$ must have some property, not all subcollections of 2^{Ω}. Therefore, we investigate the properties of a collection that can be a candidate for $\mathcal{E}$. We first present the definition of completeness proposed by Kajii et al. (2007).

Definition 7 (Kajii et al. (2007))**.** Let a collection of events T with $|T| \geq 2$, $\mathcal{E}$, be fixed. An event $T \in \mathcal{F}$ is *$\mathcal{E}$-complete* if, for any two distinct points ω_1 and ω_2 in T, there exists a set $E \in \mathcal{E}$ such that $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. The collection of all $\mathcal{E}$-complete events is called the *$\mathcal{E}$-complete collection* and is denoted by $\Upsilon(\mathcal{E})$. Moreover, $\mathcal{E}$ is said to be *complete* if all $\mathcal{E}$-complete subsets belong to $\mathcal{E}$, that is, $\mathcal{E} = \Upsilon(\mathcal{E})$. In general, because it holds that $\mathcal{E} \subseteq \Upsilon(\mathcal{E})$, $\mathcal{E}$'s completeness is equivalent to $\Upsilon(\mathcal{E}) \subseteq \mathcal{E}$.

Event T's $\mathcal{E}$-completeness means that any two points ω_i, ω_j in T are covered by some set $E \in \mathcal{E}$ with $E \subseteq T$. Suppose that each ω_i is a vertex (or a node) and each E is an edge (or a line). Then, $\mathcal{E}$-completeness means that any two vertices are connected by some edge. Because this interpretation corresponds to the notion of complete graph in graph theory, the term "complete" is adopted. If all $\mathcal{E}$-complete sets belong to $\mathcal{E}$, then the collection $\mathcal{E}$ is called complete. If $\mathcal{E}$ is complete, then there are two cases: (i) the property of completeness does not hold if some set $E \in \mathcal{E}$ is deleted from $\mathcal{E}$ and (ii) the property of completeness still holds even if any set

$E \in \mathcal{E}$ is deleted from $\mathcal{E}$. The latter case implies that the degree of intersections over sets in $\mathcal{E}$ is not complicated, which is called *simple-complete*. The notion of simple completeness was first proposed by Asano and Kojima (2015). Formally, the notion of simple completeness can be defined as follows.

Definition 8. Let $\mathcal{E}$ be a collection of events T with $|T| \geq 2$. A collection $\mathcal{E}$ is said to be simple-complete if
(i) $\mathcal{E}$ is complete, and
(ii) for all $E \in \mathcal{E}$, $\mathcal{E}\backslash\{E\}$ is complete.

In the following, let $\mathcal{C}$ be the set of all simple complete collections $\mathcal{E}$ with $\mathcal{E} \subseteq 2^{\Omega}\backslash\mathcal{F}_1$, that is, $\mathcal{C} = \{\mathcal{E}|\mathcal{E}$ is simple-complete$\}$.

Example 1. (1) Let $\Omega = \{1,2,3,4\}$ and let $\mathcal{E} = \{\{1,2\},\{2,3\},\{1,3\}\}$. Then, because $\Upsilon(\mathcal{E}) = \{\{1,2\},\{2,3\},\{1,3\},\{1,2,3\}\}$, $\mathcal{E}$ is not complete. Let $\mathcal{E}' = \{\{1,2\},\{2,3\},\{1,3\},\{1,2,3\}\}$. Then, $\mathcal{E}'$ is complete but not simple-complete because $\mathcal{E}'\backslash\{1,2,3\}$ is not complete. Conversely, $\mathcal{E}'' = \{\{1,2\},\{2,3\}\}$ is complete and simple-complete.
(2) Let $|\Omega| \geq 3$ and let $\mathcal{E} = \{T \mid |T| \geq 2, T \subseteq \Omega\}$. Then, $\mathcal{E}$ is complete. However, $\mathcal{E}$ is not simple-complete because $\mathcal{E}\backslash\{\Omega\}$ is not complete.
(3) Let $E_1, E_2, \ldots, E_m$ be a partition of Ω. Then, $\mathcal{E} = \{E_1, E_2, \ldots, E_m\}$ is complete and simple-complete.
(4) Let $\Omega = \{1,2,\ldots,n\}$, and let $\mathcal{E} = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}$. Then, $\mathcal{E}$ is complete and simple-complete.

The following lemma plays an important role in applying the notion of simple-completeness to the analyses of Choquet integrals.

Lemma 5. *The following conditions are equivalent:*
(i) $\mathcal{E}$ is simple-complete.
(ii) $\mathcal{E}$ is complete and $E \notin \Upsilon(\mathcal{E}\backslash E)$ for all $E \in \mathcal{E}$.
(iii) for any $T \subseteq \Omega$ with $|T| \geq 2$, there exists a two-point set $\{\omega_1,\omega_2\} \subseteq T$ such that for all S with $\{\omega_1,\omega_2\} \subseteq S \subsetneq T$, it holds that $S \notin \mathcal{E}$.

Proof. See Appendix. □

The following proposition states that the collection of events derived by operator I on which the signs of Möbius inversions are strictly positive is equal to the collection of all simple-complete collections. Because our model does not generate all subcollections of $2^{\Omega}\backslash\mathcal{F}_1$, this might be a limitation of our model. However, as we see in Example 1, the collection of all simple-complete collections $\mathcal{E}$ is rich enough to analyze meaningful collections. Moreover, because our model generates all collections in $\mathcal{C}$, a collection of events satisfying simple-completeness plays essential roles in our model.

Proposition 5. *Let $\mathfrak{I}$ be the set of all Choquet integrals I satisfying TPC, and let $\mathcal{C} = \{\mathcal{E} | \mathcal{E}$ is simple-complete$\}$. Define the mapping $\Xi : \mathfrak{I} \to \mathcal{C}$ by $\Xi(I) = \{T | \beta_T > 0\} \in \mathcal{C}$, where β_T is obtained in Proposition 4. Then, the mapping Ξ is well defined and an onto mapping.*

Proof. See Appendix. □

In summary, the following theorem holds.

Theorem 1. *Let an operator $I : \mathbb{R}^{\Omega} \to \mathbb{R}$ be a Choquet integral. Then, the following are equivalent.*
(i) Operator $I : \mathbb{R}^{\Omega} \to \mathbb{R}$ satisfies TPC.
(ii) There exist a unique simple-complete collection $\mathcal{E} \in \mathcal{C}$, a unique set of positive coefficients $\{\beta_E\}_{E\in\mathcal{E}}$, and a unique set of nonnegative coefficients $\{\beta_{\{\omega\}}\}_{\omega\in\Omega}$ such that

$$I(f) = \sum_{i=1}^{n} \beta_{\{\omega_i\}} f(\omega_i) + \sum_{E\in\mathcal{E}} \beta_E \min_{\omega\in E} f(\omega).$$

In particular, because $I(f) = \int_{\Omega} f dv$ with $v = \sum_{i=1}^{n} \beta_{\{\omega_i\}} u_{\{\omega_i\}} + \sum_{E\in\mathcal{E}} \beta_E u_E$, v is a belief function.

4. Discussion

It is worthwhile comparing our results with those of the existing literature (Kajii et al. (2007) and Asano and Kojima (2015)).[6] First, whereas Asano and Kojima (2015) assumed the convexity of capacity v, we assume the I-coconvexity, which is weaker than v's convexity. Second, Kajii et al. (2007) and Asano and Kojima (2015) did not pin down an event T such that the Möbius inversion β_T is not

[6]Kajii et al. (2007) proposed the notions of $\mathcal{E}$-cominimum functions and $\mathcal{E}$-cominimum additivity. Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. Two functions $x, y \in \mathbb{R}^{\Omega}$ are *$\mathcal{E}$-cominimum* if $\operatorname{argmin}_E x \cap \operatorname{argmin}_E y \neq \emptyset$ for all $E \in \mathcal{E}$. An operator $I : \mathbb{R}^{\Omega} \to \mathbb{R}$ is *$\mathcal{E}$-cominimum additive* if $I(x + y) = I(x) + I(y)$ for any $\mathcal{E}$-cominimum functions $x, y \in \mathbb{R}^{\Omega}$.

equal to zero. In other words, they stated that only T with $T \in \mathcal{E}$ could satisfy $\beta_T \neq 0$. Our results provide a clear distinction between $\beta_T > 0$ and $\beta_T = 0$. Third, whereas Asano and Kojima (2015) obtained $\beta_T = 0$ and $\beta_T \geq 0$ by the Cominimum Independence Axiom (I's cominimum additivity) and the Uncertainty Aversion Axiom (v's convexity), respectively, in this paper, we obtain $\beta_T = 0$ and $\beta_T \geq 0$ by TPC with respect to invariant weights. Fourth, we derive the mapping Ξ on the set of all operators satisfying TPC and show that its codomain is exactly equal to the set of all simple-complete collections, which is proposed in Asano and Kojima (2015). Finally, we show that Choquet integrals satisfying TPC derive a class of belief functions.

The importance of mapping Ξ should be mentioned. This has been pointed out in the literature. For instance, Zhang (1999) noted that the DM's beliefs should be defined not based on an algebra or a σ-algebra, but on a λ-system, and defined this collection as a set of *unambiguous events.*[7] Zhang (2002) proposed a new axiomatization of the Choquet Expected Utility in which the DM's beliefs are captured by an inner measure and his or her preferences are represented by the Choquet integral with respect to the inner measure. While Zhang (1999) and Zhang (2002) assumed that a collection of events that is a λ-system is given, Epstein and Zhang (2001) derived the domain on which the DM's beliefs are defined, and showed that the set of unambiguous events is a λ-system. In another related work, by a binary relation $\succeq$, Nehring (1999) characterized the collection of unambiguous events that is a λ-system. In addition, Nehring (1999) characterized the collection of unambiguous events by restricted additivity of Choquet integrals. In contrast to the abovementioned papers, in this paper, the codomain (or range) of Ξ does not coincide with $2^{\mathcal{F}}$. However, as discussed above, the derivation of a collection of events $\mathcal{E}$ is important because this collection $\mathcal{E}$ plays a crucial role in determining whether Möbius inversions are strictly positive or zero.

Finally, Chateauneuf and Rébillé (2004) provided a characterization related to this paper. Let $\Omega = \{1, 2, \ldots, n\}$ and let $\mathcal{E} = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}$. Then, $\mathcal{E}$ is complete and simple-complete by Example 1 (4). Keeping this in mind, the following representation derived by Chateauneuf and Rébillé (2004, Theorem 1) can

[7] A collection $\mathcal{A}$ is a λ-system if (1) $\emptyset \in \mathcal{A}$, (2) for any $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$, $A \cup B \in \mathcal{A}$, and (3) for any $A \in \mathcal{A}$, $A^c \in \mathcal{A}$.

be obtained by this paper:

$$I(f) = \sum_{i=1}^{n} \beta_i f(i) + \sum_{i=1}^{n-1} \beta_{i,i+1} \min\{f(i), f(i+1)\},$$

where $\beta_i \geq 0$ for $i = 1, \ldots, n$ and $\beta_{i,i+1} \geq 0$ for $i = 1, \ldots, n-1$.

5. Conclusion

The main contributions of this paper are threefold. First, using the notion of *invariant weights*, we characterized the Choquet integral with respect to belief functions along the lines of Schmeidler (1986). Second, we directly derived a class of Shafer-type belief functions on a state space and a collection of events that determines whether Möbius inversions are strictly positive or zero. It is plausible for us to derive collections where the signs of Möbius inversions are strictly positive. The strict positivity of the Möbius inversions implies that the capacity v determined by the Möbius inversions is a belief function. Third, we showed that the codomain of the mapping Ξ is simple-complete collections.

Some works remain to be solved. We provided a characterization of a class of belief functions, but we did not provide characterizations of DM's preferences under uncertainty. Our characterization theorem can be applied to a Leontief preference, a multiperiod decision model in Gilboa (1989), and an inequality aversion model in Rohde (2010), which would be intriguing applications of our analysis to economic issues in future.

Appendix

Lemma 6. *Let Ω be a finite set, and let $h(\omega, \alpha) : \Omega \times [0, 1] \to \mathbb{R}$ be a continuous function with respect to α. Then, the Choquet integral with respect to a capacity v on Ω defined by*

$$I(\alpha) = \int_{\Omega} h(\omega, \alpha) dv$$

is continuous with respect to α.

Proof. Let $h(\omega_1, \alpha) > h(\omega_2, \alpha) > \ldots > h(\omega_k, \alpha)$, and let $\langle \Omega_i \rangle_{i=1}^{k}$ be a partition of Ω such that $\Omega_i = \{\omega \in \Omega | h(\omega_i, \alpha) = h(\omega, \alpha)\}$. Note that $h(\omega, \alpha)$ is a constant function on each Ω_i. Then,

$$I(\alpha) = \sum_{j=1}^{k} (h(\omega_j, \alpha) - h(\omega_{j+1}, \alpha)) v(\Omega_1 \cup \ldots \cup \Omega_j),$$

where $h(\omega_{k+1}, \alpha) \coloneqq 0$. Let $d = \min_{1 \le j \le k-1} (h(\omega_j, \alpha) - h(\omega_{j+1}, \alpha))$. Because $h(\omega, \alpha)$ is continuous with respect to α, there exists sufficiently small $\delta > 0$ such that $|h(\omega_j, \alpha') - h(\omega_j, \alpha)| < d/2$ for all α' with $|\alpha' - \alpha| < \delta$ and for all j. By this inequality, it holds that $h(\omega_j, \alpha') > h(\omega_j, \alpha) - d/2$ and $h(\omega_{j+1}, \alpha) + d/2 > h(\omega_{j+1}, \alpha')$, which implies that

$$h(\omega_j, \alpha') - h(\omega_{j+1}, \alpha') > h(\omega_j, \alpha) - h(\omega_{j+1}, \alpha) - d \ge 0.$$

Therefore, for α', $h(\omega_1, \alpha') > h(\omega_2, \alpha') > \ldots > h(\omega_k, \alpha')$. With this in mind, it holds that

$$I(\alpha') = \sum_{j=1}^{k} (h(\omega_j, \alpha') - h(\omega_{j+1}, \alpha')) v(\Omega_1 \cup \ldots \cup \Omega_j),$$

where $h(\omega_{k+1}, \alpha') \coloneqq 0$. Then, because $h(\omega_j, \alpha)$ is continuous with respect to α for each ω_j, it can be shown that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\alpha' - \alpha| < \delta \Rightarrow |I(\alpha') - I(\alpha)| < \varepsilon$. $\square$

Proof of Lemma 2

Proof. Without loss of generality, let $f(\omega_i) \ge f(\omega_j)$. By the definition of $f^{\alpha}_{\omega_i, \omega_j}$ and the monotonicity of Choquet integrals, $I(f^{1}_{\omega_i, \omega_j}) \ge I(f) \ge I(f^{0}_{\omega_i, \omega_j})$. By Lemma 6, it follows that $I(f^{\alpha}_{\omega_i, \omega_j})$ is continuous with respect to α. Thus, there exists an $\alpha \in [0, 1]$ such that $I(f^{\alpha}_{\omega_i, \omega_j}) = I(f)$. $\square$

Proof of Remark 2

Proof. It follows from Lemma 2 that there exists $\alpha \in [0,1]$ such that $I(f^{\alpha}_{\omega_i,\omega_j}) = I(f)$. Because we assume that the operator $I(f)$ is equal to $E^p(f)$, it follows that

$$\begin{aligned}
& E^p(f^{\alpha}_{\omega_i,\omega_j}) = E^p(f) \\
\Leftrightarrow \quad & (\alpha f(\omega_i) + (1-\alpha) f(\omega_j))(p(\{\omega_i\}) + p(\{\omega_j\})) = f(\omega_i)p(\omega_i) + f(\omega_j)p(\omega_j) \\
\Leftrightarrow \quad & \alpha f(\omega_i) + (1-\alpha) f(\omega_j) \\
= \quad & \frac{p(\{\omega_i\})}{p(\{\omega_i\}) + p(\{\omega_j\})} f(\omega_i) + \frac{p(\{\omega_j\})}{p(\{\omega_i\}) + p(\{\omega_j\})} f(\omega_j). \qquad (2)
\end{aligned}$$

Thus, we can take a common $\alpha = p(\omega_i)/(p(\{\omega_i\}) + p(\{\omega_j\}))$ without depending on f, and this α is the only value common to all f. □

Proof of Lemma 3

Proof. Let $\{s_1, s_2, \ldots, s_k\} = T\backslash\{\omega_i, \omega_j\}$ and $\{s_{k+1}, \ldots, s_{n-2}\} = T^c$ such that $f(s_1) \geq f(s_2) \geq \cdots \geq f(\omega_i) \geq f(\omega_j) \geq f(s_{k+1}) \geq \cdots \geq f(s_{n-2})$. Then, it follows that

$$\begin{aligned}
& I(f^{\alpha}_{\omega_i,\omega_j}) \\
= \quad & (f(s_1) - f(s_2))v(\{s_1\}) + (f(s_2) - f(s_3))v(\{s_1, s_2\}) + \cdots \\
+ \quad & (f(s_{k-1}) - f(s_k))v(\{s_1, \ldots, s_{k-1}\}) + (f(s_k) - \alpha f(\omega_i) - (1-\alpha) f(\omega_j))v(T\backslash\{\omega_i, \omega_j\}) \\
+ \quad & 0 \times v(T\backslash\{\omega_j\}) + (\alpha f(\omega_i) + (1-\alpha) f(\omega_j) - f(s_{k+1}))v(T) \\
+ \quad & (f(s_{k+1}) - f(s_{k+2}))v(T \cup \{s_{k+1}\}) + \cdots + f(s_{n-2}),
\end{aligned}$$

and

$$\begin{aligned}
& I(f) \\
= \quad & (f(s_1) - f(s_2))v(\{s_1\}) + (f(s_2) - f(s_3))v(\{s_1, s_2\}) + \cdots \\
+ \quad & (f(s_{k-1}) - f(s_k))v(\{s_1, \ldots, s_{k-1}\}) + (f(s_k) - f(\omega_i))v(T\backslash\{\omega_i, \omega_j\}) \\
+ \quad & (f(\omega_i) - f(\omega_j))v(T\backslash\{\omega_j\}) + (f(\omega_j) - f(s_{k+1}))v(T) \\
+ \quad & (f(s_{k+1}) - f(s_{k+2}))v(T \cup \{s_{k+1}\}) + \cdots + f(s_{n-2}).
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
& I(f^{\alpha}_{\omega_i,\omega_j}) - I(f) \\
= \quad & ((1-\alpha) f(\omega_i) - (1-\alpha) f(\omega_j))v(T\backslash\{\omega_i, \omega_j\}) - (f(\omega_i) - f(\omega_j))v(T\backslash\{\omega_j\})
\end{aligned}$$

$$
\begin{aligned}
&+ \quad (\alpha f(\omega_i) - \alpha f(\omega_j))v(T) \\
&= \quad (f(\omega_i) - f(\omega_j))\left\{(1-\alpha)v(T\backslash\{\omega_i,\omega_j\}) + \alpha v(T) - v(T\backslash\{\omega_j\})\right\}.
\end{aligned}
$$

Similar to the case where $f(\omega_i) \geq f(\omega_j)$, for $f(\omega_j) \geq f(\omega_i)$, it follows that

$$
\begin{aligned}
& I(f^{\alpha}_{\omega_i,\omega_j}) - I(f) \\
= \quad & (f(\omega_j) - f(\omega_i))\left\{\alpha v(T\backslash\{\omega_i,\omega_j\}) + (1-\alpha)v(T) - v(T\backslash\{\omega_i\})\right\},
\end{aligned}
$$

which proves Lemma 3. □

Proof of Proposition 2

Proof. (i) $\Rightarrow$ (ii). Let ω_i and ω_j be I-comodular on T. Take $f \in B^T_{\omega_i,\omega_j}$ with $f(\omega_i) > f(\omega_j)$. By Definition 5, we can take $\bar{\alpha} \in \cap_{f \in B^T_{\omega_i,\omega_j}} \{\alpha_f^{\omega_i,\omega_j}\}$. Because $I(f^{\alpha}_{\omega_i,\omega_j}) - I(f) = 0$, it follows from Lemma 3 that

$$
(1-\bar{\alpha})v(T\backslash\{\omega_i,\omega_j\}) + \bar{\alpha}v(T) = v(T\backslash\{\omega_j\}). \tag{3}
$$

Next, take $g \in B^T_{\omega_i,\omega_j}$ with $g(\omega_j) > g(\omega_i)$. Then, the number $\bar{\alpha}$ mentioned above is also an element of $\{\alpha_g^{\omega_i,\omega_j}\}$. Therefore, $I(g^{\alpha}_{\omega_i,\omega_j}) - I(g) = 0$ and Lemma 3 implies that

$$
\bar{\alpha}v(T\backslash\{\omega_i,\omega_j\}) + (1-\bar{\alpha})v(T) = v(T\backslash\{\omega_i\}). \tag{4}
$$

By adding (3) and (4), it holds that

$$
v(T) + v(T\backslash\{\omega_i,\omega_j\}) = v(T\backslash\{\omega_i\}) + v(T\backslash\{\omega_j\}).
$$

(ii) $\Rightarrow$ (i). Suppose that

$$
v(T) + v(T\backslash\{\omega_i,\omega_j\}) = v(T\backslash\{\omega_i\}) + v(T\backslash\{\omega_j\}). \tag{5}
$$

Case 1: $v(T) \neq v(T\backslash\{\omega_i,\omega_j\})$.

Let us consider $\bar{\alpha} \in [0,1]$ satisfying (3). Because we assume $v(T) \neq v(T\backslash\{\omega_i,\omega_j\})$, such $\bar{\alpha}$ can be uniquely taken. Then, for $f \in B^T_{\omega_i,\omega_j}$ with $f(\omega_i) \geq f(\omega_j)$, it follows from Lemma 3 and (3) that $I(f^{\bar{\alpha}}_{\omega_i,\omega_j}) = I(f)$. On the other hand, by subtracting (3) from (5), (4) holds. Thus, for $g \in B^T_{\omega_i,\omega_j}$ with $g(\omega_j) \geq g(\omega_i)$, it follows from Lemma 3 and (4) that $I(g^{\bar{\alpha}}_{\omega_i,\omega_j}) = I(g)$. Therefore, for all $f \in B^T_{\omega_i,\omega_j}$, it holds that

$\bar{\alpha} \in \{\alpha_f^{\omega_i,\omega_j}\}$, which implies that ω_i and ω_j are I-comodular on T.
Case 2: $v(T) = v(T\backslash\{\omega_i, \omega_j\})$.

Because v is a capacity, it holds that $v(T) = v(T\backslash\{\omega_i, \omega_j\}) = v(T\backslash\{\omega_i\}) = v(T\backslash\{\omega_j\})$. Then, (3) and (4) hold for any $\bar{\alpha} \in [0,1]$. Therefore, by Lemma 3, it holds that $\{\alpha_f^{\omega_i,\omega_j}\} = [0,1]$ for all $f \in B^T_{\omega_i,\omega_j}$, which implies that ω_i and ω_j are I-comodular on T.
(ii) $\Leftrightarrow$ (iii). Let $T_1 = T\backslash\{\omega_i\}$ and $T_2 = T\backslash\{\omega_j\}$. Then, $T_1 \cup T_2 = T$ and $T_1 \cap T_2 = T\backslash\{\omega_i, \omega_j\}$. Then, it follows that

$$\begin{aligned}
& v(T) + v(T\backslash\{\omega_i, \omega_j\}) - v(T\backslash\{\omega_i\}) - v(T\backslash\{\omega_j\}) \\
= \;& v(T_1 \cup T_2) + v(T_1 \cap T_2) - v(T_1) - v(T_2) \\
= \;& v(\cup_{1\le i\le 2} T_i) - \sum_{\emptyset \neq S \subseteq \{1,2\}} (-1)^{|S|+1} v(\cap_{j\in S} T_j) \\
= \;& \sum_{\{\omega_i,\omega_j\}\subseteq S\subseteq T} \beta_S,
\end{aligned}$$

where the last equality holds by Lemma 4. Thus, the proof is completed. □

Proof of Proposition 3

Proof. (i) $\Rightarrow$ (ii).
Case 1: $v(T) = v(T\backslash\{\omega_i, \omega_j\})$.

Because v is a capacity, it holds that $v(T) = v(T\backslash\{\omega_i, \omega_j\}) = v(T\backslash\{\omega_i\}) = v(T\backslash\{\omega_j\})$. Then, $v(T) + v(T\backslash\{\omega_i, \omega_j\}) = v(T\backslash\{\omega_i\}) + v(T\backslash\{\omega_j\})$ regardless of Condition (i).
Case 2: $v(T) \neq v(T\backslash\{\omega_i, \omega_j\})$.

Take $f, g \in B^T_{\omega_i,\omega_j}$ with $f(\omega_i) > f(\omega_j)$ and $g(\omega_j) > g(\omega_i)$. Because we assume Condition (i), there exists $\bar{\alpha} \in \{\alpha_f^{\omega_i,\omega_j}\}$ such that $I(g^{\bar{\alpha}}_{\omega_i,\omega_j}) \geq I(g)$. Because $\bar{\alpha} \in \{\alpha_f^{\omega_i,\omega_j}\}$ implies $I(f^{\bar{\alpha}}_{\omega_i,\omega_j}) = I(f)$ and we assume $f(\omega_i) > f(\omega_j)$, it follows from Lemma 3 that

$$(1-\bar{\alpha})v(T\backslash\{\omega_i, \omega_j\}) + \bar{\alpha}v(T) = v(T\backslash\{\omega_j\}). \tag{6}$$

Because $v(T) \neq v(T\backslash\{\omega_i, \omega_j\})$, such $\bar{\alpha}$ is unique. Therefore, by assumption, this $\bar{\alpha}$ must satisfy $I(g^{\bar{\alpha}}_{\omega_i,\omega_j}) \geq I(g)$. Because we assume $g(\omega_j) > g(\omega_i)$, it follows from Lemma 3 that

$$\bar{\alpha}v(T\backslash\{\omega_i, \omega_j\}) + (1-\bar{\alpha})v(T) \geq v(T\backslash\{\omega_i\}). \tag{7}$$

By adding (6) and (7), it holds that $v(T)+v(T\backslash\{\omega_i,\omega_j\}) \geq v(T\backslash\{\omega_i\})+v(T\backslash\{\omega_j\})$.
(ii) $\Rightarrow$ (i). Fix $f, g \in B^T_{\omega_i,\omega_j}$ arbitrarily.
Case 1: $f(\omega_i) = f(\omega_j)$ or $v(T) = v(T\backslash\{\omega_i,\omega_j\})$.

By Lemma 3, it holds that $\{\alpha_f^{\omega_i,\omega_j}\} = [0,1]$. Therefore, $\{\alpha_g^{\omega_i,\omega_j}\} \subseteq \{\alpha_f^{\omega_i,\omega_j}\}$, which implies that an element $\bar{\alpha} \in \{\alpha_g^{\omega_i,\omega_j}\}$ is also in $\{\alpha_f^{\omega_i,\omega_j}\}$ and $I(g^{\bar{\alpha}}_{\omega_i,\omega_j}) = I(g)$ for such $\bar{\alpha}$. Thus, Condition (i) holds.
Case 2: $f(\omega_i) \neq f(\omega_j)$ and $v(T) \neq v(T\backslash\{\omega_i,\omega_j\})$.

It follows from Lemma 3 that

$$\bar{\alpha} \in \{\alpha_f^{\omega_i,\omega_j}\} \Leftrightarrow (1-\bar{\alpha})v(T\backslash\{\omega_i,\omega_j\}) + \bar{\alpha}v(T) = v(T\backslash\{\omega_j\}), \tag{8}$$

and that such $\bar{\alpha}$ is uniquely determined. If $g(\omega_i) \geq g(\omega_j)$, then $I(g^{\bar{\alpha}}_{\omega_i,\omega_j}) = I(g)$ by (8) and Lemma 3. Suppose that $g(\omega_j) > g(\omega_i)$. Then, by Condition (ii), it holds that

$$v(T) + v(T\backslash\{\omega_i,\omega_j\}) \geq v(T\backslash\{\omega_i\}) + v(T\backslash\{\omega_j\}). \tag{9}$$

By subtracting (9) from (8), it holds that

$$\bar{\alpha}v(T\backslash\{\omega_i,\omega_j\}) + (1-\bar{\alpha})v(T) \geq v(T\backslash\{\omega_i\}). \tag{10}$$

It follows from (10) and Lemma 3 that

$$I(g^{\bar{\alpha}}_{\omega_i,\omega_j}) - I(g) \geq 0.$$

(ii) $\Leftrightarrow$ (iii). Similar to the proof of Proposition 2, it follows from Lemma 4 that

$$\begin{aligned} & v(T) + v(T\backslash\{\omega_i,\omega_j\}) - v(T\backslash\{\omega_i\}) - v(T\backslash\{\omega_j\}) \\ = & \sum_{\{\omega_i,\omega_j\}\subseteq S\subseteq T} \beta_S, \end{aligned}$$

which implies that $v(T)+v(T\backslash\{\omega_i,\omega_j\}) \geq v(T\backslash\{\omega_i\})+v(T\backslash\{\omega_j\}) \Leftrightarrow \sum_{\{\omega_i,\omega_j\}\subseteq S\subseteq T} \beta_S \geq 0$. Thus, the proof is completed. $\square$

Proof of Proposition 4

Proof. Step 1: We show that $\beta_{\{\omega_i\}} \geq 0$ for all $i = 1,\dots,n$. Because v is a capacity, $\beta_{\{\omega_i\}} = v(\{\omega_i\}) \geq 0$ for all $i = 1,\dots,n$.
Step 2: We construct the collection $\mathcal{E}$. Suppose that TPC holds. Then, for any

$T \in 2^\Omega$ with $|T| \geq 2$, there exist $\omega_i, \omega_j \in T$ satisfying TPC. We fix any pair ω_i, ω_j for T. Our argument does not depend on how we choose the pair. By TPC, ω_i and ω_j are I-coconvex on T. By Proposition 3, $v(T) + v(T\backslash\{\omega_i, \omega_j\}) \geq v(T\backslash\{\omega_i\}) + v(T\backslash\{\omega_j\})$. Here, if $v(T) + v(T\backslash\{\omega_i, \omega_j\}) = v(T\backslash\{\omega_i\}) + v(T\backslash\{\omega_j\})$ for all T, then set $\mathcal{E} = \{\emptyset\}$. Otherwise, define the collection $\mathcal{E} = \{E_1, \ldots, E_m\}$ of T such that $v(T) + v(T\backslash\{\omega_i, \omega_j\}) > v(T\backslash\{\omega_i\}) + v(T\backslash\{\omega_j\})$. It follows from Proposition 3 and the definition of $\mathcal{E}$ that for each $E \in \mathcal{E}$,

$$\sum_{\{\omega_i,\omega_j\}\subseteq S\subseteq E} \beta_S > 0, \tag{11}$$

where ω_i and ω_j are some pair satisfying TPC for E. On the other hand, for $T \notin \{E_1, \ldots, E_m\}$, it follows from Proposition 2 and the definition of $\mathcal{E}$ that

$$\sum_{\{\omega_i,\omega_j\}\subseteq S\subseteq T} \beta_S = 0. \tag{12}$$

Step 3: We show that for any S with $\{\omega_i, \omega_j\} \subseteq S \subsetneq T$, $\beta_S = 0$, where $T \in 2^\Omega$ with $|T| \geq 2$ and $\omega_i, \omega_j \in T$ are any pair satisfying TPC. If $|T| = 2$, then such S does not exist. So, we assume that $|T| \geq 3$. We show this claim by induction. Let $|S| = 2$, that is, $S = \{\omega_i, \omega_j\}$. It follows from Proposition 2 and TPC that

$$0 = \sum_{\{\omega_i,\omega_j\}\subseteq S'\subseteq S} \beta_{S'} = \beta_S.$$

Suppose that the claim holds for $2 \leq |S| \leq k$. Let $|S| = k+1$. Then, it also follows from Proposition 2 and TPC that

$$0 = \sum_{\{\omega_i,\omega_j\}\subseteq S'\subseteq S} \beta_{S'} = \sum_{\{\omega_i,\omega_j\}\subseteq S'\subsetneq S} \beta_{S'} + \beta_S = \beta_S,$$

where $\sum_{\{\omega_i,\omega_j\}\subseteq S'\subsetneq S} \beta_{S'} = 0$ by the assumption of induction.

Step 4: We show that $\beta_{E_j} > 0$ for $j = 1, \ldots, m$ and $\beta_T = 0$ for $T \notin \{E_1, \ldots, E_m\}$ with $|T| \geq 2$. By the definition of E_j at Step 2, it holds that $\sum_{\{\omega_i,\omega_j\}\subseteq S\subseteq E} \beta_S > 0$. From Step 3, it follows that $\beta_S = 0$ for $\{\omega_i, \omega_j\} \subseteq S \subsetneq E$. Thus, $\beta_{E_j} > 0$. Next, let $T \notin \{E_1, \ldots, E_m\}$ and $|T| \geq 2$. By the definition of $\mathcal{E}$ at Step 2, it holds that $\sum_{\{\omega_i,\omega_j\}\subseteq S\subseteq T} \beta_S = 0$. From Step 3, it follows that $\beta_S = 0$ for $\{\omega_i, \omega_j\} \subseteq S \subsetneq T$. Thus, $\beta_T = 0$. $\square$

Proof of Lemma 5. (i) $\Leftrightarrow$ (ii)

First, suppose (i). By assumption, $\mathcal{E}$ is complete, and for any $E_j \in \mathcal{E}$, $\mathcal{E}\backslash\{E_j\}$ is also complete, the latter of which implies $\Upsilon(\mathcal{E}\backslash\{E_j\}) = \mathcal{E}\backslash\{E_j\}$. Thus, it holds that $E_j \notin \Upsilon(\mathcal{E}\backslash\{E_j\})$ for all $E_j \in \mathcal{E}$, which shows that (ii) holds.

Next, suppose (ii). We show that for all $E_j \in \mathcal{E}$, $\mathcal{E}\backslash\{E_j\}$ is complete. Let T be an $\mathcal{E}\backslash\{E_j\}$-complete set. Because $E_j \notin \Upsilon(\mathcal{E}\backslash\{E_j\})$, it holds that $T \neq E_j$. On the other hand, because $\mathcal{E}\backslash\{E_j\} \subseteq \mathcal{E}$, T is also $\mathcal{E}$-complete. Because $\mathcal{E}$ is complete, $T \in \mathcal{E}$. Therefore, $T \in \mathcal{E}\backslash\{E_j\}$, and it holds that $\Upsilon(\mathcal{E}\backslash\{E_j\}) \subseteq \mathcal{E}\backslash\{E_j\}$. Thus, $\Upsilon(\mathcal{E}\backslash\{E_j\}) = \mathcal{E}\backslash\{E_j\}$.

(ii) $\Leftrightarrow$ (iii)

Suppose (ii). When $T \notin \Upsilon(\mathcal{E})$ (that is, T is not $\mathcal{E}$-complete), by $\mathcal{E}$'s completeness and definition, there exists a two-point set $\{\omega_1, \omega_2\} \subseteq T$ such that $\{\omega_1, \omega_2\} \subseteq E_j \subseteq T$ does not hold for any $E_j \in \mathcal{E}$. Thefore, (iii) holds for such T. When $T \in \Upsilon(\mathcal{E})$ (that is, T is $\mathcal{E}$-complete), $T \in \mathcal{E}$ by $\mathcal{E}$'s completeness. Therefore, this set T is equal to some set $E_j \in \mathcal{E}$. By (ii), $E_j \notin \Upsilon(\mathcal{E}\backslash\{E_j\})$, that is, E_j is not $\mathcal{E}\backslash\{E_j\}$-complete. Therefore, there exists a two-point set $\{\omega_i, \omega_j\} \subseteq T$ such that for all S with $\{\omega_i, \omega_j\} \subseteq S \subseteq T$, $S \notin \mathcal{E}\backslash\{E_j\}$. Thus, for such T, (iii) holds because any S with $\{\omega_i, \omega_j\} \subseteq S \subsetneq E_j (= T)$ does not belong to $\mathcal{E}$.

Next, suppose (iii). When $T \in \Upsilon(\mathcal{E})$ (that is, T is $\mathcal{E}$-complete), for any two-point set $\{\omega_p, \omega_q\} \subseteq T$, there exists a set $E_j \in \mathcal{E}$ such that $\{\omega_p, \omega_q\} \subseteq E_j \subseteq T$. By (iii), there exists a two-point set $\{\omega_1, \omega_2\} \subseteq T$ such that for all S with $\{\omega_1, \omega_2\} \subseteq S \subsetneq T$, it holds that $S \notin \mathcal{E}$. Therefore, for some E_j, it holds that $T = E_j$, which means $\Upsilon(\mathcal{E}) \subseteq \mathcal{E}$. Thus, $\mathcal{E}$ is complete. Next, let T be any set $E_j \in \mathcal{E}$. By (iii), there exists a two-point set $\{\omega_1, \omega_2\} \subseteq T$ such that for all S with $\{\omega_1, \omega_2\} \subseteq S \subsetneq T$, it holds that $S \notin \mathcal{E}$. Therefore, E_j is not $\mathcal{E}\backslash\{E_j\}$-complete, which implies that $E_j \notin \Upsilon(\mathcal{E}\backslash\{E_j\})$ for all $E_j \in \mathcal{E}$. Thus, (ii) is proved. $\square$

Proof of Proposition 5. **Well-definedness:** We show that the collection $\Xi(I) = \{T|\beta_T > 0\}$ derived in Proposition 4 is simple-complete, where $I(f) = \sum_{i=1}^{n} \beta_{\omega_i} f(\omega_i) + \sum_{E\in\mathcal{E}} \beta_E \min_{\omega\in E} f(\omega)$. Take any T with $|T| \geq 2$, and let $\omega_i, \omega_j \in T$ satisfy TPC. From Step 3 in the proof of Proposition 4, it holds that $\beta_S = 0$ for any S with $\{\omega_i, \omega_j\} \subseteq S \subsetneq T$. By the definition of $\Xi(I)$, it holds that $S \notin \Xi(I)$. Thus, by (iii) in Lemma 5, the collection $\Xi(I)$ is simple-complete.

Onto Mapping: For $\mathcal{E} = \emptyset$, let I be an expectation (that is, the usual expectation with respect to probability). Then, $\Xi(I) = \emptyset$. Therefore, let $\mathcal{E} = \{E_1, \ldots, E_m\}$ be any simple-complete collection, and let $I(f) = \sum_{i=1}^{n} \beta_{\{\omega_i\}} f(\omega_i) + \sum_{j=1}^{m} \beta_{E_j} \min_{\omega \in E_j} f(\omega)$, where $\beta_{\{\omega_i\}} \geq 0$ for $i = 1, \ldots, n$ and $\beta_{E_j} > 0$ for $j = 1, \ldots, m$. For $v = \sum_{i=1}^{n} \beta_{\{\omega_i\}} u_{\{\omega_i\}} + \sum_{j=1}^{m} \beta_{E_j} u_{E_j}$, $I(f)$ is equal to the Choquet integral of f with respect to v, that is, $I(f) = \int_\Omega f(\omega) dv$.

Next, we show that I satisfies TPC. Let $T \in 2^\Omega$ with $|T| \geq 2$ be fixed. Because $\mathcal{E}$ is simple-complete, there exist two distinct points $\omega_i, \omega_j \in T$ such that for all S with $\{\omega_i, \omega_j\} \subseteq S \subsetneq T$, it holds that $S \notin \mathcal{E}$. Therefore, $\beta_S = 0$ and $\beta_T \geq 0$. For these ω_i, ω_j, it holds that $\sum_{\{\omega_i, \omega_j\} \subseteq S \subseteq T} \beta_S \geq 0$. Thus, Proposition 3 implies that ω_i and ω_j are I-coconvex on T. Moreover, for S with $\{\omega_i, \omega_j\} \subseteq S \subsetneq T$, any S' with $\{\omega_i, \omega_j\} \subseteq S' \subseteq S$ implies $S' \notin \mathcal{E}$ because $\mathcal{E}$ is simple-complete. For these ω_i and ω_j, it holds that $\sum_{\{\omega_i, \omega_j\} \subseteq S' \subseteq S} \beta_{S'} = 0$. Thus, Proposition 2 implies that ω_i and ω_j are I-comodular on T. Therefore, TPC holds. Hence, there exists $I \in \mathfrak{I}$ such that $\Xi(I) = \mathcal{E}$. $\square$

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