On the Invertibility of EGARCH(p,q)*

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Revised: February 2015

* The authors are grateful to the Editor-in-Chief, Rob Taylor, an Associate Editor and two referees for very helpful comments and suggestions, and to Christian Hafner for insightful discussions. For financial support, the first author wishes to thank the National Science Council, Taiwan, and the second author is most grateful to the Australian Research Council and the National Science Council, Taiwan.

Abstract

Of the two most widely estimated univariate asymmetric conditional volatility models, the exponential GARCH (or EGARCH) specification can capture asymmetry, which refers to the different effects on conditional volatility of positive and negative effects of equal magnitude, and leverage, which refers to the negative correlation between the returns shocks and subsequent shocks to volatility. However, the statistical properties of the (quasi-) maximum likelihood estimator (QMLE) of the EGARCH parameters are not available under general conditions, but only for special cases under highly restrictive and unverifiable conditions, such as EGARCH(1,0) or EGARCH(1,1), and possibly only under simulation. A limitation in the development of asymptotic properties of the QMLE for the EGARCH(p,q) model is the lack of an invertibility condition for the returns shocks underlying the model. It is shown in this paper that the EGARCH(p,q) model can be derived from a stochastic process, for which the invertibility conditions can be stated simply and explicitly. This will be useful in re-interpreting the existing properties of the QMLE of the EGARCH(p,q) parameters.

Keywords: Leverage, asymmetry, existence, stochastic process, asymptotic properties, invertibility.

JEL classifications: C22, C52, C58, G32.

1. Introduction

In addition to modeling and forecasting volatility, and capturing clustering, two key characteristics of univariate time-varying conditional volatility models in the GARCH class of Engle (1982) and Bollerslev (1986) are asymmetry and leverage. Asymmetry refers to the different impacts on volatility of positive and negative shocks of equal magnitude, whereas leverage, as a special case of asymmetry, captures the negative correlation between the returns shocks and subsequent shocks to volatility. Black (1976) defined leverage in terms of the debt-to-equity ratio, with increases in volatility arising from negative shocks to returns and decreases in volatility arising from positive shocks to returns.

The two most widely estimated asymmetric univariate models of conditional volatility are the exponential GARCH (or EGARCH) model of Nelson (1990, 1991), and the GJR (alternatively, asymmetric or threshold) model of Glosten, Jagannathan and Runkle (1992). As EGARCH is a discrete-time approximation to a continuous-time stochastic volatility process, and is expressed in logarithms, conditional volatility is guaranteed to be positive without any restrictions on the parameters. In order to capture leverage, the EGARCH model requires parametric restrictions to be satisfied. Leverage is not possible for GJR, unless the short run persistence parameter is negative, which is not consistent with the standard sufficient condition for conditional volatility to be positive, or for the process to be consistent with a random coefficient autoregressive model (see McAleer (2014)).

As GARCH can be obtained from random coefficient autoregressive models (see Tsay (1987)), and similarly for GJR (see McAleer et al. (2007) and McAleer (2014)), the statistical properties for the (quasi-) maximum likelihood estimator (QMLE) of the GARCH and GJR parameters are straightforward to establish. However, the statistical properties for the QMLE of the EGARCH parameters are not available under general conditions. A limitation in the development of asymptotic properties of the QMLE for EGARCH is the lack of an invertibility condition for the returns shocks underlying the model.

McAleer and Hafner (2014) showed that EGARCH(1,1) could be derived from a random coefficient complex nonlinear moving average (RCCNMA) process. The reason for the lack of statistical properties of the QMLE of EGARCH(p,q) under general conditions is that the stationarity and invertibility conditions for the RCCNMA process are not known, except possibly under simulation, in part because the RCCNMA process is not in the class of random coefficient linear moving average models (for further details, see Marek (2005)).

The recent literature on the asymptotic properties of the QMLE of EGARCH shows that such properties are available only for some special cases, and under highly restrictive and unverifiable conditions. For example, Straumann and Mikosch (2006) derive some asymptotic results for the simple EARCH(∞) model, but their regularity conditions are difficult to interpret or verify. Wintenberger (2013) proves consistency and asymptotic normality for the QMLE of EGARCH(1,1) under the non-verifiable assumption of invertibility of the model. Demos and Kyriakopoulou (2014) present sufficient conditions for asymptotic normality under a highly restrictive conditions that are difficult to verify.

It is shown in this paper that the EGARCH(p,q) model can, in fact, be derived from a stochastic process, for which the invertibility conditions can be stated simply and explicitly. This will be useful in re-interpreting the existing properties of the QMLE of the EGARCH(p,q) parameters.

The remainder of the paper is organized as follows. In Section 2, the EARCH(∞) model is discussed, together with notation and lemmas. Section 3 presents a new stochastic process and regularity conditions, from which EARCH(∞) is derived, without proofs of existence and uniqueness. Section 4 develops a key result for the invertibility of the EARCH(∞) model. Section 5 analyses the EGARCH(p,q) specification, while Section 6 develops the regularity conditions for the invertibility of EGARCH(p,q). Section 7 considers the special case of the N(0,1) distribution. Section 8 provides a summary of the invertibility conditions for EGARCH(p,q). Some concluding comments are given in Section 9. Proofs of the lemmas and propositions are given in the Appendix.

2. EARCH(∞), Notation and Lemmas

Instead of using a recursive equation for conditional volatility, which would require proofs of existence and uniqueness, we will work on a direct definition of the stochastic process that drives the so-called innovation, ε_t. By definition, the new process will define uniquely the stochastic process that drives the innovation, as follows:

$$\varepsilon_t = \eta_t . \exp\left(\frac{\omega}{2} + \sum_{i=1}^{+\infty} \beta_i \left[\frac{\alpha}{2}\left|\eta_{t-i}\right| + \frac{\gamma}{2}\eta_{t-i}\right]\right), \tag{0}$$

where $\omega \in \Re$, $(\alpha, \gamma) \in \Re^2$, $\sum_i |\beta_i| < \infty$, and η_t~(0,1), so that $\eta_t \in L^2$. Thus, we have the EARCH(∞) model, as introduced by Nelson (1990, 1991):

$$\begin{cases} \log(\sigma_t^2) \equiv \omega + \sum_{i=1}^{\infty} \beta_i \left[\alpha\left|\eta_{t-i}\right| + \gamma\eta_{t-i}\right] \\ \varepsilon_t = \eta_t \sigma_t \end{cases}$$

The primary purpose of this paper is to establish the invertibility of the model, where invertibility refers to the fact that the normalized shocks (η_t) may be written in terms of the previous observed values, that is, η_t is $\sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$-adapted. Note that this definition is equivalent to that used by Wintenberger (2013) and Straumann and Mikosch (2006), namely that σ_t is $\sigma(\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots)$- adapted.

In a similar manner to proving invertibility for the Moving Average (MA) case, we will express recursively all the independently and identically distributed (*iid*) shocks in terms of the past observed shocks and some arbitrary fixed constant, and then prove that this backward recursion converges almost surely to the real value of η_t.

Consider the following notation:

$$\delta_t \equiv \frac{\alpha}{2} + \frac{\gamma}{2} sign(\eta_t),$$

so that:

$$\varepsilon_t = \eta_t . \exp\left(\frac{\omega}{2} + \sum_{i=1}^{\infty} \beta_i \delta_{t-i} |\eta_{t-i}| \right). \tag{1}$$

As sign(η_t) = sign(ε_t), δ_t is indeed $\sigma(\varepsilon_t)$-adapted. Therefore, by proving that $|\eta_t|$ is $\sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$-adapted, it will follow automatically that the model is invertible.

By assuming that the distribution of η_t does not admit a probability mass at 0, we can take the absolute value and then the logarithm of ε_t. In order to be rigorous in the development below, we assume that $\eta_t \neq 0$, almost surely. By rewriting the equation, we have:

$$\log|\eta_t| = \log|\varepsilon_t| - \frac{\omega}{2} - \sum_{i=1}^{\infty} \beta_i \delta_{t-i} |\eta_{t-i}| . \tag{2}$$

Define the following function:

$$g_{\alpha,\gamma}(x, y) \equiv -\frac{\alpha + sign(y).\gamma}{2} \exp(x) ,$$

so that we have:

$$\log|\eta_t| = \log|\varepsilon_t| - \frac{\omega}{2} + \sum_{i=1}^{\infty} \beta_i . g_{\alpha,\gamma}\left(\log|\eta_{t-i}|, \varepsilon_{t-i}\right).$$

This function is not Lipschitzian, so that we should find some results about variability, as in the Lyapunov coefficient in other invertibility proofs. Lemma 1.1 gives a solution, which will be used widely in several proofs below:

Lemma 1.1

(1) $\left|g_{\alpha,\gamma}(x_1, y) - g_{\alpha,\gamma}(x_2, y)\right| \leq \left|\frac{\alpha + sign(y).\gamma}{2}\right| \exp\left(\max(x_1, x_2)\right)\left|x_1 - x_2\right|$

(2) $\left|g_{\alpha,\gamma}(x_1, y) - g_{\alpha,\gamma}(x_2, y)\right| \geq \left|\frac{\alpha + sign(y).\gamma}{2}\right| \exp\left(\frac{x_1 + x_2}{2}\right)\left|x_1 - x_2\right|$

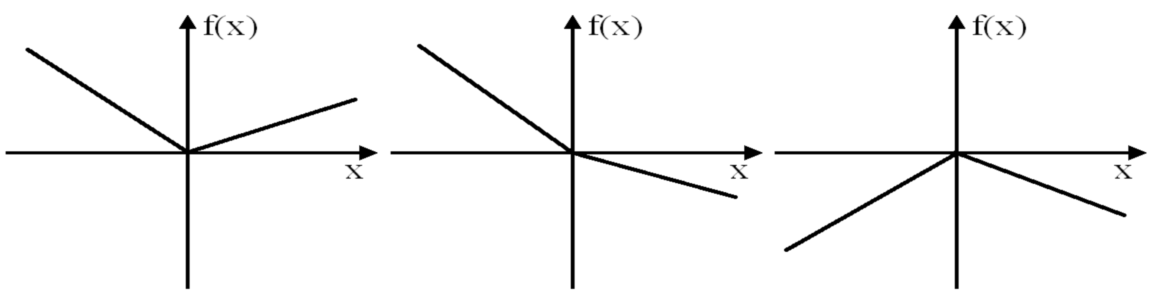
The proof of Lemma 1.1 is given in the Appendix (part 1). Moreover, we will also use the Borel-Cantelli Lemma and one of its corollaries, namely Lemma 1.2 (which is also given in the Appendix (part 1)).

3. EARCH(1): A New Stochastic Specification and Regularity Conditions

By ensuring positivity, the EGARCH model allows the possibility of leverage, namely that positive shocks lead to a decrease in volatility and negative shocks lead to an increase in volatility. Therefore, leverage occurs when $|\alpha| < |\gamma|$ and $\gamma < 0$. We will also examine two other cases where shocks lead to either an increase in volatility ($\alpha \geq |\gamma|$) or a decrease in volatility ($\alpha < -|\gamma|$). A fourth possibility is symmetric to the leverage case, and hence need not be considered in detail.

All of these cases allows asymmetry as there are still two coefficients. The three cases are summarized in these graphs, where $f(x) = \alpha|x| + \gamma x$:

$\alpha \geq |\gamma|$ $\quad\quad$ $(|\alpha| < |\gamma|) \wedge (\gamma < 0)$ $\quad\quad$ $\alpha < -|\gamma|$



Before examining the invertibility of EARCH(∞) and EGARCH(p,q), we will examine briefly the simple EARCH(1) model to provide a justification for restricting the analysis to one of the above cases as a pre-condition for invertibility. This is also motivated by two other reasons: (i) it will allow us to introduce a novel approach; and (ii) the conditions for EARCH(1) are slightly different and less restrictive than those found in Section 6 for EGARCH(p,q) when $p = 1$ and $q = 0$ because of the concavity of log(.).

Consider the equation induced from (2) above for the special case of EARCH(1), that is, where $\beta_1 = 1$ and $\forall i \geq 2,\ \beta_i = 0$:

$$\log|\eta_t| = \log|\varepsilon_t| - \frac{\omega}{2} + g_{\alpha,\gamma}\left(\log|\eta_{t-1}|, \varepsilon_{t-1}\right). \tag{3}$$

We now introduce the following recursive series for a fixed $n \in \mathrm{N}^*$:

$$\begin{cases} u_1^{(n)} = \log|\varepsilon_{t-n+1}| - \dfrac{\omega}{2} + g_{\alpha,\gamma}\left(\log|\eta_{t-n}|, \varepsilon_{t-n}\right) \\ u_{k+1}^{(n)} = \log|\varepsilon_{t-n+k+1}| - \dfrac{\omega}{2} + g_{\alpha,\gamma}\left(u_k^{(n)}, \varepsilon_{t-n+k}\right) \end{cases} \tag{4}$$

It follows by recursion that:

$$u_k^{(n)} = \log|\eta_{t-n+k}|, \forall n \in \mathrm{N}^*, \forall k \in \mathrm{N}^*,$$

so that:

$$u_n^{(n)} = \log|\eta_t|, \forall n \in \mathrm{N}^*.$$

Define for any $c_0 \in \Re \cup \{-\infty\}$:

$$\begin{cases} v_1^{(n)} = \log|\varepsilon_{t-n+1}| - \dfrac{\omega}{2} + g_{\alpha,\gamma}(c_0, \varepsilon_{t-n}) \\ v_{k+1}^{(n)} = \log|\varepsilon_{t-n+k+1}| - \dfrac{\omega}{2} + g_{\alpha,\gamma}\left(v_k^{(n)}, \varepsilon_{t-n+k}\right) \end{cases} \quad (5)$$

These series $\forall n$ are $\sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$-adapted. In order to prove invertibility, we examine the convergence of the series $\left|v_n^{(n)} - u_n^{(n)}\right|$ toward zero, as the series defined in (5) is simply the natural backward recursion defined in (4), but conditionally on some constant value for previous shocks, namely $|\eta_{t-n}| = \exp(c_0)$.

(i) First case: $\alpha \geq |\gamma|$

By using Lemma 1.1, as $\delta_t \geq 0$ in this case:

$$\left|v_n^{(n)} - \log|\eta_t|\right| = \left|u_n^{(n)} - v_n^{(n)}\right| \leq \delta_{t-1} \exp\left(\max\left(u_{n-1}^{(n)}, v_{n-1}^{(n)}\right)\right)\left|u_{n-1}^{(n)} - v_{n-1}^{(n)}\right|.$$

Dealing with a sum of max(. , .), as it would be the case if we expand the recursion further, is difficult, so linearization yields:

$$\max(u_{n-1}^{(n)}, v_{n-1}^{(n)}) = u_{n-1}^{(n)} + \left(v_{n-1}^{(n)} - u_{n-1}^{(n)}\right)^+ \quad \text{where} \quad (x)^+ = \max(0, x).$$

But we have: $u_{n-1}^{(n)} = \log|\eta_{t-1}|$ and $v_{n-1}^{(n)} - u_{n-1}^{(n)} = \log|\varepsilon_{t-1}| - \dfrac{\omega}{2} + g_{\alpha,\gamma}\left(v_{n-2}^{(n)}, \varepsilon_{t-2}\right) - \log|\eta_{t-1}|$.

By using the fact that:

$$\log|\varepsilon_{t-1}| = \log|\eta_{t-1}| + \frac{\omega}{2} + \frac{\alpha}{2}|\eta_{t-2}| + \frac{\gamma}{2}\eta_{t-2}, \text{ and } \quad g_{\alpha,\gamma}(v_{n-2}^{(n)}, \varepsilon_{t-2}) \leq 0,$$

by assumption, we have:

$$\max\left(u_{n-1}^{(n)}, v_{n-1}^{(n)}\right) \le \log|\eta_{t-1}| + \frac{\alpha}{2}|\eta_{t-2}| + \frac{\gamma}{2}\eta_{t-2}.$$

By recursion we have:

$$\left|v_n^{(n)} - \log|\eta_t|\right| \le \exp\left(\sum_{i=1}^{n-1} \log\left(\delta_{t-i}|\eta_{t-i}|\right) + \delta_{t-i-1}|\eta_{t-i-1}|\right)\delta_{t-n}\left(|\eta_{t-n}| + \exp(c_0)\right). \qquad (6)$$

From the upper bound, the invertibility conditions based on the Law of Large Number (LLN) are given as:

$$\begin{cases} \log|\eta_t| \in L^1 \\ \mathrm{E}\left[\log\left(\delta_t|\eta_t|\right) + \delta_t|\eta_t|\right] < 0 \end{cases} \qquad \text{(Conditions 1)}$$

The proof of invertibility under these conditions (Proposition 2.1) is given in the Appendix (part 2). The proposition is given as:

Proposition 2.1

If Conditions 1 are verified when $\alpha \ge |\gamma|$, then the model EARCH(1) is invertible, that is, we have :

$$\left|v_n^{(n)} - u_n^{(n)}\right| = \left|v_n^{(n)} - \log|\eta_t|\right| \underset{n\to\infty}{\overset{a.s.}{\to}} 0.$$

Therefore, when α and γ satisfy $\alpha \ge |\gamma|$ and $\mathrm{E}\left[\log\left(\frac{\alpha|\eta_t| + \gamma\eta_t}{2}\right) + \frac{\alpha|\eta_t| + \gamma\eta_t}{2}\right] < 0$ (which is a non-empty set), we have invertibility. This condition is the same as in Remark 3.10 of Straumann and Mikosch (2006), so that our approach will not necessarily lead to more restrictive conditions than those already known.

Remark: For purposes of rigour in the proof, we had to assume that $\log|\eta_t| \in L^1$, or that the shocks η_t do not admit a mass at zero. However, in our backward recursion, $u_n^{(n)}$, if we had found η_t = 0 (which is equivalent to ε_t = 0, and is therefore a $\sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$-adapted event), we would have obtained directly the invertibility of the model. Actually, only $(\log|\eta_t|)^+ \in L^1$ is required, but it is already implied by the fact that $\eta_t \in L^2$.

(ii) Second case: $\alpha < -|\gamma|$

This case is the third case in the graphs above, namely where a shock leads to a decrease in volatility. For this case, we provide a counter-example to show that we cannot have the case of invertibility under the same general conditions and approach as stated above, but perhaps under more restrictive conditions (such as the normalized shocks are uniformly bounded).

Assume $\eta_t \sim \mathrm{N}(0,1)$, although any other distribution with thicker tails would lead to a similar result as given below.

Proposition 2.2

If $\eta_t \overset{i.i.d.}{\sim} \mathrm{N}(0,1)$ and $\alpha < -|\gamma|$, then we cannot prove invertibility with our method as $\left|v_n^{(n)} - u_n^{(n)}\right|$ does not converge to 0, and even admits an extracted series that diverges almost surely toward infinity.

The proof of this proposition can be found in the Appendix (part 2). More precisely, this result indicates that the backward recursion will behave too erratically to allow us to prove invertibility. It indicates also that the past tends to have a persistent effect on the time series induced by this model, and could be quite divergent. For this reason, the model here might not be invertible, and so it will be assumed that $\alpha < -|\gamma|$ does not hold.

(iii) Third case: $|\alpha| < |\gamma|$ and $\gamma < 0$

We now examine leverage. We can also consider for this case the counter-example used for the previous case (see Appendix (part 2)). Given the previous results, we cannot use inequality (1) in Lemma 1 to reach a conclusion regarding invertibility. Specifically, we would not be able to obtain an upper bound for $\left|v_n^{(n)} - u_n^{(n)}\right|$ that converges to zero. Moreover, we would also not be able to use inequality (2) of Lemma 1 recursively to prove the divergence like in Proposition 2.2 as we could obtain a lower bound that would tend to zero. Actually, it would be difficult to conclude in this case, but as this is a combination of the two first cases, we are also likely to find a very erratic asymptotic behavior for $\left|v_n^{(n)} - u_n^{(n)}\right|$.

Thus, as a conclusion of this part, our approach could lead to a proof of invertibility for the case $\alpha \geq |\gamma|$, and possibly lead to non-invertibility for the other two cases. Accordingly, in order to examine a more general case than the simple EARCH(1) model, it will be necessary to assume that $\alpha \geq |\gamma|$.

4. Key Result for the Invertibility of EARCH(∞)

Given the previous analysis, in the following it will be assumed that $\alpha \geq |\gamma|$ and that all the β_i are non-negative. The following was derived from equation (2):

$$\log|\eta_t| = \log|\varepsilon_t| - \frac{\omega}{2} + \sum_{i=1}^{\infty} \beta_i . g_{\alpha,\gamma}\left(\log|\eta_{t-i}|, \varepsilon_{t-i}\right). \tag{7}$$

Define the $u_k^{(n)}$ and $v_k^{(n)}$ series as:

$$\begin{cases} u_1^{(n)} = \log|\varepsilon_{t-n+1}| - \dfrac{\omega}{2} + \displaystyle\sum_{i=0}^{\infty} \beta_{i+1} g_{\alpha,\gamma}\left(\log|\eta_{t-n-i}|, \varepsilon_{t-n-i}\right) \\ u_{k+1}^{(n)} = \log|\varepsilon_{t-n+k+1}| - \dfrac{\omega}{2} + \displaystyle\sum_{j=1}^{k} \beta_j g_{\alpha,\gamma}\left(u_{k+1-j}^{(n)}, \varepsilon_{t-n+k+1-j}\right) + \displaystyle\sum_{i=0}^{\infty} \beta_{i+1+k} g_{\alpha,\gamma}\left(\log|\eta_{t-n-i}|, \varepsilon_{t-n-i}\right) \end{cases} \quad (8)$$

As before, it follows that:

$$u_k^{(n)} = \log|\eta_{t-n+k}|, \forall n \in \mathrm{N}^*, \forall k \in \mathrm{N}^*.$$

As it is not as straightforward as the EARCH(1) case, Lemma 3.1 will be useful (the proof of which is given in the Appendix (part 3)):

Lemma 3.1

$$u_k^{(n)} = \log|\eta_{t-n+k}|, \forall n \in \mathrm{N}^*, \forall k \in \mathrm{N}^*.$$

Now we define the $v_k^{(n)}$ series:

$$\begin{cases} v_1^{(n)} = \log|\varepsilon_{t-n+1}| - \dfrac{\omega}{2} \\ v_{k+1}^{(n)} = \log|\varepsilon_{t-n+k+1}| - \dfrac{\omega}{2} + \displaystyle\sum_{j=1}^{k} \beta_j g_{\alpha,\gamma}\left(v_{k+1-j}^{(n)}, \varepsilon_{t-n+k+1-j}\right) \end{cases} \quad (9)$$

We remark that $v_k^{(n)}$ is established like $u_k^{(n)}$, but by assuming that all the η_i for $i \le t-n$ are equal to zero. Here, we have chosen these “initial values” in order to simplify the development, but one can also check our further results for any kind of values for η_i before *t* - *n*, as long as the sum does not diverge. In any event, the proof of invertibility will be based on the $v_k^{(n)}$ as $\sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$-adapted.

It is essential to prove that:

$$\left|v_n^{(n)} - \log|\eta_t|\right| = \left|v_n^{(n)} - u_n^{(n)}\right| \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} 0 .$$

Consider the upper bound for $\left|v_n^{(n)} - u_n^{(n)}\right|$ in inequality (1) of Lemma 1.1, from which it can be shown that:

$$\left|v_n^{(n)} - u_n^{(n)}\right| \le \sum_{i=0}^{+\infty} \beta_{i+n}\delta_{t-n-i}\left|\eta_{t-n-i}\right| + \sum_{j=1}^{n-1} \beta_j \delta_{t-j} \exp\left(\max\left(v_{n-j}^{(n)}, u_{n-j}^{(n)}\right)\right)\left|v_{n-j}^{(n)} - u_{n-j}^{(n)}\right| .$$

so that:

$$\max\left(u_{n-j}^{(n)}, v_{n-j}^{(n)}\right) = \log\left|\eta_{t-j}\right| + \left(v_{n-j}^{(n)} - \log\left|\eta_{t-j}\right|\right)^+ ,$$

$$v_{n-j}^{(n)} = \log\left|\eta_{t-j}\right| + \sum_{i=1}^{+\infty} \beta_i \delta_{t-j-i}\left|\eta_{t-j-i}\right| + \sum_{i=1}^{n-j-1} \beta_i g_{\alpha,\gamma}\left(v_{n-j-i}^{(n)}, \varepsilon_{t-j-i}\right)$$

$$\Rightarrow v_{n-j}^{(n)} \le \log\left|\eta_{t-j}\right| + \sum_{i=1}^{+\infty} \beta_i \delta_{t-j-i}\left|\eta_{t-j-i}\right| ,$$

as $g_{\alpha,\gamma}$ is non-positive function, so that:

$$\boxed{\max\left(u_{n-j}^{(n)}, v_{n-j}^{(n)}\right) \le \xi_{t-j} \equiv \log\left|\eta_{t-j}\right| + \sum_{i=1}^{\infty} \beta_i \delta_{t-j-i}\left|\eta_{t-j-i}\right|} .$$

Therefore. It follows that:

$$\left|u_n^{(n)} - v_n^{(n)}\right| \le \sum_{i=0}^{\infty} \beta_{i+n}\delta_{t-n-i}\left|\eta_{t-n-i}\right| + \sum_{j=1}^{n-1} \beta_j \delta_{t-j} \exp\left(\xi_{t-j}\right)\left|u_{n-j}^{(n)} - v_{n-j}^{(n)}\right|.$$

The recursion may be extended, as follows:

Define:

$$a_k \equiv \sum_{i=0}^{+\infty} \delta_{t-n-i}\left|\eta_{t-n-i}\right| \left[\beta_{i+n} + \sum_{p=1}^{k-1} \sum_{i_1,\dots,i_p \in A_p^{(n)}} \hat{\Pi}_p \hat{D}_p \exp\left(\sum_{j=1}^{p} \xi_{t-\hat{S}_j}\right) \times \beta_{i+n-\hat{S}_p} \right] +$$
$$+ \sum_{i_1,\dots,i_k \in A_k^{(n)}} \hat{\Pi}_k \hat{D}_k \exp\left(\sum_{i=1}^{k} \xi_{t-\hat{S}_j}\right) \left| u_{n-\hat{S}_k}^{(n)} - v_{n-\hat{S}_k}^{(n)} \right|$$

where:

- $\hat{S}_l = \sum_{j=1}^{l} i_j$
- $A_p^{(n)} = \left\{ i_1 \ge 1, \dots, i_p \ge 1 : \hat{S}_p \le n-1 \right\}$
- $\hat{\Pi}_l = \prod_{j=1}^{l} \beta_{i_j}$
- $\hat{D}_l = \prod_{j=1}^{l} \delta_{t-\hat{S}_j}$

The above leads to Lemma 3.2, the proof of which is given in the Appendix (part 3):

Lemma 3.2

$$\left|v_n^{(n)} - u_n^{(n)}\right| \le a_k, \forall k \in [1, n[$$

By taking $k = n - 1$, by using the inequality $\left|u_1^{(n)} - v_1^{(n)}\right| \le \sum_{i=0}^{\infty} \beta_{i+1}\delta_{t-n-i}\left|\eta_{t-n-i}\right|$, we have the following general result for EARCH(∞):

Proposition 3.1

If $\alpha \geq |\gamma|$, $\beta_i \geq 0, \forall i$, then we have the following inequality for the series u and v for EARCH(∞):

$$\left|u_n^{(n)} - v_n^{(n)}\right| \leq \sum_{i=0}^{+\infty} \delta_{t-n-i} |\eta_{t-n-i}| \left[\beta_{i+n} + \sum_{p=1}^{n-1} \sum_{i_1,\ldots,i_p \in A_p^{(n)}} \hat{\Pi}_p \hat{D}_p \exp\left(\sum_{j=1}^{p} \xi_{t-\hat{S}_j} \right) \times \beta_{i+n-\hat{S}_p} \right].$$

An examination of invertibility for a general EARCH(∞) would use this upper bound. In our case, as it could be difficult if we do not assume a minimum on the behavior of the beta coefficients, we will examine the case of EGARCH(p,q).

5. EGARCH(*p*,*q*) Specification

Consider the general EGARCH(p,q) model:

$$\log \sigma_t = \frac{\omega}{2} + \sum_{i=1}^{p} a_i \log \sigma_{t-i} + \sum_{i=1}^{q} b_i \delta_{t-i} |\eta_{t-i}|, \quad a_i \in \Re, \quad b_i \in \Re. \tag{10}$$

In order to be able to use the previous result for EARCH(∞), this model should admit an EARCH(∞) representation. By using the backward lag operator L, this model can be rewritten as:

$$\left(1 - \sum_{i=1}^{p} a_i L^i\right) \log \sigma_t = \frac{\omega}{2} + \sum_{i=1}^{q} b_i \delta_{t-i} |\eta_{t-i}|, \quad a_i \in \Re, \quad b_i \in \Re. \tag{11}$$

In order to have an EARCH(∞) representation, the polynomial $\left(1 - \sum_{i=1}^{p} a_i L^i\right)$ should have roots outside the unit circle. If we set $\theta_i \in C$, $|\theta_i| < 1$, we can rewrite the model as:

$$(1-\theta_1 L)\ldots(1-\theta_p L)\log\sigma_t = \frac{\omega}{2} + \sum_{i=1}^{q} b_i \delta_{t-i}|\eta_{t-i}|, \quad |\theta_i| \in [0,1[, \quad b_i \in \Re . \qquad (12)$$

In order to consider invertibility, we should have $\alpha \geq |\gamma|$ and the β_i coefficients of the EARCH(∞) representation to be non-negative. This could easily be achieved if all the coefficients a_i and b_i are non-negative. Indeed, if we rename $y_i \equiv \delta_{t-i}|\eta_{t-i}|$, one can easily check the positivity of the β_i coefficients by taking the partial differential of $\log\sigma_t$ with respect to y_t :

$$\log\sigma_t = \frac{\omega}{2} + \sum_{i\geq 1} \beta_i y_i \Rightarrow \forall i \;\; \frac{\partial \log\sigma_t}{\partial y_i} = \beta_i$$

$$\frac{\partial \log\sigma_t}{\partial y_i} = \sum_{j=1}^{p} a_j \frac{\partial \log\sigma_{t-j}}{\partial y_i} + \sum_{k=1}^{q} b_k 1_{k=i}$$

where 1 represents the index function. From the above equation, one can easily check recursively the positivity of the β_i coefficients.

<u>Remark</u>: In the following, it will be assumed that all the coefficients a_i and b_i are non-negative, so the β_i of the EARCH(∞) representation are also non-negative.

As the β_i coefficients are assumed to be non-negative, we wish to find an appropriate upper bound that can be used in Proposition 3.1, specifically an upper bound such as $\beta_i \leq C.\beta^{i-1}$**,** where C is a positive real number and $\beta \in]0,1[$. As long as such a bound can be found, this can be used in the inequality in Proposition 3.1 by redefining the coefficients as:

$$\alpha \leftarrow C \times \alpha$$
$$\gamma \leftarrow C \times \gamma$$
$$\beta_i \leftarrow \beta^{i-1}$$

and to reduce examination of invertibility of an EGARCH(p,q) model to a simple EGARCH(1,1) model of this following specification:

$$\log \sigma_t = \frac{\omega}{2} + \beta \log \sigma_{t-1} + \delta_{t-1} |\eta_{t-1}|.$$

These "updated" coefficients will be given as α^*, γ^*, β^* below.

From equation (12), in the EARCH(∞) representation the above β^* would be greater than the maxima of the absolute values of the θ_i. When all the $|\theta_i|$ are different, we could choose β^* as being the maximum value. However, the polynomial $\left(1 - \sum_{i=1}^{p} a_i L^i\right)$ may have double roots, or at least, as it is a polynomial with real coefficients, admits couples of complex roots and their conjugates, thereby having the same absolute value. In these case, we would not be able to find an upper bound like $\beta_i \leq C.\beta^{*i-1}$ if we use $\beta^* = \max_i |\theta_i|$. Therefore, in our "general" analysis, consider a coefficient such as $\beta_{\sup} > \max_i |\theta_i|$. This coefficient can be chosen arbitrarily as long as it is strictly less than 1 and above the absolute values of the θ_i. Order these parameters such that $|\theta_1| \geq ... \geq |\theta_p|$. As shown in the analysis of EARCH(1), it will be recalled that the parameter ω had no influence on invertibility.

In order to find the appropriate α^*, γ^*, β^* values, we present a recursion. Starting with $(1-\theta_1 L)^{-1} \times \left(\sum_{i=1}^{q} b_i \delta_{t-i} |\eta_{t-i}|\right)$:

$$(1-\theta_1 L)^{-1} \times \left(\sum_{i=1}^{q} b_i \delta_{t-i} |\eta_{t-i}| \right) = \sum_{l=0}^{+\infty} \theta_1^{\,l} \sum_{i=1}^{q} b_i \delta_{t-l-i} |\eta_{t-l-i}| = \sum_{m=1}^{+\infty} \theta_1^{\,m-1} \left(\sum_{i=1}^{\min(q,m)} b_i \theta_1^{\,1-i} \right) \delta_{t-m} |\eta_{t-m}| .$$

By taking $m = i + l$, we can introduce $\beta_{\sup}$:

$$\sum_{m=1}^{+\infty} \theta_1^{\,m-1} \left(\sum_{i=1}^{\min(q,m)} b_i \theta_1^{\,1-i} \right) \delta_{t-m} |\eta_{t-m}| = \sum_{m=1}^{+\infty} \beta_{\sup}^{\ \ m-1} \left(\frac{\theta_1}{\beta_{\sup}} \right)^{m-1} \left(\sum_{i=1}^{\min(q,m)} b_i \theta_1^{\,1-i} \right) \delta_{t-m} |\eta_{t-m}| = \sum_{m=1}^{+\infty} \beta_{\sup}^{\ \ m-1} C_m \delta_{t-m} |\eta_{t-m}|$$

where:

$$C_m \equiv \left(\frac{\theta_1}{\beta_{\sup}} \right)^{m-1} \left(\sum_{i=1}^{\min(q,m)} b_i \theta_1^{\,1-i} \right),$$

so that:

$$|C_m| \le \left(\sum_{i=1}^{\min(q,m)} b_i \left(\frac{|\theta_1|}{\beta_{\sup}} \right)^{m-i} \beta_{\sup}^{\ \ 1-i} \right) \le \sum_{i=1}^{q} b_i \beta_{\sup}^{\ \ 1-i} .$$

Consider:

$$(1-\theta_i L)^{-1} \times \left(\sum_{m=1}^{+\infty} \beta_{\sup}^{m-1} C_m \delta_{t-m} |\eta_{t-m}| \right), \text{ and for any other } \theta_i,\ i \ge 2,$$

so that:

$$(1-\theta_i L)^{-1}\times\left(\sum_{m=1}^{+\infty}\beta_{\sup}^{m-1}C_m\delta_{t-m}|\eta_{t-m}|\right)=\sum_{l=0}^{+\infty}\theta_i^l\sum_{m=1}^{+\infty}\beta_{\sup}^{m-1}C_m\delta_{t-l-m}|\eta_{t-l-m}|=\sum_{s=1}^{+\infty}\beta_{\sup}^{s-1}\left(\sum_{l=0}^{s-1}\left(\frac{\theta_i}{\beta_{\sup}}\right)^l C_{s-l}\right)\delta_{t-s}|\eta_{t-s}|.$$

It follows by assumption that: $\frac{|\theta_i|}{\beta_{\sup}}<1$, and by definition that: $|C_m|\leq\sum_{i=1}^{q}b_i\beta_{\sup}^{1-i}$. If we redefine recursively:

$$C_s:=\sum_{l=0}^{s-1}\left(\frac{|\theta_i|}{\beta_{\sup}}\right)^l C_{s-l},$$

we can see that:

$$|C_s|\leq\frac{\sum_{i=1}^{q}b_i\beta_{\sup}^{1-i}}{\left(1-\frac{|\theta_i|}{\beta_{\sup}}\right)}$$

from which it follows that:

$$(1-\theta_i L)^{-1}\times(1-\theta_1 L)^{-1}\times\left(\sum_{i=1}^{q}b_i\delta_{t-i}|\eta_{t-i}|\right)=\sum_{s=1}^{+\infty}\beta_{\sup}^{s-1}C_s\delta_{t-s}|\eta_{t-s}|. \tag{13}$$

Therefore, one can easily check by following the above recursion that:

$$(1-\theta_1 L)^{-1}\times\ldots\times(1-\theta_p L)^{-1}\times\left(\sum_{i=1}^{q}b_i\delta_{t-i}|\eta_{t-i}|\right)=\sum_{u=1}^{+\infty}\beta_{\sup}^{u-1}C_u\delta_{t-u}|\eta_{t-u}|, \tag{14}$$

where:

$$|C_u|\leq\frac{\sum_{i=1}^{q}b_i\beta_{\sup}^{1-i}}{\prod_{p\geq i\geq 2}\left(1-\frac{|\theta_i|}{\beta_{\sup}}\right)}\equiv C \quad \text{(and } C_u \text{ is a positive number).} \tag{15}$$

From (14), we obtain:

$$(1-\theta_1 L)^{-1}\times\ldots\times(1-\theta_p L)^{-1}\times\left(\sum_{i=1}^{q} b_i\delta_{t-i}|\eta_{t-i}|\right)=\sum_{u=1}^{+\infty}\beta_{\sup}^{u-1}\frac{C_u}{C}C\delta_{t-u}|\eta_{t-u}|.$$

Therefore, the EGARCH(p,q) model has an EARCH(∞) representation with positive $\beta_i=\beta_{\sup}^{i-1}\dfrac{C_i}{C}\le\beta_{\sup}^{i-1}$, and coefficients $\alpha^*=C\alpha$ and $\gamma^*=C\gamma$. If we consider the inequality in Proposition 3.1, we can see that we can also use the $\beta_i\le\beta_{\sup}^{i-1}$ inequality to obtain the new upper bound :

$$\left|u_n^{(n)}-v_n^{(n)}\right|\le\sum_{i=0}^{+\infty}\beta^{*i}\delta^*_{t-n-i}|\eta_{t-n-i}|\left[\beta^{*n-1}+\sum_{p=1}^{n-1}\beta^{*n-1-p}\sum_{1\le s_1<\ldots<s_p\le n-1}\exp\left(\sum_{j=1}^{p}\log\left(\delta^*_{t-s_j}\right)+\xi^*_{t-s_j}\right)\right] \quad (16)$$

where the previous parameters are replaced by the following coefficients:

- $$\alpha^*=\frac{\sum_{i=1}^{q} b_i\beta_{\sup}^{1-i}}{\prod_{2\le i\le p}\left(1-\dfrac{|\theta_i|}{\beta_{\sup}}\right)}\alpha$$
- $$\gamma^*=\frac{\sum_{i=1}^{q} b_i\beta_{\sup}^{1-i}}{\prod_{2\le i\le p}\left(1-\dfrac{|\theta_i|}{\beta_{\sup}}\right)}\gamma$$
- $$\beta^*=\beta_{\sup}$$

6. Invertibility of EGARCH(p,q)

It can be seen that our approach has the distinct advantage of reducing the problem of the invertibility of EGARCH(p,q) to the simpler case of an EGARCH(1,1) model, using the above coefficients. The inequality in (16) can be rewritten to make the proof of invertibility more straightforward. Note that we have:

$$\sum_{j=1}^{p}\xi^{*}_{t-s_j} = \sum_{j=1}^{p}\log\left|\eta_{t-s_j}\right| + \sum_{j=1}^{p}\sum_{i=1}^{+\infty}\beta^{*i-1}\delta^{*}_{t-s_j-i}\left|\eta_{t-s_j-i}\right|$$

$$= \sum_{j=1}^{p}\log\left|\eta_{t-s_j}\right| + \sum_{l=1}^{+\infty}\sum_{\substack{1\le j\le p\\ i\ge 1\\ i+s_j=l}}\beta^{*i-1}\delta^{*}_{t-l}\left|\eta_{t-l}\right|$$

$$= \sum_{j=1}^{p}\log\left|\eta_{t-s_j}\right| + \sum_{l=1}^{n-1}\sum_{\substack{1\le j\le p\\ i\ge 1\\ i+s_j=l}}\beta^{*i-1}\delta^{*}_{t-l}\left|\eta_{t-l}\right| + \sum_{l=n}^{+\infty}\sum_{\substack{1\le j\le p\\ i\ge 1\\ i+s_j=l}}\beta^{*i-1}\delta^{*}_{t-l}\left|\eta_{t-l}\right|.$$

As $1\le s_1<\ldots<s_p\le n-1$, $\sum_{\substack{1\le j\le p\\ i\ge 1\\ i+s_j=l}}\beta^{*i-1}\le\frac{1}{1-\beta^{*}}$ if $l<n$, and $\sum_{\substack{1\le j\le p\\ i\ge 1\\ i+s_j=l}}\beta^{*i-1}\le\frac{\beta^{*l-n}}{1-\beta^{*}}$ if $l\ge n$,

so that:

$$\sum_{j=1}^{p}\xi^{*}_{t-s_j} \le \sum_{j=1}^{p}\log\left|\eta_{t-s_j}\right| + \sum_{l=1}^{n-1}\frac{\delta^{*}_{t-l}\left|\eta_{t-l}\right|}{1-\beta^{*}} + \sum_{l=n}^{+\infty}\frac{\beta^{*l-n}}{1-\beta^{*}}\delta^{*}_{t-l}\left|\eta_{t-l}\right|. \quad (17)$$

It follows that:

$$\left|u_n^{(n)}-v_n^{(n)}\right| \le B_n\exp\left(\sum_{l=1}^{n-1}\frac{\delta^{*}_{t-l}\left|\eta_{t-l}\right|}{1-\beta^{*}}\right)\left[\beta^{*n-1}+\sum_{p=1}^{n-1}\beta^{*n-1-p}\sum_{1\le s_1<\ldots<s_p\le n-1}\exp\left(\sum_{j=1}^{p}\log\left(\delta^{*}_{t-s_j}\left|\eta_{t-s_j}\right|\right)\right)\right] \quad (18)$$

where:

$$B_n = \sum_{i=0}^{+\infty} \beta^{*i} \delta^*_{t-n-i} |\eta_{t-n-i}| \exp\left(\sum_{l=n}^{+\infty} \frac{\beta^{*l-n}}{1-\beta^*} \delta^*_{t-l} |\eta_{t-l}| \right).$$

We now provide sufficient conditions for the invertibility of the EGARCH(p,q) specification. It is assumed that the conditions hold, and we then prove some lemmas before proving invertibility under these conditions:

$$\left| \mathrm{E}\left[\frac{\delta^*_t |\eta_t|}{1-\beta^*} \right] \right| + \log\left(\beta^* + E\left[\delta^*_t |\eta_t|\right]\right) < 0 \qquad \text{(Conditions 2)}$$

If $\beta^* = 0$, we find a condition that is deduced by concavity of log(.) from the conditions for EARCH(1) (in part 3), which is more restrictive. Moreover, by using the fact that $\mathrm{E}[\eta_t] = 0$ and $\mathrm{E}[|\eta_t|] \leq 1$ (as $\mathrm{E}[\eta_t{}^2] = 1$), we can obtain the following simpler sufficient condition:

$$\boxed{\frac{\alpha^*}{2(1-\beta^*)} + \log\left(\beta^* + \frac{\alpha^*}{2}\right) < 0}. \qquad (19)$$

We notice also that when we set β^* toward 0, the condition $\alpha^* < 1$ proposed by Straumann and Mikosch (2006) in their Remark 3.10 is also verified.

<u>Remark:</u> We continue to assume that $\mathrm{P}(\eta_t = 0) = 0$ in order to retain rigour in the proofs. However, as in the case of examining the simple EARCH(1) model, it may be also possible to relax the constraint here, even if it is less straightforward to prove the result. In the following proofs, the condition $\log|\eta_t| \in L^1$ is no longer necessary.

The proof of Lemma 4.1 is given in the Appendix (part 4):

Lemma 4.1

For any $\nu > 1/2$, we have, with probability 1:

$B_n = \exp\left(o\left(n^{\nu}\right)\right)$.

Inside the larger brackets in inequality (18), we have sums of independent variables, $1 \le s_1 < ... < s_p \le n-1$, which is more difficult to control than a sum from 1 to *p*, for instance. So we cannot simply use (LLN) as it was the case with the EARCH(1) model. Therefore, we will simply take the expectation in the proof to return to a sum over consecutive indexes (we also take expectations in order to use Lemma 1.2 with the Markov inequality to obtain convergence toward zero of $\left|v_n^{(n)} - u_n^{(n)}\right|$).

The following proposition proves invertibility, the proof of which can be found in the Appendix (part 4):

Proposition 4.1

If $\alpha \ge |\gamma|$, the a_i and b_i are non-negative, the roots of $\left(1 - \sum_{i=1}^{p} a_i L^i\right)$ are outside the unit circle and, if the Conditions 2 are verified, then EGARCH(*p*,*q*) is invertible as:

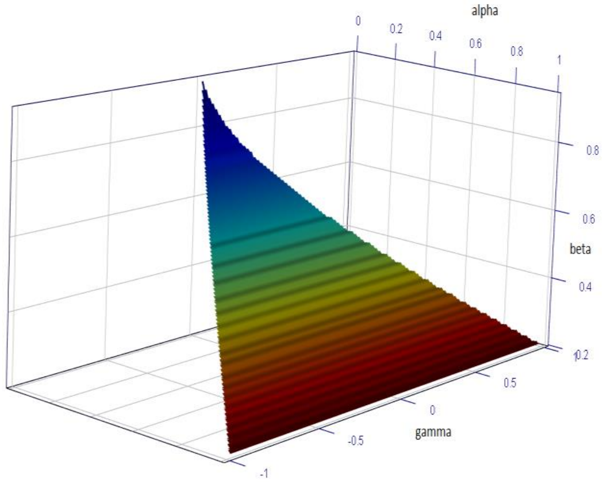
$$\left|v_n^{(n)} - u_n^{(n)}\right| = \left|v_n^{(n)} - \log|\eta_t|\right| \underset{n\to\infty}{\overset{a.s.}{\to}} 0.$$

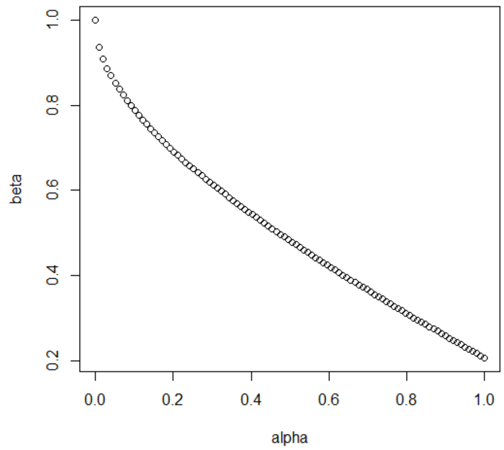
7. Special case of the N(0,1) distribution

In the case of the Gaussian distribution, the Conditions 2 can be re-written as:

$$\frac{\alpha^*}{\sqrt{2\pi}\left(1-\beta^*\right)} + \log\left(\beta^* + \frac{\alpha^*}{\sqrt{2\pi}}\right) < 0.$$

Therefore, if we calculate the maximum beta for several values of alpha (and gamma) under this condition, we obtain the following graphs:





It would seem that our domain of possible parameters is more restrictive, in the case of a Gaussian distribution for the normalized shocks, and for the case of EGARCH(1,1), than those given in Wintenberger (2013).

However, under further restrictions on the distribution of η_t, the condition could be extended to a slightly less restrictive condition, as follows:

$$\mathrm{E}\left[\frac{\delta_t^*|\eta_t|}{1-\beta^*}\right] + \log\left(\beta^* + \exp\left(E\left[\log\left(\delta_t^*|\eta_t|\right)\right]\right)\right) < 0 .$$

By the convexity of the exp(.) function, the last condition is indeed implied by Conditions 2. Moreover, when $\beta^* = 0$, this yields the condition in the case of EARCH(1), which is also the condition given in Straumann and Mikosch (2006).

8. Summary of the Invertibility Conditions for EGARCH(*p,q*)

It is instructive to summarize the conditions we have derived for the invertibility of any EGARCH(p,q) model, namely:

$$\log\sigma_t = \frac{\omega}{2} + \sum_{i=1}^{p} a_i \log\sigma_{t-i} + \sum_{i=1}^{q} b_i \delta_{t-i} \left|\eta_{t-i}\right|, \quad a_i \in \Re, \quad b_i \in \Re,$$

where:

$$\delta_t \equiv \frac{\alpha}{2} + \frac{\gamma}{2} sign(\eta_t).$$

The conditions for the invertibility of the EGARCH(p,q) specification are as follows:

- η_t ~ (0,1), and so $\eta_t \in L^2$;
- $\mathrm{P}(\eta_t = 0) = 0$ (it is highly probable that such condition can be ignored);
- $\alpha \geq |\gamma|$;
- the a_i and b_i coefficients are non-negative;
- the roots of $\left(1 - \sum_{i=1}^{p} a_i L^i\right)$ lie outside the unit circle;
- if $\left(1 - \sum_{i=1}^{p} a_i L^i\right) = \prod_{i=1}^{p}\left(1 - \theta_i L^i\right)$ and an arbitrary chosen parameter $\beta_{\sup}$, such that $1 > \beta_{\sup} > \max_i |\theta_i|$, then we consider the parameters:

$$\alpha^* = \frac{\sum_{i=1}^{q} b_i \beta_{\sup}^{1-i}}{\prod_{2\leq i\leq p}\left(1 - \frac{|\theta_i|}{\beta_{\sup}}\right)}\alpha \quad ; \quad \gamma^* = \frac{\sum_{i=1}^{q} b_i \beta_{\sup}^{1-i}}{\prod_{2\leq i\leq p}\left(1 - \frac{|\theta_i|}{\beta_{\sup}}\right)}\gamma \quad ; \quad \beta^* = \beta_{\sup} \quad ; \quad \delta_t^* \equiv \frac{\alpha^*}{2} + \frac{\gamma^*}{2} sign(\eta_t);$$

- $\mathrm{E}\left[\frac{\delta_t^*|\eta_t|}{1-\beta^*}\right]+\log\left(\beta^*+E\left[\delta_t^*|\eta_t|\right]\right)<0$, but more generally, the following condition is sufficient :

$$\boxed{\frac{\alpha^*}{2(1-\beta^*)}+\log\left(\beta^*+\frac{\alpha^*}{2}\right)<0}.$$

9. Concluding Remarks

The two most widely estimated asymmetric univariate models of conditional volatility are the exponential GARCH (or EGARCH) model and the GJR model. Asymmetry refers to the different effects on conditional volatility of positive and negative effects of equal magnitude, As EGARCH is a discrete-time approximation to a continuous-time stochastic volatility process, and is expressed in logarithms, conditional volatility is guaranteed to be positive without any restrictions on the parameters. For leverage, which refers to the negative correlation between returns shocks and subsequent shocks to volatility, EGARCH requires parametric restrictions to be satisfied. Leverage is not possible for GJR, unless the short run persistence parameter is negative, which is unlikely in practice, or if the process is to be consistent with a random coefficient autoregressive model (see McAleer (2014)).

The statistical properties for the QMLE of the GJR parameters are straightforward to establish. However, the statistical properties for the QMLE of the EGARCH(p,q) parameters are not available under general conditions, but rather only for special cases under highly restrictive and unverifiable conditions, and possibly only under simulation.

To date, a limitation in the development of asymptotic properties of the QMLE for EGARCH has been the lack of invertibility for the returns shocks underlying the model. The purpose of this paper was to establish the invertibility conditions for the

EGARCH(p,q) specification, in a more general case, and following an approach that is different from that in the literature. It was shown in the paper that the EGARCH model could be derived from a stochastic process, for which the invertibility conditions could be stated simply and explicitly (see the sets of Conditions 1 and 2). This should be useful in re-interpreting the existing properties of the QMLE of the EGARCH(p,q) parameters.

The main findings of the paper can be given as follows:

- We used a novel approach that was based directly on the stochastic process from which the EGARCH model may be derived, instead of working with the stochastic recursive equation, which requires proofs of theoretical properties, such as the existence and uniqueness of the solution.
- An examination of the simple EARCH(1) model provided a strong motivation for assuming that $\alpha > |\gamma|$, which is standard in the literature. In order to do that, we provide a proof that under this case, invertibility can be proved, as in the case of Straumann and Mikosch (2006). Moreover, we provided an alternative proof of the (possible) lack of invertibility for the symmetric case, $\alpha < -|\gamma|$. As the case of leverage is a combination of the two previous cases, we conclude that instability is highly possible in this case.
- The paper also provided a general inequality for the proof of invertibility of any EARCH(∞) model.
- We then used this inequality to derive the conditions for invertibility of the EGARCH(p,q) specification, which is a new and general result in the literature.
- Finally, our conditions, despite (possibly) being more restrictive, are more easily verified and do not require numerical simulations, as it is the case of the conditions given in Straumann and Mikosch (2006).
- The asymptotic properties of the estimated parameters, such as consistency of the QMLE or alternative estimators, may be proved using the invertibility conditions established in the paper, based on the methods given in Wintenberger (2013).

Appendix

Part 1: Proofs of the Lemmas

Lemma 1.1

(1) $\left|g_{\alpha,\gamma}(x_1,y)-g_{\alpha,\gamma}(x_2,y)\right| \leq \left|\frac{\alpha+sign(y).\gamma}{2}\right| \exp(\max(x_1,x_2))|x_1-x_2|$

(2) $\left|g_{\alpha,\gamma}(x_1,y)-g_{\alpha,\gamma}(x_2,y)\right| \geq \left|\frac{\alpha+sign(y).\gamma}{2}\right| \exp\left(\frac{x_1+x_2}{2}\right)|x_1-x_2|$

Proof:

The case $x_1 = x_2$ is obvious, so assume $x_1 \neq x_2$. We have:

$$\left|g_{\alpha,\gamma}(x_1,y)-g_{\alpha,\gamma}(x_2,y)\right| = \left|\frac{\alpha+sign(y).\gamma}{2}\right|.\left|\frac{\exp(x_1)-\exp(x_2)}{x_1-x_2}\right|.|x_1-x_2|.$$

If we note $x_{\min}$ and $x_{\max}$, respectively, the min and the max among x_1 and x_2, we know that $\exists c \in]x_{\min}, x_{\max}[$, such that :

$$\left|\frac{\exp(x_1)-\exp(x_2)}{x_1-x_2}\right| = \exp(c).$$

The first inequality is obtained by the fact that exp(.) is an increasing function. For the second inequality, some straightforward algebra leads to:

$$c = \frac{x_{\max}+x_{\min}}{2} + \log\left(\frac{\exp(x)-\exp(-x)}{2x}\right),$$

where $x = \frac{x_{\max} - x_{\min}}{2}$. By using the Taylor expansion of the function exp(.), as x > 0, we have the terms in the log(.) function are greater than 1, and therefore c > $\frac{x_1 + x_2}{2}$. This proves the second inequality.

■

Borel-Cantelli Lemma

Consider the probability space, (Ω,A,P), and $A_n \in$ A, $\forall n \geq 0$.

(1) If $\sum_{n\geq 0} \mathrm{P}(A_n) < +\infty$ then $\mathrm{P}\left(\limsup_n A_n\right) = 0$;

(2) If $(A_n)_n$ is independent, and if $\sum_{n\geq 0} \mathrm{P}(A_n) = +\infty$ then $\mathrm{P}\left(\limsup_n A_n\right) = 1$.

Lemma 1.2

If $\forall \varepsilon > 0$ and $\sum_n \mathrm{P}(|X_n - X| > \varepsilon) < +\infty$, then $X_n \underset{n\to\infty}{\overset{P.a.s.}{\to}} X$.

Part 2: Invertibility of EARCH(1)

First case: $\alpha \geq |\gamma|$

We have by recursion the following inequality:

$$\left|v_n^{(n)} - \log|\eta_t|\right| \le \exp\left(\sum_{i=1}^{n-1} \log\left(\delta_{t-i}|\eta_{t-i}|\right) + \delta_{t-i-1}|\eta_{t-i-1}|\right)\delta_{t-n}\left(|\eta_{t-n}| + \exp(c_0)\right). \quad (6)$$

The invertibility conditions in this case are:

$$\begin{cases} \log|\eta_t| \in L^1 \\ \mathrm{E}\left[\log\left(\delta_t|\eta_t|\right) + \delta_t|\eta_t|\right] < 0 \end{cases} \quad \text{(Conditions 1)}$$

<u>Proposition 2.1</u>

If the set of Conditions 1 is verified when $\alpha \ge |\gamma|$, then the model EARCH(1) is invertible as:

$$\left|v_n^{(n)} - u_n^{(n)}\right| = \left|v_n^{(n)} - \log|\eta_t|\right| \underset{n\to\infty}{\overset{a.s.}{\to}} 0.$$

<u>Proof:</u>

Note that $-\varepsilon \equiv \mathrm{E}\left[\log\left(\delta_t|\eta_t|\right) + \delta_t|\eta_t|\right] < 0$, and by the Law of Large Numbers (LLN), we have:

$$\sum_{i=1}^{n-1} \log\left(\delta_{t-i}|\eta_{t-i}|\right) + \delta_{t-i-1}|\eta_{t-i-1}| \overset{a.s.}{=} -\varepsilon n + o(n).$$

Using the Markov inequality, version (1) of the Borel-Cantelli Lemma, and η_t is *iid*:

$$\mathrm{P}\left(|\eta_{t-n}| \ge \exp\left(\frac{\varepsilon}{2}n\right)\right) \le \mathrm{E}\left[|\eta_t|\right].\exp\left(-\frac{\varepsilon}{2}n\right)$$

$$\Rightarrow \sum_n \mathrm{P}\left(\left|\eta_{t-n} \ge \exp\left(\frac{\varepsilon}{2}n\right)\right|\right) < +\infty$$

$$\Rightarrow \mathrm{P}\left(\exists n \in \mathrm{N}^{*}, \forall k \geq n : \left|\eta_{t-k}\right| < \exp\left(\frac{\varepsilon}{2} k \right) \right) = 1 .$$

Thus, by using inequality (6), we have almost surely:

$$\exists N \in \mathrm{N}, \forall n \geq N : \left| v_n^{(n)} - u_n^{(n)} \right| \leq \exp\left(-\varepsilon n + o(n) \right) \times \frac{|\alpha| + |\gamma|}{2} \left(\exp(c_0) + \exp\left(\frac{\varepsilon}{2} n \right) \right).$$

Therefore, it follows with "exponential speed", as defined in Straumann and Mikosch (2006) and Wintenberger (2013):

$$\left| v_n^{(n)} - u_n^{(n)} \right| = \left| v_n^{(n)} - \log\left|\eta_t\right| \right| \underset{n \to \infty}{\overset{a.s.}{\to}} 0 .$$

As $\eta_t = \exp\left(\lim_{n \to \infty} v_n^{(n)} \right) \times sign(\varepsilon_t)$, this proves invertibility.

■

Second case: $\alpha < -|\gamma|$

As $\eta_t \sim \mathrm{N}(0,1)$, also assume $c_0 \neq -\infty$, and consider (with $-\alpha - \gamma > 0$, by assumption):

$$A_n = \left\{ 0 \leq \eta_{t-4n} \leq \frac{\exp(c_0)}{2} ; \eta_{t-4n+1} \geq \sqrt{\log(n^{7/4})} - 1 ; \eta_{t-4n+2} \geq \frac{4}{-\alpha - \gamma} ; \eta_{t-4n+3} \geq \frac{2}{-\alpha - \gamma} \right\} .$$

Obviously, under independence, we have:

$$\mathrm{P}(A_n) = \mathrm{P}\left(0 \leq \eta_{t-4n} \leq \frac{\exp(c_0)}{2} \right) \times \mathrm{P}\left(\eta_{t-4n+1} \geq \sqrt{\log(n^{7/4})} - 1 \right) \times \mathrm{P}\left(\eta_{t-4n+2} \geq \frac{4}{-\alpha - \gamma} \right) \times \mathrm{P}\left(\eta_{t-4n+3} \geq \frac{2}{-\alpha - \gamma} \right)$$

As all the terms except the second term do not depend on *n*, and therefore are constant, we can rewrite the above equality as follows, where $\Phi(.)$ is the CDF of the normal distribution:

$$\mathrm{P}(A_n) = C_{\alpha,\gamma} \times \Phi\left(-\sqrt{\log(n^{7/4})}\right),$$

$$C_{\alpha,\gamma} \equiv \mathrm{P}\left(0 \le \eta_{t-4n} \le \frac{\exp(c_0)}{2}\right) \times \mathrm{P}\left(\eta_{t-4n+2} \ge \frac{4}{-\alpha-\gamma}\right) \times \mathrm{P}\left(\eta_{t-4n+3} \ge \frac{2}{-\alpha-\gamma}\right) \ne 0 .$$

But we can see that:

$$\Phi\left(1-\sqrt{\log(n^{7/4})}\right) = \int_{\sqrt{\log(n^{7/4})}-1}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \ge \int_{\sqrt{\log(n^{7/4})}-1}^{\sqrt{\log(n^{7/4})}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \ge \frac{1}{\sqrt{2\pi}} e^{-\frac{\log(n^{7/4})}{2}} \ge \frac{1}{\sqrt{2\pi}} \frac{1}{n^{7/8}}$$

and, by direct comparison to a Bertrand sum, we can see that $\sum_n \mathrm{P}(A_n)$ diverges. Therefore, as the A_n are independent, we can apply line (2) of the Borel-Cantelli Lemma, as stated previously and $\forall k \in \mathrm{N}, \exists n \ge k : A_n$ will occur with probability one.

Consider taking *n* sufficiently large such that the event A_n occurs. By straightforward calculus, it follows that :

- $v_1^{(4n)} = \log|\varepsilon_{t-4n+1}| - \frac{\omega}{2} + g_{\alpha,\gamma}(c_0, \varepsilon_{t-4n})$,

- $g_{\alpha,\gamma}(c_0, \varepsilon_{t-4n}) = \frac{-\alpha-\gamma}{2}\exp(c_0)$, by assumption ($sign(\varepsilon_{t-4n}) = sign(\eta_{t-4n})$),

- $\log|\varepsilon_{t-4n+1}| = \log|\eta_{t-4n+1}| + \frac{\omega}{2} + \frac{\alpha}{2}|\eta_{t-4n}| + \frac{\gamma}{2}\eta_{t-4n}$.

Given $0 \le \eta_{t-4n} \le \frac{\exp(c_0)}{2}$), it is easy to conclude that:

$$\boxed{v_1^{(4n)} \geq \log|\eta_{t-4n+1}| + \frac{-\alpha-\gamma}{4}\exp(c_0)}.$$

- $$v_2^{(4n)} \geq \log(\eta_{t-4n+2}) + \frac{-\alpha-\gamma}{2}\left(\exp\left(\frac{-\alpha-\gamma}{4}\exp(c_0)\right)-1\right)|\eta_{t-4n+1}|.$$

As we have $\eta_{t-4n+1} \geq \sqrt{\log(n^{7/4})} - 1$, for n sufficiently large, we have:

$$\eta_{t-4n+1} \geq \sqrt{\log(n^{3/2}+1)} \Big/ \left(\frac{-\alpha-\gamma}{2}\left(\exp\left(\frac{-\alpha-\gamma}{4}\exp(c_0)\right)-1\right)\right),$$

so that:

$$\boxed{v_2^{(4n)} \geq \log(\eta_{t-4n+2}) + \sqrt{\log(n^{3/2}+1)}}.$$

- By using the Taylor expansion of the exp(.) function, we have:

$$\exp\left(\sqrt{\log(n^{3/2}+1)}\right) \geq 1 + \frac{1}{2}\log(n^{3/2}+1).$$

As $\eta_{t-4n+2} \geq \frac{4}{-\alpha-\gamma}$, we obtain:

$$\boxed{v_3^{(4n)} \geq \log(\eta_{t-4n+3}) + \log(n^{3/2}+1)}.$$

- Finally, as $\eta_{t-4n+3} \geq \frac{2}{-\alpha-\gamma}$, we obtain:

$$\boxed{v_4^{(4n)} \geq \log|\eta_{t-4n+4}| + n^{3/2}}.$$

This result allows us to prove Proposition 2.2:

<u>Proposition 2.2</u>

If $\eta_t \overset{i.i.d.}{\sim} \mathrm{N}(0,1)$ and $\alpha < -|\gamma|$, then we cannot prove invertibility using our method, as $\left|v_n^{(n)} - u_n^{(n)}\right|$ does not converge to 0, and even admits an extracted series that diverges almost surely toward infinity.

<u>Proof:</u>

In order to show that $\left|v_n^{(n)} - u_n^{(n)}\right|$ diverges, we have to show that one of its extracting series diverges. Consider $\left|v_{4n}^{(4n)} - u_{4n}^{(4n)}\right|$. By applying recursively (2) of Lemma 1.2, by taking $v_0^{(4n)} \equiv c_0$, and because we have $g_{\alpha,\gamma}(.,.) > 0$, we obtain:

$$\left|v_{4n}^{(4n)} - u_{4n}^{(4n)}\right| \geq \exp\left(4n\log\left|\frac{|\alpha|-|\gamma|}{2}\right| + \sum_{\substack{i=1 \\ i\neq 4n-4}}^{4n-1}\left(\log|\eta_{t-i}| + \frac{\delta_{t-i-1}|\eta_{t-i-1}|}{2}\right) + \frac{\log|\eta_{t-4n+4}| + v_4^{(4n)}}{2}\right)\left|\,|\eta_{t-4n}| - \exp(c_0)\right|$$

.

By the assumption on the distribution, and by using (LLN), it follows that:

$$4n\log\left|\frac{|\alpha|-|\gamma|}{2}\right| + \sum_{\substack{i=1 \\ i\neq 4n-4}}^{4n-1}\left(\log|\eta_{t-i}| + \frac{\delta_{t-i-1}|\eta_{t-i-1}|}{2}\right) + \log|\eta_{t-4n+4}| = O(n).$$

From the results given above, $\forall N \in \mathrm{N}, \exists n \geq N : A_n$ occurs with probability one, so that $v_4^{(4n)} \geq \log|\eta_{t-4n+4}| + n^{3/2}$. Therefore, with probability one:

$$\forall N \in \mathrm{N}, \exists n \geq N : \left| v_{4n}^{(4n)} - u_{4n}^{(4n)} \right| \geq \exp\left(O(n) + \frac{n^{3/2}}{2} \right) \left| \left| \eta_{t-4n} \right| - \exp(c_0) \right| \geq \exp\left(O(n) + \frac{n^{3/2}}{2} \right) \frac{\exp(c_0)}{2}.$$

Therefore, we can extract a series that diverges toward infinity. Moreover, this holds for any value of c_0, except -∞. As the backward recursion, $v_k^{(n)}$, is implied conditionally on $\log|\eta_{t-n}| = c_0$, and as the probability of having $\eta_{t-n} = 0$ is equal to zero, the proposition proves that, under such conditions and with this method, we cannot prove invertibility as we will face a backward series that behaves erratically. Such an outcome would likely also hold for other distributions with thicker tails than the Gaussian.

Third case: $|\alpha| < |\gamma|$ and $\gamma < 0$

We finally look at the leverage case. We can also consider for this case the set of events $(A_n)_{n \in \mathrm{N}^*}$. Given previous results, we can see that we cannot use inequality (1) of Lemma 1 to prove invertibility, specifically because of the asymptotic properties of $(A_n)_{n \in \mathrm{N}^*}$ we would not be able to obtain an upper bound for $\left| v_n^{(n)} - u_n^{(n)} \right|$ that converges to zero. Moreover, we also would not be able to use recursively inequality (2) of Lemma 1 as each event of $(A_n)_{n \in \mathrm{N}^*}$ that occurs could be followed by a $v_5^{(4n)}$ which is negative (if η_{t-4n+4} is sufficiently negative) with a greater absolute value than $v_4^{(4n)}$, so we could obtain a lower bound that would tend to zero.

<u>Part 3 : Proofs of Lemmas and Propositions for Invertibility of EARCH(∞)</u>

It is assumed that $\alpha \geq |\gamma|$ and that all the β_i coefficients are non-negative:

$$\log|\eta_t| = \log|\varepsilon_t| - \frac{\omega}{2} + \sum_{i=1}^{\infty} \beta_i . g_{\alpha,\gamma}\left(\log|\eta_{t-i}|, \varepsilon_{t-i} \right), \tag{7}$$

$$\begin{cases} u_1^{(n)} = \log|\varepsilon_{t-n+1}| - \dfrac{\omega}{2} + \sum_{i=0}^{\infty} \beta_{i+1} g_{\alpha,\gamma}\left(\log|\eta_{t-n-i}|, \varepsilon_{t-n-i}\right) \\ u_{k+1}^{(n)} = \log|\varepsilon_{t-n+k+1}| - \dfrac{\omega}{2} + \sum_{j=1}^{k} \beta_j g_{\alpha,\gamma}\left(u_{k+1-j}^{(n)}, \varepsilon_{t-n+k+1-j}\right) + \sum_{i=0}^{\infty} \beta_{i+1+k} g_{\alpha,\gamma}\left(\log|\eta_{t-n-i}|, \varepsilon_{t-n-i}\right) \end{cases} \quad (8)$$

Lemma 3.1

$u_k^{(n)} = \log|\eta_{t-n+k}|, \forall n \in \mathrm{N}^*, \forall k \in \mathrm{N}^*.$

Proof:

We will prove the result recursively for any $n \in \mathrm{N}^*$. Fix $n > 0$ and define:

$$\left(\mathrm{H}_p\right) \equiv "\forall k \in [1, p], u_k^{(n)} = \log|\eta_{t-n+k}|".$$

According to equality (8), $\left(\mathrm{H}_1\right)$ is true. Assume $\left(\mathrm{H}_p\right)$ and prove $\left(\mathrm{H}_{p+1}\right)$:

$$u_{p+1}^{(n)} = \log|\varepsilon_{t-n+p+1}| - \frac{\omega}{2} + \sum_{j=1}^{p} \beta_j g_{\alpha,\gamma}\left(u_{p+1-j}^{(n)}, \varepsilon_{t-n+p+1-j}\right) + \sum_{i=0}^{\infty} \beta_{i+1+p} g_{\alpha,\gamma}\left(\log|\eta_{t-n-i}|, \varepsilon_{t-n-i}\right)$$

$$= \log|\varepsilon_{t-n+p+1}| - \frac{\omega}{2} + \sum_{j=1}^{p} \beta_j g_{\alpha,\gamma}\left(\log|\eta_{t-n+p+1-j}|, \varepsilon_{t-n+p+1-j}\right) + \sum_{i=p+1}^{\infty} \beta_i g_{\alpha,\gamma}\left(\log|\eta_{t-n+p+1-i}|, \varepsilon_{t-n+p+1-i}\right),$$

by using $\left(\mathrm{H}_p\right)$, then we can conclude by matching the previous equality with (7), so that $\left(\mathrm{H}_{p+1}\right)$ is true.

■

We have:

$$\left|u_n^{(n)} - v_n^{(n)}\right| \le \sum_{i=0}^{\infty} \beta_{i+n} \delta_{t-n-i} |\eta_{t-n-i}| + \sum_{j=1}^{n-1} \beta_j \delta_{t-j} \exp\left(\xi_{t-j}\right) \left|u_{n-j}^{(n)} - v_{n-j}^{(n)}\right| \equiv a_1,$$

and also:

$$\left|u_{n-j}^{(n)}-v_{n-j}^{(n)}\right| \le \sum_{i=0}^{\infty}\beta_{i+n-j}\delta_{t-n-i}\left|\eta_{t-n-i}\right| + \sum_{l=1}^{n-j-1}\beta_l\delta_{t-j-l}\exp\left(\xi_{t-j-l}\right)\left|u_{n-j-l}^{(n)}-v_{n-j-l}^{(n)}\right| .$$

so that we can write:

$$\left|u_{n}^{(n)}-v_{n}^{(n)}\right| \le \sum_{i=0}^{\infty}\beta_{i+n}\delta_{t-n-i}\left|\eta_{t-n-i}\right| + \sum_{j=1}^{n-1}\sum_{i=0}^{+\infty}\beta_j\beta_{i+n-j}\delta_{t-j}\exp\left(\xi_{t-j}\right)\delta_{t-n-i}\left|\eta_{t-n-i}\right| +$$
$$\sum_{j=1}^{n-2}\sum_{l=1}^{n-j-1}\beta_j\beta_l\delta_{t-j-l}\delta_{t-j}\exp\left(\xi_{t-j-l}+\xi_{t-j}\right)\left|u_{n-j-l}^{(n)}-v_{n-j-l}^{(n)}\right| \equiv a_2 .$$

Define:

$$a_k \equiv \sum_{i=0}^{+\infty}\delta_{t-n-i}\left|\eta_{t-n-i}\right|\left[\beta_{i+n}+\sum_{p=1}^{k-1}\sum_{i_1,\ldots,i_p\in A_p^{(n)}}\hat{\Pi}_p\,\hat{D}_p\exp\left(\sum_{j=1}^{p}\xi_{t-\hat{S}_j}\right)\times\beta_{i+n-\hat{S}_p}\right]+$$
$$+\sum_{i_1,\ldots,i_k\in A_k^{(n)}}\hat{\Pi}_k\,\hat{D}_k\exp\left(\sum_{i=1}^{k}\xi_{t-\hat{S}_j}\right)\left|u_{n-\hat{S}_k}^{(n)}-v_{n-\hat{S}_k}^{(n)}\right|$$

where:

- $\hat{S}_l = \sum_{j=1}^{l} i_j$
- $A_p^{(n)} = \left\{i_1 \ge 1,\ldots,i_p \ge 1 : \hat{S}_p \le n-1\right\}$
- $\hat{\Pi}_l = \prod_{j=1}^{l}\beta_{i_j}$
- $\hat{D}_l = \prod_{j=1}^{l}\delta_{t-\hat{S}_j}$

Lemma 4

$$\left|v_n^{(n)} - u_n^{(n)}\right| \le a_k, \forall k \in [1, n[$$

Proof:

We will prove the lemma recursively:

$$(H_k):"\left|u_n^{(n)} - v_n^{(n)}\right| \le a_k".$$

According to the first two inequalities derived above, we have (H_1) and (H_2), which are true. Assume (H_k) and prove (H_{k+1}):

$$a_k = \sum_{i=0}^{+\infty} \delta_{t-n-i}\left|\eta_{t-n-i}\right|\left[\beta_{i+n} + \sum_{p=1}^{k-1} \sum_{i_1,\dots,i_p \in A_p^{(n)}} \hat{\Pi}_p \hat{D}_p \exp\left(\sum_{j=1}^{p} \xi_{t-\hat{S}_j}\right) \times \beta_{i+n-\hat{S}_p}\right] +$$

$$+ \sum_{i_1,\dots,i_k \in A_k^{(n)}} \hat{\Pi}_k \hat{D}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\hat{S}_j}\right)\left|u_{n-\hat{S}_k}^{(n)} - v_{n-\hat{S}_k}^{(n)}\right|.$$

However:

$$\left|u_{n-\hat{S}_k}^{(n)} - v_{n-\hat{S}_k}^{(n)}\right| \le \sum_{i=0}^{\infty} \beta_{i+n-\hat{S}_k} \delta_{t-n-i}\left|\eta_{t-n-i}\right| + \sum_{l=1}^{n-\hat{S}_k-1} \beta_l \delta_{t-\hat{S}_k-l} \exp\left(\xi_{t-\hat{S}_k-l}\right)\left|u_{n-\hat{S}_k-l}^{(n)} - v_{n-\hat{S}_k-l}^{(n)}\right|,$$

so that:

$$\sum_{i_1,\dots,i_k \in A_k^{(n)}} \hat{\Pi}_k \hat{D}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\hat{S}_j}\right)\left|u_{n-\hat{S}_k}^{(n)} - v_{n-\hat{S}_k}^{(n)}\right|$$

$$\leq \sum_{\substack{i_1,\ldots,i_k \in A_k^{(n)} \\ \hat{S}_k = n-1}} \hat{D}_k \hat{\Pi}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\hat{S}_j}\right) \left|u_1^{(n)} - v_1^{(n)}\right|$$

$$+ \sum_{\substack{i_1,\ldots,i_k \in A_k^{(n)} \\ \hat{S}_k < n-1}} \hat{D}_k \hat{\Pi}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\hat{S}_j}\right) \left(\sum_{i_{k+1}=1}^{n-\hat{S}_k-1} \beta_l \delta_{t-\hat{S}_k-i_{k+1}} \exp\left(\xi_{t-\hat{S}_k-i_{k+1}}\right) \left|u^{(n)}_{n-\hat{S}_k-i_{k+1}} - v^{(n)}_{n-\hat{S}_k-i_{k+1}}\right|\right)$$

$$+ \sum_{\substack{i_1,\ldots,i_k \in A_k^{(n)} \\ \hat{S}_k < n-1}} \hat{D}_k \hat{\Pi}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\hat{S}_j}\right) \left(\sum_{i=0}^{\infty} \beta_{i+n-\hat{S}_k} \delta_{t-n-i} \left|\eta_{t-n-i}\right|\right).$$

By using the inequality:

$$\left|u_1^{(n)} - v_1^{(n)}\right| \leq \sum_{i=0}^{\infty} \beta_{i+1} \delta_{t-n-i} \left|\eta_{t-n-i}\right|,$$

and by recombining the sums above, we can see that:

$$\sum_{i_1,\ldots,i_k \in A_k^{(n)}} \hat{\Pi}_k \hat{D}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\hat{S}_j}\right) \left|u^{(n)}_{n-\hat{S}_k} - v^{(n)}_{n-\hat{S}_k}\right|$$

$$\leq \sum_{\substack{i_1,\ldots,i_k \in A_k^{(n)} \\ \hat{S}_k < n-1}} \hat{D}_k \hat{\Pi}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\hat{S}_j}\right) \left(\sum_{i_{k+1}=1}^{n-\hat{S}_k-1} \beta_l \delta_{t-\hat{S}_k-i_{k+1}} \exp\left(\xi_{t-\hat{S}_k-i_{k+1}}\right) \left|u^{(n)}_{n-\hat{S}_k-i_{k+1}} - v^{(n)}_{n-\hat{S}_k-i_{k+1}}\right|\right)$$

$$+ \sum_{i_1,\ldots,i_k \in A_k^{(n)}} \hat{D}_k \hat{\Pi}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\hat{S}_j}\right) \left(\sum_{i=0}^{\infty} \beta_{i+n-\hat{S}_k} \delta_{t-n-i} \left|\eta_{t-n-i}\right|\right)$$

$$\leq \sum_{\substack{i_1,\ldots,i_k \in A_k^{(n)} \\ \hat{S}_k < n-1}} \sum_{i_{k+1}=1}^{n-\hat{S}_k-1} \hat{D}_k \hat{\Pi}_k \beta_l \delta_{t-\hat{S}_k-i_{k+1}} \exp\left(\sum_{j=1}^{k} \xi_{t-\hat{S}_j} + \xi_{t-\hat{S}_k-i_{k+1}}\right) \left|u^{(n)}_{n-\hat{S}_k-i_{k+1}} - v^{(n)}_{n-\hat{S}_k-i_{k+1}}\right|$$

$$+ \sum_{i=0}^{\infty} \delta_{t-n-i} \left|\eta_{t-n-i}\right| \left[\sum_{i_1,\ldots,i_k \in A_k^{(n)}} \hat{D}_k \hat{\Pi}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\hat{S}_j}\right) \beta_{i+n-\hat{S}_k}\right].$$

By noticing that:

$$\left\{i_1,\dots,i_k \in A_k^{(n)}, i_{k+1} \in \left[1, n-\hat{S}_k - 1\right] : \hat{S}_k < n-1\right\} = A_{k+1}^{(n)},$$

we finally have:

$$\left|u_n^{(n)} - v_n^{(n)}\right| \le a_k \le a_{k+1} \Rightarrow \left(H_{k+1}\right) \text{ is true.}$$

■

By taking $k = n - 1$, and by using the inequality $\left|u_1^{(n)} - v_1^{(n)}\right| \le \sum_{i=0}^{\infty} \beta_{i+1}\delta_{t-n-i}\left|\eta_{t-n-i}\right|$, we have the following general result for EARCH(∞):

Proposition 3.1

If $\alpha \ge |\gamma|$, $\beta_i \ge 0, \forall i$, then we have the following inequality for the series u and v for EARCH(∞):

$$\left|u_n^{(n)} - v_n^{(n)}\right| \le \sum_{i=0}^{+\infty} \delta_{t-n-i}\left|\eta_{t-n-i}\right| \left[\beta_{i+n} + \sum_{p=1}^{n-1} \sum_{i_1,\dots,i_p \in A_p^{(n)}} \hat{\Pi}_p \hat{D}_p \exp\left(\sum_{j=1}^{p} \xi_{t-\hat{S}_j}\right) \times \beta_{i+n-\hat{S}_p}\right].$$

Part 4: Invertibility of EGARCH(*p*,*q*)

We have:

$$\left|u_n^{(n)} - v_n^{(n)}\right| \le B_n \exp\left(\sum_{l=1}^{n-1} \frac{\delta_{t-l}^*\left|\eta_{t-l}\right|}{1-\beta^*}\right)\left[\beta^{*n-1} + \sum_{p=1}^{n-1} \beta^{*n-1-p} \sum_{1\le s_1<\dots<s_p\le n-1} \exp\left(\sum_{j=1}^{p} \log\left(\delta_{t-s_j}^*\left|\eta_{t-s_j}\right|\right)\right)\right] \quad (18)$$

Lemma 4.1

For any $\nu > 1/2$, we have with probability 1:

$B_n = \exp\left(o\left(n^\nu\right)\right)$.

Proof:

We have:

$$B_n = \sum_{i=0}^{+\infty} \beta^{*i} \delta^*_{t-n-i} \left|\eta_{t-n-i}\right| \exp\left(\sum_{l=n}^{+\infty} \frac{\beta^{*l-n}}{1-\beta^*} \delta^*_{t-l} \left|\eta_{t-l}\right| \right).$$

We know that $X_n \equiv \sum_{l=n}^{+\infty} \frac{\beta^{*l-n}}{1-\beta^*} \delta^*_{t-l} \left|\eta_{t-l}\right|$ are L^2-variables as absolutely convergent sum of L^2-variables (it is assumed that $\eta_t \sim (0,1)$) as L^2 is a Hilbert space). Furthermore, by using Chebychev inequality, we obtain:

$$\mathrm{P}\left(\left|X_n\right| \geq n^\nu\right) \leq \frac{\mathrm{E}\left[\left|X_n\right|^2\right]}{n^{2\nu}} = \frac{\mathrm{E}\left[\left|X_1\right|^2\right]}{n^{2\nu}},$$

as the X_n are identically distributed. Therefore, $\sum \mathrm{P}\left(\left|X_n\right| \geq n^\nu\right) < \infty$ and, by using the Borel-Cantelli Lemma, we have with probability one that: $X_n = O\left(n^\nu\right)$. As this is true $\forall \nu > 1/2$, we also have: $X_n = o\left(n^\nu\right)$.

By using the same reasoning with $Y_n \equiv \sum_{i=0}^{+\infty} \beta^{*i} \delta^*_{t-n-i} \left|\eta_{t-n-i}\right|$, we obtain the invertibility condition.

■

Proposition 4.1

If $\alpha \geq |\gamma|$, the a_i and b_i are non-negative, the roots of $\left(1-\sum_{i=1}^{p} a_i L^i\right)$ lie outside the unit circle and, if Conditions 2 are satisfied, then EGARCH(*p*,*q*) is invertible as:

$$\left|v_n^{(n)} - u_n^{(n)}\right| = \left|v_n^{(n)} - \log|\eta_t|\right| \underset{n\to\infty}{\overset{a.s.}{\rightarrow}} 0 .$$

Proof:

According to Conditions 2 and by continuity, we know that $\exists \varepsilon_1, \varepsilon_2 > 0$, such that:

$$\mathrm{E}\left[\frac{\delta_t^*|\eta_t|}{1-\beta^*}\right] + \log\left(\beta^* + E\left[\delta_t^*|\eta_t|\right]\right) + \varepsilon_1 < -\varepsilon_2 .$$

We also have inequality (18):

$$\left|u_n^{(n)} - v_n^{(n)}\right| \leq B_n \exp\left(\sum_{l=1}^{n-1} \frac{\delta_{t-l}^*|\eta_{t-l}|}{1-\beta^*}\right)\left[\beta^{*n-1} + \sum_{p=1}^{n-1} \beta^{*n-1-p} \sum_{1\leq s_1<\ldots<s_p\leq n-1} \exp\left(\sum_{j=1}^{p} \log\left(\delta_{t-s_j}^* \left|\eta_{t-s_j}\right|\right)\right)\right].$$

If we note that:

$$Z_n = \frac{\left[\beta^{*n-1} + \sum_{p=1}^{n-1} \beta^{*n-1-p} \sum_{1\leq s_1<\ldots<s_p\leq n-1} \exp\left(\sum_{j=1}^{p} \log\left(\delta_{t-s_j}^* \left|\eta_{t-s_j}\right|\right)\right)\right]}{\exp\left((n-1)\log\left(\beta^* + E\left[\delta_t^*|\eta_t|\right]\right) + (n-1)\varepsilon_1\right)},$$

we have:

$$\left|u_n^{(n)} - v_n^{(n)}\right| \leq B_n \exp\left(\sum_{l=1}^{n-1} \frac{\delta_{t-l}^*|\eta_{t-l}|}{1-\beta^*} + (n-1)\log\left(\beta^* + E\left[\delta_t^*|\eta_t|\right]\right) + (n-1)\varepsilon_1\right) Z_n .$$

It can be shown that Z_n goes to zero almost surely, as follows. Let $\varepsilon > 0$ by the Markov inequality:

$$\mathrm{P}\left(Z_n > \varepsilon\right) \le \frac{\mathrm{E}\left[\beta^{*n-1} + \sum_{p=1}^{n-1} \beta^{*n-1-p} \sum_{1 \le s_1 < \ldots < s_p \le n-1} \exp\left(\sum_{j=1}^{p} \log\left(\delta^*_{t-s_j} \left|\eta_{t-s_j}\right|\right)\right)\right]}{\exp\left((n-1)\log\left(\beta^* + E\left[\delta^*_t \left|\eta_t\right|\right]\right) + (n-1)\varepsilon_1\right) \times \varepsilon}.$$

However:

$$\mathrm{E}\left[\beta^{*n-1} + \sum_{p=1}^{n-1} \beta^{*n-1-p} \sum_{1 \le s_1 < \ldots < s_p \le n-1} \exp\left(\sum_{j=1}^{p} \log\left(\delta^*_{t-s_j} \left|\eta_{t-s_j}\right|\right)\right)\right]$$

$$= \sum_{p=0}^{n-1} \binom{n-1}{p} \beta^{*n-1-p} \left(\mathrm{E}\left[\delta^*_t \left|\eta_t\right|\right]\right)^p ,$$

where:

$$\binom{n-1}{p} = \frac{(n-1)!}{(n-1-p)!\,p!},$$

as $\delta^*_{t-s_j} \left|\eta_{t-s_j}\right|$ are L^1 and *iid*. Using Newton's formula, it can be shown that:

$$\mathrm{E}\left[\beta^{*n-1} + \sum_{p=1}^{n-1} \beta^{*n-1-p} \sum_{1 \le s_1 < \ldots < s_p \le n-1} \exp\left(\sum_{j=1}^{p} \log\left(\delta^*_{t-s_j} \left|\eta_{t-s_j}\right|\right)\right)\right] = \left(\beta^* + \mathrm{E}\left[\delta^*_t \left|\eta_t\right|\right]\right)^{n-1}.$$

Therefore:

$$\mathrm{P}(Z_n > \varepsilon) \le \frac{\exp(-(n-1)\varepsilon_1)}{\varepsilon},$$

and, by using Lemma 1.2, we can show that:

$$Z_n \underset{n\to\infty}{\overset{a.s.}{\to}} 0 .$$

Moreover by LLN:

$$\sum_{l=1}^{n-1} \frac{\delta_{t-l}^* |\eta_{t-l}|}{1-\beta^*} = \mathrm{E}\left[\frac{\delta_t^* |\eta_t|}{1-\beta^*}\right] \times n + o(n).$$

Therefore:

$$\exp\left(\sum_{l=1}^{n-1} \frac{\delta_{t-l}^* |\eta_{t-l}|}{1-\beta^*} + n \log\left(\beta^* + E\left[\delta_t^* |\eta_t|\right]\right) + n\varepsilon_1\right) = \exp(-n\varepsilon_2 + o(n)).$$

According to Lemma 4.1, we have:

$$B_n = \exp(o(n)).$$

Therefore:

$$\left|u_n^{(n)} - v_n^{(n)}\right| \le \exp(-n\varepsilon_2 + o(n)) Z_n ,$$

$$\left|v_n^{(n)} - u_n^{(n)}\right| = \left|v_n^{(n)} - \log|\eta_t|\right| \underset{n\to\infty}{\overset{a.s.}{\to}} 0 ,$$

which proves invertibility of EGARCH(p,q).

■

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