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Abstract

The information-based asset-pricing framework of Brody, Hughston and Macrina (BHM) is extended to include a wider class of models for market information. In the BHM framework, each asset is associated with a collection of random cash flows. The price of the asset is the sum of the discounted conditional expectations of the cash flows. The conditional expectations are taken with respect to a filtration generated by a set of 'information processes'. The information processes carry imperfect information about the cash flows. To model the flow of information, we introduce in this paper a class of processes which we term *Lévy random bridges* (LRBs). This class generalises the Brownian bridge and gamma bridge information processes considered by BHM. An LRB is defined over a finite time horizon. Conditioned on its terminal value, an LRB is identical in law to a Lévy bridge. We consider in detail the case where the asset generates a single cash flow X_T occurring at a fixed date T. The flow of market information about X_T is modelled by an LRB terminating at the date T with the property that the (random) terminal value of the LRB is equal to X_T. An explicit expression for the price process of such an asset is found by working out the discounted conditional expectation of X_T with respect to the natural filtration of the LRB. The prices of European options on such an asset are calculated.

1 Introduction and Preliminaries

In financial markets, the information that traders and investors have about an asset is reflected in its price. The arrival of new information then leads to changes in asset prices. The 'information-based framework' (or 'X-factor theory') of Brody, Hughston

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and Macrina (BHM) isolates the emergence of information, and examines its role as a driver of price dynamics (see [8, 10, 11, 38, 35, 31]). In the BHM framework, each asset is associated with a collection of random cash flows. The price of the asset is the sum of the discounted conditional expectations of the cash flows. The conditional expectations are taken with respect (i) an appropriate measure, and (ii) the filtration generated by a set of so-called information processes. The information processes carry noisy or imperfect market information about the cash flows. The present paper extends the work of [10] and [11] by introducing a wider class of information processes as a basis for the generation of the market filtration. The set-up is as follows:

We fix a probability space $(\Omega, \mathbb{Q}, \mathcal{F})$; we assume that all processes and filtrations are càdlàg; and, unless otherwise stated, when discussing a stochastic process we assume that the process takes values in $\mathbb{R}$, begins at time 0, and that the filtration under consideration is the filtration generated by the process. We will also be working in a finite time horizon, and T will be used without further introduction to represent the end of this horizon.

1.1 Lévy processes

This section and the next contain a few well known results about (1-dimensional) Lévy processes and stable processes, which can be found in Bertoin [7] and Sato [39]. A Lévy process is a stochastically-continuous process that starts from the value 0, and has stationary, independent increments. An increasing Lévy process is called a *subordinator*. For $\{L_t\}$ a Lévy process, its *characteristic exponent* $\Psi : \mathbb{R} \to \mathbb{C}$ is defined by

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\lambda L_t}] = \exp(-t\Psi(\lambda)), \qquad \lambda \in \mathbb{R}. \tag{1}$$

The characteristic exponent of a Lévy process characterises its law, and its form is prescribed by the Lévy-Khintchine formula:

$$\Psi(\lambda) = \mathrm{i}a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{-\infty}^{\infty} (1 - \mathrm{e}^{\mathrm{i}x\lambda} + \mathrm{i}x\lambda\mathbb{1}_{\{|x|<1\}})\Pi(\mathrm{d}x), \tag{2}$$

where $a \in \mathbb{R}$, $\sigma > 0$, and Π is a measure (the *Lévy measure*) on $\mathbb{R}\backslash\{0\}$ such that

$$\int_{-\infty}^{\infty} (1 \wedge |x|^2)\,\Pi(\mathrm{d}x) < \infty. \tag{3}$$

There are particular subclasses of Lévy processes that we consider in this work, defined as follows:

Definition 1.1. *Let $\{L_t\}_{0\leq t\leq T}$ and $\{M_t\}_{0\leq t\leq T}$ be Lévy processes. Then we write*

1. *$\{L_t\} \in \mathcal{C}[0,T]$ if the density of L_t exists for every $t \in (0,T]$,*

2. *$\{M_t\} \in \mathcal{D}$ if the marginal law of M_t is discrete for some $t > 0$.*

Remark 1.1. *If the marginal law of M_t is discrete for some $t > 0$, then the marginal law of M_t is discrete for all $t > 0$. The density of L_t exists if and only if its law is absolutely continuous with respect to the Lebesgue measure. In general, the absolute continuity of L_t depends on t* [39, chap. 5]*, and so $\mathcal{C}[0,T_1] \subseteq \mathcal{C}[0,T_2]$ for $T_1 \leq T_2$.*

We shall reserve the notation $f_t(x)$ to represent the density of L_t for some $\{L_t\} \in \mathcal{C}[0,T]$. Hence $f_t : \mathbb{R} \to \mathbb{R}_+$ and $\mathbb{Q}[L_t \in \mathrm{d}x] = f_t(x)\,\mathrm{d}x$. We reserve $Q_t(a)$ to represent the probability mass function of M_t for some $\{M_t\} \in \mathcal{D}$. We denote the state-space of $\{M_t\}$ by $\{a_i\} \subset \mathbb{R}$; hence $Q_t : \{a_i\} \to [0,1]$ and $\mathbb{Q}[M_t = a_i] = Q_t(a_i)$. We assume that the a_i's are strictly increasing in i.

The transition probabilities of Lévy processes satisfy the convolution identities

$$f_t(x) = \int_{-\infty}^{\infty} f_{t-s}(x-y) f_s(y)\,\mathrm{d}y \qquad \text{for } \{L_t\} \in \mathcal{C}[0,T], \tag{4}$$

and

$$Q_t(a_n) = \sum_{m=-\infty}^{\infty} Q_{t-s}(a_n - a_m) Q_s(a_m) \qquad \text{for } \{M_t\} \in \mathcal{D}, \tag{5}$$

where s, t satisfy $0 \leq s < t \leq T$. These are the Chapman-Kolmogorov equations for the processes $\{L_t\}$ and $\{M_t\}$.

The law of any càdlàg stochastic process is characterised by its finite-dimensional distributions. The finite-dimensional densities of $\{L_t\}_{0\leq t\leq T}$ exist and, with the understanding that $x_0 = t_0 = 0$, they are given by

$$\mathbb{Q}[L_{t_1} \in \mathrm{d}x_1, \ldots, L_{t_n} \in \mathrm{d}x_n] = \prod_{i=1}^{n} \left[f_{t_i - t_{i-1}}(x_i - x_{i-1})\,\mathrm{d}x_i \right], \tag{6}$$

for every $n \in \mathbb{N}_+$, every $0 < t_1 < \cdots < t_n \leq T$, and every $(x_1, \ldots, x_n) \in \mathbb{R}^n$. With the understanding that $a_{k_0} = t_0 = 0$, the finite-dimensional probabilities of $\{M_t\}$ are

$$\mathbb{Q}[M_{t_1} = a_{k_1}, \ldots, M_{t_n} = a_{k_n}] = \prod_{i=1}^{n} Q_{t_i - t_{i-1}}(a_{k_i} - a_{k_{i-1}}), \tag{7}$$

for every $n \in \mathbb{N}_+$, every $0 < t_1 < \cdots < t_n$, and every $(k_1, \ldots, k_n) \in \mathbb{Z}^n$.

1.2 Lévy bridges

A bridge process is a stochastic process that is pinned to some fixed point at a fixed future time. Bridges of Markov processes were constructed and analysed by Fitzsimmons *et al.* [21] in a general setting. In this section we focus on the bridges of Lévy processes in the classes $\mathcal{C}[0,T]$ and $\mathcal{D}$. The first result that we prove is that Lévy bridges are Markov processes.

Proposition 1.1. *The bridges of processes in $\mathcal{C}[0,T]$ and $\mathcal{D}$ are Markov processes.*

Proof. We need to the show that the process $\{L_t\} \in \mathcal{C}[0,T]$ is a Markov process when we know that $L_T = x$, for some constant x such that $0 < f_T(x) < \infty$. (It will be apparent shortly that the condition that $0 < f_T(x) < \infty$ is required to ensure that the law of the bridge process is well defined.) In other words, we need to show that

$$\mathbb{Q}\left[L_t \le y \,|\, L_{t_1} = x_1, \ldots, L_{t_m} = x_m, L_T = x\right] = \mathbb{Q}\left[L_t \le y \,|\, L_{t_m} = x_m, L_T = x\right], \qquad (8)$$

for all $m \in \mathbb{N}_+$, all $(x_1, \ldots, x_m, y) \in \mathbb{R}^{m+1}$, and all $0 \le t_1 < \cdots < t_m < t \le T$. The important property that we require of $\{L_t\}$ is that it has independent increments. Let us write

$$\begin{aligned}
\Delta_0 &= 0, && (9)\\
\Delta_i &= L_{t_i} - L_{t_{i-1}}, \qquad \text{for } 1 \le i \le m, && (10)\\
\delta_0 &= 0, && (11)\\
\delta_i &= x_i - x_{i-1}, \qquad \text{for } 1 \le i \le m. && (12)
\end{aligned}$$

Then we have:

$$\begin{aligned}
&\mathbb{Q}\left[L_t \le y \,|\, L_{t_1} = x_1, \ldots, L_{t_m} = x_m, L_T = x\right]\\
&\quad = \mathbb{Q}\left[L_t - L_{t_m} \le y - x_m \,|\, \Delta_1 = \delta_1, \ldots, \Delta_m = \delta_m, L_T - L_{t_m} = x - x_m\right]\\
&\quad = \mathbb{Q}\left[L_t - L_{t_m} \le y - x_m \,|\, L_T - L_{t_m} = x - x_m\right]\\
&\quad = \mathbb{Q}\left[L_t - L_{t_m} \le y - x_m \,|\, L_T - L_{t_m} = x - x_m, L_{t_m} = x_m\right]\\
&\quad = \mathbb{Q}\left[L_t \le y \,|\, L_T = x, L_{t_m} = x_m\right]. \qquad (13)
\end{aligned}$$

The proof for processes in class $\mathcal{D}$ is similar. □

Let $\{L_t\} \in \mathcal{C}[0,T]$, and let $\{L_{tT}^{(z)}\}_{0 \le t \le T}$ be an $\{L_t\}$-bridge to the value $z \in \mathbb{R}$ at time T. For the transition probabilities of the bridge process to be well defined, we require that $0 < f_T(z) < \infty$. Then by the Bayes theorem we have

$$\begin{aligned}
\mathbb{Q}\left[L_{tT}^{(z)} \in \mathrm{d}y \,\middle|\, L_{sT}^{(z)} = x\right] &= \mathbb{Q}\left[L_t \in \mathrm{d}y \,|\, L_s = x, L_T = z\right]\\
&= \frac{\mathbb{Q}\left[L_t \in \mathrm{d}y, L_T \in \mathrm{d}z \,|\, L_s = x\right]}{\mathbb{Q}\left[L_T \in \mathrm{d}z \,|\, L_s = x\right]}\\
&= \frac{f_{t-s}(y-x) f_{T-t}(z-y)}{f_{T-s}(z-x)}\,\mathrm{d}y, \qquad (14)
\end{aligned}$$

where s, t satisfy $0 \le s < t < T$. We define the marginal bridge density $f_{tT}(y;z)$ by

$$f_{tT}(y;z) = \frac{f_t(y) f_{T-t}(z-y)}{f_t(z)}. \qquad (15)$$

In this way

$$\mathbb{Q}\left[L_{tT}^{(z)} \in \mathrm{d}y \,\middle|\, L_{sT}^{(z)} = x\right] = f_{t-s,T-s}(y-x; z-x)\,\mathrm{d}y. \qquad (16)$$

The condition $0 < f_T(z) < \infty$ is enough to ensure that

$$y \mapsto f_{t-s,T-s}(y - L_{sT}^{(z)}; z - L_{sT}^{(z)}) \tag{17}$$

is a well defined density for almost every value of $L_{sT}^{(z)}$. To see this, note that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{t-s,T-s}(y-x;z-x)\,\mathbb{Q}\left[L_{sT}^{(z)}\in \mathrm{d}x\right]\mathrm{d}y \\
&\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{t-s,T-s}(y-x;z-x) f_{sT}(x;z)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_{-\infty}^{\infty} \frac{f_{T-t}(z-y)}{f_T(z)} \int_{-\infty}^{\infty} f_{t-s}(y-x) f_s(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \frac{1}{f_T(z)} \int_{-\infty}^{\infty} f_{T-t}(z-y) f_t(y)\,\mathrm{d}y = 1.
\end{aligned} \tag{18}$$

From which it follows that

$$\mathbb{Q}\left[\int_{-\infty}^{\infty} f_{t-s,T-s}(y - L_{sT}^{(z)}; z - L_{sT}^{(z)})\,\mathrm{d}y = 1\right] = 1. \tag{19}$$

Let $\{M_t\} \in \mathcal{D}$, and let $\{M_{tT}^{(k)}\}_{0\leq t\leq T}$ be an $\{M_t\}$-bridge to the value a_k at time T, so $\mathbb{Q}[M_{TT}^{(k)} = a_k] = 1$. For the transition probabilities of the bridge process to be well defined, we require that $\mathbb{Q}[M_T = a_k] = Q_T(a_k) > 0$. Then the classical Bayes theorem gives

$$\begin{aligned}
\mathbb{Q}\left[M_{tT}^{(k)} = a_j \,\middle|\, M_{sT}^{(k)} = a_i\right] &= \mathbb{Q}\left[M_t = a_j \mid M_s = a_i, M_T = a_k\right] \\
&= \frac{\mathbb{Q}\left[M_t = a_j, M_T = a_k \mid M_s = a_i\right]}{\mathbb{Q}\left[M_T = a_k \mid M_s = a_i\right]} \\
&= \frac{Q_{t-s}(a_j - a_i) Q_{T-t}(a_k - a_j)}{Q_{T-s}(a_k - a_i)},
\end{aligned} \tag{20}$$

where s, t satisfy $0 \leq s < t < T$. Note that if $Q_T(a_k) = 0$, then the ratio (20) is not well defined when $s = 0$.

2 Lévy random bridges

The idea of information-based asset pricing is to model the flow of information in financial markets and to hence construct the market filtration explicitly. Let X_T be a random variable (a market factor), with some given *a priori* distribution, whose value will be revealed to a market at time T. We wish to construct an information process $\{\xi_{tT}\}$ such that $\xi_{TT} = X_T$. In this way we can use the filtration generated by $\{\xi_{tT}\}$ to model the information that market participants have about X_T. One problem that must be overcome is how to ensure that the marginal law of ξ_{TT} is the same as the *a priori* law of X_T.

Two explicit forms for the information process have been considered in the literature. The first is

$$\xi_{tT} = \frac{t}{T} X_T + \beta_{tT} \qquad (0 \le t \le T), \tag{21}$$

where $\{\beta_{tT}\}_{0\le t\le T}$ is a Brownian bridge starting and ending at the value 0 (see [8, 10, 31, 35, 9, 38]). The second is

$$\xi_{tT} = X_T \gamma_{tT} \qquad (0 \le t \le T), \tag{22}$$

where $\{\gamma_{tT}\}_{0\le t\le T}$ is a gamma bridge starting at the value 0 and ending at the value 1 (in this case we require $X_T > 0$) (see [11]). These forms share the property that each is identical in law to a Lévy process conditioned to have the *a priori* law of X_T at time T. The Brownian bridge information process is identical in law to a conditioned Brownian motion, and the gamma bridge information process is identical in law to a conditioned gamma process.

With this as motivation, in this section we define a class of processes that we call Lévy random bridges (LRBs). An LRB is identical in law to a Lévy process conditioned to have a prespecified marginal law at time T. Later in this work we will use LRBs as information processes in information-based models.

2.1 Defining LRBs

An LRB can be described as a process whose bridge laws are Lévy bridge laws. In the definitions below we prefer to define LRBs by reference to their finite-dimensional distributions rather than as conditioned Lévy processes. This proves convenient in future calculations.

Definition 2.1. *We say that the process $\{L_{tT}\}_{0\le t\le T}$ has law $LRB_{\mathcal{C}}([0,T],\{f_t\},\nu)$ if the following conditions are satisfied:*

1. *L_{TT} has marginal law ν.*
2. *There exists a Lévy process $\{L_t\} \in \mathcal{C}[0,T]$ such that L_t has density $f_t(x)$ for all $t \in (0,T]$.*
3. *ν concentrates mass where $f_T(z)$ is positive and finite, i.e. $0 < f_T(z) < \infty$ for ν-a.e. z.*
4. *For every $n \in \mathbb{N}_+$, every $0 < t_1 < \cdots < t_n < T$, every $(x_1, \ldots, x_n) \in \mathbb{R}^n$, and ν-a.e. z, we have*

$$\mathbb{Q}\left[L_{t_1,T} \le x_1, \ldots, L_{t_n,T} \le x_n \,|\, L_{TT} = z\right] = \mathbb{Q}\left[L_{t_1} \le x_1, \ldots, L_{t_n} \le x_n \,|\, L_T = z\right].$$

Definition 2.2. *We say that the process* $\{M_{tT}\}_{0\leq t\leq T}$ *has law* $LRB_{\mathcal{D}}([0,T],\{Q_t\},P)$ *if the following conditions are satisfied:*

1. M_{TT} *has probability mass function* P.
2. *There exists a Lévy process* $\{M_t\}\in\mathcal{D}$ *such that* M_t *has marginal probability mass function* $Q_t(a)$ *for all* $t\in(0,T]$.
3. *The law of* M_{TT} *is absolutely continuous with respect to the law of* M_T, *i.e.*

$$\text{if } P(a)>0 \text{ then } Q_T(a)>0.$$

4. *For every* $n\in\mathbb{N}_+$, *every* $0<t_1<\cdots<t_n<T$, *every* $(k_1,\ldots,k_n)\in\mathbb{Z}^n$, *and every* b *such that* $P(b)>0$, *we have*

$$\mathbb{Q}\left[L_{t_1,T}=a_{k_1},\ldots,L_{t_n,T}=a_{k_n}\,|\,L_{TT}=b\right]=\mathbb{Q}\left[L_{t_1}=a_{k_1},\ldots,L_{t_n}=a_{k_n}\,|\,L_T=b\right].$$

Definition 2.3. *For a fixed time* $s<T$, *if the law of the process* $\{\eta_{s+t}\}_{0\leq t\leq T-s}$ *is of the type* $LRB_{\cdot}([0,T-s],\cdot,\cdot)$, *then we say that* $\{\eta_t\}_{s\leq t\leq T}$ *has law* $LRB_{\cdot}([s,T],\cdot,\cdot)$.

If the law of a process is one of the *LRB*-types defined above, then we say that it is a Lévy random bridge (LRB).

2.2 Finite-dimensional distributions

For the rest of this section we assume that $\{L_{tT}\}$ and $\{M_{tT}\}$ are LRBs with laws $LRB_{\mathcal{C}}([0,T],\{f_t\},\nu)$ and $LRB_{\mathcal{D}}([0,T],\{Q_t\},P)$, respectively. We also assume that $\{L_t\}$ is a Lévy process such that L_t has density $f_t(x)$ for $t\leq T$, and $\{M_t\}$ is a Lévy process such that M_t has probability mass function $Q_t(a_i)$ for $t\leq T$.

The finite dimensional distributions of $\{L_{tT}\}$ are given by

$$\mathbb{Q}\left[L_{t_1,T}\in \mathrm{d}x_1,\ldots,L_{t_n,T}\in\mathrm{d}x_n,L_{TT}\in\mathrm{d}z\right]=\prod_{i=1}^{n}\left[f_{t_i-t_{i-1}}(x_i-x_{i-1})\,\mathrm{d}x_i\right]\psi_{t_n}(\mathrm{d}z;x_n), \tag{23}$$

where the (un-normalised) measure $\psi_t(\mathrm{d}z;\xi)$ is given by

$$\psi_0(\mathrm{d}z;\xi)=\nu(\mathrm{d}z), \tag{24}$$

$$\psi_t(\mathrm{d}z;\xi)=\frac{f_{T-t}(z-\xi)}{f_T(z)}\nu(\mathrm{d}z) \tag{25}$$

for $0<t<T$. It follows from the definition of $LRB_{\mathcal{C}}([0,T],\{f_t\},\nu)$ and equation (19) that

$$f_{tT}(x;z)=\frac{f_t(x)f_{T-t}(z-x)}{f_T(z)} \tag{26}$$

is a well-defined density (as a function of x) for $t < T$ and ν-a.e. z. Then from (23) the marginal law of L_{tT} is given by

$$\mathbb{Q}[L_{tT} \in \mathrm{d}x] = f_t(x)\psi_t(\mathbb{R}; x)\,\mathrm{d}x = \int_{z=-\infty}^{\infty} f_{tT}(x; z)\,\nu(\mathrm{d}z)\,\mathrm{d}x. \tag{27}$$

Hence the density of L_{tT} exists for $t < T$, and

$$0 \leq \psi_t(\mathbb{R}; x) < \infty \qquad \text{for Lebesgue-a.e. } x \in \text{Support}(f_t). \tag{28}$$

In particular, we have

$$0 < \psi_t(\mathbb{R}; L_{tT}) < \infty \qquad \text{and} \qquad 0 < f_{T-t}(x - L_{tT}) < \infty \tag{29}$$

for a.e. value of L_{tT}. If $\nu(\{z\}) = 1$ for some point $z \in \mathbb{R}$, i.e. $\mathbb{Q}[L_{TT} = z] = 1$, then $\{L_{tT}\}$ is a Lévy bridge. If $\nu(\mathrm{d}z) = f_T(z)\,\mathrm{d}z$, then $\{L_{tT}\} \overset{\text{law}}{=} \{L_t\}$ for $t \in [0, T]$.

In the discrete case, the finite dimensional probabilities of $\{M_{tT}\}$ are

$$\mathbb{Q}\left[M_{t_1,T} = a_{k_1}, \ldots, M_{t_n,T} = a_{k_n}, M_{TT} = z\right] = \prod_{i=1}^{n} \left[Q_{t_i - t_{i-1}}(a_{k_i} - a_{k_{1-1}})\right] \phi_{t_n}(z; a_{k_n}), \tag{30}$$

where the function $\phi_t(z; \xi)$ is given by

$$\phi_0(z; \xi) = P(z), \tag{31}$$

$$\phi_t(z; \xi) = \frac{Q_{T-t}(z - \xi)}{Q_T(z)} P(z) \tag{32}$$

for $0 < t < T$. If P is identical to Q_T, then $\{M_{tT}\} \overset{\text{law}}{=} \{M_t\}$ for $t \in [0, T]$.

The existing literature on information-based asset pricing exploits special properties Brownian and gamma bridges. See Émery & Yor [17] for an insight into how remarkable these bridges are. The methods we use do not require special properties of particular Lévy bridges. However, we will often use the Brownian and gamma cases as examples, and the results we obtain agree with previous work.

Many of the results that follow are proved for the LRB $\{L_{tT}\}$, which has a continuous state-space. Analogous results are provided for the discrete state-space process $\{M_{tT}\}$; details of proofs are omitted since they are similar to the continuous case.

2.3 LRBs as conditioned Lévy processes

For some purposes it is useful to interpret an LRB as a Lévy process conditioned to have a specified marginal law ν at time T. Suppose that the random variable Z has law ν; then we have

$$\begin{aligned}
&\mathbb{Q}\left[L_{t_1} \in \mathrm{d}x_1, \ldots, L_{t_n} \in \mathrm{d}x_n, L_T \in \mathrm{d}z \,|\, L_T = Z\right] \\
&\quad = \mathbb{Q}\left[L_{t_1} \in \mathrm{d}x_1, \ldots, L_{t_n} \in \mathrm{d}x_n \,|\, L_T = z\right] \nu(\mathrm{d}z) \\
&\quad = \frac{f_{T-t_{n-1}}(z - x_{n-1})}{f_T(z)} \prod_{i=1}^{n} \left[f_{t_i - t_{i-1}}(x_i - x_{i-1})\,\mathrm{d}x_i\right] \nu(\mathrm{d}z).
\end{aligned} \tag{33}$$

Hence the conditioned Lévy process has law $LRB_{\mathcal{C}}([0,T],\{f_t\},\nu)$.

2.4 The Markov property

In this section we show that LRBs are Markov processes. The Markov property will be a key tool in the application of LRBs to information-based asset pricing. As will be seen in the proof below, the Markov property of an LRB follows from the Markov property from the associated Lévy bridge processes.

2.4.1 Continuous state-space

Proposition 2.1. *The process $\{L_{tT}\}_{0\leq t\leq T}$ is a Markov process with transition law*

$$\begin{aligned}\mathbb{Q}[L_{tT}\in \mathrm{d}y\,|\,L_{sT}=x]&=\frac{\psi_t(\mathbb{R};y)}{\psi_s(\mathbb{R};x)}f_{t-s}(y-x)\,\mathrm{d}y,\\ \mathbb{Q}[L_{TT}\in \mathrm{d}y\,|\,L_{sT}=x]&=\frac{\psi_s(\mathrm{d}y;x)}{\psi_s(\mathbb{R};x)},\end{aligned}\tag{34}$$

where $0\leq s<t<T$.

Proof. To show that $\{L_{tT}\}$ is Markov, it is sufficient to show that

$$\mathbb{Q}\left[L_{tT}\leq y\,|\,L_{t_1,T}=x_1,\ldots,L_{t_m,T}=x_m\right]=\mathbb{Q}\left[L_{tT}\leq y\,|\,L_{t_m,T}=x_m\right],\tag{35}$$

for all $m\in\mathbb{N}_+$, all $(x_1,\ldots,x_m,y)\in\mathbb{R}^{m+1}$, and all $0\leq t_1<\cdots<t_m<t\leq T$. When $t=T$ we apply the Bayes theorem to (23) and obtain

$$\mathbb{Q}\left[L_{TT}\in \mathrm{d}y\,|\,L_{t_1,T}=x_1,\ldots,L_{t_m,T}=x_m\right]=\frac{\psi_{t_m}(\mathrm{d}y;x_m)}{\psi_{t_m}(\mathbb{R};x_m)}.\tag{36}$$

We need now only consider the case $t<T$. Proposition 1.1 shows that Lévy bridges are Markov processes; therefore,

$$\mathbb{Q}\left[L_t\leq y\,|\,L_{t_1}=x_1,\ldots,L_{t_m}=x_m,L_T=x\right]=\mathbb{Q}\left[L_t\leq y\,|\,L_{t_m}=x_m,L_T=x\right].\tag{37}$$

It is straightforward by Definition 2.2 part 4 to show that LRBs are Markov processes. Indeed we have:

$$\begin{aligned}&\mathbb{Q}\left[L_{tT}\leq y\,|\,L_{t_1,T}=x_1,\ldots,L_{t_m,T}=x_m\right]\\ &\quad=\int_{-\infty}^{\infty}\mathbb{Q}\left[L_{tT}\leq y\,|\,L_{t_1,T}=x_1,\ldots,L_{t_m,T}=x_m,L_{T,T}=x\right]\nu(\mathrm{d}x)\\ &\quad=\int_{-\infty}^{\infty}\mathbb{Q}\left[L_{t}\leq y\,|\,L_{t_1}=x_1,\ldots,L_{t_m}=x_m,L_{T}=x\right]\nu(\mathrm{d}x)\\ &\quad=\int_{-\infty}^{\infty}\mathbb{Q}\left[L_{t}\leq y\,|\,L_{t_m}=x_m,L_{T}=x\right]\nu(\mathrm{d}x)\\ &\quad=\int_{-\infty}^{\infty}\mathbb{Q}\left[L_{tT}\leq y\,|\,L_{t_m,T}=x_m,L_{T,T}=x\right]\nu(\mathrm{d}x)\\ &\quad=\mathbb{Q}\left[L_{tT}\leq y\,|\,L_{t_m,T}=x_m\right].\end{aligned}\tag{38}$$

The form of the transition law of $\{L_{tT}\}$ appearing in (34) follows from (23). $\square$

Example. In the Brownian case we set

$$f_t(z) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{z^2}{2t}\right] \tag{39}$$

for $t > 0$. Thus $f_t(x)$ is the marginal density of standard Brownian motion at time t. Then we have

$$\mathbb{Q}[L_{tT} \in \mathrm{d}y \,|\, L_{sT} = x] = \sqrt{\frac{T-s}{T-t}} \frac{\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\left[\frac{(z-y)^2}{T-t} - \frac{z^2}{T}\right]} \nu(\mathrm{d}z)}{\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\left[\frac{(z-x)^2}{T-s} - \frac{z^2}{T}\right]} \nu(\mathrm{d}z)} \frac{\mathrm{e}^{-\frac{1}{2}\frac{(y-x)^2}{t-s}}}{\sqrt{2\pi(t-s)}} \, \mathrm{d}y, \tag{40}$$

and

$$\mathbb{Q}[L_{TT} \in \mathrm{d}y \,|\, L_{sT} = x] = \frac{\mathrm{e}^{-\frac{1}{2}\left[\frac{(y-x)^2}{T-s} - \frac{y^2}{T}\right]} \nu(\mathrm{d}y)}{\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\left[\frac{(z-x)^2}{T-s} - \frac{z^2}{T}\right]} \nu(\mathrm{d}z)} = \frac{\mathrm{e}^{\frac{1}{T-s}\left[xy - \frac{1}{2}\frac{s}{T}y^2\right]} \nu(\mathrm{d}y)}{\int_{-\infty}^{\infty} \mathrm{e}^{\frac{1}{T-s}\left[xz - \frac{1}{2}\frac{s}{T}z^2\right]} \nu(\mathrm{d}z)}. \tag{41}$$

Example. In the gamma case we consider a one-parameter family of processes indexed by $m > 0$. We set

$$f_t(z) = \mathbb{1}_{\{z>0\}} \frac{z^{mt-1}}{\Gamma[mt]} \mathrm{e}^{-z}, \tag{42}$$

where $\Gamma[z]$ is the gamma function. These densities are the increment densities of the gamma process with mean rate m and variance rate m (see Brody *et al.* [11]). Then

$$\begin{aligned} &\mathbb{Q}[L_{tT} \in \mathrm{d}y \,|\, L_{sT} = x] \\ &\quad = \frac{\mathbb{1}_{\{y>x\}}}{\mathrm{B}[m(T-t), m(t-s)]} \frac{\int_y^{\infty} (z-y)^{m(T-t)-1} z^{1-mT} \, \nu(\mathrm{d}z)}{\int_x^{\infty} (z-x)^{m(T-s)-1} z^{1-mT} \, \nu(\mathrm{d}z)} (y-x)^{m(t-s)-1} \, \mathrm{d}y, \end{aligned} \tag{43}$$

and

$$\mathbb{Q}[L_{TT} \in \mathrm{d}y \,|\, L_{sT} = x] = \frac{\mathbb{1}_{\{y>x\}} (y-x)^{m(T-s)-1} y^{1-mT} \, \nu(\mathrm{d}y)}{\int_x^{\infty} (z-x)^{m(T-s)-1} z^{1-mT} \, \nu(\mathrm{d}z)}, \tag{44}$$

where

$$\mathrm{B}[\alpha, \beta] = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \, \mathrm{d}x = \frac{\Gamma[\alpha]\Gamma[\beta]}{\Gamma[\alpha+\beta]} \tag{45}$$

is the Beta function.

2.4.2 Discrete state-space

The analogous result to Proposition 2.1 for the discrete case is provided below—the proof is similar.

Proposition 2.2. *The process $\{M_{tT}\}_{0\leq t\leq T}$ has the Markov property, with transition probabilities given by*

$$\begin{aligned}\mathbb{Q}\left[M_{tT}=a_j \,|\, M_{sT}=a_i\right] &= \frac{\sum_{k=-\infty}^{\infty}\phi_t(a_k;a_j)}{\sum_{k=-\infty}^{\infty}\phi_s(a_k;a_i)}Q_{t-s}(a_j-a_i),\\ \mathbb{Q}\left[M_{TT}=a_j \,|\, M_{sT}=a_i\right] &= \frac{\phi_s(a_j;a_i)}{\sum_{k=-\infty}^{\infty}\phi_s(a_k;a_i)},\end{aligned} \tag{46}$$

where $0\leq s<t<T$.

2.5 Conditional terminal distributions

Let $\{\mathcal{F}_t^L\}$ and $\{\mathcal{F}_t^M\}$ be the filtrations generated by $\{L_{tT}\}$ and $\{M_{tT}\}$, respectively.

Definition 2.4. *Let ν_s to be the $\mathcal{F}_s^L$-conditional law of the terminal value L_{TT}, and let P_s to be the $\mathcal{F}_s^M$-conditional probability mass function of the terminal value M_{TT}.*

We have $\nu_0(A)=\nu(A)$, and $P_0(a)=P(a)$. Furthermore, when $s>0$, it follows from the results of the previous section that

$$\nu_s(\mathrm{d}z)=\frac{\psi_s(\mathrm{d}z;L_{sT})}{\psi_s(\mathbb{R};L_{sT})}, \tag{47}$$

and

$$P_s(a_k)=\frac{\phi_s(a_k;M_{sT})}{\sum_{j=-\infty}^{\infty}\phi_s(a_j;M_{sT})}. \tag{48}$$

When the *a priori* qth moment of L_{TT} is finite, the $\mathcal{F}_s^L$-conditional qth moment is finite and given by

$$\int_{-\infty}^{\infty}|z|^q\,\nu_s(\mathrm{d}z). \tag{49}$$

Similarly, when the *a priori* qth moment of M_{TT} is finite, the $\mathcal{F}_s^M$-conditional qth moment is finite and given by

$$\sum_{k=-\infty}^{\infty}|a_k|^q\,P_s(a_k). \tag{50}$$

When they are finite, the quantities in equations (49) and (50) are martingales (with respect to $\{\mathcal{F}_t^L\}$ and $\{\mathcal{F}_t^M\}$, respectively). If $q\in\mathbb{Z}$ then $\int|z|^q\,\nu(\mathrm{d}z)<\infty$ ensures that $\int z^q\,\nu(\mathrm{d}z)$ is a martingale, and $\sum |a_k|^q P(a_k) < \infty$ ensures that $\sum a_k^q\,P(a_k)$ is a martingale.

When the terminal law ν admits a density, we denote the density by $p(z)$, i.e. $\nu(\mathrm{d}z) = p(z)\,\mathrm{d}z$. In this case the L_{tT}-conditional density of L_{TT} exists, and we denote it by

$$p_t(z) = \frac{\nu_t(\mathrm{d}z)}{\mathrm{d}z} = \frac{f_{T-t}(z - L_{tT})p(z)}{\psi_t(\mathbb{R}; L_{tT}) f_T(z)}. \tag{51}$$

2.6 Measure changes

In this section we assume that there exists a measure $\mathbb{L}$ under which $\{L_{tT}\}$ is a Lévy process, and that the density of L_{tT} is $f_t(x)$. Writing $\psi_t = \psi_t(\mathbb{R}; L_{tT})$, we can show that $\{\psi_t\}_{0\leq t<T}$ is an $\mathbb{L}$-martingale (with respect to the filtration generated by $\{L_{tT}\}$). In particular, for times s, t satisfying $0 \leq s < t$, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{L}}\left[\psi_t \,\middle|\, \mathcal{F}_s^L\right] &= \mathbb{E}_{\mathbb{L}}\left[\int_{-\infty}^{\infty} \frac{f_{T-t}(z - L_{tT})}{f_T(z)}\,\nu(\mathrm{d}z) \,\middle|\, \mathcal{F}_s^L\right] \\
&= \mathbb{E}_{\mathbb{L}}\left[\int_{-\infty}^{\infty} \frac{f_{T-t}(z - L_{sT} - (L_{tT} - L_{sT}))}{f_T(z)}\,\nu(\mathrm{d}z) \,\middle|\, L_{sT}\right] \\
&= \int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty} \frac{f_{T-t}(z - L_{sT} - y)}{f_T(z)}\,\nu(\mathrm{d}z)\, f_{t-s}(y)\,\mathrm{d}y \\
&= \int_{z=-\infty}^{\infty} \frac{1}{f_T(z)} \int_{y=-\infty}^{\infty} f_{T-t}(z - L_{sT} - y) f_{t-s}(y)\,\mathrm{d}y\,\nu(\mathrm{d}z) \\
&= \int_{z=-\infty}^{\infty} \frac{f_{T-s}(z - L_{sT})}{f_T(z)}\,\nu(\mathrm{d}z) \\
&= \psi_s.
\end{aligned} \tag{52}$$

Since $\psi_0 = 1$, we can define a probability measure $\mathbb{L}^{\mathrm{rb}}$ by the Radon-Nikodým derivative

$$\left.\frac{\mathrm{d}\mathbb{L}^{\mathrm{rb}}}{\mathrm{d}\mathbb{L}}\right|_{\mathcal{F}_t^L} = \psi_t \qquad \text{for } 0 \leq t < T. \tag{53}$$

It was noted in Section 2.2 that $0 < \psi_t < \infty$, so $\mathbb{L}^{\mathrm{rb}}$ is equivalent to $\mathbb{L}$ for $t < T$. For s, t satisfying $0 \leq s < t < T$, the transition law of $\{L_{tT}\}$ under $\mathbb{L}^{\mathrm{rb}}$ is

$$\begin{aligned}
\mathbb{L}^{\mathrm{rb}}\left[L_{tT} \in \mathrm{d}y \,|\, L_{sT}\right] &= \mathbb{E}_{\mathbb{L}^{\mathrm{rb}}}\left[\mathbb{1}_{\{L_{tT}\in \mathrm{d}y\}} \,|\, L_{sT}\right] \\
&= \psi_s^{-1}\,\mathbb{E}_{\mathbb{L}}\left[\psi_t \mathbb{1}_{\{L_{tT}\in \mathrm{d}y\}} \,|\, L_{sT}\right] \\
&= \psi_s^{-1} \int_{-\infty}^{\infty} \frac{f_{T-t}(z - y)}{f_T(z)}\,\nu(\mathrm{d}z)\, f_{t-s}(y - L_{sT})\,\mathrm{d}y \\
&= \frac{\psi_t}{\psi_s} f_{t-s}(y - L_{sT})\,\mathrm{d}y.
\end{aligned} \tag{54}$$

We see that $\{L_{tT}\}_{0\leq t<T}$ is a Markov process under the measure $\mathbb{L}^{\mathrm{rb}}$. Furthermore, by virtue of Proposition 2.1, $\{L_{tT}\}$ is an LRB with law $LRB_{\mathcal{C}}([0, T], \{f_t\}, \nu)$.

We can restate this result with reference to the measure $\mathbb{Q}$ as the following:

Proposition 2.3. *Let* $\mathbb{L}$ *be defined by*

$$\left.\frac{\mathrm{d}\mathbb{L}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_t^L} = \psi_t(\mathbb{R}; L_{tT})^{-1} \tag{55}$$

for $t \in [0, T)$*. Then* $\mathbb{L}$ *is a probability measure. Under* $\mathbb{L}$*,* $\{L_{tT}\}_{0\le t<T}$ *is a Lévy process, and* L_{tT} *has density* $f_t(x)$*.*

In the case of a discrete state-space a similar result is obtained.

Proposition 2.4. *Let* $\mathbb{L}$ *be defined by*

$$\left.\frac{\mathrm{d}\mathbb{L}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_t^M} = \left[\sum_{k=-\infty}^{\infty} \phi_t(a_k; M_{tT})\right]^{-1} \tag{56}$$

for $t \in [0, T)$*. Then* $\mathbb{L}$ *is a probability measure. Under* $\mathbb{L}$*,* $\{M_{tT}\}_{0\le t<T}$ *is a Lévy process, and* M_{tT} *has mass function* $Q_t(a)$*.*

2.7 Dynamic consistency

Fix a time s less than T. Given L_{sT}, we define a process $\{\eta_t\}$ by setting

$$\eta_t = L_{tT} - L_{sT} \qquad (s \le t \le T). \tag{57}$$

We shall show that $\{\eta_t\}$ is an LRB. At time s, the law of η_T is

$$\nu^*(A) = \nu_s(A + L_{sT}) \qquad \text{for all } A \in \mathcal{B}(\mathbb{R}), \tag{58}$$

where $A + y$ denotes the shifted set given by

$$A + y = \{x : x - y \in A\}\,. \tag{59}$$

Given the terminal value η_T, the finite-dimensional distributions of $\{\eta_t\}$ are given by

$$\begin{aligned}
&\mathbb{Q}\left[\eta_{s+t_1} \in \mathrm{d}x_1, \dots, \eta_{s+t_n} \in \mathrm{d}x_n \,|\, L_{sT}, \eta_T = z\right] \\
&\quad = \mathbb{Q}\left[L_{s+t_1,T} - L_{sT} \in \mathrm{d}x_1, \dots, L_{s+t_n,T} - L_{sT} \in \mathrm{d}x_n \,|\, L_{sT}, L_{TT} - L_{sT} = z\right] \\
&\quad = \mathbb{Q}\left[L_{s+t_1} - L_s \in \mathrm{d}x_1, \dots, L_{s+t_n} - L_s \in \mathrm{d}x_n \,|\, L_s, L_T - L_s = z\right] \\
&\quad = \mathbb{Q}\left[L_{t_1} \in \mathrm{d}x_1, \dots, L_{t_n} \in \mathrm{d}x_n \,|\, L_{T-s} = z\right] \\
&\quad = \frac{f_{T-s-t_n}(z - x_n)}{f_{T-s}(z)} \prod_{i=1}^{n} f_{t_i - t_{i-1}}(x_i - x_{i-1})\,,
\end{aligned} \tag{60}$$

for every $n \in \mathbb{N}_+$, every $0 = t_0 < t_1 < \cdots < t_n < T - s$, and every $(x_1, \dots, x_n) \in \mathbb{R}^n$, where $x_0 = 0$. Then we have

$$\begin{aligned}
&\mathbb{Q}\left[\eta_{s+t_1} \in \mathrm{d}x_1, \dots, \eta_{s+t_n} \in \mathrm{d}x_n, \eta_T \in \mathrm{d}z \,|\, L_{sT}\right] \\
&\qquad\qquad = \frac{f_{T-s-t_n}(z - x_n)}{f_{T-s}(z)} \prod_{i=1}^{n} f_{t_i - t_{i-1}}(x_i - x_{i-1})\; \nu^*(\mathrm{d}z).
\end{aligned} \tag{61}$$

Comparison of this expression to (23) shows that $\{\eta_{s+t}\}_{0\leq t\leq T-s}$ has law $LRB_{\mathcal{C}}([0,T-s],\{f_t\},\nu^*)$, and so the law of $\{\eta_t\}_{s\leq t\leq T}$ is $LRB_{\mathcal{C}}([s,T],\{f_t\},\nu^*)$.

In the discrete case, we define $\{\eta_t\}$ by

$$\eta_t = M_{tT} - M_{sT} \qquad (s \leq t \leq T). \tag{62}$$

Then, given M_{sT}, $\{\eta_t\}$ has law $LRB_{\mathcal{D}}([s,T],\{Q_t\},P^*)$, where P^* is defined by

$$P^*(a) = P_s(a + M_{sT}). \tag{63}$$

2.8 Increments of LRBs

The form of the transition law in Proposition 2.1 shows that in general the increments of an LRB are not independent. The special cases of LRBs with independent increments are discussed later. A result that holds for all LRBs is that they have stationary increments:

Proposition 2.5. *For s,t,u satisfying $0\leq s<u<T$ and $0<t<T-u$, we have*

$$\mathbb{Q}\left[L_{u+t,T} - L_{uT} \leq z \,|\, L_{sT}\right] = \mathbb{Q}[L_{s+t,T} - L_{sT} \leq z \,|\, L_{sT}], \tag{64}$$

and

$$\mathbb{Q}\left[M_{u+t,T} - M_{uT} \leq z \,|\, M_{sT}\right] = \mathbb{Q}[M_{s+t,T} - M_{sT} \leq z \,|\, M_{sT}]. \tag{65}$$

Proof. We only provide the proof for $\{L_{tT}\}$ since the proof for $\{M_{tT}\}$ is similar. First we assume that $s=0$. From (34), we have

$$\mathbb{Q}[L_{u+t,T} \in \mathrm{d}y, L_{uT} \in \mathrm{d}x] = \psi_{u+t}(\mathbb{R};y) f_t(y-x) f_u(x)\,\mathrm{d}x\,\mathrm{d}y. \tag{66}$$

Then we have

$$\begin{aligned}\mathbb{Q}[L_{u+t,T} - L_{uT} \in \mathrm{d}z, L_{uT} \in \mathrm{d}x] &= \psi_{u+t}(\mathbb{R}; z+x) f_t(z) f_u(x)\,\mathrm{d}x\,\mathrm{d}z \\ &= \int_{w=-\infty}^{\infty} \frac{f_{T-(u+t)}(w-z-x)}{f_T(w)}\,\mathrm{d}w\, f_t(z) f_u(x)\,\mathrm{d}x\,\mathrm{d}z.\end{aligned} \tag{67}$$

Integrating over x and changing the order of integration yields

$$\begin{aligned}\mathbb{Q}[L_{u+t,T} - L_{uT} \in \mathrm{d}z] &= \int_{w=-\infty}^{\infty}\int_{x=-\infty}^{\infty} f_{T-(u+t)}(w-z-x) f_u(x)\,\mathrm{d}x\,\frac{\mathrm{d}w}{f_T(w)}\, f_t(z)\,\mathrm{d}z \\ &= \int_{w=-\infty}^{\infty} \frac{f_{T-t}(w-z)}{f_T(w)}\,\mathrm{d}w\, f_t(z)\,\mathrm{d}z \\ &= \psi_t(\mathbb{R},z) f_t(z)\,\mathrm{d}z \\ &= \mathbb{Q}[L_{tT} \in \mathrm{d}z].\end{aligned} \tag{68}$$

For the case where $s > 0$, we use the dynamic consistency property. For s fixed and L_{sT} given, the process $\{\eta_{uT}\}_{s\leq u\leq T} = \{L_{uT} - L_{sT}\}_{s\leq u\leq T}$ is an LRB with law $LRB_{\mathcal{C}}([s,T],\{f_t\},\nu^*)$, where $\nu^*(A) = \nu_s(A + L_{sT})$. We have

$$
\begin{aligned}
\mathbb{Q}\left[L_{u+t,T} - L_{uT} \in \mathrm{d}z \,|\, L_{sT}\right] &= \mathbb{Q}\left[\eta_{u+t,T} - \eta_{uT} \in \mathrm{d}z \,|\, L_{sT}\right] \\
&= \mathbb{Q}\left[\eta_{tT} \in \mathrm{d}z \,|\, L_{sT}\right] \\
&= \int_{-\infty}^{\infty} \frac{f_{T-t}(w-z)}{f_{T-s}(w)} \nu^*(\mathrm{d}w)\, f_{t-s}(z)\,\mathrm{d}z \\
&= \int_{-\infty}^{\infty} \frac{f_{T-t}(w-z+L_{sT})}{f_{T-s}(w-L_{sT})} \nu_s(\mathrm{d}w)\, f_{t-s}(z)\,\mathrm{d}z \\
&= \frac{1}{\psi_s(\mathbb{R};L_{sT})} \int_{-\infty}^{\infty} \frac{f_{T-t}(w-z+L_{sT})}{f_T(w)} \nu(\mathrm{d}w)\, f_{t-s}(z)\,\mathrm{d}z \\
&= \frac{\psi_t(\mathbb{R};z+L_{sT})}{\psi_s(\mathbb{R};L_{sT})} f_{t-s}(z)\,\mathrm{d}z \\
&= \mathbb{Q}[L_{tT} - L_{sT} \in \mathrm{d}z \,|\, L_{sT}]\,. \qquad (69)
\end{aligned}
$$

□

When the expected terminal value is finite, the stationary increments property offers enough structure to allow the calculation of the expected value of L_{tT} for all $t < T$.

Corollary 2.1. *If* $\mathbb{E}[|L_{tT}|] < \infty$ *for* $t \in (0,T]$ *then*

$$\mathbb{E}\left[L_{tT} \,|\, L_{sT}\right] = \frac{T-t}{T-s} L_{sT} + \frac{t-s}{T-s} \mathbb{E}\left[L_{TT} \,|\, L_{sT}\right], \qquad (70)$$

and if $\mathbb{E}[|M_{tT}|] < \infty$ *for* $t \in (0,T]$ *then*

$$\mathbb{E}\left[M_{tT} \,|\, M_{sT}\right] = \frac{T-t}{T-s} M_{sT} + \frac{t-s}{T-s} \mathbb{E}\left[M_{TT} \,|\, M_{sT}\right], \qquad (71)$$

for $0 \leq s < t$.

Proof. We provide the proof for $\{L_{tT}\}$. The proof for $\{M_{tT}\}$ is similar. First we assume that $s = 0$. Suppose that $t = mT/n$, where $m, n \in \mathbb{N}_+$ and $m < n$. In this case we wish to show that

$$\mathbb{E}[L_{tT}] = \frac{m}{n} \mathbb{E}[L_{TT}]. \qquad (72)$$

Writing $L(t,T) = L_{tT}$ for clarity, define the random variables $\{\Delta_i\}$ by

$$\Delta_i = L\left(\tfrac{i}{n}T, T\right) - L\left(\tfrac{(i-1)}{n}T, T\right). \qquad (73)$$

It follows from Proposition 2.5 that the Δ_i's are identically distributed. Hence

$$\mathbb{E}[\Delta_i] = \frac{1}{n}\,\mathbb{E}\left[\sum_{i=1}^{n} \Delta_i\right] = \frac{1}{n}\,\mathbb{E}[L_{TT}]. \qquad (74)$$

Then we have

$$\mathbb{E}\left[L\left(\tfrac{m}{n}T,T\right)\right]=\mathbb{E}\left[\sum_{i=1}^{m}\Delta_i\right]=\frac{m}{n}\,\mathbb{E}[L_{TT}], \tag{75}$$

as required.

For general t, we can pick an increasing sequence of positive rational numbers $\{q_i\}$ such that $\lim_{i\to\infty} q_i = t/T$. Then by use of the monotone convergence theorem one obtains

$$\mathbb{E}[L(t,T)]=\mathbb{E}\left[\lim_{i\to\infty}L\left(q_iT,T\right)\right]=\lim_{i\to\infty}\mathbb{E}\left[L\left(q_iT,T\right)\right]=\frac{t}{T}\,\mathbb{E}[L_{TT}]. \tag{76}$$

For the case where $s>0$, we use the dynamic consistency property. For s fixed and L_{sT} given, the process

$$\eta_{tT}=L_{tT}-L_{sT} \qquad (s\leq t\leq T) \tag{77}$$

is an LRB with law $LRB_{\mathcal{C}}([s,T],\{f_t\},\nu^*)$, where $\nu^*(A)=\nu_s(A+L_{sT})$. Then we have

$$\begin{aligned}\mathbb{E}\left[L_{tT}\,|\,L_{sT}\right]&=L_{sT}+\mathbb{E}[\eta_{tT}\,|\,L_{sT}]\\&=L_{sT}+\frac{t-s}{T-s}\int_{-\infty}^{\infty}z\,\nu^*(\mathrm{d}z)\\&=L_{sT}+\frac{t-s}{T-s}\int_{-\infty}^{\infty}(z-L_{sT})\,\nu_s(\mathrm{d}z)\\&=\frac{T-s}{T-s}L_{sT}+\frac{t-s}{T-s}\mathbb{E}\left[L_{TT}\,|\,L_{sT}\right].\end{aligned} \tag{78}$$

□

We have shown that the increments of LRBs are stationary, so it is natural to ask when the increments are independent, i.e. when is an LRB a Lévy process? The answer lies in the functional form of $\psi_t(\mathbb{R};y)$.

For s,t satisfying $0\leq s<t<T$, the likelihood that $L_{tT}=y$ given that $L_{sT}=x$ is

$$q(t,y;s,x)=\frac{\psi_t(\mathbb{R};y)}{\psi_s(\mathbb{R};x)}f_{t-s}(y-x). \tag{79}$$

If $\{L_{tT}\}$ has stationary, independent increments then

$$q(t,y;s,x)=q(t-s,y-x;0,0). \tag{80}$$

Therefore the ratio

$$\frac{\psi_t(\mathbb{R};y)}{\psi_s(\mathbb{R};x)} \tag{81}$$

is a function only of the differences $t-s$ and $y-x$. Thus if we have

$$\psi_t(\mathbb{R};y)=a\exp(by+ct), \tag{82}$$

for some constants a, b and c, then $\{L_{tT}\}$ is a Lévy process. There are constraints on a, b and c since (79) is a probability density. When $b = c = 0$ we have $\nu(\mathrm{d}z) = f_T(z)\,\mathrm{d}z$ which is the case where $\{L_{tT}\} \overset{\text{law}}{=} \{L_t\}$.

Example. In the Brownian case we consider a process $\{W_{tT}\}$ with law

$$LRB_{\mathcal{C}}([0,T], \{f_t\}, f_T(z-\theta T)\,\mathrm{d}z),$$

where $f_t(x)$ is the normal density with zero mean and variance t, given by (39). In other words, $\{W_{tT}\}$ is a standard Brownian motion that has been conditioned so that W_{TT} is a normal random variable with mean θT and variance T. In this case, we have

$$\begin{aligned}\psi_t(\mathbb{R}; y) &= \int_{-\infty}^{\infty} \frac{f_{T-t}(z-y)}{f_T(z)} f_T(z-\theta T)\,\mathrm{d}z \\ &= \exp\left(\theta y - \frac{\theta}{2}t\right). \end{aligned} \tag{83}$$

Simplifying the expression for the transition densities of the $\{W_{tT}\}$, we verify that this process is a Brownian motion with drift θ. It is notable that by Girsanov's theorem the process $\{\psi_t(\mathbb{R}; W_t)\}$ is the Radon-Nikodým density process that transforms a standard Brownian motion into a Brownian motion with drift θ. Hence we can also deduce from the analysis in Section 2.6 that $\{W_{tT}\}$ is a Brownian motion with drift θ.

Example. In the gamma case, we consider a process $\{\Gamma_{tT}\}$ with law

$$LRB_{\mathcal{C}}([0,T], \{f_t\}, \kappa^{-1} f_T(z/\kappa)\,\mathrm{d}z),$$

where $f_t(x)$ is the gamma density with mean mt and variance mt defined by (42), and $\kappa > 0$ is constant. Then $\{\Gamma_{tT}\}$ is a gamma process with mean rate m and variance rate m, conditioned so that Γ_{TT} has a gamma distribution with mean $\kappa m T$ and variance $\kappa^2 m T$. We have

$$\begin{aligned}\psi_t(\mathbb{R}; y) &= \int_{-\infty}^{\infty} \frac{f_{T-t}(z-y)}{f_T(z)} \frac{f_T(z/\kappa)}{\kappa}\,\mathrm{d}z \\ &= \kappa^{-mt} \exp\left((1-\kappa^{-1})y\right). \end{aligned} \tag{84}$$

The transition density of $\{\Gamma_{tT}\}$ is

$$\mathbb{Q}[\Gamma_{tT} \in \mathrm{d}y \,|\, \Gamma_{sT} = x] = \mathbb{1}_{\{y>x\}} \frac{(y-x)^{m(t-s)-1} \mathrm{e}^{-(y-x)/\kappa}}{\kappa^{m(t-s)} \Gamma(m(t-s))}\,\mathrm{d}y. \tag{85}$$

Hence $\{\Gamma_{tT}\}$ is a gamma process with mean rate κm and variance rate $\kappa^2 m$.

2.8.1 Increment distributions

Partition the time interval $[0, T]$ by $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$. Then define the increments $\{\Delta_i\}_{i=1}^n$ and $\{\alpha_i\}_{i=1}^n$ by

$$\Delta_i = L_{t_i,T} - L_{t_{i-1},T} \tag{86}$$

$$\alpha_i = t_i - t_{i-1}. \tag{87}$$

Assume that ν has no *continuous singular part* [39]. Denoting the Dirac delta function centred at z by $\delta_z(x)$, we can write

$$\nu(\mathrm{d}z) = \sum_{i=-\infty}^{\infty} v_i \delta_{z_i}(z)\,\mathrm{d}z + p(z)\,\mathrm{d}z, \tag{88}$$

for some $\{a_i\} \subset \mathbb{R}$, $\{z_i\} \subset \mathbb{R}_+$, and $p : \mathbb{R} \to \mathbb{R}_+$. Here $p(z)$ is the density of the continuous part of ν, and v_i is a point mass of ν located at z_i. By (23), the joint law of the random vector $(\Delta_1, \ldots, \Delta_n)^{\mathrm{T}}$ is given by

$$\mathbb{Q}[\Delta_1 \in \mathrm{d}y_1 \ldots, \Delta_n \in \mathrm{d}y_n] = \widetilde{f}\left(\sum_{i=1}^n y_i\right) \prod_{i=1}^n f_{\alpha_i}(y_i)\,\mathrm{d}y_i, \tag{89}$$

where

$$\widetilde{f}(z) = \frac{p(z) + \sum_{i=-\infty}^{\infty} v_i \delta_{z_i}(z)}{f_T(z)}. \tag{90}$$

Equation (89) shows that $(\Delta_1, \ldots, \Delta_n)^{\mathrm{T}}$ has a generalized multivariate Liouville distribution as defined by Gupta & Richards [29]. The classical multivariate Liouville distribution is obtained when $f_t(x)$ is the density of a gamma distribution (see [26, 27, 28, 18]). A survey of Liouville distributions can be found in Gupta & Richards [25]. Barndoff-Nielsen & Jørgensen [4] construct a generalized Liouville distribution by conditioning a vector of independent inverse Gaussian random variables on their sum.

In the discrete case, the joint distribution of increments also has a generalized Liouville distribution. Define the increments $\{D_i\}$ by

$$D_i = M_{t_i,T} - M_{t_{i-1},T}. \tag{91}$$

Then we can write

$$\mathbb{Q}[D_1 \in \mathrm{d}y_1 \ldots, D_n \in \mathrm{d}y_n] = \widetilde{Q}\left(\sum_{i=1}^n y_i\right) \prod_{i=1}^n \mathrm{d}Q_{\alpha_i}(y_i), \tag{92}$$

where

$$\widetilde{Q}(z) = \frac{\sum_{i=-\infty}^{\infty} P(a_i)\delta_{a_i}(z)}{Q_T(z)}. \tag{93}$$

2.8.2 The reordering of increments

We are able to extend the Markov property of LRBs. If we partition the path of an LRB into increments, then the Markov property means that future increments depend on the past only through the *sum* of past increments. We shall show that for LRBs the ordering of the increments does not matter for this to hold—given the values of any set of increments of an LRB (past or future), the other increments depend on this subset only through the sum of its elements.

Let π be a permutation of $\{1, 2, \ldots, n\}$. We define the partial sum S^{π}_m by

$$S^{\pi}_m = \sum_{i=1}^{m} \Delta_{\pi(i)} \qquad \text{for } m = 1, 2, \ldots, n, \tag{94}$$

where the $\{\Delta_i\}$ are defined as in (86); and we define the partition $0 = t^{\pi}_0 < t^{\pi}_1 < \cdots < t^{\pi}_n = T$ by

$$t^{\pi}_{j+1} = \sum_{i=1}^{j} \alpha_{\pi(i)} \qquad \text{for } j = 1, 2, \ldots, n-1. \tag{95}$$

Proposition 2.6. *One can extend the Markov property of $\{L_{tT}\}$ to the following:*

$$\mathbb{Q}\left[\Delta_{\pi(m+1)} \leq y_{m+1}, \ldots, \Delta_{\pi(n)} \leq y_n \,\middle|\, \Delta_{\pi(1)}, \ldots, \Delta_{\pi(m)}\right] = \\ \mathbb{Q}\left[\Delta_{\pi(m+1)} \leq y_{m+1}, \ldots, \Delta_{\pi(n)} \leq y_n \,\middle|\, S^{\pi}_m\right]. \tag{96}$$

If ν has no singular continuous part, then

$$\mathbb{Q}\left[\Delta_{\pi(m+1)} \in \mathrm{d}y_{m+1}, \ldots, \Delta_{\pi(n)} \in \mathrm{d}y_n \,\middle|\, S^{\pi}_m\right] = \\ \frac{\widetilde{f}\left(S^{\pi}_m + \sum_{i=m+1}^{n} y_i\right)}{\psi_{t^{\pi}_m}(\mathbb{R}; S^{\pi}_m)} \prod_{i=m+1}^{n} f_{\alpha_{\pi(i)}}(y_i)\, \mathrm{d}y_i. \tag{97}$$

Proof. Define the increments $\{\Delta^{\pi}_i\}$ by

$$\Delta^{\pi}_i = L_{t^{\pi}_n, T} - L_{t^{\pi}_{n-1}, T}. \tag{98}$$

The law of the random vector $(\Delta^{\pi}_1, \ldots, \Delta^{\pi}_{n-1}, \sum_1^n \Delta^{\pi}_i)^{\mathrm{T}}$ is given by

$$\mathbb{Q}\left[\Delta^{\pi}_1 \in \mathrm{d}y_1, \ldots, \Delta^{\pi}_{n-1} \in \mathrm{d}y_{n-1}, \sum_{i=1}^{n} \Delta^{\pi}_i \in \mathrm{d}z\right] = \\ \frac{\nu(\mathrm{d}z)}{f_T(z)} f_{\alpha_{\pi(n)}}\left(z - \sum_{i=1}^{n-1} y_i\right) \prod_{i=1}^{n-1} f_{\alpha_{\pi(i)}}(y_i)\, \mathrm{d}y_i. \tag{99}$$

This is also the law of $(\Delta_{\pi(1)}, \ldots, \Delta_{\pi(n-1)}, \sum_1^n \Delta_{\pi(i)})^{\mathrm{T}}$; hence

$$(\Delta_{\pi(1)}, \ldots, \Delta_{\pi(n)}) \overset{\text{law}}{=} (\Delta^{\pi}_1, \ldots, \Delta^{\pi}_n). \tag{100}$$

The Markov property of LRBs gives

$$\mathbb{Q}\left[\Delta^{\pi}_{m+1} \leq y_{m+1}, \dots, \Delta^{\pi}_{n} \leq y_n \,\middle|\, \Delta^{\pi}_{1}, \dots, \Delta^{\pi}_{m}\right] = \\ \mathbb{Q}\left[\Delta^{\pi}_{m+1} \leq y_{m+1}, \dots, \Delta^{\pi}_{n} \leq y_n \,\middle|\, \sum_{i=1}^{m} \Delta^{\pi}_{i}\right], \quad (101)$$

and so we have

$$\mathbb{Q}\left[\Delta_{\pi(m+1)} \leq y_{m+1}, \dots, \Delta_{\pi(n)} \leq y_n \,\middle|\, \Delta_{\pi(1)}, \dots, \Delta_{\pi(m)}\right] = \\ \mathbb{Q}\left[\Delta_{\pi(m+1)} \leq y_{m+1}, \dots, \Delta_{\pi(n)} \leq y_n \,\middle|\, S^{\pi}_{m}\right]. \quad (102)$$

This proves the first part of the proposition.

For the second part of the proof we assume that ν takes the form (88). Note that

$$L_{t^{\pi}_{m},T} = \sum_{i=1}^{m} \Delta^{\pi}_{i}, \quad (103)$$

and that the density of $L_{t^{\pi}_{m},T}$ is

$$x \mapsto f_{t^{\pi}_{m}}(x)\psi_{t^{\pi}_{m}}(\mathbb{R}; x) = \int_{z=-\infty}^{\infty} \frac{f_{t^{\pi}_{m}}(x) f_{T-t^{\pi}_{m}}(z-x)}{f_T(z)} \, \nu(\mathrm{d}z). \quad (104)$$

The elements of the vector $(L_{t^{\pi}_{m},T}, \Delta^{\pi}_{m+1}, \dots, \Delta^{\pi}_{n})^{\mathrm{T}}$ are non-overlapping increments of $\{L_{tT}\}$, and the law of the vector is given by

$$\mathbb{Q}\left[L_{t^{\pi}_{m},T} \in \mathrm{d}x, \Delta^{\pi}_{m+1} \in \mathrm{d}y_{m+1}, \dots, \Delta^{\pi}_{n} \in \mathrm{d}y_n\right] = \\ \widetilde{f}\left(x + \sum_{i=m+1}^{n} y_i\right) f_{t^{\pi}_{m}}(x)\,\mathrm{d}x \prod_{i=m+1}^{n} f_{\alpha_{\pi(i)}}(y_i)\,\mathrm{d}y_i. \quad (105)$$

Thus we have

$$\begin{aligned} &\mathbb{Q}\left[\Delta^{\pi}_{m+1} \in \mathrm{d}y_{m+1}, \dots, \Delta^{\pi}_{n} \in \mathrm{d}y_n \,\middle|\, L_{t^{\pi}_{m},T} = x\right] \\ &\quad = \frac{\mathbb{Q}\left[\Delta^{\pi}_{m+1} \in \mathrm{d}y_{m+1}, \dots, \Delta^{\pi}_{n} \in \mathrm{d}y_n, L_{t^{\pi}_{m},T} \in \mathrm{d}x\right]}{\mathbb{Q}\left[L_{t^{\pi}_{m},T} \in \mathrm{d}x\right]} \\ &\quad = \frac{\widetilde{f}\left(x + \sum_{i=m+1}^{n} y_i\right) \prod_{i=m+1}^{n} f_{\alpha_{\pi(i)}}(y_i)}{\psi_{t^{\pi}_{m}}(\mathbb{R}; S^{\pi}_{m})}. \end{aligned} \quad (106)$$

□

We note that Gupta & Richards [29] prove that if $(\Delta_1, \Delta_2, \dots, \Delta_n)$ has a generalized Liouville distribution then equation (96) holds.

We can use Proposition 2.6 to extend the dynamic consistency property.

Corollary 2.2. *1. Fix times s_1, T_1 satisfying $0 < T_1 \leq T - s_1$. The time-shifted, space-shifted partial process*

$$\eta^{(1)}_{t,T_1} = L_{s_1+t,s_1+T_1} - L_{s_1,T} \qquad (0 \leq t \leq T_1) \tag{107}$$

is an LRB with the law $LRB_{\mathcal{C}}([0,T_1], \{f_t\}, \nu^{(1)})$, where $\nu^{(1)}$ is a probability law on $\mathbb{R}$ with density $f_{T_1}(x)\psi_{T_1}(\mathbb{R}; x)$.

2. Construct the partial processes $\{\eta^{(i)}_{t,T_i}\}$, $i = 1, \ldots, n$ from non-overlapping portions of $\{L_{tT}\}$ in a similar way to part 1. The intervals $[s_i, s_i + T_i]$, $i = 1, \ldots, n$, are non-overlapping except possibly at the endpoints. Set $\eta^{(i)}_{t,T_i} = \eta^{(i)}_{T_i,T_i}$ when $t > T_i$. If $u > t$, then

$$\mathbb{Q}\left[\eta^{(1)}_{u,T_1} - \eta^{(1)}_{t,T_1} \leq x_1, \ldots, \eta^{(n)}_{u,T_n} - \eta^{(n)}_{t,T_n} \leq x_n \,\middle|\, \mathcal{F}^\eta_t\right] = \mathbb{Q}\left[\eta^{(1)}_{u,T_1} - \eta^{(1)}_{t,T_1} \leq x_1, \ldots, \eta^{(n)}_{u,T_n} - \eta^{(n)}_{t,T_n} \leq x_n \,\middle|\, \sum_{i=1}^n \eta^{(i)}_{t,T_i}\right], \tag{108}$$

where

$$\mathcal{F}^\eta_t = \sigma\left(\left\{\eta^{(i)}_{s,T_i}\right\}_{0 \leq s \leq t}, i = 1, 2, \ldots, n\right). \tag{109}$$

Remark 2.1. *The partial processes of Corollary 2.2 are dependent, and*

$$\mathbb{Q}\left[\eta^{(i)}_{tT} \in \mathrm{d}x \,\middle|\, \mathcal{F}^\eta_s\right] = \mathbb{Q}\left[\eta^{(i)}_{tT} \in \mathrm{d}x \,\middle|\, \eta^{(i)}_{sT}, \sum_{j=1}^n \eta^{(j)}_{sT}\right], \tag{110}$$

for $0 \leq s < t \leq T$.

We state but do not prove the discrete analogue of Proposition 2.6:

Proposition 2.7. *One can extend the Markov property of $\{M_{tT}\}$ to the following:*

$$\mathbb{Q}\left[D_{\pi(m+1)} \leq y_{m+1}, \ldots, D_{\pi(n)} \leq y_n \,\middle|\, D_{\pi(1)}, \ldots, D_{\pi(m)}\right] = \mathbb{Q}\left[D_{\pi(m+1)} \leq y_{m+1}, \ldots, D_{\pi(n)} \leq y_n \,\middle|\, R^\pi_m\right], \tag{111}$$

where $R^\pi_m = \sum_{i=1}^m D_{\pi(i)}$. Furthermore,

$$\mathbb{Q}\left[D_{\pi(m+1)} = y_{m+1}, \ldots, D_{\pi(n)} = y_n \,\middle|\, D^\pi_m\right] = \frac{\widetilde{Q}\left(R^\pi_m + \sum_{i=m+1}^n y_i\right)}{\sum_{k=-\infty}^{\infty} \phi_{t^\pi_m}(a_k; R^\pi_m)} \prod_{i=m+1}^n Q_{\alpha_{\pi(i)}}(y_i). \tag{112}$$

Corollary 2.2 can be extended to include LRBs with discrete state-spaces.

3 Information-based asset pricing

3.1 BHM framework

This section contains an overview of the BHM framework. The approach was applied to credit risk in Brody *et al.* [8], and this was extended to include stochastic interest rates in Rutkowski & Yu [38]. A general asset pricing framework was proposed in Brody *et al.* [10] (see also Macrina [35]), and there have also been applications to inflation modelling (Hughston & Macrina [31]), insider trading (Brody *et al.* [9]), insurance (Brody *et al.* [11]), and interest rate theory (Hughston & Macrina [30]).

We fix a finite time horizon $[0, T]$ and a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. We assume that the risk-free rate of interest $\{r_t\}$ is deterministic, and that $r_t > 0$ and $\int_t^\infty r_u \, \mathrm{d}u = \infty$, for all $t > 0$. Then the time-s (no-arbitrage) price of a risk-free, zero-coupon bond maturing at time t (paying a nominal amount of unity) is

$$P_{st} = \exp\left(-\int_s^t r_u \, \mathrm{d}u\right) \qquad (s \leq t). \tag{113}$$

For $t < T$, we define the time-t price of a contingent cash flow H_T, due at time T, to be

$$H_{tT} = P_{tT} \, \mathbb{E}[H_T \,|\, \mathcal{F}_t], \tag{114}$$

where $\{\mathcal{F}_t\}$ is the *market filtration.* The sigma-algebra $\mathcal{F}_t$ represents the information available to market participants at time-t. In order for equation (114) to be consistent with the theory of no-arbitrage pricing, we interpret $\mathbb{Q}$ to be the risk-neutral measure.

In such a set-up, the dynamics of the price process $\{H_{tT}\}$ are implicitly determined by the evolution of the market filtration $\{\mathcal{F}_t\}$. We assume the existence of a (possibly multi-dimensional) *information process* $\{\xi_{tT}\}_{0 \leq t \leq T}$ such that

$$\mathcal{F}_t = \sigma\left(\{\xi_{sT}\}_{0 \leq s \leq t}\right). \tag{115}$$

Thus $\{\xi_{tT}\}$ is responsible for the delivery of all information to the market participants. The task of modelling the emergence of information in the market is reduced to that of specifying the law of the information process $\{\xi_{tT}\}$.

3.1.1 Single X-factor market

We assume that the cash flow H_T can be written in the form

$$H_T = h(X_T), \tag{116}$$

for some function $h(x)$, and some market factor X_T. We call X_T an X-factor. We assume that $\{\xi_{tT}\}$ is a one-dimensional process such that $\xi_{TT} = X_T$. Then we have

$$H_{tT} = P_{tT} \, \mathbb{E}[h(X_T) \,|\, \mathcal{F}_t] = P_{tT} \, \mathbb{E}[h(\xi_{TT}) \,|\, \mathcal{F}_t]. \tag{117}$$

This construction ensures that $H_{TT} = H_T$. In the case where $\{\xi_{tT}\}$ is a Markov process, we have

$$H_{tT} = P_{tT} \, \mathbb{E}[h(\xi_{TT}) \,|\, \xi_{tT}]. \tag{118}$$

3.1.2 Multiple X-factor market

In the more general framework, we model an asset which generates N cash flows $H_{T_1}, H_{T_2}, \ldots, H_{T_N}$, which are to be received on the dates $T_1 \leq T_2 \leq \cdots \leq T_N$, respectively. At time T_k, we assume that the vector of X-factors $X_{T_k} \in \mathbb{R}^{n_k}$ $(n_k \in \mathbb{N}_+)$ is revealed to the market, and we write

$$X_{T_k} = \left(X_{T_k}^{(1)}, X_{T_k}^{(2)}, \ldots, X_{T_k}^{(n_k)} \right)^{\mathrm{T}}. \tag{119}$$

We assume the X-factors are mutually independent, and that

$$H_{T_k} = h_k(X_{T_1}, X_{T_2}, \ldots, X_{T_k}), \tag{120}$$

for some function $h_k : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k} \to \mathbb{R}$. For each X-factor $X_{T_j}^{(i)}$, there is a factor information process $\{\xi_t^{(i,j)}\}$ such that $\xi_t^{(i,j)} = X_{T_j}^{(i)}$ for $t \geq T_j$, and the factor information processes are mutually independent. Setting $T = T_N$, we define the market information process $\{\xi_{tT}\}$ to be an $\mathbb{R}^{n_1+n_2+\cdots+n_N}$ valued process with each of its elements being a factor information process. The market filtration $\{\mathcal{F}_t\}$ is generated by $\{\xi_{tT}\}$. By construction, we have that H_{T_k} is $\mathcal{F}_t$-measurable for $t \geq T_k$. The time-t price of cash flow H_{T_k} is

$$H_{tT}^{(k)} = \begin{cases} P_{t,T_k}\, \mathbb{E}\left[h_k(X_{T_1}, X_{T_2}, \ldots, X_{T_k}) \,|\, \mathcal{F}_t \right] & \text{for } t \leq T_k, \\ 0 & \text{for } t > T_k. \end{cases} \tag{121}$$

The asset price process is then

$$H_{tT} = \sum_{k=1}^{n} H_{tT}^{(k)} \qquad (0 \leq t \leq T). \tag{122}$$

3.2 Lévy bridge information

We consider a market with a single X-factor, denoted X_T. This X-factor is the size of a contingent cash flow to be received at time $T > 0$, so we take $h(x) = x$. For example, X_T could be the redemption amount of a credit risky bond. *A priori*, X_T is assumed to be integrable and to have probability law ν (we also exclude the case where X_T is constant). Information is supplied to the market by an information process $\{\xi_{tT}\}$. The law of $\{\xi_{tT}\}$ is $LRB_{\mathcal{C}}([0,T], \{f_t\}, \nu)$, and we set $\xi_{TT} = X_T$. We assume throughout this section that the information process has a continuous state-space, the results can be extended to include LRB information processes with discrete state-spaces.

Since the information process has the Markov property, the price process of the cash flow X_T is

$$X_{tT} = P_{tT}\, \mathbb{E}\left[X_T \,|\, \xi_{tT} \right] \qquad (0 \leq t \leq T). \tag{123}$$

We note that X_T is $\mathcal{F}_T$-measurable and $X_{TT} = X_T$, but X_T is not $\mathcal{F}_t$-measurable for $t < T$ since we have excluded the case where X_T is constant. For $t \in (0, T)$, the $\mathcal{F}_t$-conditional law of X_T as given by equation (47) is

$$\nu_t(\mathrm{d}z) = \frac{\psi_t(\mathrm{d}z; \xi_{tT})}{\psi_t(\mathbb{R}; \xi_{tT})}, \tag{124}$$

where

$$\psi_t(\mathrm{d}z; \xi) = \frac{f_{T-t}(z-\xi)}{f_T(z)}\,\mathrm{d}z. \tag{125}$$

Then we have

$$X_{tT} = P_{tT} \int_{-\infty}^{\infty} z\, \nu_t(\mathrm{d}z). \tag{126}$$

When ν admits a density $p(z)$, the $\mathcal{F}_t$-conditional density of X_T exists and is given by

$$p_t(z) = \frac{f_{T-t}(z-\xi_{tT})p(z)}{\psi_t(\mathbb{R}; \xi_{tT}) f_T(z)}. \tag{127}$$

Example. In the Brownian case the price process is

$$X_{tT} = P_{tT} \frac{\int_{-\infty}^{\infty} z\, \mathrm{e}^{\frac{1}{T-t}\left[\xi_{tT} z - \frac{1}{2}\frac{t}{T} z^2\right]}\, \nu(\mathrm{d}z)}{\int_{-\infty}^{\infty} \mathrm{e}^{\frac{1}{T-t}\left[\xi_{tT} z - \frac{1}{2}\frac{t}{T} z^2\right]}\, \nu(\mathrm{d}z)}. \tag{128}$$

The following SDE can be derived for $\{X_{tT}\}$ (see [8, 10, 35, 38]):

$$\mathrm{d}X_{tT} = r_t X_{tT}\,\mathrm{d}t + \frac{P_{tT}\mathrm{Var}[X_T \,|\, \xi_{tT}]}{T-t}\,\mathrm{d}W_t, \tag{129}$$

where $\{W_t\}$ is an $\{\mathcal{F}_t\}$-Brownian motion.

Example. In the gamma case we have

$$X_{tT} = P_{tT} \frac{\int_{\xi_{tT}}^{\infty} (z-\xi_{tT})^{m(T-t)-1} z^{2-mT}\, \nu(\mathrm{d}z)}{\int_{\xi_{tT}}^{\infty} (z-\xi_{tT})^{m(T-t)-1} z^{1-mT}\, \nu(\mathrm{d}z)}. \tag{130}$$

3.3 European option pricing

We consider the problem of pricing a European option on the price process $\{X_{tT}\}$. For a strike price K and $0 \leq s < t < T$, the time-s price of a call option on X_{tT} is

$$C_{st} = P_{st}\, \mathbb{E}\left[(X_{tT} - K)^+ \,\middle|\, \xi_{sT} \right]. \tag{131}$$

The expectation can be expanded as

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[(X_{tT}-K)^+ \,\middle|\, \xi_{sT}\right] &= \mathbb{E}_{\mathbb{Q}}\left[(P_{tT}\,\mathbb{E}_{\mathbb{Q}}[X_T \,|\, \xi_{tT}]-K)^+ \,\middle|\, \xi_{sT}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\left(\int_{-\infty}^{\infty}(P_{tT}z-K)\,\nu_t(\mathrm{d}z)\right)^+ \,\middle|\, \xi_{sT}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\psi_t(\mathbb{R};\xi_{tT})}\left(\int_{-\infty}^{\infty}(P_{tT}z-K)\,\psi_t(\mathrm{d}z;\xi_{tT})\right)^+ \,\middle|\, \xi_{sT}\right].
\end{aligned} \tag{132}$$

Recall that the Radon-Nikodym density process

$$\left.\frac{\mathrm{d}\mathbb{L}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_t} = \psi_t(\mathbb{R};\xi_{tT})^{-1} \tag{133}$$

defines a measure $\mathbb{L}$ under which $\{\xi_{tT}\}_{0\le t<T}$ is a Lévy process. By changing measure, we find that the expectation is

$$\begin{aligned}
&\frac{1}{\psi_s(\mathbb{R};\xi_{sT})}\,\mathbb{E}_{\mathbb{L}}\left[\left(\int_{-\infty}^{\infty}(P_{tT}z-K)\,\psi_t(\mathrm{d}z;\xi_{tT})\right)^+ \,\middle|\, \xi_{sT}\right] = \\
&\qquad \frac{1}{\psi_s(\mathbb{R};\xi_{sT})}\,\mathbb{E}_{\mathbb{L}}\left[\left(\int_{-\infty}^{\infty}(P_{tT}z-K)\frac{f_{T-t}(z-\xi_{tT})}{f_T(z)}\,\nu(\mathrm{d}z)\right)^+ \,\middle|\, \xi_{sT}\right].
\end{aligned} \tag{134}$$

Equation (29) states that $0<f_{T-s}(z-\xi_{sT})<\infty$. Thus we can write the expectation in terms of the ξ_{sT}-conditional terminal law ν_s as

$$\begin{aligned}
&\mathbb{E}_{\mathbb{L}}\left[\left(\int_{-\infty}^{\infty}(P_{tT}z-K)\frac{f_{T-t}(z-\xi_{tT})}{f_{T-s}(z-\xi_{sT})}\,\nu_s(\mathrm{d}z)\right)^+ \,\middle|\, \xi_{sT}\right] = \\
&\qquad \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}(P_{tT}z-K)\frac{f_{T-t}(z-x)}{f_{T-s}(z-\xi_{sT})}\,\nu_s(\mathrm{d}z)\right)^+ f_{t-s}(x-\xi_{sT})\,\mathrm{d}x.
\end{aligned} \tag{135}$$

We defined the (marginal) Lévy bridge density $f_{tT}(x;z)$ by

$$f_{tT}(x;z) = \frac{f_{T-t}(z-x)f_t(x)}{f_T(z)}. \tag{136}$$

From this we can define the ξ_{sT}-dependent law $\mu_{st}(\mathrm{d}x;z)$ by

$$\mu_{st}(\mathrm{d}x;z) = f_{t-s,T-s}(x-\xi_{sT},z-\xi_{sT})\,\mathrm{d}x. \tag{137}$$

So $\mu_{st}(\mathrm{d}x;z)$ is the time-t marginal law of a Lévy bridge starting at the value ξ_{sT} at time s, and terminating at the value z at time T. Defining the set B_t by

$$B_t = \left\{x\in\mathbb{R} : \int_{-\infty}^{\infty}(P_{tT}z-K)\frac{f_{T-t}(z-x)}{f_T(z)}\,\nu(\mathrm{d}z)>0\right\}, \tag{138}$$

the expectation reduces to

$$\int_{-\infty}^{\infty} (P_{tT} z - K) \mu_{st}(B_t; z) \, \nu_s(\mathrm{d}z). \tag{139}$$

And so the option price is

$$C_{st} = P_{st} \int_{-\infty}^{\infty} (P_{tT} z - K) \mu_{st}(B_t; z) \, \nu_s(\mathrm{d}z). \tag{140}$$

We can write $X_{tT} = \Lambda(t, \xi_{tT})$, for Λ a deterministic function. The set B_t can then be written

$$B_t = \{\xi \in \mathbb{R} : \Lambda(t, \xi) > K\}. \tag{141}$$

We see that if Λ is increasing in its second argument then $B_t = (\xi_t^*, \infty)$ for some critical value ξ_t^* of the information process. Λ is monotonic if the information process is a Lévy process.

Example. In the Brownian case we have

$$\Lambda(t, x) = P_{tT} \frac{\int_{-\infty}^{\infty} z \, \mathrm{e}^{\frac{1}{T-t}\left[xz - \frac{1}{2}\frac{t}{T}z^2\right]} \, \nu(\mathrm{d}z)}{\int_{-\infty}^{\infty} \mathrm{e}^{\frac{1}{T-t}\left[xz - \frac{1}{2}\frac{t}{T}z^2\right]} \, \nu(\mathrm{d}z)}. \tag{142}$$

It can be shown that the function Λ is increasing in its second argument (see [10, 38]), so $B_t = (\xi_t^*, \infty)$ for the unique ξ_t^* satisfying $\Lambda(t, \xi_t^*) = K$. A short calculation verifies that $\mu_{st}(\mathrm{d}x; z)$ is the normal law with mean $M(z)$ and variance V given by

$$M(z) = \frac{T-t}{T-s} \xi_{sT} + \frac{t-s}{T-s} z, \qquad V = \frac{t-s}{T-s}(T-t). \tag{143}$$

This is the time-t marginal law of a Brownian bridge starting from the value ξ_{sT} at time s, and finishing at the value z at time T. We have

$$\mu_{st}(B_t; z) = 1 - \Phi\left[\frac{\xi_t^* - M(z)}{\sqrt{V}}\right] = \Phi\left[\frac{M(z) - \xi_t^*}{\sqrt{V}}\right], \tag{144}$$

where $\Phi[x]$ is the standard normal distribution function. The option price is then

$$C_{st} = P_{sT} \int_{-\infty}^{\infty} z \, \Phi\left[\frac{M(z) - \xi_t^*}{\sqrt{V}}\right] \nu_s(\mathrm{d}z) + P_{st} K \int_{-\infty}^{\infty} \Phi\left[\frac{M(z) - \xi_t^*}{\sqrt{V}}\right] \nu_s(\mathrm{d}z). \tag{145}$$

Example. In the gamma case we have

$$\Lambda(t, x) = P_{tT} \frac{\int_x^{\infty} (z - x)^{m(T-t)-1} z^{2-mT} \, \nu(\mathrm{d}z)}{\int_x^{\infty} (z - x)^{m(T-t)-1} z^{1-mT} \, \nu(\mathrm{d}z)}. \tag{146}$$

The monotonicity of $\Lambda(t,x)$ in x was proved for $m(T-t)>1$ by Brody *et al.* [11]. The authors also give a numerical example where $\Lambda(t,x)$ was not monotonic in x for $m(T-t)<1$. For all $t<T$, we have

$$\mu_{st}(\mathrm{d}x;z) = \mathbb{1}_{\{\xi_{sT}<x<z\}}\, k(z)^{-1} \left(\frac{x-\xi_{st}}{z-\xi_{sT}}\right)^{m(t-s)-1} \left(\frac{z-y}{z-\xi_{sT}}\right)^{m(T-t)-1} \mathrm{d}x, \tag{147}$$

where $k(z)$ is the normalising constant

$$k(z) = (z-\xi_{sT})\,\mathrm{B}[m(t-s),m(T-t)]. \tag{148}$$

So $\mu_{st}(\mathrm{d}x;z)$ is an $(z-\xi_{sT})$-scaled, ξ_{sT}-shifted, beta law with parameters $\alpha=m(t-s)$ and $\beta=m(T-t)$. This is the time-t marginal law of a gamma bridge starting at the value ξ_{sT} at time s, and terminating at the value x at time T. When $m(T-t)>1$, a critical ξ_t^* exists such that $\Lambda(t,\xi_t^*)=K$. Then $B_t=(\xi_t^*,\infty)$, and

$$\begin{aligned}\mu_{st}(B_t;z) &= 1 - I\left[\frac{\xi_t^*-\xi_{sT}}{z-\xi_{sT}}; m(t-s), m(T-t)\right] \\ &= I\left[\frac{z-\xi_t^*}{z-\xi_{sT}}; m(T-t), m(t-s)\right],\end{aligned} \tag{149}$$

where

$$I[z;\alpha,\beta] = \frac{1}{\mathrm{B}[\alpha,\beta]}\int_0^z x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x \tag{150}$$

is the regularized incomplete beta function. The option price is

$$\begin{aligned}C_{st} = P_{sT}\int_{\xi_{sT}}^{\infty} z\, I\left[\frac{z-\xi_t^*}{z-\xi_{sT}}; m(T-t), m(t-s)\right]\nu_s(\mathrm{d}z) \\ + P_{st}K\int_{\xi_{sT}}^{\infty} I\left[\frac{z-\xi_t^*}{z-\xi_{sT}}; m(T-t), m(t-s)\right]\nu_s(\mathrm{d}z).\end{aligned} \tag{151}$$

3.4 Binary bond

The simplest (non-trivial) contingent cash flow is when $X_T\in\{k_0,k_1\}$, for two values $k_0<k_1$. This is the pay-off from a zero-coupon, credit-risky bond that has a nominal value k_1, and a fixed recovery rate k_0/k_1 on default. Assume that, *a priori*, $\mathbb{Q}[X_T=k_0]=p>0$ and $\mathbb{Q}[X_T=k_1]=1-p$. Then we have

$$\mathbb{Q}[X_T=k_0\,|\,\xi_{tT}] = \left(1+\frac{f_T(k_0)}{f_T(k_1)}\frac{f_{T-t}(k_1-\xi_{tT})}{f_{T-t}(k_0-\xi_{tT})}\frac{1-p}{p}\right)^{-1}, \tag{152}$$

and

$$\mathbb{Q}[X_T=k_1\,|\,\xi_{tT}] = \left(1+\frac{f_T(k_1)}{f_T(k_0)}\frac{f_{T-t}(k_0-\xi_{tT})}{f_{T-t}(k_1-\xi_{tT})}\frac{p}{1-p}\right)^{-1}. \tag{153}$$

The price process $\{X_{tT}\}$ associated with the cash flow is given by

$$X_{tT} = P_{tT}\left(k_0\,\mathbb{Q}[X_T = k_0 \,|\, \xi_{tT}] + k_1\,\mathbb{Q}[X_T = k_1 \,|\, \xi_{tT}]\right) \qquad (0 \leq t \leq T). \tag{154}$$

Example. In the Brownian case we have

$$\mathbb{Q}[X_T = k_0 \,|\, \xi_{tT}] = \left(1 + \exp\left[-\frac{1}{2}\frac{k_1 - k_0}{T - t}(\tfrac{t}{T}(k_0 + k_1) - 2\xi_{tT})\right]\frac{1-p}{p}\right)^{-1}, \tag{155}$$

and

$$\mathbb{Q}[X_T = k_1 \,|\, \xi_{tT}] = \left(1 + \exp\left[\frac{1}{2}\frac{k_1 - k_0}{T - t}(\tfrac{t}{T}(k_0 + k_1) - 2\xi_{tT})\right]\frac{p}{1-p}\right)^{-1}. \tag{156}$$

Writing $\rho_i = \mathbb{Q}[X_T = k_i \,|\, \xi_{tT}]$, note that

$$\begin{aligned}\mathrm{Var}[X_T \,|\, \xi_{tT}] &= (k_1 - k_0)^2 \rho_1 \rho_0 \\ &= -(k_0 - k_0\rho_0 - k_1\rho_1)(k_1 - k_0\rho_0 - k_1\rho_1) \\ &= -(k_0 - X_{tT})(k_1 - X_{tT}).\end{aligned} \tag{157}$$

Thus, recalling (129), we see that the SDE of $\{X_{tT}\}$ is

$$\mathrm{d}X_{tT} = r_t X_{tT}\,\mathrm{d}t - \frac{P_{tT}(k_0 - X_{tT})(k_1 - X_{tT})}{T - t}\,\mathrm{d}W_t, \tag{158}$$

with the initial condition $X_{0T} = k_0 p + k_1(1-p)$. For $K \in (P_{tT}k_0, P_{tT}k_1)$, we are able to solve the equation $\Lambda(t,x) = K$ for x. We have

$$\begin{aligned}\Lambda(t,x) &= P_{tT}\left(k_0\,\mathbb{Q}[X_T = k_0 \,|\, \xi_{tT} = x] + k_1\,\mathbb{Q}[X_T = k_1 \,|\, \xi_{tT} = x]\right) \\ &= P_{tT}\left(k_1 - (k_1 - k_0)\,\mathbb{Q}[X_T = k_0 \,|\, \xi_{tT} = x]\right),\end{aligned} \tag{159}$$

so the solution to $\Lambda(t,x) = K$ is

$$\xi^*_t = \frac{t}{2T}(k_0 + k_1) - \frac{T - t}{k_1 - k_0}\,\log\left[\frac{p}{1-p}\frac{K - P_{tT}k_0}{P_{tT}k_1 - K}\right], \tag{160}$$

and the price of a call option on X_{tT} is

$$C_{st} = P_{st}\sum_{i=0}^{1}(P_{tT}k - K)\,\Phi\left[\frac{M(k_i) - \xi^*_t}{\sqrt{V}}\right]\mathbb{Q}[X_T = k_i \,|\, \xi_{sT}]. \tag{161}$$

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