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Representative Valuations"

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Price-Based Combinatorial Auction Design: Representative Valuations*

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Abstract

We investigate price-based mechanisms with connectedness in combinatorial auctions, where with restrictions of privacy and complexity, the auctioneer asks a limited number of prices to buyers who provide demand responses. Consistent with the price-based property, several necessary and sufficient conditions are presented for the existence of the *VCG* mechanism, strategy-proofness with participation constraints, approximate strategy-proofness, Nash equilibrium, efficiency, core, and others. In all cases, the concept of the representative valuation function, which assigns the minimal valuation in both absolute and relative terms to any revealed package, plays the central role in determining whether these conditions are satisfied.

Keywords: Combinatorial Auctions, Price-Based Mechanisms, VCG Mechanisms, Connectedness, Representative Valuation Functions.

JEL Classification Numbers: D44, D61, D82.

1. Introduction

This paper investigates the problem of *combinatorial auction design*, where multiple indivisible items with multiple units are sold to multiple buyers who have private and quasi-linear valuations. On the basis of the information about the buyers' valuation functions that is collected through an auction, the auctioneer divides these items and units into multiple packages to be purchased by the respective buyers. Consistent with the buyers' incentive, the possibility for achieving efficiency for suitable allocations of packages is examined when there is a limit to the range within which the auctioneer can collect such information. We introduce a new concept—the *representative valuation function*. This concept plays the central role in demonstrating simple and tractable methods for examining the possibility of achieving efficiency.

The revelation principle addressed by Myerson (1979) implies that under a condition about the buyer's incentive, any well-behaved indirect mechanism can be replaced with a direct mechanism that requires each buyer to announce his/her entire valuation function truthfully. This principle plays the central role in rendering the study of incentives tractable; it is without loss of generality that researchers can confine their attention to direct mechanisms that are *strategy-proof* in the sense that each buyer regards making truthful announcements as the dominant strategy. Many researchers focused on a specific direct mechanism called the *VCG* (Vickery-Clarke-Groves) mechanism[1]; it is the only mechanism that is efficient, strategy-proof, and ex-post individually rational in the sense that the resulting payoffs in the ex-post term for the seller and the buyers are non-negative at all times.

A real buyer, however, is afraid that any information that is confidential and is not necessary for the auctioneer's decisions, such as the absolute valuations on desired packages, could leak to his/her rivals. Moreover, it might be too complicated for any buyer who has normal limitations on his/her cognitive ability to assess valuations on all possible packages simultaneously. The revelation principle, however, does not address concerns over issues such as privacy and complexity mentioned above, even if there exists a possibility that the auctioneer's decisions are substantially limited by the

[1] See Vickery (1961), Clarke (1971), and Groves (1973).

concerns. Hence, in the field of auction theory, it is meaningful to reexamine various types of indirect mechanisms and search for the possibility of replacing a direct mechanism with an alternative indirect auction format that can relax the constraints of privacy and complexity while maintaining the buyers' incentive and transparency in decisions.[2]

On the basis of this motivation, many researchers have investigated one class of auction formats, which can be called *price-based* mechanisms, where the auctioneer asks a limited number of price vectors and each buyer reveals packages as his/her demand responses to them. An example is the clock auction a la Walrasian tatonnement, in which the auctioneer starts with a low price vector and ascends slowly until there is no item for which the aggregate demand exceeds the supply. Several authors such as Kelso and Crawford (1982), Gul and Stacchetti (1999, 2000), and Milgrom (2000) have shown that when the buyers have substitutes preferences and behave as price takers, the auctioneer can identify the competitive equilibrium price vector and suggest it to the buyers through the procedure of the clock auction, thereby achieving efficiency. There are various modifications of the clock auction that should be studied, wherein the ask prices of the auctioneer might be non-linear and non-anonymous when the buyers have complements preferences.[3]

The clock auction has a desirable property in terms of privacy and complexity; what the auctioneer needs to know for decisions is only partial information about the buyers' valuation functions, which is collected from the observation of the buyers' demand responses. Hence, the buyers do not have to assess valuations about all possible packages. This property can facilitate the buyers' decision making.

Despite these merits, a naïve format of the clock auction has a serious limitation: the buyers do not have an incentive in behaving as price takers. Hence, the auctioneer generally fails to discover the correct competitive equilibrium, i.e., fails to achieve efficiency. Gul and Stacchetti (2000) have indicated that the clock auction and its

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[2] For related points of view about the VCG mechanisms and direct mechanisms in general, see, for instance, Rothkopf, Teisberg, and Kahn (1990), Ausubel and Milgrom (2002), and Parkes (2006).

[3] See Ausubel and Milgrom (2002), Milgrom (2004, Chapters 7 and 8), Ausubel (2004, 2006), Ausubel and Cramton (2004), Ausubel, Cramton, and Milgrom (2006), Parkes (2006), Mishra and Parkes (2007), and others.

variants can never be strategy-proof as long as the determination of the buyers' payments is dependent only on the information that is available in the process when the ask price vector of the auctioneer converges on that of the competitive equilibrium.

For the buyer's incentive to be compatible with privacy and complexity, this paper examines a more general class of price-based mechanisms, in which the auctioneer does not necessarily attempt to discover the competitive equilibrium. This class of mechanism is needed for the buyer's incentive to be compatible with privacy and complexity. The auctioneer instead infers the range of profiles of the buyers' valuation functions that are consistent with the observed *price-demand sets*, i.e., the whole data about how the buyers provide their demand responses through the auction's procedure. In this case, the allocation and payments that are induced by the mechanism must be the same across all possible profiles in this range. Since the range of such profiles is not generally single-valued, it is quite important to search for a simple and easy method to confirm whether the mechanism is implemented as being price-based, even if the scope in which he/she can collect information is limited beforehand.

This paper imposes a restriction on the class of price-based mechanisms, which can be called *connectedness*, meaning that the auctioneer is prohibited from making his/her ask price vector jump discontinuously to a price vector that he/she has not previously asked for. This restriction makes real buyers' demand responses easier to provide, because they can refer to their previous demand responses. With connectedness, we demonstrate a necessary and sufficient condition under which there exists a price-based VCG mechanism in spite of the limited scope in which the auctioneer can collect information. In this case, we can also use a simple and easy method to confirm whether this condition is satisfied.

Without loss of generality, after observing the price-demand set for each buyer, the auctioneer can focus only on a particular valuation function named the *representative valuation function*. In the consistency with the observed price-demand set, the representative valuation function is defined as assigning *the minimal valuation to any revealed package in both absolute and relative terms*. The representative valuation function is easily calculated from the observed price-demand set.

We can show that the efficient allocation and the efficient allocations without any single buyer that are induced by the profile of representative valuation functions can also be induced by any profile of valuation functions that are consistent with the observed price-demand sets, if, and only if, these allocations are all revealed in these sets. Based on this property, our necessary and sufficient condition implies that *the efficient allocation and the efficient allocations without any single buyer that are induced by the profile of representative valuation functions are revealed in the observed price-demand sets at all times*. Therefore, the auctioneer can stop his/her price asking activity as soon as it is confirmed that these allocations are revealed. On these accounts, the auctioneer can keep a breach of privacy and complexity to a minimum.

The connectedness plays an important role for calculating the payments of the VCG mechanism; the auctioneer can calculate the relative valuation between any pair of packages as the summation of price differences, if and only if these packages are revealed. This property of connectedness, along with the revelation of the efficient allocation and the efficient allocations without any single buyer, guarantees that the auctioneer can calculate the payments of the VCG mechanism.

There is a difficulty in the implementation of the VCG mechanism as being price-based; the auctioneer has to make the buyers reveal all the efficient allocations without any single buyer as well as the overall efficient allocation. Gul and Stacchetti (2000) and Ausubel (2006) have indicated that it is impossible for the buyers to reveal the efficient allocations without any single buyer through any procedure of ascending auction that uses the single linear price trajectory to discover the competitive equilibrium. In order to implement the VCG mechanism as price-based, Ausubel (2006) designed an alternative, rather complicated, clock auction that uses multiple linear price trajectories that converge in not only the competitive equilibrium, but also the competitive equilibria without any single buyer, provided buyers have substitutes preferences. Parkes (2006) and Mishra and Parkes (2007) have investigated the primal-dual algorithm in the general environments with complements preferences to discover the more involved concept of universal competitive equilibrium, which reveals the efficient allocations without any single buyer and the efficient allocation all together.[4]

Based on this difficulty for implementing the VCG mechanism as price-based, the

[4] These auction formats do not consider appropriate solutions of privacy and complexity mentioned above enough.

latter part of this paper replaces the constraint of ex-post individual rationality with weaker constraints named *participation constraints*, implying that the seller and the buyers have incentive to participate in the auction in the ex-ante and interim terms, respectively. With participation constraints, we examine the possibility of the existence of an efficient, strategy-proof, and price-based mechanism.

We confine our attention to mechanisms in which the auctioneer always starts with a very low price vector, for which any buyer demands all items and units. We assume that the buyers' valuation functions are randomly and independently determined. In this setting, we can show that there exists an efficient, strategy-proof, and price-based mechanism with participation constraints if and only if *the efficient allocation that is induced by the profile of representative valuation functions is revealed in the observed price-demand sets at all times*. In contrast to the VCG mechanisms, we do *not* need the efficient allocations without any single buyer to be revealed. Hence, the auctioneer can stop his/her price asking activity as soon as it is confirmed that the efficient allocation is revealed; the replacement of ex-post individual rationality with the participation constraints can dramatically decrease the privacy infringement and simplify the procedure of auction.

We also investigate the situation in which the limitation of the range in which the auctioneer can collect information is too severe to achieve efficiency. We examine the possibility for an inefficient price-based mechanism to be strategy-proof in an *approximate* sense that there is a positive but small upper limit to each buyer's possible gains from deviation. We require a mechanism to be *strictly* price-based in the sense that the allocation induced by this mechanism must be revealed at all times. We show a necessary and sufficient condition for a strictly price-based mechanism to be approximately strategy-proof. The concept of representative valuation function plays the central role; it is sufficient to examine only the buyers' incentive when they have representative valuation functions.

Finally, we investigate a general class of *indirect* mechanisms, and show characterization results for the consistency of the Nash equilibrium, efficiency, and core with the price-based property. Even in these cases, what we have to do for these results is to examine how the representative valuation functions. Based on these characterization results, we consider *core-selecting* mechanisms, which were addressed by Bernheim and Whinston (1986), Day and Raghavan (2007), and Day and Milgrom (2008). We show a characterization result for the existence of price-based Nash equilibria that induce core outcomes at all times; it is sufficient for this characterization that the efficient allocation is revealed in the price-demand sets whenever the buyers have representative valuation functions.

The rest of this paper is organized as follows. Section 2 models the problem of combinatorial auction design. Sections 3 and 4 introduce the concepts of price-demand set and price-based mechanism, respectively. Section 5 introduces the concept of representative valuation function, and shows the necessary and sufficient condition under which there exists the price-based VCG mechanism, where a simple and easy method is shown to confirm whether this condition is satisfied. Section 6 shows a necessary and sufficient condition for the existence of efficient, strategy-proof, and price-based mechanisms. Section 7 indicates that even with participation constraints, the same condition as that in Section 6 is sufficient for the existence of such mechanisms. Section 8 considers mechanisms that are approximately strategy-proof. Section 9 investigates indirect mechanisms in general, and shows characterization results for the compatibility of the Nash equilibrium, efficiency, and core with the price-based property. Section 10 investigates core-selecting mechanisms. Finally, Section 11 provides concluding remarks.

2. Model

The present paper investigates the allocation problem in which there exist multiple items with multiple units that a single seller supplies to multiple buyers. Let us denote $N \equiv \{1,...,n\}$ as the non-empty and finite set of *buyers*. A *package for each buyer* $i \in N$ is denoted by $a_i = (a_{i1},...,a_{il})$, where a positive integer m_z implies the amount of the $z-th$ item that the seller supplies, and $a_{iz} \in \{0,...,m_z\}$ for each $z \in \{1,...,l\}$ implies the amount of the $z-th$ item that buyer i demands or obtains. The set of all packages for buyer i is denoted by $A_i \equiv \underset{z\in\{1,...,l\}}{\times}\{0,...,m_z\}$. Let us denote by $\underline{a}_i \in A_i$ the *null* package for buyer i, where $\underline{a}_{iz} = 0$ for all $z \in \{1,...,l\}$. We define an *allocation* as a profile of packages $a \equiv (a_1,...,a_n)$, where we assumed that

$$\sum_{i\in N} a_{iz} \le m_z \quad \text{for all} \quad z \in \{1,...,l\}.$$

Let us denote by $A \subset \underset{i\in N}{\times} A_i$ the set of all allocations. We define an *allocation without a buyer* $i \in N$ as $a^i \equiv (a_j)_{j\in N\setminus\{i\}}$, where we assume that

$$\sum_{j\in N\setminus\{i\}} a_{jz} \le m_z \quad \text{for all} \quad z \in \{1,...,l\}.$$

Let us denote by $A^i \subset \underset{j\ne i}{\times} A_j$ the set of all allocations without a buyer $i \in N$. According to $a^i \in A^i$, each buyer $j \in N\setminus\{i\}$ demands or obtains package a_j, whereas buyer i demands or obtains nothing.

A *valuation function for buyer* $i \in N$ is denoted by $u_i : A_i \to R$, which is quasi-linear, and satisfies that

$$u_i(\underline{a}_i) = 0,$$

and that any increase in the amount of items has a positive value, i.e.,

(1) $$u_i(a_i) > u_i(\tilde{a}_i) \quad \text{whenever} \quad a_i \ne \tilde{a}_i \quad \text{and} \quad a_i \ge \tilde{a}_i.$$

Let us denote by U_i the set of all such valuation functions for buyer i. Let $U \equiv \underset{i\in N}{\times} U_i$, $U^i \equiv \underset{j\in N\setminus\{i\}}{\times} U_j$, $u \equiv (u_i)_{i\in N} \in U$, and $u^i \equiv (u_j)_{j\in N\setminus\{i\}} \in U^i$.

An allocation $a \in A$ is said to be *efficient for a profile* $u \in U$ *of valuation* *functions* if

$$\sum_{i \in N} u_i(a_i) \geq \sum_{i \in N} u_i(\tilde{a}_i) \quad \text{for all} \quad \tilde{a} \in A.$$

Let us denote by $A^*(u) \subset A$ the set of all efficient allocations for $u \in U$. An allocation $a^i \in A^i$ without a buyer i is said to be *efficient for a profile* $u^i \in U^i$ *of the valuation functions without buyer* $i \in N$ if

$$\sum_{j \in N \setminus \{i\}} u_j(a_i) \geq \sum_{j \in N \setminus \{i\}} u_j(\tilde{a}_i) \quad \text{for all} \quad \tilde{a}^i \in A^i.$$

Let us denote by $A^{i*}(u) \subset A^i$ the set of all efficient allocations for $u^i \in U^i$.

A direct mechanism, or shortly a *mechanism*, is defined as $G = (g, (q_i)_{i \in N})$, where $g : U \to A$ implies the allocation function, and $q_i : U \to R$ implies the payment function for each buyer $i \in N$. For every $u \in U$, let us denote $g(u) = (g_i(u))_{i \in N}$, where $g_i(u) \in A_i$ for each $i \in N$. We can interpret any mechanism as a naïve direct revelation where each buyer $i \in N$ is required to announce a valuation function $\tilde{u}_i \in U_i$ to the auctioneer, based on which the auctioneer selects the allocation $g(\tilde{u}) \in A$ and transfers the monetary payment $q_i(\tilde{u}) \in R$ from each buyer i to the seller. In this case, the resulting payoff for each buyer $i \in N$ is given by

$$u_i(g_i(\tilde{u})) - q_i(\tilde{u}),$$

where we assume that the true valuation function for buyer i was given by $u_i \in U_i$. This is not necessarily the same as his/her announced function $\tilde{u}_i$. The resulting revenue that the seller obtains is given by $\sum_{i \in N} q_i(\tilde{u})$.

A mechanism G is said to be *efficient* if for every $u \in U$, $g(u) \in A$ is efficient, i.e., $g(u) \in A^*(u)$. A mechanism G is said to be *strategy-proof* if for every $u \in U$, any buyer $i \in N$ has incentive to truthfully reveal information about his/her valuation function $u_i \in U_i$ as a dominant strategy, i.e.,

$$u_i(g_i(u)) - q_i(u) \geq u_i(g_i(\tilde{u}_i, u^i)) - q_i(\tilde{u}_i, u^i) \quad \text{for all} \quad \tilde{u}_i \in U_i.$$

A mechanism G is said to be *of the VCG type*, if it is efficient, and for every $u \in U$,

$$q_i(u) = \max_{a^i \in A^i} \sum_{j \in N \setminus \{i\}} u_j(a_j) - \sum_{j \in N \setminus \{i\}} u_j(g_j(u)) \quad \text{for all} \quad i \in N.$$[5]

It must be noted that any VCG mechanism G satisfies *ex-post individual rationality* in the sense that the resulting revenue for the seller is always non-negative, i.e.,

(2) $$\sum_{i \in N} q_i(u) \geq 0 \quad \text{for all} \quad u \in U,$$

and the resulting payoff for each buyer is always non-negative, i.e.,

(3) $$u_i(g_i(u)) - q_i(u) \geq 0 \quad \text{for all} \quad u \in U \quad \text{and all} \quad i \in N.$$

It is clear from the mechanism design literature that any VCG mechanism is strategy-proof, and that any efficient and strategy-proof mechanism G that satisfies ex-post individual rationality must be VCG.[6]

This paper investigates the possibility that a VCG mechanism is regarded as being *price-based* in the sense that, instead of directly announcing the entire valuation function, each buyer reveals partial information by making demand responses to a limited number of price vectors that the auctioneer asks him/her.

[5] See Vickery (1961), Clarke (1971), and Groves (1973).

[6] See Rothkopf, Teisberg, and Kahn (1990), Milgrom (2004), and Ausubel and Milgrom (2006).

3. Price-Demand Sets

A *price vector for buyer* $i \in N$ is denoted by $p_i = (p_i(a_i))_{a_i \in A_i} \in R^{|A_i|}$, where we assume that $p_i(\underline{a}_i) = 0$, and

(4) $$p_i(a_i) > p_i(\tilde{a}_i) \quad \text{if} \quad a_i \neq \tilde{a}_i \quad \text{and} \quad a_i \geq \tilde{a}_i .$$

It is appropriate to assume that any price vector p_i that the auctioneer asks of any buyer i will satisfy inequalities (4), because any increase in amount of items has a positive value, i.e., because any valuation function $u_i \in U_i$ satisfies inequalities (1). Let us denote P_i as the set of all such price vectors for buyer i.

A *price-demand set for buyer* $i \in N$ is defined as a non-empty and compact subset, denoted by

$$E_i \subset P_i \times A_i .$$

Each element (p_i, a_i) of price-demand set E_i implies that the auctioneer asks price vector p_i to buyer i, and in response buyer i reveals his/her demand a_i. Instead of requesting any buyer $i \in N$ to directly announce his/her entire valuation function u_i, the auctioneer collects partial information about it through the observation of price-demand set E_i.

Let us denote by $U_i(E_i) \subset U_i$ the set of valuation functions for buyer i that is consistent with price-demand set E, i.e., according to which, for every $(p_i, a_i) \in E_i$, buyer $i's$ demand a_i maximizes his/her payoff. This can be expressed as

$$U_i(E_i) \equiv \{u_i \in U_i \mid a_i \in \arg\max_{\tilde{a}_i \in A_i} \{u_i(\tilde{a}_i) - p_i(\tilde{a}_i)\} \textit{ for all } (p_i, a_i) \in E_i\},$$

where we assume that he behaves as a price taker. By observing any price-demand set E_i, the auctioneer recognizes that the valuation function of buyer i is included in $U_i(E_i)$, but he/she does not know which valuation function in $U_i(E_i)$ is the correct one. Note that for every $u_i \in U_i(E_i)$,

$$a_i \in \arg\max_{\tilde{a}_i \in A_i} \{u_i(\tilde{a}_i) - p_i(\tilde{a}_i)\} \quad \text{for all} \quad p_i \in P_i(a_i, E_i) .$$

Throughout this paper, we confine our attention to price-demand sets E_i such that $U_i(E_i)$ is non-empty, i.e., $U_i(E_i) \neq \phi$. This implies a *revealed-preference activity rule*, according to which, any buyer is restricted to make his/her demand responses consistent with a single valuation function.

Let us define $P_i(a_i, E_i) \subset P_i$ as the convex hull of the closure of the set $\{p_i \in P_i \,|\, (p_i, a_i) \in E_i\}$ of all price vectors for buyer i that the auctioneer asks of him/her in price-demand set E_i. Note that given any valuation function in $U_i(E_i)$ and any price vector $p_i \in P_i(a_i, E_i)$, buyer $i's$ demand a_i maximizes his/her payoff. Let us denote by $A_i(E_i) \subset A_i$ the set of all allocations for buyer i such that $P_i(a_i, E_i) \neq \phi$, i.e., the set of all demands that buyer i reveals in price-demand set E_i. Let us denote $E \equiv \underset{i \in N}{\times} E_i$, $E^i \equiv \underset{j \in N \setminus \{i\}}{\times} E_j$, $A(E) \equiv \underset{i \in N}{\times} A_i(E_i)$ and $A^i(E^i) \equiv \underset{j \in N \setminus \{i\}}{\times} A_j(E_j)$. Moreover, let us define

$$P_i(E_i) \equiv \bigcup_{a_i \in A_i} P_i(a_i, E_i).$$

A price-demand set E_i for buyer $i \in N$ is said to be *connected* if for every $\{p_i, \tilde{p}_i\} \subset P_i(E_i)$, there exists a continuous function $\rho_i : [0,1] \to P_i(E_i)$ such that $\rho_i(0) = p_i$ and $\rho_i(1) = \tilde{p}_i$. Throughout this paper, we confine our attention to price-demand sets that are connected.

A price-demand set satisfies the connectedness if the auctioneer changes his/her ask price vector continuously. Hence, the clock auction that traces a single ascending linear price trajectory, to which Gul and Stacchetti (2000) have limited their attention, always induces connected price-demand sets. More generally, a price-demand set satisfies the connectedness if the auctioneer never makes his/her ask price vector jump discontinuously to any price vector that he has never asked before; i.e., if he/she either changes his/her ask price vector continuously or jumps to any price vector that he/she has asked before. Hence, the dynamical clock auction studied by Ausubel (2006), which traces multiple ascending price trajectories, always induces connected price-demand sets, provided that these trajectories start with the same price vector.

Let us denote by $\Phi_i \subset 2^{P_i \times A_i}$ the set of all connected price-demand sets for buyer

$i \in N$ such that $U_i(E_i) \neq \phi$. Let $U(E) \equiv \times_{i \in N} U_i(E_i)$, $U^i(E^i) \equiv \times_{j \in N \backslash \{i\}} U_j(E_j)$, and $\Phi \equiv \times_{i \in N} \Phi_i$. The following lemma shows that the auctioneer can correctly calculate the difference in valuation between any pair of packages whenever these packages are revealed in the connected price-demand set.

Lemma 1: *For every* $i \in N$, *every* $E_i \in \Phi_i$, *and every* $\{a_i, \tilde{a}_i\} \subset A_i(E_i)$, *there uniquely exists* $x_i(a_i, \tilde{a}_i, E_i) \in R$ *such that*

$$x_i(a_i, \tilde{a}_i, E_i) = u_i(a_i) - u_i(\tilde{a}_i) \quad \textit{for all} \quad u_i \in U_i(E_i).$$

Proof: Since E_i is connected, we can select a continuous function $\rho_i : [0,1] \to P_i(E_i)$ such that

$$\rho_i(0) \in P_i(a_i, E_i) \quad \text{and} \quad \rho_i(1) \in P_i(\tilde{a}_i, E_i).$$

In this case, we can select finite sequences $(t(m))_{m=1}^{\overline{m}}$ and $(a_i(m))_{m=1}^{\overline{m}}$ such that

$$t(m) \in [0,1] \quad \text{and} \quad a_i(m) \in A_i(E_i) \quad \text{for all} \quad m \in \{1, ..., \overline{m}\},$$

$$t(1) = t(2) = 0, \quad t(\overline{m} - 1) = t(\overline{m}) = 1,$$

$$t(m+1) \geq t(m) \quad \text{for all} \quad m \in \{1, ..., \overline{m} - 1\},$$

$$a_i(1) = a_i, \quad a_i(\overline{m}) = \tilde{a}_i,$$

and

$$\rho_i(t) \in P_i(a_i(m), E_i) \quad \text{for all} \quad m \in \{2, ..., \overline{m} - 1\} \quad \text{and all} \quad t \in [t(m), t(m+1)].$$

Let us specify

$$x_i(a_i, \tilde{a}_i, E_i) \equiv \sum_{m=2}^{\overline{m}} \{\rho_i(t(m))(a_i(m-1)) - \rho_i(t(m))(a_i(m))\}.$$

Let us consider an arbitrary valuation function $u_i \in U_i(E_i)$. For every $m \in \{2, ..., \overline{m}\}$, since

$$\rho_i(t(m)) \in P_i(a_i(m), E_i) \quad \text{and} \quad \rho_i(t(m)) \in P_i(a_i(m-1), E_i),$$

it follows that

$$u_i(a_i(m)) - \rho_i(t(m))(a_i(m)) = u_i(a_i(m-1)) - \rho_i(t(m))(a_i(m-1)).$$

Hence, the difference in valuation between $a_i(m)$ and $a_i(m-1)$ is equivalent to the difference in price between them, i.e.,

$$u_i(a_i(m)) - u_i(a_i(m-1)) = \rho_i(t(m))(a_i(m)) - \rho_i(t(m))(a_i(m-1)),$$

which implies that

$$x_i(a_i, \tilde{a}_i, E_i) \equiv \sum_{m=2}^{\bar{m}} \{u_i(a_i(m-1)) - u_i(a_i(m))\}$$

$$= u_i(a_i(1)) - u_i(a_i(m)) = u_i(a_i) - u_i(\tilde{a}_i).$$

Q.E.D.

In the proof of Lemma 1, the connectedness of the price-demand set plays the central role for the unique determination of difference in valuation; the connectedness guarantees that the difference in valuation is equivalent to the summation of price differences. Concepts that are related to connectedness can be found in the dynamical clock auctions studied by Ausubel (2006) and in the universal competitive equilibrium studied by Parkes (2006) and Mishra and Parkes (2007).

Any price-demand set reveals only partial information about a buyer's valuation function; the absolute term of valuations is not revealed unless the null demand $\underline{a}_i$ is revealed. Provided that the auctioneer never asks extremely high price vectors, it is a typical thing for any winning buyer $i \in N$ that $\underline{a}_i \notin A_i(E_i)$ That is, buyer i does not reveal the null demand $\underline{a}_i$. In this case, the auctioneer can collect information only about the *relative* term of valuations; no buyer ever reveals his/her absolute valuations. Provided that the null demand is not revealed, for every $u_i \in U_i(E_i)$ and every $\varepsilon > 0$, it holds that $u_{i,\varepsilon}$ is always included in $U_i(E_i)$, where $u_{i,\varepsilon}$ is defined as

$$u_{i,\varepsilon}(\underline{a}_i) = 0,$$

and

$$u_{i,\varepsilon}(a_i) = u_i(a_i) + \varepsilon \quad \text{for all} \quad a_i \in A_i \setminus \{\underline{a}_i\}.$$

Hence, the auctioneer cannot distinguish how strongly each buyer prefers any non-null package to the null package.

4. Price-Based Mechanisms

A *price-demand scheme* is defined as $\alpha = (\alpha_i)_{i \in N} : U \to \Phi$, where for every $u \in U$ and every $\tilde{u} \in U \setminus \{u\}$,

$$\alpha(u) = \alpha(\tilde{u}) \text{ if } \tilde{u} \in U(\alpha(u)).$$

If each buyer $i \in N$ behaves as if his/her true valuation function is given by $\tilde{u}_i$, then the auctioneer observes the price-demand set $E_i = \alpha_i(\tilde{u})$, recognizes that buyer $i's$ true valuation function is included in $U_i(E_i)$, but does not know which valuation function in $U_i(E_i)$ is the correct one. The concept of the price-demand scheme should be distinct from the naïve message space reduction, because information feedback is allowed in that the price-demand set $\alpha_i(\tilde{u})$ for each buyer $i \in N$ is generally dependent on not only $\tilde{u}_i$ but also $\tilde{u}^i$.

The concept of the price-demand scheme applies to many price-based auction formats for a wide area without remaining in the already examined formats such as the dynamical clock auction by Ausubel (2006) or the primal-dual algorithm by Parkes (2006) and Mishra and Parkes (2007).[7] Most studies have investigated the possibility that the auctioneer discovers the competitive equilibrium price vector and asks this to the buyers. In contrast, the present paper studies price-demand schemes that are not necessarily aimed at discovering the competitive equilibrium price vector; the auctioneer can achieve efficiency without suggesting the competitive equilibrium price vector to the buyers.

A mechanism G is said to be *price-based for a price-demand scheme* α if for every $u \in U$ and every $\tilde{u} \in U \setminus \{u\}$,

$$(g(u), (q_i(u))_{i \in N}) = (g(\tilde{u}), (q_i(\tilde{u}))_{i \in N}) \text{ whenever } \alpha(u) = \alpha(\tilde{u}).$$

[7] With the restriction of revealed preference activity rule, a basic characteristic of a price-based auction format can be almost summarized by a price-demand scheme. In order to connect the already examined auction formats with the formulations of this paper in a more precise manner, it might be appropriate to define the concept of price-demand scheme, not as a single-valued function, but as a set-valued function on Φ. The contents of this paper do not change basically even if we change the definition of price-demand scheme in this manner, unless the analysis of this paper becomes complicated.

The determination of $(g(\tilde{u}),(q_i(\tilde{u}))_{i\in N})$ depends only on the observed profile $\alpha(\tilde{u})$ of price-demand sets. In order to implement a mechanism that is price-based for α, the auctioneer does not need to know any more information than $\alpha(u)$. Let us denote

$$\alpha^i(u)=(\alpha_j(u))_{j\in N\setminus\{i\}},$$

$$\alpha(U)\equiv\{E\in\Phi\mid\alpha(u)=E \text{ for some } u\in U\},$$

$$\alpha_i(U)\equiv\{E_i\in\Phi_i\mid\alpha_i(u)=E_i \text{ for some } u\in U\},$$

and

$$\alpha^i(U)\equiv\{E^i\in\Phi^i\mid\alpha^j(u)=E^j \text{ for some } u\in U\}.$$

The following proposition shows a necessary and sufficient condition for the existence of the price-based VCG mechanism; it is necessary and sufficient that *the efficient allocation and the efficient allocations without any single buyer are all revealed in the observed profile of price-demand sets.*

Proposition 2: *For every price-demand scheme α, there exists a VCG mechanism G that is price-based for α if and only if for every $E\in\alpha(U)$, there exist $a(E)\in A(E)$ and $a^i(E^i)\in A^i(E^i)$ for each $i\in N$ such that*

(5) $$a(E)\in A^*(u) \text{ for all } u\in U(E),$$

and for every $i\in N$,

(6) $$a^i(E^j)\in A^{i*}(u^i) \text{ for all } u^i\in U^i(E^i).$$

Proof: We prove the "if" part as follows. Suppose that for every $E\in\alpha(U)$, there exist $a(E)\in A(E)$ and $a^i(E^i)\in A^i(E^i)$ for each $i\in N$ that satisfy properties (5) and (6). Then, we can specify $g:U\to A$ by

$$g(u)=a(\alpha(u)) \text{ for all } u\in U.$$

From Lemma 1, for every $i\in N$, we can specify $q_i:U\to R$ by

$$q_i(u)=\sum_{j\in N\setminus\{i\}}x_j(a^i_j(\alpha^j(u)),a_j(\alpha(u)),\alpha^j(u)) \text{ for all } u\in U.$$

Note from Lemma 1 and property (6) that

$$q_i(u)=\max_{a^i\in A^i}\sum_{j\in N\setminus\{i\}}u_j(a_j)-\sum_{j\in N\setminus\{i\}}u_j(g(u)).$$

Hence, the correspondingly specified mechanism $G=(g,(q_i)_{i\in N})$ is VCG.

We prove the "only if" part as follows. Suppose that $G=(g,(q_i)_{i\in N})$ is VCG and price-based for α, where we assumed that

$$g(u)\in A^*(u) \text{ for all } u\in U.$$

From inequalities (1) and (4), it follows that whenever $\{a_i,\tilde{a}_i\}\not\subset A_i(E_i)$, then there exists $\{u_i,\tilde{u}_i\}\subset U_i(E_i)$ such that

$$u_i(a_i)-u_i(\tilde{a}_i)\neq \tilde{u}_i(a_i)-\tilde{u}_i(\tilde{a}_i).$$

Hence, for every $u\in U$, if either $g(u)\notin A(\alpha(u))$ or $A^j(\alpha^j(u))\bigcap A^{j*}(u^j)=\phi$ for some $j\in N$, then there exist $j\in N$ and $\tilde{u}_j\in U_j$ such that

$$\alpha(\tilde{u}_j,u^j)=\alpha(u),$$

and for every $i\in N\setminus\{j\}$,

$$q_i(\tilde{u}_j,u^j)=\max_{a^i\in A^i}\{\tilde{u}_j(a_j)+\sum_{h\in N\setminus\{i,j\}}u_h(a_h)\}-\{\tilde{u}_j(g_j(u))+\sum_{h\in N\setminus\{i,j\}}u_h(g_h(u))\}$$

$$\neq \max_{a^i\in A^i}\sum_{h\in N\setminus\{i\}}u_h(a_h)-\sum_{h\in N\setminus\{i\}}u_h(g_h(u))$$

$$=q_i(u).$$

This contradicts the supposition that G is price-based for α. Hence, we have proved that for every $u\in U$,

$$g(u)\in A(\alpha(u)),\text{ and } A^j(\alpha^j(u))\bigcap A^{j*}(u^j)\neq\phi \text{ for all } j\in N.$$

Suppose that there exist $u\in U$, $\tilde{u}\in U$, $j\in N$, and $a^j\in A^j$ such that

$$\alpha(u)=\alpha(\tilde{u}),\quad a^j\in A^j(\alpha^j(u))\bigcap A^{j*}(u^j),\text{ and } a^j\notin A^{j*}(\tilde{u}^j).$$

In this case, without loss of generality, we can select $\tilde{u}\in U(\alpha(u))$ satisfying that

$$q_j(\tilde{u})=\max_{\tilde{a}^j\in A^j}\sum_{i\in N\setminus\{j\}}\tilde{u}_i(\tilde{a}_i)-\sum_{i\in N\setminus\{j\}}\tilde{u}_i(g_i(u))$$

$$>\sum_{i\in N\setminus\{j\}}\tilde{u}_i(a_i^j)-\sum_{i\in N\setminus\{j\}}\tilde{u}_i(g_i(u)).$$

Since $g(u)\in A(\alpha(u))$ and $a^j\in A^j(\alpha^j(u))$, it follows that

$$\sum_{i\in N\setminus\{j\}}\tilde{u}_i(a_i^j)-\sum_{i\in N\setminus\{j\}}\tilde{u}_i(g_i(u))=\sum_{i\in N\setminus\{j\}}u_i(a_i^j)-\sum_{i\in N\setminus\{j\}}u_i(g_i(u))=q_i(u),$$

which implies that $q_i(\tilde{u})\neq q_i(u)$. This contradicts the supposition that G is price-based for α. Hence, we have proved that for every $u \in U$ and every $j \in N$,

$$A^j(\alpha^j(u)) \bigcap (\bigcup_{u^j \in U^j(\alpha^j(u))} A^{j*}(u^j)) \neq \phi .$$

From the above observations, we have proved the "only if" part, i.e., properties (5) and (6) are necessary for the existence of the price-based VCG mechanism.

Q.E.D.

Because of Lemma 1, the connectedness of price-demand set guarantees that the difference in valuation between any pair of packages in the efficient allocation and the efficient allocations without any single buyer can be calculated as being equivalent to the summation of price differences, provided all these allocations are revealed. Hence, the observation of the profile of price-demand sets is sufficient for implementing the VCG mechanism.

5. Representative Valuations

The drawback of the necessary and sufficient condition in Proposition 2 is that for any observed price-demand set $E_i \in \alpha_i(U)$, the auctioneer has to examine all the possibilities of valuation functions in $U_i(E_i)$ about whether the efficient allocation and the efficient allocations without any single buyer are revealed. However, we can show that this rather intractable condition can be replaced with a much simpler one; what we need to do for this sufficiency is to examine just a particular single valuation function named the *representative valuation function*. For buyer $i \in N$ and every price-demand set $E_i \in \Phi_i$, we specify the representative valuation function, denoted by $u_i^{[E_i]} \in U_i$, as follows, where we assume that

$$u_i^{[E_i]}(\underline{a}_i) = 0,$$

and we fix an arbitrary allocation $\tilde{a}_i \in A_i(E_i)$ for buyer i; for every $a_i \in A_i(E_i) \setminus \{\tilde{a}_i\}$,

$$u_i^{[E_i]}(a_i) = u_i^{[E_i]}(\tilde{a}_i) - x_i(\tilde{a}_i, a_i, E_i),$$

and for every $a_i \notin A_i(E_i)$,

$$u_i^{[E_i]}(a_i) = \min_{(p_i, a_i') \in E_i} \{u_i^{[E_i]}(a_i') - p_i(a_i') + p_i(a_i)\}.$$

Note that the representative valuation function associated with $E_i \in \Phi_i$, i.e., $u_i^{[E_i]}$, uniquely exists, and is included in $U_i(E_i)$. As the following lemma shows, the representative valuation function $u_i^{[E_i]}$ assigns any revealed demand $a_i \in A_i(E_i)$ with *the minimal possible valuation in both absolute and relative terms*.

Lemma 3: *For every* $u_i \in U_i(E_i)$ *and every* $a_i \in A_i(E_i)$,

$$u_i(a_i) \geq u_i^{[E_i]}(a_i),$$

and for every $a_i' \in A_i$,

$$u_i(a_i) - u_i(a_i') \geq u_i^{[E_i]}(a_i) - u_i^{[E_i]}(a_i'),$$

where

$$u_i(a_i) = u_i^{[E_i]}(a_i) \quad \text{if} \quad \underline{a}_i \in A_i(E_i),$$

and

$$u_i(a_i) - u_i(a_i') = u_i^{[E_i]}(a_i) - u_i^{[E_i]}(a_i') \quad \textit{if} \quad a_i' \in A_i(E_i).$$

Proof: It is clear from the specification of $u_i^{[E_i]}$ that

$$u_i(a_i) - u_i(a_i') \geq u_i^{[E_i]}(a_i) - u_i^{[E_i]}(a_i'),$$

and

$$u_i(a_i) - u_i(a_i') = u_i^{[E_i]}(a_i) - u_i^{[E_i]}(a_i') \quad \text{if} \quad a_i' \in A_i(E_i).$$

By letting $a_i' = \underline{a}_i$, it is clear from $u_i(\underline{a}_i) = u_i^{[E_i]}(\underline{a}_i) = 0$ that

$$u_i(a_i) \geq u_i^{[E_i]}(a_i),$$

and

$$u_i(a_i) = u_i^{[E_i]}(a_i) \quad \text{if} \quad \underline{a}_i \in A_i(E_i).$$

Q.E.D.

Let us denote $u^{[E]} \equiv (u_i^{[E_i]})_{i \in N}$. The following theorem shows that the necessary and sufficient condition (5) and (6) in Proposition 2 can be replaced with a simpler condition, which implies that, associated with the profile $u^{[E]}$ of the representative valuation functions, there exist an efficient allocation $a(E) \in A(E)$ and efficient allocations without any single buyer, i.e., $a^j(E^j) \in A^j(E^j)$ for each $j \in N$, that are revealed in the observed profile E of price-demand sets. Hence, what we have to do for this sufficiency is to examine just about $u^{[E]}$.

Theorem 4: *For every price-demand scheme α, there exists a VCG mechanism that is price-based for α if and only if for every $E \in \alpha(U)$, there exist $a(E) \in A(E)$ and $a^j(E^j) \in A^j(E^j)$ for each $j \in N$ such that*

(7) $$a(E) \in A^*(u^{[E]}),$$

and for every $j \in N$,

(8) $$a^j(E^j) \in A^{j*}(u^{[E]j}).$$

Proof: It is clear from Lemma 3 the specification of $u^{[E]}$ that for every $i \in N$, every

$a_i \in A_i(E_i)$, and every $\tilde{a}_i \in A_i$,

$$u_i^{[E_i]}(a_i) - u_i^{[E_i]}(\tilde{a}_i) \le u_i(a_i) - u_i(\tilde{a}_i) \quad \text{for all} \quad u_i \in U_i(E_i).$$

Hence, for every $a \in A(E)$,

$$a \in A^*(u) \quad \text{for all} \quad u \in U(E) \quad \text{whenever} \quad a \in A^*(u^{[E]}).$$

Since $u^{[E]} \in U(E)$, property (5) is equivalent to property (7). Moreover, it follows in the same manner that for every $a^j \in A^j(E^j)$,

$$a^j \in A^{j*}(u^j) \quad \text{for all} \quad u^j \in U^j(E^j) \quad \text{whenever} \quad a^j(E^j) \in A^{j*}(u^{[E]j}).$$

Since $u^{[E]j} \in U^j(E^j)$, property (6) is equivalent to property (8).

Q.E.D.

The proof of Theorem 4 showed that the efficient allocation and the efficient allocations without any single buyer that are induced by the profile $u^{[E]}$ of representative valuation functions can be induced also by any profile $u \in U(E)$ of valuation functions that are consistent with E, if, and only if, these allocations are all revealed in E. This implies to check the existence of the price-based VCG mechanism, we should examine just about $u^{[E]}$.

Practically, after continuing to ask price vectors to the buyers and observing any price-demand set $E \in \Phi$ in consequences, the auctioneer calculates the representative valuation $u_i^{[E_i]}$ for each buyer $i \in N$. The auctioneer calculates the efficient allocation and the efficient allocations without any single buyer that are induced by $u^{[E]}$, and finds out whether these allocations are all revealed. If the auctioneer ascertains that these allocations are revealed, then he/she stops asking price vectors promptly, and achieves the efficient allocation. In this case, he/she also calculates the VCG payments, and transfers them from the buyers to the seller. On the other hand, if he/she ascertains that some of these allocations are not revealed, then he/she continues to ask price vectors until it is ascertained that these allocations are all revealed.

6. Strategy-Proofness

This section investigates the possibility that there exists a price-based mechanism that is efficient and strategy-proof, but is not necessarily VCG, i.e., does not necessarily satisfy ex-post individual rationality. Let us denote by $\underline{p}_i = (\underline{p}_i(a_i))_{a_i \in A_i}$ the *zero price vector*, where $\underline{p}_i(a_i) = 0$ for all $a_i \in A_i$. In this section and the next one, we focus on price-demand schemes α such that the auctioneer asks very low price vectors that are close to $\underline{p}_i$, i.e.,

(9) $$\underline{p}_i \in P_i(\alpha_i(u)) \text{ for all } u \in U \text{ and all } i \in N.$$

Let us denote by $\overline{a}_i \in A_i$ the *maximal package for buyer* i, where

$$\overline{a}_{iz} = m_z \text{ for all } z \in \{1,...,l\}.$$

Note from inequalities (1) that any buyer i reveals the maximal package $\overline{a}_i$ as his/her demand response to any virtually zero price vector, which along with property (9) implies that it is certain that he/she reveals the maximal package $\overline{a}_i$, i.e.,

$$\overline{a}_i \in A_i(\alpha_i(u)) \text{ for all } u \in U.$$

The following lemma shows that for any price-based mechanism to be efficient, it is necessary that the allocation that is induced by the mechanism is revealed in the resulting price-demand sets at all times.

Lemma 5: *Suppose that* α *satisfies property (9) and that a mechanism* G *is price-based for* α. *If* G *is efficient, then*

$$g(u) \in A(\alpha(u)) \text{ for all } i \in N.$$

Proof: Suppose that there exist $u \in U$ and $i \in N$ such that

$$g_i(u) \notin A_i(\alpha_i(u)).$$

Note that $g_i(u) \neq \underline{a}_i$ in this case; if not, then, it follows from $g_i(u) = \underline{a}_i \notin A_i(\alpha_i(u))$ that

$$g(u) \in A^*(u_{i,\varepsilon}, u^i) \text{ for all } \varepsilon > 0,$$

which is a contradiction, because any efficient allocation $a \in A^*(u_{i,\varepsilon}, u^i)$ satisfies $a_i \neq \underline{a}_i$ whenever ε is selected to be sufficiently large.

Since $g_i(u) \neq \underline{a}_i$, we can select $a_i \in A_i \setminus \{g_i(u)\}$ such that $a_i \leq g_i(u)$, and for every $a_i' \in A_i \setminus \{a_i, g_i(u)\}$,

$$u_i(g_i(u)) - u_i(a_i) \leq u_i(g_i(u)) - u_i(a_i') \text{ if } a_i' \leq g_i(u).$$

Note from inequalities (1) that we can select $u_i' \in U_i(\alpha_i(u)) \setminus \{u_i\}$ and $j \in N \setminus \{i\}$ such that $u_i'(g_i(u)) - u_i'(a_i)$ is close to zero enough to satisfy that

(10) $$u_i'(g_i(u)) - u_i'(a_i) < u_j(g_j(u) + g_i(u) - a_i) - u_j(g_j(u)).$$

Let us specify $\hat{a} \in A$ by

$$\hat{a}_i = a_i,$$

$$\hat{a}_j = g_j(u) + g_i(u) - a_i, \text{ and}$$

$$\hat{a}_h = g_h(u) \text{ for all } h \in N \setminus \{i, j\}.$$

From inequality (10),

$$u_i'(g_i(u)) + \sum_{h \in N \setminus \{i\}} u_h(g_h(u)) < u_i'(\hat{a}_i) + \sum_{h \in N \setminus \{i\}} u_h(\hat{a}_h),$$

which contradicts the fact that $g(u)$ is efficient for (u_i', u^i).

Q.E.D.

The following proposition shows that for an efficient, strategy-proof, and price-based mechanism to exist, it is necessary and sufficient that any resulting price-demand set E reveals the efficient allocation for the profile $u^{[E]}$ of representative valuation functions. In contrast to the VCG mechanisms, we do not require the efficient allocations without any single buyer to be revealed.

Proposition 6: *Suppose that price-demand scheme α satisfies property (9). Then, there exists an efficient and strategy-proof mechanism that is price-based for α if and only if for every $E \in \alpha(U)$, there exists $a(E) \in A(E)$ such that*

(11) $$a(E) \in A^*(u^{[E]}).$$

Proof: The proof of the "only if" part of this theorem is straightforward from Lemma 5, because the equality of $a(\alpha(u)) = g(u)$ satisfies property (11). We prove the "if" part as follows. Suppose that for every $E \in \alpha(U)$, there exists $a(E) \in A(E)$ that satisfies property (11). We specify an efficient mechanism G by

$$g(u) = a(\alpha(u)) \quad \text{for all} \quad u \in U,$$

and

$$q_i(u) = \sum_{j \in N \setminus \{i\}} x_j(\bar{a}_j, a_j(\alpha(u)), \alpha_j(u)) \quad \text{for all} \quad u \in U.$$

From Lemma 1 and $\{\bar{a}_i, g_i(u)\} \subset A_i(\alpha_i(u))$, $x_i(\bar{a}_i, g_i(u), \alpha_i(u))$ is well-defined. From Lemma 1, it follows that

$$q_i(u) = \sum_{j \in N \setminus \{i\}} \{u_j(\bar{a}_j) - u_j(g_j(u))\},$$

and therefore,

$$\begin{aligned} & u_i(g(u)) - q_i(u) - \{u_i(g(\tilde{u}_i, u^j)) - q_i(\tilde{u}_i, u^j)\} \\ &= \sum_{j \in N} u_j(g_j(u)) - \sum_{j \in N \setminus \{i\}} u_j(\bar{a}_j) - \{\sum_{j \in N} u_j(g_j(\tilde{u}_i, u^i)) - \sum_{j \in N \setminus \{i\}} u_j(\bar{a}_j)\} \\ &= \sum_{j \in N} u_j(g_j(u)) - \sum_{j \in N} u_j(g_j(\tilde{u}_i, u^i)) \geq 0 \quad \text{for all} \quad \tilde{u}_i \in U_i. \end{aligned}$$

This implies that G is strategy-proof.

Q.E.D.

7. Participation Constraints

This section considers the situation where the profile of valuation functions $u \in U$ is randomly determined according to a probability measure f on U. Let us denote by $E^f[\cdot]$ and $E^f[\cdot | u_i]$ the expectation operator in the ex-ante term and the expectation operator in the interim term conditional on $u_i \in U_i$, respectively.

A mechanism G is said to satisfy *participation constraints* if each buyer has incentive to participate in the allocation problem in the interim term, i.e.,

(12) $$E^f[u_i(g_i(u_i, \tilde{u}^i)) - q_i(u_i, \tilde{u}^i) | u_i] \geq 0 \quad \text{for all } i \in N \text{ and all } u_i \in U_i,$$

and the seller has incentive to participate in the allocation problem in the ex-ante term, i.e.,

(13) $$E^f[\sum_{i \in N} q_i(\tilde{u})] \geq 0.$$

The efficient and strategy-proof price-based mechanism that was specified in the proof of Theorem 6 does not satisfy participation constraints. The following theorem shows that whenever the buyers' valuation functions are independently distributed, we can design an alternative, efficient, strategy-proof, and price-based mechanism that satisfies participation constraints.

Theorem 7: *Suppose that price-demand scheme α satisfies property (9), and that the buyers' valuation functions are independently distributed. Then, there exists an efficient and strategy-proof mechanism that is price-based for α and satisfies participation constraints if and only if for every $E \in \alpha(U)$, there exists $a(E) \in A(E)$ that satisfies property (11).*

Proof: From Theorem 6, all we have to do is to prove the "if" part of this theorem. Since the buyers' valuation functions are independently distributed, it follows that $E^f[\cdot | u_i]$ is independent of u_i. Hence, for every $i \in N$, we can specify a real number $D_i^f \in R$ by

$$D_i^f = E^f[\sum_{j \in N \setminus \{i\}} \{u_j(a_j^{i*}) - u_j(\overline{a}_j)\} | u_i] \quad \text{for all } u_i \in U_i.$$

Let us specify a mechanism G by

$$g(u) = a(\alpha(u)) \text{ for all } u \in U,$$

and

$$q_i(u) = \sum_{j \in N \setminus \{i\}} x_j(\bar{a}_j, a_j(\alpha(u)), \alpha_j(u)) + D_i^f \text{ for all } u \in U.$$

In the same manner as in the proof of Theorem 6, we can prove that the specified mechanism G is efficient and strategy-proof. Note that

$$E^f[q_i(u) \mid u_i] = E^f[\sum_{j \in N \setminus \{i\}} x_j(\bar{a}_j, a_j(\alpha(u)), \alpha_j(u)) \mid u_i] + D_i^f$$

$$= E^f[\sum_{j \in N \setminus \{i\}} \{u_j(\bar{a}_j) - u_j(g_j(u))\} \mid u_i] + D_i^f$$

$$= E^f[\sum_{j \in N \setminus \{i\}} \{u_j(a_j^{i*}) - u_j(g_j(u))\} \mid u_i],$$

which is nonnegative. Hence,

$$E^f[u_i(g_i(u)) - q_i(u) \mid u_i] = E^f[\sum_{j \in N} u_j(g_j(u)) - \sum_{j \in N \setminus \{i\}} u_j(a_j^{i*}) \mid u_i] \geq 0,$$

and

$$E^f[\sum_{i \in N} q_i(u)] = E^f[\sum_{i \in N}[\sum_{j \in N \setminus \{i\}} \{u_j(a_j^{i*}) - u_j(g_j(u))\}]] \geq 0,$$

which imply inequalities (12) and (13), respectively

Q.E.D.

By replacing ex-post individual rationality with participation constraints, we can dramatically simplify the manner of designing an efficient and strategy-proof mechanism; we do not need to require the allocations without any single buyer to be revealed. For instance, when the buyers have substitutes preferences, we can design a clock auction that is strategy-proof, satisfies participation constraints, and traces just a single ascending linear price trajectory. This is in contrast with the case of ex-post individual rationality; as Gul and Stacchetti (2000) and Ausubel (2006) have explained, with the restriction of ex-post individual rationality, it is generally impossible for any single ascending trajectory to collect sufficient information to implement the VCG mechanism.

8. Approximate Strategy-Proofness

A mechanism G is said to be *strictly price-based for a price-demand scheme* α if it is price-based for α, and

$$g(u) \in A(\alpha(u)) \text{ for all } u \in U .$$

These inequalities imply that the allocation $g(u)$ induced by the mechanism G is revealed in the profile $\alpha(u)$ of price-demand sets at all times. Note from Theorem 4 that if a VCG mechanism is price-based for a price-demand scheme α, then it is strictly price-based for α. Note also from Lemma 5 that if α satisfies property (9) and mechanism G is efficient and price-based for α, then G is strictly price-based for α.

This section investigates a general class of strictly price-based mechanisms that are not necessarily efficient or strategy-proof. For every $\varepsilon \geq 0$, a mechanism G is said to be $\varepsilon -$*strategy-proof* if for every $u \in U$ and every $i \in N$,

(14) $$u_i(g_i(u)) - q_i(u) \geq u_i(g_i(\tilde{u}_i, u^i)) - q_i(\tilde{u}_i, u^i) - \varepsilon \text{ for all } \tilde{u}_i \in U_i .$$

With the selection of ε to be close to zero, the $\varepsilon -$strategy-proofness implies that the mechanism is, not exactly, but approximately, strategy-proof. The following proposition shows a necessary and sufficient condition for a strictly price-based mechanism to be $\varepsilon -$strategy-proof.

Proposition 8: *Suppose that a mechanism* G *is strictly price-based for a price-demand scheme* α. *Then, it is* $\varepsilon -$*strategy-proof if and only if for every* $E \in \alpha(U)$, *every* $i \in N$, *every* $u^i \in U^i(E^i)$, *and every* $\tilde{u}_i \in U_i$,

$$u_i^{[E_i]}(g_i(u_i^{[E_i]}, u^i)) - q_i(u_i^{[E_i]}, u^i) \geq u_i^{[E_i]}(g_i(\tilde{u}_i, u^i)) - q_i(\tilde{u}_i, u^i) - \varepsilon .$$

Proof: The proof of the "only if" part is straightforward from the definition of $\varepsilon -$ strategy-proofness. We prove the "only if" part as follows. Note from the specification of $u_i^{[E_i]}$ that for every $u_i \in U_i(E_i)$, every $a_i \in A_i(E_i)$, and every $a_i' \in A_i$,

$$u_i(a_i) - u_i(a_i') \geq u_i^{[E_i]}(a_i) - u_i^{[E_i]}(a_i') ,$$

which, along with $g_i(u_i^{[E_i]}, u^i) = g_i(u_i, u^i) \in A_i(E_i)$ and $q_i(u_i^{[E_i]}, u^i) = q_i(u_i, u^i)$, implies that

$$u_i(g_i(u_i, u^i)) - q_i(u_i, u^i) - \{u_i(g_i(\tilde{u}_i, u^i)) - q_i(\tilde{u}_i, u^i)\}$$
$$= u_i(g_i(u_i^{[E_i]}, u^i)) - q_i(u_i^{[E_i]}, u^i) - \{u_i(g_i(\tilde{u}_i, u^i)) - q_i(\tilde{u}_i, u^i)\}$$
$$\geq u_i^{[E_i]}(g_i(u_i^{[E_i]}, u^i)) - q_i(u_i^{[E_i]}, u^i) - \{u_i^{[E_i]}(g_i(\tilde{u}_i, u^i)) - q_i(\tilde{u}_i, u^i)\}$$
$$\geq -\varepsilon .$$

Hence, we have proved that for every $u \in U$ and every $i \in N$,

$$u_i(g_i(u)) - q_i(u) \geq u_i(g_i(\tilde{u}_i, u^i)) - q_i(\tilde{u}_i, u^i) - \varepsilon \quad \text{for all } \tilde{u}_i \in U_i .$$

Q.E.D.

Proposition 8 implies that it is sufficient to examine whether the incentive constraint (14) is satisfied for the case of representative valuation functions. The representative valuation function assigns the minimal *relative* valuations for revealed packages. This makes the incentive constraint for the representative valuation function the severest among all possible valuation functions that are consistent with the observed price-demand set. The following proposition shows a necessary and sufficient condition for the existence of approximately strategy-proof and strictly price-based mechanism. The necessary and sufficient condition, which is given by inequalities (16), implies that a buyer cannot necessarily increase his/her gain by interchanging his/her manner of making demand responses between any pair of distinct representative valuation functions.

Proposition 9: *Suppose that a price-demand scheme* α *and an allocation function* g *satisfy that for every* $u \in U$ *and every* $\tilde{u} \in U \setminus \{u\}$,

$$g(u) \in A(\alpha(u)),$$

and

(15) $$g(u) = g(\tilde{u}) \quad \textit{for all } \tilde{u} \in U(\alpha(u)).$$

Then, for every $\varepsilon \geq 0$, *there exists a profile of payment functions* $(q_i)_{i \in N}$ *such that the associated mechanism* $G = (g, (q_i)_{i \in N})$ *is strictly price-based for* α *and is*

$\varepsilon-$strategy-proof, if and only if for every $E \in \alpha(U)$, every $i \in N$, every $u^i \in U^i(E^i)$, and every $\tilde{E}_i \in \alpha_i(U)$ such that $\tilde{E}_i = \alpha_i(\tilde{u}_i, u^i)$ for some $\tilde{u}_i \in U_i$, it holds that

(16) $$u_i^{[E_i]}(g_i(u_i^{[E_i]}, u^i)) + u_i^{[\tilde{E}_i]}(g_i(u_i^{[\tilde{E}_i]}, u^i)).$$
$$\geq u_i^{[E_i]}(g_i(u_i^{[\tilde{E}_i]}, u^i)) + u_i^{[\tilde{E}_i]}(g_i(u_i^{[E_i]}, u^i)) - 2\varepsilon.$$

Proof: Suppose that $G = (g, (q_i)_{i\in N})$ is strictly price-based for α and is $\varepsilon-$strategy-proof. Then, for every $E \in \alpha(U)$, every $i \in N$, every $u^i \in U^i(E^i)$, and every $\tilde{E}_i \in \alpha_i(U)$ such that $\tilde{E}_i = \alpha_i(\tilde{u}_i, u^i)$ for some $\tilde{u}_i \in U_i$,

(17) $$u_i^{[E_i]}(g_i(u_i^{[E_i]}, u^i)) - q_i(u_i^{[E_i]}, u^i) \geq u_i^{[E_i]}(g_i(u_i^{[\tilde{E}_i]}, u^i)) - q_i(u_i^{[\tilde{E}_i]}, u^i) - \varepsilon,$$

and

(18) $$u_i^{[\tilde{E}_i]}(g_i(u_i^{[\tilde{E}_i]}, u^i)) - q_i(u_i^{[\tilde{E}_i]}, u^i) \geq u_i^{[\tilde{E}_i]}(g_i(u_i^{[E_i]}, u^i)) - q_i(u_i^{[E_i]}, u^i) - \varepsilon.$$

By summing these inequalities, we have inequality (16).

Suppose that inequalities (16) hold. Then, there exist $q_i(u_i^{[E_i]}, u^i)$ and $q_i(u_i^{[\tilde{E}_i]}, u^i)$ such that

$$u_i^{[E_i]}(g_i(u_i^{[E_i]}, u^i)) - u_i^{[E_i]}(g_i(u_i^{[\tilde{E}_i]}, u^i)) + \varepsilon \geq q_i(u_i^{[E_i]}, u^i) - q_i(u_i^{[\tilde{E}_i]}, u^i)$$
$$\geq -\{u_i^{[\tilde{E}_i]}(g_i(u_i^{[\tilde{E}_i]}, u^i)) - u_i^{[\tilde{E}_i]}(g_i(u_i^{[E_i]}, u^i)) + \varepsilon\},$$

which implies inequalities (17) and (18). Without loss of generality, from equalities (15), we can select $q_i(u_i^{[E_i]}, u^i)$ for each $E_i \in \alpha_i(U)$ and each $u^i \in U^i$ such that for every $\tilde{u}^i \in U^i$,

$$q_i(u_i^{[E_i]}, u^i) = q_i(u_i^{[E_i]}, \tilde{u}^i) \quad \text{whenever} \quad \alpha(u_i^{[E_i]}, u^i) = \alpha(u_i^{[E_i]}, \tilde{u}^i).$$

Hence, we can specify q_i by

$$q_i(u) = q_i(u^{[\alpha(u)]}) \quad \text{for all} \quad u \in U.$$

It is clear that the specified mechanism $(g, (q_i)_{i\in N})$ is strictly price-based for α. From Proposition 8 and inequalities (17), it is clear that $(g, (q_i)_{i\in N})$ is $\varepsilon-$strategy-proof.

Q.E.D.

9. Indirect Mechanisms

This section considers a general class of *indirect* mechanisms, and examines whether the price-based property is consistent with the Nash equilibrium, efficiency, and core. The aspect of representative valuation function that the minimal valuations in the relative term are assigned to any revealed package plays the central role even in this examination. An *indirect mechanism* is defined as $H=(h,(S_i,r_i)_{i\in N})$, where S_i denotes the set of *messages* for buyer i, $S\equiv\underset{i\in N}{\times}S_i$, $h:S\to A$ implies the allocation function, and $r_i:S\to R$ implies the payment function for each buyer $i\in N$. Let us denote $h(s)=(h_i(s))_{i\in N}$. Let us denote a message profile by $s=(s_i)_{i\in N}\in S$. Note that a direct mechanism $G=(g,(q_i)_{i\in N})$ is regarded as a special case of indirect mechanism $H=(h,(S_i,r_i)_{i\in N})$, where $S_i=U_i$, $h_i=g_i$, and $r_i=q_i$ for all $i\in N$.

A *strategy for buyer* i is defined as a function $\sigma_i:U\to S_i$, according to which, buyer i announces message $\sigma_i(u)\in S_i$ when the profile of the buyers' valuation functions is given by $u\in U$. Here, we take into account the case of complete information, where $\sigma_i(u)$ depends on not only u_i but also u^i. Let $\sigma(u)\equiv(\sigma_i(u))_{i\in N}$ and $\sigma^i(u)\equiv(\sigma_j(u))_{j\in N\setminus\{i\}}$. Let Σ_i denote the set of all strategies for buyer i. Let $\sigma=(\sigma_i)_{i\in N}\in S$ denote a strategy profile. Let $\Sigma\equiv\underset{i\in N}{\times}\Sigma_i$ denote the set of all strategy profiles.

A combination of an indirect mechanism and a strategy profile (H,σ) is said to be *price-based for a price-demand scheme* α if for every $u\in U$ and every $\tilde{u}\in U$,

$$\sigma(u)=\sigma(\tilde{u}) \text{ whenever } \alpha(u)=\alpha(\tilde{u}).$$

A combination of an indirect mechanism and a strategy profile (H,σ) is said to be *strictly price-based for a price-demand scheme* α if it is price-based for α, and for every $u\in U$,

$$h(\sigma(u))\in A(\alpha(u)).$$

9.1 Nash equilibrium

A message profile $s \in S$ is said to be a *Nash equilibrium* in the game given by a combination of an indirect mechanism and a profile of valuation functions (H,u) if for every $i \in N$,

$$u_i(h(s)) - r_i(s) \geq u_i(h(\tilde{s}_i, s^i)) - r_i(\tilde{s}_i, s^i) \text{ for all } \tilde{s}_i \in S_i .$$

A strategy profile $\sigma \in \Sigma$ is said to be a *universal Nash equilibrium* in indirect mechanism H if for every $u \in U$, the message profile $\sigma(u)$ is a Nash equilibrium in (H,u). The following proposition shows that under the constraint of the strictly price-based property, for a universal Nash equilibrium strategy profile, it is sufficient to examine just about profiles of representative valuation functions.

Proposition 10: *Suppose that a combination of an indirect mechanism and a strategy profile (H,σ) is strictly price-based for a price-demand scheme α. Then, σ is a universal Nash equilibrium in H if and only if for every $E \in \alpha(U)$, the message profile $\sigma(u^{[E]})$ is a Nash equilibrium in $(H,u^{[E]})$.*

Proof: The "only if" part is straightforward from the definition of the universal Nash equilibrium. Let us consider any $u \in U(E)$ and $i \in N$. Note from Lemma 3 that for every $a_i \in A_i(E_i)$, and every $a_i' \in A_i$,

$$u_i(a_i) - u_i(a_i') \geq u_i^{[E_i]}(a_i) - u_i^{[E_i]}(a_i') ,$$

which, along with $\sigma(u^{[E]}) = \sigma(u)$ and $h(\sigma(u^{[E]})) \in A(E)$, implies that

$$\begin{aligned}
& u_i(h(\sigma(u))) - r_i(\sigma(u)) - \{u_i(h(s_i, \sigma^i(u))) - r_i(s_i, \sigma^i(u))\} \\
&= u_i(h(\sigma(u^{[E]}))) - r_i(\sigma(u^{[E]})) - \{u_i(h(s_i, \sigma^i(u^{[E]}))) - r_i(s_i, \sigma^i(u^{[E]}))\} \\
&\geq u_i^{[E_i]}(h(\sigma(u^{[E]}))) - r_i(\sigma(u^{[E]})) - \{u_i^{[E_i]}(h(s_i, \sigma^i(u^{[E]}))) - r_i(s_i, \sigma^i(u^{[E]}))\} \\
&\geq 0 .
\end{aligned}$$

Hence, we have proved that $\sigma(u)$ is a Nash equilibrium in (H,u) for all $u \in U$.

Q.E.D.

9.2. Efficiency and Core

A strategy profile $\sigma \in \Sigma$ is said to be *efficient* in an indirect mechanism H if for every $u \in U$, the allocation $h(\sigma(u)) \in A$ that is induced by the message profile $\sigma(u)$ is efficient for u. The following proposition shows that under the constraint of the strictly price-based property, for an efficient strategy profile, it is sufficient to examine just about profiles of representative valuation functions.

Proposition 11: *Suppose that a combination of an indirect mechanism and a strategy profile (H,σ) is strictly price-based for a price-demand scheme α. Then, σ is efficient in H if and only if for every $E \in \alpha(U)$, the allocation $\sigma(u^{[E]})$ is efficient in $(H,u^{[E]})$.*

Proof: The "only if" part is straightforward from the definition of efficiency in terms of strategy profile. We can prove the "if" part in the same manner as the proof of Theorem 4. It is clear from Lemma 3 that for every $i \in N$, every $a_i \in A_i(E_i)$, every $\tilde{a}_i \in A_i$, and $u_i \in U_i(E)$,

$$u_i(a_i) - u_i(\tilde{a}_i) \geq u_i^{[E_i]}(a_i) - u_i^{[E_i]}(\tilde{a}_i).$$

Hence, for every $a \in A(E)$,

$$a \in A^*(u) \text{ for all } u \in U(E) \text{ whenever } a \in A^*(u^{[E]}),$$

which, along with $h(\sigma(u^{[E]})) \in A(E)$, $h(\sigma(u^{[E]})) \in A^*(u^{[E]})$, and $h(\sigma(u^{[E]})) = h(\sigma(u))$ for all $u \in U(E)$, implies that for every $E \in \alpha(u)$ and every $u \in U(E)$,

$$h(\sigma(u)) \in A^*(u).$$

Q.E.D.

Let us define a *characteristic function* $W_u : 2^N \to R$ for a profile of valuation functions $u \in U$ by

$$W_u(\tilde{N}) = \max_{a \in A} \sum_{i \in \tilde{N}} u_i(a_i) \quad \text{for all} \quad \tilde{N} \subset N,$$

which implies that the maximal aggregate value that a coalition $\tilde{N} \bigcup \{0\}$ can achieve by excluding any buyer who does not belong to this coalition. Let us denote by $v = (v_i)_{i \in N \bigcup \{0\}} \in R$ a payoff vector, where v_0 implies the seller's payoff, and for each $i \in N$, v_i implies buyer $i's$ payoff. A payoff vector v is said to be *in the core for a profile of valuation functions* $u \in U$, if it is induced by an efficient allocation, and it is not blocked by any coalition, i.e.,

$$\sum_{i \in N \bigcup \{0\}} v_i = W_u(N),$$

and

$$\sum_{i \in \tilde{N} \bigcup \{0\}} v_i \geq W_u(\tilde{N}) \quad \text{for all} \quad \tilde{N} \subset N.$$

Let us denote by $v(H,u,s) = (v_i(H,u,s))_{i \in N \bigcup \{0\}} \in R$ the payoff vector induced by a message profile $s \in S$ in the game (H,u), where

$$v_0(H,u,s) \equiv \sum_{i \in N} v_i(h(s)),$$

and

$$v_i(H,u,s) \equiv u_i(h(s)) - v_i(h(s)) \quad \text{for all} \quad i \in N.$$

A strategy profile $\sigma \in \Sigma$ is said to be *compatible with the core in an indirect mechanism* H if for every $u \in U$, the induced payoff vector $v(H,u,\sigma(u))$ is in the core. The following proposition shows that under the restriction of the strictly price-based property, for a strategy profile compatible with the core, it is sufficient to examine the profiles of representative valuation functions.

Proposition 12: *Suppose that a combination of an indirect mechanism and a strategy profile* (H,σ) *is strictly price-based for a price-demand scheme* α*. Then,* σ *is compatible with the core in* H *if and only if for every* $E \in \alpha(U)$, $v(H,u^{[E]},\sigma(u^{[E]}))$ *is in the core for* $u^{[E]}$.

Proof: The "only if" part is straightforward from the definition of efficiency in terms of strategy profile. We can prove the "if" part in the same manner as the proof of Theorem 4. It is clear from Lemma 3 that for every $i \in N$, every $a_i \in A_i(E_i)$, every $\tilde{a}_i \in A_i$, and every $u_i \in U_i(E_i)$,

$$u_i^{[E_i]}(a_i) - u_i^{[E_i]}(\tilde{a}_i) \leq u_i(a_i) - u_i(\tilde{a}_i).$$

Hence, for every $a \in A(E)$, every $\tilde{a} \in A$, and every $\tilde{N} \subset N$, if

$$\sum_{i \in \tilde{N} \cup \{0\}} u_i^{[E_i]}(a_i) \geq \sum_{i \in \tilde{N}} u_i^{[E_i]}(\tilde{a}_i),$$

then

$$\sum_{i \in \tilde{N} \cup \{0\}} u_i(a_i) \geq \sum_{i \in \tilde{N}} u_i(\tilde{a}_i) \quad \text{for all} \quad u \in U(E).$$

This, along with $h(\sigma(u^{[E]})) \in A(E)$ and $h(\sigma(u^{[E]})) = h(\sigma(u))$ for all $u \in U(E)$, implies that for every $E \in \alpha(u)$, $v(H, u, \sigma(u))$ is in the core for any $u \in U(E)$ whenever $v(H, u^{[E]}, \sigma(u^{[E]}))$ is in the core for $u^{[E]}$.

Q.E.D.

10. Core-Selecting Mechanisms

An indirect mechanism H is said to be *core-selecting* if it is a direct mechanism and $v(H,u,u)$ is in the core for every $u \in U$. An example of a core-selecting mechanism is the first-price package auction addressed by Bernheim and Whinston (1986). There are many recent works in the combinatorial auction literature, such as Day and Raghavan (2007) and Day and Milgrom (2008) that investigated the general framework of core-selecting mechanism. It is clear from this literature that in any core-selecting mechanism, there exists a universal Nash equilibrium strategy profile that is compatible with the core. This strategy profile, however, is not necessarily price-based for a price-demand scheme. Hence, it is important to examine whether there exists a universal Nash equilibrium that is not only compatible with the core, but also price-based for the price-demand scheme. This section provides an affirmative answer.

For any universal Nash equilibrium strategy profile σ and any price-demand scheme α, let us specify an alternative strategy profile $\sigma^{[\alpha]}$ by

$$\sigma^{[\alpha]}(u) = \sigma^{[\alpha]}(u^{[E]}) \text{ for all } E \in \alpha(U) \text{ and all } u \in U(E).$$

Note that this specified strategy profile $\sigma^{[\alpha]}$ satisfies the price-based property for α. The following proposition shows a sufficient condition, under which, $\sigma^{[\alpha]}$ is also a universal Nash equilibrium that is compatible with the core. This condition, which is given by property (19) below, implies that the induced allocation $\sigma(u^{[E]}) \in A$ is always revealed in any profile of price-demand sets E.

Proposition 13: *Consider any price-demand scheme α, any core-selecting mechanism H, and any universal Nash equilibrium σ that is compatible with the core. Suppose that*

(19) $\qquad \sigma(u^{[E]}) \in A(E)$ *for all* $E \in \alpha(U)$.

Then, strategy profile $\sigma^{[\alpha]}$ is also a universal Nash equilibrium that is strictly price-based for α, and it is compatible with the core.

Proof: Property (19), along with the price-based property of $\sigma^{[\alpha]}$, implies that

$(H, \sigma^{[\alpha]})$ is strictly price-based for α. Since for every $E \in \alpha(U)$, $\sigma(u^{[E]})$ is a Nash equilibrium in $(H, u^{[E]})$, and $v(H, u^{[E]}, \sigma(u^{[E]}))$ is in the core for $u^{[E]}$, it follows from Propositions 10 and 12 that $\sigma^{[\alpha]}$ is a universal Nash equilibrium that is compatible with the core.

Q.E.D.

11. Conclusion

We investigated the problem of combinatorial auction design, where multiple items with multiple units are sold to the buyers who have quasi-linear and private valuations. Because of privacy and complexity, the auctioneer can only use partial information, which is collected through a price-based auction format. The auctioneer collects it by asking a limited number of price vectors to which the buyers provide their demand responses.

In a general setting with connectedness, we showed that for the existence of the price-based VCG mechanism, it is necessary and sufficient that the efficient allocation and the efficient allocations without any single buyer are all revealed in the observed price-demand sets. We demonstrated that for the existence of an efficient, strategy-proof, and price-based mechanism with participation constraints, it is necessary and sufficient that the efficient allocation is revealed. We showed a necessary and sufficient condition for the existence of strictly price-based mechanism that is approximately strategy-proof.

We also investigated a general class of indirect mechanisms, and showed sufficient conditions for the consistency of the price-based property with universal Nash equilibrium, efficiency, and core. Finally, we investigated core-selecting mechanisms, and presented a sufficient condition for the existence of a universal Nash equilibrium that is compatible with the core.

For making it tractable to examine whether these conditions are satisfied, the concept of representation valuation function played the central role. Consistent with any observed price-demand set, the representative valuation function assigns the minimal relative valuation to any revealed package. Because of this relative minimization, it is sufficient to examine the representative valuation functions in every aspect of this paper.

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