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A Dynamic Two Country Heckscher-Ohlin Model with Non-Homothetic Preferences

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Summary. We examine the properties of a two country dynamic Heckscher-Ohlin model that allows for preferences to be non-homothetic. We show that the model has a continuum of steady state equilibria under free trade, with the initial conditions determining which equilibrium will be attained. We establish conditions under which a static Heckscher-Ohlin theorem will hold in the steady state, and also conditions for a dynamic H-O theorem to hold. If both goods are normal, each country will have a unique autarkic steady state, and all steady state equilibria are saddle points. We also consider the case in which one good is inferior, and show that this can lead to multiple autarkic steady states, violations of the static H-O theorem in the steady state. Furthermore, there may exist steady state equilibria that Pareto dominate other steady states. These steady states will be unstable if discount factors are the same in each country, although they may exhibit dynamic indeterminacy if discount factors differ.

Key words: two-country model, Heckscher-Ohlin, inferior good, multiple equilibria, indeterminacy

JEL Classification Numbers: E13, E32, F11, F43

1 Introduction

In this paper we examine the role of tastes in determining the steady state capital stocks, the pattern of trade and the local dynamic properties of steady state equilibria in a dynamic Heckscher-Ohlin (H-O) model of international trade. It is well known that the predictions of the static H-O model may be overturned if the tastes of a country are strongly biased toward the good that uses its abundant factor intensively. The role of tastes is more complex in a dynamic model, because they can affect both the allocation of spending at a point in time as well as the intertemporal allocation of spending.

The static model with identical and homothetic preferences generates demands for goods that are independent of the international distribution of income. The analog of this assumption in the dynamic model, which makes demands for goods independent of the international distribution of wealth, is to assume identical and homothetic preferences with a constant intertemporal elasticity of substitution (CIES). Chen [10] examined the dynamic H-O model with preferences of that type and showed that it yields a continuum of steady state equilibria under free trade, with the world capital stock being constant across all of the steady states. Which steady state the world economy converges to is determined by the initial distribution of capital across countries because a country's wealth affects its savings behavior. Initial differences in wealth will thus have persistent effects on capital accumulation and will determine steady state capital stocks. Chen also showed that the pattern of trade in the steady state will be determined by relative factor supplies, and is thus consistent with the predictions of the static H-O theorem. Ventura [25] has utilized a similar model, which allows for the possibility of endogenous growth as well as a steady state, to study the implications of international trade for the convergence of per capita incomes across countries.[1]

Most of the existing literature on taste differences has focused on differences in rates of time preference across countries. Stiglitz [24] analyzed the case with infinitely lived consumers and showed that the patient country will export the capital intensive good in the steady state. The assumption of different rates of time preference means that countries will have different autarkic steady states, and that at least one country will specialize in production in the steady state. Galor and Lin [13] examine the effect of differences in time preference in an overlapping generations model, and also obtain the result that the patient country will export the capital intensive good in the steady state. They assume that the investment good be capital intensive in order to ensure a unique path to the steady state.[2]

Our analysis will focus on the case of infinitely lived consumers with identical preferences across countries, but we relax the requirement that they be homothetic. We will also assume that the

[1] Bond et al [8] analyze a Heckscher-Ohlin model in which there is accumulation of both capital and labor (through accumulation of human capital). They establish existence of a balanced growth path for the world economy, but show that the accumulation paths of individual countries are indeterminate because households are indifferent between physical and human capital accumulation at the margin.

[2] Galor [12] shows that there will be indeterminacy if the consumption good is capital intensive.

sum of the rate of time preference and the depreciation rate is the same across countries, which ensures that the autarkic steady state prices of the two goods are the same. This means that taste differences do not serve as a source of long run comparative advantage.

One goal of our analysis is to show how the non-homotheticity of tastes affects the characteristics of the steady state equilibria and the pattern of trade. We show that there will be a continuum of steady states for the world economy in the general case, but the potential for non-homothetic preferences alters the characteristics of the set of steady state equilibria. We first consider the case in which all goods are normal, and show that the H-O theorem must hold as long as the discount parameter is the same across countries. However, it can be violated if the discount factors are not equal across countries. The world capital stock will not be constant across the potential steady states because the distribution of world income affects demands. We also consider the case in which one of the goods is inferior, and show that this leads to the possibility that some steady states that yield higher capital stocks for both countries than other steady states.

A second objective of our analysis is to analyze the implications of non-homotheticity in demand for the dynamics of the international economy in the neighborhood of the steady state. We first establish a sufficient condition for a steady state to be a saddle point, and show that this condition is satisfied as long as both goods are normal. We also show that the condition will be satisfied when one good is inferior for one of the countries as long as the effect of inferiority is not too large. In the case discount rates are the same across countries, the steady state equilibrium must be either a source or a saddle point. However, there is a possibility of dynamic indeterminacy in the case where discount rates differ across countries.

The possibility of dynamic indeterminacy in a two country trade model typically arises due to the existence of a distortion in markets or due to market incompleteness. For example, Nishimura and Shimomura [18] derived the possibility of local indeterminacy by introducing factor-generated externalities into a dynamic two-country H-O model. Shimomura [23] and Doi et al. [11] showed that indeterminacy can arise around one of the steady states in an endowment model of international trade in which one of the goods is durable and there is a negative income effect. Bond and Driskill [9] showed that multiple steady states and indeterminacy can arise in the model with a durable good when both goods are normal as long as the exporting country has the higher marginal propensity to consume a good. The indeterminacy in the trade models with durable goods arises due to market incompleteness due to the assumption that there is no international lending or borrowing. The same market incompleteness arises in the H-O model of trade we examine. However, if there is factor price equalization along the optimal path, international lending and borrowing is redundant if the discount factors of the two countries are the same. As a result, a difference in discount factors between countries and inferiority in consumption are both necessary conditions for indeterminacy to occur.

2 The Dynamic Two-Country Heckscher-Ohlin Model

In this section we formulate the continuous-time version dynamic optimization problem for a representative country in a dynamic H-O model. By dynamic H-O model, we mean that each country has access to the same technology for producing two goods using a fixed factor (labor) and a reproducible factor (capital) under conditions of perfect competition and constant returns to scale. Good 1 is a pure consumption good, and the second good is a consumable capital good. Factors of production are assumed to be mobile between sectors within a country, but immobile internationally, and there are no markets for international borrowing and lending. We refer to the representative country as the home country: the corresponding behavioral relations for the other (foreign) country will be denoted by a "$*$."

We assume that the home (foreign) country is made up of H (H^*) households, with each household having an endowment of labor, l, and a concave utility function u defined over consumption of goods 1 and 2, c_1 and c_2. The restriction that the labor endowment per household and the household utility function are common across countries defines a sense in which each country has the same preferences, although the household preferences are not necessarily homothetic. It also means that we can distinguish between variations in the scale of the economy, which changes the number of households but keeps wealth per household constant, and variations in the wealth of a household. This distinction is useful in the case where preferences are not homothetic.

2.1 The Production Side

As to technologies, we will assume that

Assumption 1: The production function in each sector is quasi-concave and linearly homogeneous. Pure consumption good 1 is labor intensive.

Letting w denote the wage rate and r the rental on capital, the technology in sector i can be characterized by the unit cost function $\chi_i(w,r)$, $i = 1,2$. The competitive profit conditions require that

$$p \leq \chi_1(w,r), \tag{1}$$

$$1 \leq \chi_2(w,r), \tag{2}$$

where good 2 is chosen as numeraire. The aggregate stocks of labor and capital are denoted by L and K respectively. Factor market equilibrium requires that

$$1 = v_1 + v_2, \tag{3}$$

$$K/L = v_1\kappa_1(w/r) + v_2\kappa_2(w/r), \tag{4}$$

where v_i is the fraction of labor devoted to sector i and $\kappa_i(w/r) = \chi_{ir}(w,r)/\chi_{iw}(w,r)$.

Solving for w and r when (1) and (2) hold with equality, we obtain the factor prices $(w(p), r(p))$ that are consistent with production of both goods. These factor prices will satisfy full employment for $K/L \in [\kappa_1(w/r), \kappa_2(w/r)]$. We will make assumptions below regarding tastes and discount factors and depreciation rates that will ensure that any steady state equilibrium must involve a price consistent with incomplete specialization in each country. Since our analysis will focus on properties of the steady state and the behavior in the neighborhood of the steady state, we will limit our presentation of the production side to the case of incomplete specialization.

With incomplete specialization, we can express GNP as $w(p)L + r(p)K$. Our assumption that households have identical labor endowments implies that $L = Hl$. It will further be assumed that the initial endowment of capital is equally distributed across households, so we can denote the initial per household stock of capital by $k_0 = K_0/H$. This will imply identical holdings of capital across households at each point in time, $k = K/H$, which makes it convenient to express the production side on a per household basis. Applying the envelope theorem, we obtain the per household output of good i, y_i to be

$$y_1(p,k,l) = w'(p)l + r'(p)k, \quad y_2(p,k,l) = w(p)l + r(p)k - p[w'(p)l + r'(p)k]. \tag{5}$$

The supply functions are linear in k and l with incomplete specialization, where $r'(p) < 0$ since good 1 is labor intensive.

2.2 The Consumption Side

We analyze the optimization problem for a representative household that owns l units of labor under the assumption that the initial endowment of capital of a household is $k_0 = K_0/H$. We will impose the following restrictions on this utility function:

Assumption 2: The utility function is concave, with $u_{11} < 0$ and $D \equiv u_{11}u_{22} - u_{12}u_{21} > 0$ for any $(c_1, c_2) \in \{(c_1, c_2) \in \mathbb{R}^2_+ | u_i(c_1, c_2) > 0,\ i = 1, 2\}$.

The representative household is assumed to maximize the discounted sum of its utilities

$$\max \int_0^\infty u(c_1, c_2) e^{-\rho t} dt, \tag{6}$$

subject to its flow budget constraint

$$w(p)l + r(p)k = pc_1 + c_2 + \dot{k} + \delta k, \qquad k_0 \text{ given}, \tag{7}$$

where δ is the rate of depreciation on home country capital and ρ is the home country discount rate. The budget constraint reflects the assumed absence of an international capital market, since it requires that $pz_1 + z_2 = 0$, where $z_1 = c_1 - y_1$ ($z_2 = c_2 + \dot{k} + \delta k - y_2$) is the per household excess demand for good 1 (2).

Solving the current value Hamiltonian for this problem yields the necessary conditions for the choice of consumption levels, the differential equation describing the evolution of the costate variable, λ, and the transversality conditions:

$$u_1(c_1, c_2) = \lambda p, \quad u_2(c_1, c_2) = \lambda, \tag{8}$$

$$\dot{\lambda} = \lambda[\rho + \delta - r(p)], \tag{9}$$

$$\lim_{t\to\infty} k(t)\lambda(t)e^{-\rho t} = 0. \tag{10}$$

It will be useful for the subsequent analysis to invert the necessary conditions for choice of consumption levels to obtain consumption relations $c_i(p, \lambda)$ for $i = 1, 2$ and an expenditure relation $E(p, \lambda) \equiv pc_1(p, \lambda) + c_2(p, \lambda)$. The following lemma, which is proven in the Appendix, establishes some properties of these functions.

Lemma 1 (i) $\lambda c_{1\lambda} = pc_{1p} + c_{2p}$. (ii) $E_\lambda = pc_{1\lambda} + c_{2\lambda} < 0$. (iii) $c_{1p} < 0$. (iv) $E_p = c_1 + \lambda c_{1\lambda}$.

Our expenditure relation differs from the standard expenditure function in that it holds constant the marginal utility of income, rather than the level of utility. Good i is normal if $c_{i\lambda} < 0$, so (ii) establishes that goods must be normal in total. For the case in which the utility function is homothetic and has CIES of ς, we have $E(p, \lambda) = e(p)\lambda^{-\varsigma}$.

Using (7), (9), and the expenditure function, we can express $\dot{k}$ and $\dot{\lambda}$ as functions of (k, λ, p) :

$$\dot{k} = w(p)\, l + r(p)\, k - E(p, \lambda) - \delta k, \tag{11}$$

$$\dot{\lambda} = \lambda[\rho + \delta - r(p)]. \tag{12}$$

In the case of autarky, the system is closed by adding the market clearing condition for good 1 at home,

$$z_1(p, k, l, \lambda) \equiv c_1(p, \lambda) - y_1(p, k, l) = 0. \tag{13}$$

Equations (11), (12), and (13) govern the evolution of (k, λ, p) under autarky.

2.3 The Foreign Country and World Market Equilibrium

The optimization problem for a foreign household is analogous to that for the home country. The technologies of the two countries are assumed to be the same, so $w^*(p) = w(p)$ and $r^*(p) = r(p)$ for $k^*/l \in [\kappa_1(w(p)/r(p)), \kappa_2(w(p)/r(p))]$. Since household utility functions are the same across countries, we have $c_i^*(.) = c_i(.)$ and $E^*(.) = E(.)$. Substituting these relations into the solution of the foreign country's household optimization problem yields

$$\dot{k}^* = w(p^*)\,l + r(p^*)\,k^* - E(p^*, \lambda^*) - \delta^* k^* \tag{14}$$

$$\dot{\lambda}^* = \lambda^*[\rho^* + \delta^* - r(p^*)]. \tag{15}$$

The foreign autarkic equilibrium can be described by (14), (15), and $z_1(p^*, k^*, l, \lambda^*) = 0$.

In a free trade equilibrium, the price of good 1 will be equalized across countries and will be determined by the world market clearing condition for good 1,

$$Hz_1(p, k, l, \lambda) + H^* z_1(p, k^*, l, \lambda^*) = 0. \tag{16}$$

The free trade equilibrium can be solved for the evolution of $(k, k^*, \lambda, \lambda^*, p)$ using (11), (12), (14), (15), and (16). In our analysis of the free trade equilibrium, we will assume that

Assumption 3: $\theta \equiv \rho + \delta = \rho^* + \delta^*, \qquad \rho \geq \rho^*.$

This condition ensures that $\dot{\lambda}/\lambda = \dot{\lambda}^*/\lambda^*$ at each point in time as long as the conditions for factor price equalization are satisfied. This will result in $\lambda^* = m\lambda$ for some $m > 0$ along the optimal path, which simplifies the analysis by reducing the dimensionality of the system. In the case where $\rho = \rho^*$, the solution to the competitive equilibrium will also be Pareto optimal because free trade equates both the marginal rates of substitution between goods at a point and the marginal rate of substitution between goods at different points in time in this case. Therefore, opening international capital markets is unnecessary if discount factors are equal. This is the case in which all of the technology and taste parameters are identical across countries, so that countries differ only in their initial endowments of factors of production. If $\rho > \rho^*$, the home country households are more impatient and have less of a desire to accumulate capital, but the home country is a better location to place capital (because $\delta < \delta^*$), which implies that the real interest rate differs across countries under factor price equalization $(r(p) - \delta > r(p) - \delta^*)$. Without international capital markets, therefore, this solution is not Pareto optimal. Indeed, there will exist additional gains from opening international capital markets to allow foreign households to own capital located in the home country.

3 World Market Equilibrium

We begin our analysis of the world market equilibrium by deriving conditions for existence of a steady state equilibrium price, and showing that this price is the only one consistent with a steady state equilibrium under autarky or free trade. We then derive the steady state relationship between the marginal utility of income and excess demand for good 1, which can be used to characterize the steady state trade patterns and determine the conditions under which the Heckscher-Ohlin theorem holds. We conclude by analyzing the dynamics of the world equilibrium in the neighborhood of the steady state, and establishing conditions under which the initially capital abundant country remains capital abundant along the path to the steady state.

We first establish sufficient conditions for existence and uniqueness of a steady state price at which $\dot{k} = \dot{k}^* = \dot{\lambda} = \dot{\lambda}^* = 0$. Condition (12) requires that $r = \theta$ in order to have $\dot{\lambda} = 0$, and Assumption 3 ensures that this will also yield $\dot{\lambda}^* = 0$. In order for this to be consistent with incomplete specialization, the competitive profit conditions (1) and (2) must hold with equality when $r = \theta$. So, we assume that[3]

Assumption 4: $\inf\{r|\chi_2(w,r) = 1\} < \theta < \sup\{r|\chi_2(w,r) = 1\}$.

Then, (2) can be inverted to obtain a unique steady state wage rate $\tilde{w}(\theta)$ that is consistent with competitive production of good 2. Competitive production of good 1 will require a price $\tilde{p}(\theta) = \chi_1(\tilde{w}(\theta), \theta)$. This will be the only possible price consistent with a steady state when technologies are the same in all countries, since there would be no production of good 1 (2) with $r = \theta$ for $p < \tilde{p}(\theta)$ ($p > \tilde{p}(\theta)$). This steady state relative factor price, $\tilde{w}(\theta)/\theta$, will determine a unique capital labor ratio employed in production in sector i in the steady state, $\kappa_i(\tilde{w}(\theta)/\theta)$. Letting $\tilde{\kappa}_{\min}(\theta) \equiv \kappa_1(\tilde{w}(\theta)/\theta)$ and $\tilde{\kappa}_{\max}(\theta) \equiv \kappa_2(\tilde{w}(\theta)/\theta)$, full employment will require that $k/l, k^*/l \in [\tilde{\kappa}_{\min}(\theta), \tilde{\kappa}_{\max}(\theta)]$.

Lemma 2 *Let Assumption 4 hold. Then, there will exist a unique steady state wage, $\tilde{w}(\theta)$, and price, $\tilde{p}(\theta)$, consistent with a steady state equilibrium with incomplete specialization.*

The household budget constraint (11) requires that the per household steady state capital stock satisfy[4]

$$\tilde{k}(\lambda, l, \rho) = \frac{E(\tilde{p}, \lambda) - \tilde{w}l}{\rho}, \tag{17}$$

if $\dot{k} = 0$. Equation (17) illustrates a negative relationship between the steady state values of λ and k. Higher levels of the capital stock are associated with higher income and expenditure, which requires a lower marginal utility of income. Since $\tilde{k}_\lambda(\lambda, l, \rho) < 0$, we can invert (17) to define a range of feasible steady state values of λ, $\Lambda(l, \rho) = (\tilde{\lambda}_{\min}(l, \rho), \tilde{\lambda}_{\max}(l, \rho))$, where $\tilde{\lambda}_{\min}$ and $\tilde{\lambda}_{\max}$ are defined as the solutions to $\tilde{k}(\lambda, l, \rho)/l = \tilde{\kappa}_{\max}$ and $\tilde{k}(\lambda, l, \rho)/l = \tilde{\kappa}_{\min}$, respectively.[5]

[3] The right hand inequality would fail in the case of a fixed coefficients production function in sector 2 where the output per unit capital is less than θ. In that case the productivity of capital is too low to justify replacement and the existing stock of capital would be allowed to depreciate. The left hand inequality could fail if the marginal product of capital in sector 2 has a lower bound exceeding θ, as could arise with a CES production function where the elasticity of substitution exceeds 1.

[4] In order to simplify the presentation, we suppress the dependence of steady state values on θ, which is fixed, in the following discussion.

[5] In the following, we suppress l and ρ as aruguments of $\tilde{\lambda}_{\min}$ and $\tilde{\lambda}_{\max}$ when there is no ambiguity.

Substituting (17) into the per household excess demand for good 1, we obtain a steady state (per household) excess demand function

$$\tilde{z}_1(\lambda, l, \rho) = c_1(\tilde{p}, \lambda) - y_1(\tilde{p}, \tilde{k}(\lambda, l, \rho), l). \tag{18}$$

The autarkic steady state equilibrium is obtained by solving $\tilde{z}_1(\lambda, l, \rho) = 0$. Since good 1 is labor intensive, $y_1(\tilde{p}, \tilde{k}(\tilde{\lambda}_{\min}, l, \rho), l) = 0$ and $y_1(\tilde{p}, \tilde{k}(\tilde{\lambda}_{\max}, l, \rho), l) = [E(\tilde{p}, \tilde{\lambda}_{\max}) + \delta\tilde{k}(\tilde{\lambda}_{\max}, l, \rho)]/\tilde{p}$. The former implies that $\tilde{z}_1(\tilde{\lambda}_{\min}, l, \rho) = c_1(\tilde{p}, \tilde{\lambda}_{\min}) > 0$ and the latter that $\tilde{z}_1(\tilde{\lambda}_{\max}, l, \rho) = -[c_2(\tilde{p}, \tilde{\lambda}_{\max}) + \delta\tilde{k}(\tilde{\lambda}_{\max}, l, \rho)]/\tilde{p} < 0$, which ensures the existence of a steady state equilibrium.[6]

To establish conditions for the uniqueness of the autarkic steady state equilibrium, we differentiate (18) with respect to λ and use (5) to obtain

$$\tilde{z}_{1\lambda} = c_{1\lambda} - \frac{r'(\tilde{p})E_\lambda}{\rho}, \tag{19}$$

which is negative when good 1 is normal, since good 1 is labor intensive ($r' < 0$). With the normality in consumption, therefore, the equilibrium will be unique. In the rest of this section except for subsection 3.2 on equilibrium dynamics, we will assume that

Assumption 5: Both goods are normal at all levels of income.

Figure 1 illustrates the steady state per household excess demand function. If $\rho = \rho^*$, the foreign excess demand function will coincide with that of the home country and the autarkic steady states for the two countries will exhibit the same prices and the same marginal utility of income. To see the effect of differences in the rate of time preference across countries (holding $\rho + \delta$ constant), we totally differentiate $\tilde{z}_1(\lambda, l, \rho) = 0$ and to obtain

$$\frac{d\lambda^A}{d\rho} = -\frac{k^A r'(\tilde{p})}{\rho \tilde{z}_{1\lambda}} < 0, \qquad \frac{\partial k^A}{\partial \rho} = -\frac{k^A c_{1\lambda}}{\rho \tilde{z}_{1\lambda}} < 0. \tag{20}$$

Since both the numerator and denominator in the first expression are negative, an increase in the discount factor will be associated with a higher steady state utility level in autarky. This result can be understood by noting that (17) requires that $\tilde{k}(\lambda, l, \rho)$ is decreasing in ρ, because a given expenditure level can be sustained with a lower capital stock. Referring to Figure 1, the decrease in $\tilde{k}(\lambda, l, \rho)$ will shift $\tilde{z}_1$ downward, resulting in a lower autarkic value of λ.

[6] Ventura [25] utilizes a model in which there are two traded intermediate goods (one produced using labor only, the other using capital only) that are combined to produce a non-traded final good. The final good can be either consumed or used as a capital good, and preferences have a constant intertemporal elasticity of substitution. Although the strucutre is slightly different from the one assumed here, it generates a steady state excess demand with properties virtually identical to those derived here. Letting the final good be chosen as numeraire and assuming good 1 is the labor intensive intermediate, we have $E(1, \lambda) = \lambda^{-\varsigma}$ and $\tilde{k}(\lambda, l, \rho) = (\lambda^{-\varsigma} - \tilde{w}l)/\rho$. Letting $\chi(w, r)$ denote the unit cost function for the final good, we have $s_1 = l$ and $d_1 = \chi_w(\tilde{w}, \theta)\lambda^{-\varsigma}$, where s_1 and d_1 are the supply and demand for intermediate good 1 ($\delta = 0$ is assumed), respectively.

The second result shows that if good 1 is normal, an increase in the discount factor will decrease the steady state capital stock. Note that the increase in ρ has two effects on the autarkic steady state capital stock k^A. One is a direct negative effect due to $\partial\tilde{k}/\partial\rho < 0$ and the other originates from the decrease in λ^A ($d\lambda^A/d\rho < 0$), which is positive from $\partial\tilde{k}/\partial\lambda < 0$. The negative effect dominates the positive one.

These results can be summarized as

Proposition 1 *:(a) There is unique autarkic steady state values* (λ^A, k^A) *for the home country satisfying* $\tilde{z}_1(\lambda^A, l, \rho) = 0$ *and* $k^A = \tilde{k}(\lambda^A, l, \rho)$. *The foreign country will have the same autarkic steady state price as the home country.*

(b) If $\rho = \rho^*$, *the home and foreign countries will have the same autarkic capital stocks and utility levels. If* $\rho > \rho^*$, *the home country will have a higher autarkic utility level than the foreign country. The home capital stock will be lower.*

3.1 Free Trade Steady States with Normal Goods

A steady state equilibrium with trade exists for any values of λ and λ^* at which the world market for good 1 clears,

$$Z(\lambda, \lambda^*, l) \equiv H\tilde{z}_1(\lambda, l, \rho) + H^*\tilde{z}_1(\lambda^*, l, \rho^*) = 0. \tag{21}$$

It follows from Proposition 1 that $(\lambda^A, \lambda^{A*})$ is a solution to (21), and that these solutions are in the interior of the respective regions. Since the excess demand functions are continuous, it follows that there will be a continuum of pairs (λ, λ^*) satisfying (21). If goods are normal, these pairs must satisfy $d\lambda^*/d\lambda|_{dZ=0} = -[H\tilde{z}_{1\lambda}(\lambda, l, \rho)]/[H^*\tilde{z}_{1\lambda}(\lambda^*, l, \rho^*)] < 0$. Since there is a one to one relationship between k and λ from (17), the following Proposition holds.[7]

Proposition 2 *: There is a continuum of per household capital stocks* (k, k^*) *consistent with a steady state equilibrium. These steady states can be described by a continuous function* $\varphi(k)$ *defined on a non-empty interval, where the pair of per household capital stocks* $(k, \varphi(k))$ *is a steady state equilibrium and*

$$\varphi'(k) = -\frac{H}{H^*}\left[\frac{\rho\left(\frac{\tilde{p}c_{1\lambda}}{E_\lambda} - \frac{\tilde{p}r'(\tilde{p})}{\rho}\right)}{\rho^*\left(\frac{\tilde{p}c^*_{1\lambda}}{E^*_\lambda} - \frac{\tilde{p}r'(\tilde{p})}{\rho^*}\right)}\right] < 0. \tag{22}$$

The world capital stock is the same across all of the steady states if preferences are homothetic and $\rho = \rho^*$. *If the marginal propensity to consume good 1 is increasing (decreasing) in* λ *and* $\rho = \rho^*$, $\varphi''(k) > (<)\ 0$.

[7] Hereafter, we attach "*" to the values of functions for foreign country when we omit their arguments, e.g. $\tilde{z}^*_{1\lambda}$ denotes $\tilde{z}_{1\lambda}(\lambda^*, l, \rho^*)$.

When preferences are homothetic and $\rho = \rho^*$ (hence $\delta = \delta^*$), the bracketed expression equals 1 and the world capital stock is constant in all of the trading equilibria. This is due to the fact that a transfer of capital from one country to another has no effect on world outputs as long as both countries remain incompletely specialized. This transfer will also leave world demand unaffected if tastes are identical and homothetic and $\rho = \rho^*$, so the world stock of capital is constant across all of the potential steady states for the world economy. This is the result obtained by Chen [10] and Ventura [25]. However, if good 1 is a necessity (i.e. the income elasticity of good 1 is less than one), the marginal propensity to consume good 1, $\tilde{p}c_{1\lambda}/E_\lambda$, will be increasing in λ. Since an increase in k in the steady state is associated with a lower value of λ and a higher value of λ^*, the bracketed expression in (22) will be decreasing in k when good 1 is a necessity. This yields $\varphi''(k) > 0$, as illustrated in Figure 2 (curve (i)), so that a transfer of income from the poor country to the rich country will reduce demand for (labor intensive) good 1. As a result, the world capital stock will be higher the greater the difference in income between the two countries. This effect is reversed when good 1 is a luxury good, and the world capital stock is smaller the greater the difference in income between the countries (curve (ii) in Figure 2).

The following result on steady state trade patterns can be established using Figure 1.

Proposition 3 *If $\rho = \rho^*$, then the steady state trade pattern must satisfy the static H-O theorem. If $\rho > \rho^*$, there will exist some steady states for which the static H-O theorem is violated.*

If $\rho = \rho^*$, then the steady state trade pattern must satisfy the static H-O theorem, because a country will export the capital intensive good in a steady state iff $\lambda < \lambda^A$. If $\lambda < \lambda^A$ in a trading equilibrium it must also be the case that the home country is capital abundant relative to the foreign country when they have the same discount factor (i.e. $\lambda < \lambda^A < \lambda^*$). The marginal utility of income of the two countries must always lie on opposite sides of the common autarky value in a free trade equilibrium, as can be seen from Figure 1 for the case where $\rho = \rho^* = \rho_0$. It should be noted that a similar result would be obtained in a static model where household preferences are identical and the per household stock of capital is exogenously given, since the per household excess demand functions for the two countries could be characterized using a single static excess demand function as in Figure 1.[8] Differences in per household consumption in this case must arise from differences in per household capital stocks, and the resulting differences in demand cannot be larger than the supply differences when goods are normal in consumption.

If $\rho > \rho^*$, on the other hand, it is possible to observe violations of the H-O theorem in the steady state trade pattern. Consider the case where $\rho = \rho_1 > \rho^* = \rho_0$ in Figure 1. At the autarkic states

[8] The household budget constraint for the static model is $wl + rk = E(p, \lambda)$, which yields a common excess demand function $z_1(\lambda, l)$ at a given world price p if factor prices are equalized. Thus, relaxing the assumption of homotheticity at the household level does not lead to violations of the H-O theorem in the static model. Violations could arise in the static model if utility functions differ across countries or if the per capita labor endowment differs in the case where utility functions are identical but not homothetic. Similarly, violations of the H-O theorem in the steady state of the present model would arise in this case as well.

for each country we have $\lambda^A < \lambda^{A*}$ from (20) and there are steady state equilibria with trade where $\lambda < \lambda^A < \lambda^{A*} < \lambda^*$ and $\tilde{z}_1 > 0 > \tilde{z}_1^*$ hold. Note that (20) implies that $k^A < k^{A*}$ holds since good 1 is labor intensive. We see that, therefore, at trading equilibria where the steady state value of λ is smaller but sufficiently close to λ^A, the capital abundant foreign country will be exporting labor intensive good 1: $k^A < k < k^* < k^{A*}$ and $\tilde{z}_1 > 0 > \tilde{z}_1^*$.

3.2 Equilibrium Dynamics

The equilibrium path of $(p, \lambda, \lambda^*, k, k^*)$ is described by (11), (12), (14), (15), and (16). In this section we show how the system can be reduced to a 3 equation system in (k, k^*, λ) and then analyze the dynamics of this system in the neighborhood of the steady state.

We can first simplify the dynamic system by noting that as long as the country's relative factor supplies are consistent with incomplete specialization, factor price equalization will imply $\lambda^* = m\lambda$. We then use the world market clearing condition (16) to solve for $p(k, k^*, \lambda)$. We can invert (16) because $z_{1p}, z_{1p}^* < 0$ follows from Lemma 1 (iii) and the concavity of the production function. This function expresses the world price as a function of the remaining state variables, and has the property that

$$\frac{\partial p}{\partial \lambda} = -\frac{Hc_{1\lambda} + H^* m c_{1\lambda}^*}{Hz_{1p} + H^* z_{1p}^*}, \qquad \frac{\partial p}{\partial k} = \frac{\partial p}{\partial k^*}\frac{H}{H^*} = \frac{Hr'}{Hz_{1p} + H^* z_{1p}^*} > 0. \tag{23}$$

The first comparative static result in (23) shows that an increase in λ, which is equivalent to a decrease in utility in each country (with $\lambda^* = m\lambda$), will reduce the price of good 1 if it is a normal good in world demand. The second result shows that an increment of capital has the same impact on the relative price of good 1 regardless of where it is located as a result of the factor price equalization property, and will raise the relative price.

Using these results, the system of differential equations can be expressed as

$$\dot{k} = w\left(p(k, k^*, \lambda)\right) l + r\left(p(k, k^*, \lambda)\right) k - E(p(k, k^*, \lambda), \lambda) - \delta k, \tag{24}$$

$$\dot{k}^* = w\left(p(k, k^*, \lambda)\right) l + r\left(p(k, k^*, \lambda)\right) k^* - E(p(k, k^*, \lambda), m\lambda) - \delta^* k^*, \tag{25}$$

$$\dot{\lambda} = \lambda[\rho + \delta - r(p(k, k^*, \lambda))]. \tag{26}$$

We will use this system to analyze the trade and capital accumulation on the equilibrium path, and to derive results on the dynamics in the neighborhood of the steady state equilibria.

We evaluate the elements of a Jacobian of the dynamical system (equations (24)-(26)), given m, to study the local dynamics around the stationary state. Differentiating this system and using the comparative statics results from (23), we obtain the Jacobian J for the dynamic system and the characteristic equation,

$$\det[xI - J]$$

$$= \det \begin{bmatrix} x - \left[\rho - (z_1 + \lambda c_{1\lambda})\frac{\partial p}{\partial k}\right] & (z_1 + \lambda c_{1\lambda})\frac{\partial p}{\partial k}\frac{H^*}{H} & (z_1 + \lambda c_{1\lambda})\frac{\partial p}{\partial \lambda} + E_\lambda \\ (z_1^* + m\lambda c_{1\lambda}^*)\frac{\partial p}{\partial k} & x - \left[\rho^* - (z_1^* + m\lambda c_{1\lambda}^*)\frac{\partial p}{\partial k}\frac{H^*}{H}\right] & (z_1^* + m\lambda c_{1\lambda}^*)\frac{\partial p}{\partial \lambda} + mE_\lambda^* \\ \lambda r'\frac{\partial p}{\partial k} & \lambda r'\frac{\partial p}{\partial k}\frac{H^*}{H} & x + \lambda r'\frac{\partial p}{\partial \lambda} \end{bmatrix}. \tag{27}$$

Let $\Gamma \equiv \left(-r'\frac{\partial p}{\partial k}\right)^{-1}$, which is positive from (23) and reflects the fact that an increase in capital reduces the world return to capital by lowering the relative price of the capital intensive good. Defining $J(x) \equiv \Gamma \det[xI - J]$, it is shown in the Appendix that we can use the world market equilibrium condition and the fact that $r'\frac{\partial p}{\partial \lambda} + (c_{1\lambda} + m\frac{H^*}{H}c_{1\lambda}^*)\frac{\partial p}{\partial k} = 0$ to obtain

$$\begin{aligned} J(x) &= \Gamma x^3 - \Gamma(\rho + \rho^*)x^2 \\ &+ \left[\frac{(\rho^* - \rho)z_1}{r'} + \Gamma\rho\rho^* - \frac{\lambda}{r'}\left(\rho\tilde{z}_{1\lambda} + \rho^* m\frac{H^*}{H}\tilde{z}_{1\lambda}^*\right)\right]x \\ &+ \frac{\lambda\rho\rho^*}{r'}\left(\tilde{z}_{1\lambda} + m\frac{H^*}{H}\tilde{z}_{1\lambda}^*\right). \end{aligned} \tag{28}$$

This characteristic equation can be used to derive the local dynamics of the system in the neighborhood of a steady state trading equilibrium.

We begin with the following Lemma, which establishes conditions for determining the number of negative roots.

Lemma 3 *If $J(0)$ is positive, then the characteristic equation has one negative root. On the other hand, if $J(0)$ is negative, then the equation has two roots with negative real parts when $J(\rho + \rho^*)$ is negative, and it has no roots with negative real parts otherwise.*

Proof. $J(x)$ can be rewritten as

$$J(x) = \Gamma x^3 - \Gamma(\rho + \rho^*)x^2 + J'(0)x + J(0).$$

From application of the Routh theorem (1905, p. 226), the number of the roots of $J(x) = 0$ with positive real parts equals the number of changes in signs in the following sequence:

$$\Gamma, \quad -\Gamma(\rho + \rho^*), \quad J'(0) + \frac{J(0)}{(\rho + \rho^*)}, \quad J(0).$$

Let $J(0) > 0$. Then the number of changes is two irrespective of the sign of the third term and the characteristic equation has one negative root. Let $J(0) < 0$. Note that the sign of the third term is equal to the sign of $J(\rho + \rho^*)$, since $J(\rho + \rho^*) = J'(0)(\rho + \rho^*) + J(0)$. Therefore, if $J(\rho + \rho^*) < 0$, the number of changes is one and the equation has two roots with negative real parts. On the other hand, if $J(\rho+\rho^*) > 0$, the number is three and it has no roots with negative real parts. ■

Lemma 3 establishes that a steady state equilibrium will be a saddle point if $J(0) > 0$, which yields the following result using (28) and (19).

Proposition 4 *A free trade steady state equilibrium is a saddle point.*

Proof. $J(0) = \frac{\lambda\rho\rho^*}{r'}\left(\tilde{z}_{1\lambda} + m\frac{H^*}{H}\tilde{z}^*_{1\lambda}\right) > 0$ since $r' < 0$ and $(\tilde{z}_{1\lambda} + m\frac{H^*}{H}\tilde{z}^*_{1\lambda}) < 0$ by the normality assumption. ■

3.3 Capital Accumulation and Trade on the Optimal Path

The analysis of the steady state trade pattern examined whether the country that is capital abundant in the steady state would export the capital intensive good. An alternative approach to the question of comparative advantage is to ask whether the country that is capital abundant at an arbitrary point in time will export the capital intensive good along the optimal path and/or in the steady state. In this section we establish that this result must hold for initial conditions sufficiently close to the steady state in the normal good case.

Proposition 5 *If $\rho = \rho^*$ and factor price equalization holds along the optimal path with $r(p(t)) > \delta$, then the country that is capital abundant at $t = 0$ will be capital abundant and will export the capital intensive good for $t > 0$.*

Proof. The analysis of the characteristic equation established that the optimal path converges to the steady state when goods are normal. The assumption that $r(p) > \delta$ will hold for initial conditions sufficiently close to the steady state, since $r(p) = \rho + \delta$ in the steady state. We first show that if $k_0 < k_0^*$ and $\rho = \rho^*$, then $m < 1$ and $k(t) < k^*(t)$ for all $t > 0$. Suppose that $m \geq 1$. Then, capital stock in home is greater than or equal to that in foreign at the steady state: $\tilde{k}(\lambda, l, \rho) \geq \tilde{k}(m\lambda, l, \rho)$. Using (24) and (25) we obtain

$$\dot{k} - \dot{k}^* = [r(p) - \delta](k - k^*) + [E(p, m\lambda) - E(p, \lambda)] + (\rho - \rho^*)k^*. \tag{29}$$

Therefore, for all t, $\dot{k}(t) < \dot{k}^*(t)$ and $k(t) < k^*(t)$, since $k_0 < k_0^*$, $\rho = \rho^*$, and $E(p, \lambda) \geq E(p, m\lambda)$ if $m \geq 1$. This is a contradiction with $\tilde{k}(\lambda, l, \rho) \geq \tilde{k}(m\lambda, l, \rho)$. Thus, $m < 1$ holds if $k_0 < k_0^*$ and $\rho = \rho^*$. Now suppose that there is a time $t' \in (0, \infty)$ at which $k(t') \geq k^*(t')$. Then, for all $t \geq t'$, we have $\dot{k}(t) > \dot{k}^*(t)$ and $k(t) > k^*(t)$ from (29), which contradicts with $\tilde{k}(\lambda, l, \rho) < \tilde{k}(m\lambda, l, \rho)$. Therefore, the country with an initial capital abundance will have a higher steady state expenditure level when $\rho = \rho^*$. To establish the trade pattern along the path, note that the foreign country will import labor intensive good 1 at time $t \geq 0$ if $z_1(p, k, l, \lambda) < z_1(p, k^*, l, m\lambda)$, which requires

$$c_1(p, \lambda) - c_1(p, m\lambda) < y_1(p, k, l) - y_1(p, k^*, l). \tag{30}$$

With $m < 1$, the left hand side of (30) will be negative if good 1 is normal. The right hand side will be positive since good 1 is labor intensive and $k(t) < k^*(t)$. ∎

In order for the initially capital scarce country to leapfrog the initially capital abundant country, it must accumulate capital more rapidly by spending less than the higher income country (since its current income must also be lower if $r(p(t)) > \delta$). However, this is impossible when $\rho = \rho^*$ because the expenditure levels are based on permanent income.

Note however that initial capital abundance might be consistent with a lower steady state utility if $\rho > \rho^*$.[9]

4 An Example with Inferior Goods

The assumption that goods are normal for all levels of expenditure played a key role in establishing both the existence and uniqueness of the steady state equilibrium. The proof of existence relied on being able to invert (17) to obtain $0 < \tilde{\lambda}_{\min} < \tilde{\lambda}_{\max} < \infty$, which may not be possible if preferences exhibit either a satiation level or a minimum subsistence level. A second role of the normality assumption is to ensure the monotonicity of the steady state excess demand functions in (19), which guaranteed that the steady state equilibrium is unique. If the steady state excess demands are non-monotonic, we have the possibility that there are multiple autarkic steady states, and that the relationship $\varphi(k)$ identifying possible foreign country capital stocks is a correspondence. Finally, the monotonicity of the excess demand functions guaranteed that the steady state equilibrium is a saddle point.

In this section, we illustrate how the results of Propositions 1-5 may be altered by considering a specific functional form for the utility function that allows for satiation and inferiority in consumption.

Assumption 5': Household preferences are represented by

$$u(c_1, c_2) = \frac{\alpha(c_1^{1-\sigma} - 1)}{1-\sigma} + \frac{\beta(c_2^{1-\eta} - 1)}{1-\eta} - \gamma c_1 c_2, \quad \text{for } (c_1, c_2) \in \mathbb{R}_+^2, \tag{31}$$

where parameters satisfy the following restrictions $\alpha, \beta, \gamma, \sigma$ and $\eta > 0$ and $\sigma\eta > 1$.

The parameter restrictions ensure that the utility function is strictly concave for all (c_1, c_2) for which $u_i(c_1, c_2) > 0$ for $i = 1, 2$. The following Lemma establishes that this utility function exhibits satiation at finite consumption levels, and that good 1 is inferior in the neighborhood of the satiation point.[10]

[9] Suppose that $k_0 < k_0^*$ and $m > 1$. The possibility of $k(t) > k^*(t)$ for some $t > 0$ cannot be ruled out because the existence of a time s such that $k(s) - k^*(s) = 0$ and $\dot{k}(s) - \dot{k}^*(s) > 0$ cannot be ruled out using (29) because the second term is negative and the third term is positive.

[10] For more details see Appendix and Iwasa and Shimomura [14].

Lemma 4 *The utility function (31) yields unique solutions $c_i(p,\lambda)$ to (8) for any positive p and λ. These solutions have the following properties:*

(i) For any $p>0$, $\lim_{\lambda\to 0} c_1(p,\lambda)=\bar{c}_1$ and $\lim_{\lambda\to 0} c_2(p,\lambda)=\bar{c}_2$, while $\lim_{\lambda\to\infty} c_i(p,\lambda)=0$, $i=1,2$, where $\bar{c}_1 \equiv \left(\frac{\alpha^\eta}{\beta\gamma^{\eta-1}}\right)^{\frac{1}{\sigma\eta-1}}$ and $\bar{c}_2 \equiv \left(\frac{\beta^\sigma}{\alpha\gamma^{\sigma-1}}\right)^{\frac{1}{\sigma\eta-1}}$;

(ii) If $\bar{c}_2/\eta\bar{c}_1 > \tilde{p}$, then there is some $\lambda^0>0$ such that $c_{1\lambda}(\tilde{p},\lambda)$ is positive (negative) when λ is smaller (greater) than λ^0, while $c_{2\lambda}(\tilde{p},\lambda)$ is always negative.

It can be seen from (19) that in order for the excess demand function to be non-monotonic in λ, labor intensive good 1 must be inferior and its inferiority must be sufficiently large.

From (5) and (17), the excess demand function (18) can be written as follows:

$$\tilde{z}_1(\lambda,l,\rho) = -\frac{r'(\tilde{p})\left[\zeta_1(\rho)c_1(\tilde{p},\lambda)+c_2(\tilde{p},\lambda)-\zeta_2(\rho)l\right]}{\rho}, \tag{32}$$

where

$$\zeta_1(\rho) \equiv \tilde{p} - \frac{\rho}{r'(\tilde{p})} > 0 \quad \text{and} \quad \zeta_2(\rho) \equiv \tilde{w} - \frac{\rho w'(\tilde{p})}{r'(\tilde{p})} > 0.$$

Since this utility function exhibits a satiation level of consumption, $\bar{k}(l,\rho) = \lim_{\lambda\to 0}\tilde{k}(\lambda,l,\rho)$ is finite and decreasing in l (See Figure 3). Note that $\bar{l} \equiv (\tilde{p}\bar{c}_1+\bar{c}_2)/(\rho\tilde{\kappa}_{\max}+\tilde{w})$ is a solution to $\bar{k}(l,\rho)/l = \tilde{\kappa}_{\max}$. If $l>\bar{l}$, then $\tilde{\kappa}_{\max} > \bar{k}(l,\rho)/l$. It implies that the satiation level is reached before the capital labor ratio at which specialization in the capital intensive good occurs. Therefore, the set of feasible steady state values of λ is restricted by the possibility of satiation, $\Lambda(l,\rho) = (0,\tilde{\lambda}_{\max}(l,\rho))$. So, we redefine $\tilde{\lambda}_{\min}$ and $\tilde{\lambda}_{\max}$ as follows:

$$\tilde{\lambda}_{\min}(l,\rho) = \min\{\lambda\geq 0 | \tilde{k}(\lambda,l,\rho)/l \leq \tilde{\kappa}_{\max}\} \text{ and } \tilde{\lambda}_{\max}(l,\rho) = \max\{\lambda \geq 0|\tilde{k}(\lambda,l,\rho)/l \geq \tilde{\kappa}_{\min}\}.$$

In order for the steady state excess demand to be non-monotonic, $\zeta_1(\rho)c_{1\lambda}(\tilde{p},\lambda)+c_{2\lambda}(\tilde{p},\lambda)$ must change sign on $\Lambda(l,\rho)$. It follows from Lemma 4 that if $\bar{c}_2/\eta\bar{c}_1 > \tilde{p}$, the steady state excess demand for good 1 must be decreasing in λ for $\lambda \geq \lambda^0$. The following Lemma (proven in the Appendix) establishes a set of parameter values for the preferences under which there will be a critical value $\hat{\lambda}(\rho) < \lambda^0$ such that the excess demand function $\tilde{z}_1(\lambda,l,\rho)$ defined in (32) is increasing (decreasing) in λ for $\lambda < (>) \hat{\lambda}(\rho)$.

Assumption 6: $\xi \equiv \bar{c}_2/(\eta\bar{c}_1) > \tilde{p}$ and

$$\rho > -r'(\tilde{p}) \max\left\{\frac{\sigma\eta\xi^2-\tilde{p}^2}{\tilde{p}}, \frac{\sigma\eta\xi^2-2\tilde{p}\xi+\tilde{p}^2}{\xi-\tilde{p}}\right\}. \tag{33}$$

Lemma 5 *If Assumptions 5' and 6 hold, $\tilde{z}_1(\lambda,l,\rho)$ is strictly concave in λ for $\lambda\in[0,\lambda^0]$ and there exists a critical value $\hat{\lambda}(\rho)\in(0,\lambda^0)$ such that $\tilde{z}_1(\lambda,l,\rho)$ is increasing (decreasing) in λ for $\lambda < (>)$ $\hat{\lambda}(\rho)$.*

The first inequality in (33) ensures that the excess demand function is strictly concave in λ for $\lambda \in [0, \lambda^0]$, while the second is required for $\tilde{z}_{1\lambda}(0, l, \rho) > 0$. Taken together, these restrictions imply the existence of $\hat{\lambda}(\rho)$, where $\hat{\lambda}(\rho)$ is increasing in ρ.[11] The fact that the steady state excess demand function is increasing (decreasing) for λ values less (greater) than the critical value leads to the possibility of two autarkic steady state equilibria. Since excess demand is linearly decreasing in l, there will exist a value l_1 satisfying $\tilde{z}_1(\hat{\lambda}(\rho), l, \rho) = 0$ and a value l_0 such that $\lim_{\lambda \to 0} \tilde{z}_1(\lambda, l, \rho) = 0$. Figure 4 shows how the excess demand functions shift downward with increases in l. For $l \in (l_0, l_1)$, there will be two steady state equilibria, denoted by $\lambda^L(l)$ and $\lambda^H(l)$ with $\lambda^L(l) < \lambda^H(l)$, while there will be a unique autarkic steady state equilibrium when $l < l_0$ (See Figure 4). Notice that for all $l \in (l_0, l_1)$, $\tilde{\lambda}_{\min}(l, \rho) = 0$ and $\tilde{\lambda}_{\max}(l, \rho) > \hat{\lambda}(\rho)$ hold,[12] and hence $\lambda^L(l), \lambda^H(l) \in \Lambda(l, \rho)$.

Proposition 6 *For $l \in (l_0, l_1)$, there will be two autarkic steady state equilibria and for $l > l_1$ there will exist no autarkic steady state equilibria.*

The failure of an autarkic steady state equilibrium to exist results from the fact that when l is sufficiently high, the output of the labor intensive good per household is so high that it exceeds the demand for all values of k at which households are not satiated.

4.1 Steady State Equilibria with Trade

We now turn to a characterization of the steady state capital stocks and trade patterns that are consistent with a free trade equilibrium when Assumptions 5' and 6 are satisfied.

Let

$$A(l) \equiv \left\{ (\lambda, \lambda^*) \in \mathbb{R}_+^2 | Z(\lambda, \lambda^*, l) \geq 0 \right\}.$$

Then, given l, the set of steady state pairs, (λ, λ^*), lies on the boundary of $A(l)$. Lemma 5 ensures

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[11] From (32), we obtain

$$\tilde{z}_{1\lambda}(\lambda, l, \rho) = -\frac{r'(\tilde{p})}{\rho} E_\lambda(\tilde{p}, \lambda) + c_{1\lambda}(\tilde{p}, \lambda).$$

Then, totally differentiating of $\tilde{z}_{1\lambda}(\hat{\lambda}, l, \rho) = 0$ with respect to ρ and $\hat{\lambda}$ yields

$$\frac{r'(\tilde{p})}{\rho^2} E_\lambda(\tilde{p}, \hat{\lambda}) d\rho + \tilde{z}_{1\lambda\lambda}(\hat{\lambda}, l, \rho) d\hat{\lambda} = 0.$$

Therefore,

$$\frac{\partial \hat{\lambda}}{\partial \rho} = -\frac{r'(\tilde{p}) E_\lambda(\tilde{p}, \hat{\lambda})}{\rho^2 \tilde{z}_{1\lambda\lambda}(\hat{\lambda}, l, \rho)}.$$

It is clear from Lemmas 1 and 5 that the numerator on the right-hand side of the equation above is positive, while the denominator is negative, i.e. $\partial \hat{\lambda} / \partial \rho > 0$.

[12] Since $\bar{l}$ is the solution to $\tilde{\kappa}_{\max} = \bar{k}(l, \rho)/l$ with $\bar{k}(l, \rho) = [E(\tilde{p}, 0) - \tilde{w}l]/\rho$, $\tilde{\lambda}_{\min}(\bar{l}, \rho) = 0$ and $\tilde{z}_1(0, \bar{l}, \rho) = \bar{c}_1 > 0$ from the arguments below (18). Since $\tilde{z}_1(0, l_0, \rho) = 0$, we have $l_0 > \bar{l}$, which implies $\bar{k}(l_0, \rho)/l_0 < \tilde{\kappa}_{\max}$. So, for $l \in (l_0, l_1)$, $\bar{k}(l, \rho)/l < \tilde{\kappa}_{\max}$, i.e., $\tilde{\lambda}_{\min}(l, \rho) = 0$. It is clear from $\tilde{z}_1(\hat{\lambda}(\rho), l_1, \rho) = 0$ that for $l \leq l_1$, $\tilde{z}_1(\hat{\lambda}(\rho), l, \rho) \geq 0$, and hence $\tilde{\lambda}_{\max}(l, \rho) > \hat{\lambda}(\rho)$.

that, given $l > 0$,

$$Z_{\lambda\lambda}, Z_{\lambda^*\lambda^*} < 0 \text{ for } (\lambda, \lambda^*) \in B \equiv \left\{(\lambda, \lambda^*) \in \mathbb{R}^2_+ | \lambda, \lambda^* \leq \lambda^0\right\}$$
$$\text{and } Z_{\lambda\lambda^*} = 0,$$

that is, the function Z is strictly concave in (λ, λ^*) on B and achieves its maximum at $(\lambda, \lambda^*) = (\hat{\lambda}(\rho), \hat{\lambda}(\rho^*))$.

First, we consider the symmetric case ($\rho = \rho^*$ and $H = H^*$). Since the foreign excess demand function coincides with that of the home country when $\rho = \rho^*$, the free trade equilibria can be found as in Figure 4. For $l \in (l_0, l_1)$, the pairs, $(\lambda, \lambda^*) = (\lambda^L(l), \lambda^L(l))$, $(\lambda^H(l), \lambda^L(l))$, $(\lambda^L(l), \lambda^H(l))$, and $(\lambda^H(l), \lambda^H(l))$, are all autarkic free trade equilibria, and hence there will be a continuum of free trade equilibria. Figure 5 illustrates the set of equilibrium pairs. It is the solid locus for $l = l_0$, the dashed locus for $l = \hat{l} \equiv (l_0 + l_1)/2$, and the point, $(\lambda, \lambda^*) = (\hat{\lambda}(\rho), \hat{\lambda}(\rho))$ for $l = l_1$. Notice that for $l \in (l_0, \hat{l})$, the positively sloped curve intersects the horizontal or vertical axis, because $\tilde{z}_1(0, l, \rho) + \tilde{z}_1(\hat{\lambda}(\rho), l, \rho) > 0$ holds for such l values.[13]

Corresponding to the slopes of the excess demand functions of the respective countries at the free trade steady state, we define three types of steady state pairs in the following.

Type (i) (λ, λ^*) with $\tilde{z}_{1\lambda}$ and $\tilde{z}^*_{1\lambda} > 0$ (i.e. $\lambda < \hat{\lambda}(\rho)$ and $\lambda^* < \hat{\lambda}(\rho^*)$);

Type (ii) (λ, λ^*) with $\tilde{z}_{1\lambda}\tilde{z}^*_{1\lambda} < 0$ (i.e. $\lambda > \hat{\lambda}(\rho)$ and $\lambda^* < \hat{\lambda}(\rho^*)$, or $\lambda < \hat{\lambda}(\rho)$ and $\lambda^* > \hat{\lambda}(\rho^*)$);

Type (iii) (λ, λ^*) with $\tilde{z}_{1\lambda}$ and $\tilde{z}^*_{1\lambda} < 0$ (i.e. $\lambda > \hat{\lambda}(\rho)$ and $\lambda^* > \hat{\lambda}(\rho^*)$),

where $(\lambda, \lambda^*) \in \Lambda(l, \rho) \times \Lambda(l, \rho^*)$.

The type (iii) equilibria have basically similar properties to those with normality assumption in Section 3. However, type (i) and (ii) equilibria may have different properties from type (iii) equilibria.

The pair λ^T and λ^{T*} in Figures 4 and 5 represent a type (i) equilibrium.[14] Note that the H-O theorem must be violated at this equilibrium (as it must be at any type (i) equilibrium when $\rho = \rho^*$), since the country with the larger capital stock (i.e. lower marginal utility of income) will be exporting the labor intensive good. This occurs because the richer country demands less of the inferior labor

[13] Since $\tilde{z}_1$ is linear in l, $\tilde{z}_1(0, l_0, \rho) - \tilde{z}_1(0, l_1, \rho) = \tilde{z}_1(\hat{\lambda}(\rho), l_0, \rho) - \tilde{z}_1(\hat{\lambda}(\rho), l_1, \rho)$ holds. Since $\hat{l} = (l_0 + l_1)/2$ and $\tilde{z}_1$ is linear in l,

$$\tilde{z}_1(0, \hat{l}, \rho) = \frac{1}{2}[\tilde{z}_1(0, l_0, \rho) + \tilde{z}_1(0, l_1, \rho)],$$
$$\tilde{z}_1(\hat{\lambda}(\rho), \hat{l}, \rho) = \frac{1}{2}[\tilde{z}_1(\hat{\lambda}(\rho), l_0, \rho) + \tilde{z}_1(\hat{\lambda}(\rho), l_1, \rho)],$$

also hold. Considering $\tilde{z}_1(0, l_0, \rho) = \tilde{z}_1(\hat{\lambda}(\rho), l_1, \rho) = 0$, we obtain $\tilde{z}_1(0, \hat{l}, \rho) + \tilde{z}_1(\hat{\lambda}(\rho), \hat{l}, \rho) = 0$. This implies $\tilde{z}_1(0, l, \rho) + \tilde{z}_1(\hat{\lambda}(\rho), l, \rho) > 0$ for $l \in (l_0, \hat{l})$.

[14] Since $d\lambda^*/d\lambda|_{dZ=0} = -H\tilde{z}_{1\lambda}/H^*\tilde{z}^*_{1\lambda}$, the boundary of $A(l)$ is negatively sloped for type (i) and type (iii) equilibria, and positively sloped for type (ii) equilibria.

intensive good, and this effect dominates its relatively lower supply of the labor intensive good at these equilibria. The pair ($\lambda^{T\prime}$, λ^{T*}) in Figures 4 and 5 is an example of a type (ii) equilibrium. Note that the H-O theorem is also violated in this equilibrium, although it is not necessarily violated at all type (ii) equilibria (e.g. the type (ii) equilibrium in which the home country is at λ^T).[15]

We know from Proposition 4 that a steady state equilibrium will be a saddle point if $(\tilde{z}_{1\lambda} + m\frac{H^*}{H}\tilde{z}^*_{1\lambda}) < 0$. This condition is satisfied for type (iii) equilibria, but must clearly fail for type (i) equilibria. In contrast, type (ii) equilibria with $\lambda^* < \hat{\lambda}(\rho^*)$ ($\lambda^* > \hat{\lambda}(\rho^*)$) will be saddle-point stable if and only if the frontier of $A(l)$ is steeper (more gradual) than the ray from the origin at that point:

$$\left.\frac{d\lambda^*}{d\lambda}\right|_{dZ=0} = -\frac{H\tilde{z}_{1\lambda}}{H^*\tilde{z}^*_{1\lambda}} \begin{cases} > \lambda^*/\lambda = m & \text{if } \tilde{z}^*_{1\lambda} > 0, \\ < \lambda^*/\lambda = m & \text{if } \tilde{z}^*_{1\lambda} < 0. \end{cases} \tag{34}$$

Point S in Figure 5 is one example of type (ii) equilibrium where (34) holds.

For the equilibria where $(\tilde{z}_{1\lambda} + m\frac{H^*}{H}\tilde{z}^*_{1\lambda}) > 0$ ($J(0) < 0$), Lemma 3 shows that the local dynamics will be determined by the sign of $J(\rho + \rho^*)$. Evaluating this expression at $\rho = \rho^*$ using (28), we obtain $J(2\rho) = 2\Gamma\rho^3 - J(0) > 0$, which implies that a steady state is a source when it is not a saddle point and discount rates are identical.[16]

Based on the above, we obtain the following Proposition, which shows that the H-O theorem will be violated at some steady states with the saddle-point stability, even if discount rates are identical.

Proposition 7 *Let $\rho = \rho^*$ and $H = H^*$ hold. For $l \in (l_0, \hat{l}]$, there exist type (ii) equilibria where the H-O theorem is violated while the saddle-point stability holds.*

Proof. Let $m'(l) \equiv \lambda^L(l)/\lambda^H(l)$, which is positive for $l \in (l_0, l_1)$. Since the positively sloped curve intersects the horizontal axis for $l \in (l_0, \hat{l}]$, there is at least one intersection between the curve and the ray from the origin $\lambda^* = m\lambda$ with $m < m'(l)$ where (34) holds (e.g. point S in Figure 5). Since $\lambda \in (\hat{\lambda}(\rho), \lambda^H(l))$ and $\lambda^* \in (0, \lambda^L(l))$ holds here, $\tilde{k} < \tilde{k}^*$ and $\tilde{z}_1 > 0 > \tilde{z}^*_1$ are satisfied, that is, the capital abundant foreign exports labor intensive good 1 at the equilibrium: the H-O theorem is violated. ■

In the rest of this section, we shall consider the asymmetric case with $\rho > \rho^*$. And we redefine l_0 and l_1 as[17]

$$\lim_{\lambda,\lambda^*\to 0} Z(\lambda, \lambda^*, l_0) = 0 \text{ and } Z(\hat{\lambda}(\rho), \hat{\lambda}(\rho^*), l_1) = 0 \tag{35}$$

[15] It can be easily shown that when $\rho = \rho^*$ the H-O theorem holds if and only if the steady state value of λ or λ^* is greater than $\lambda^H(l)$.

[16] This is consistent with findings of Kehoe et al [15], who have shown that dynamic indeterminacy cannot arise in a one sector growth model with a finite number of infinitely lived agents and complete markets. If $\rho = \rho^*$, the equilibrium under factor price equalization will result in a Pareto optimal allocation in the H-O model even without the existence of markets for borrowing and lending.

[17] If $\rho = \rho^*$, these values correspond to the previous ones, which is associated with the existence of two autarkic steady states in Proposition 6.

and modify Assumption 6 as follows:

Assumption 6':

$$\rho^* > -r'(\tilde{p}) \max\left\{\frac{\sigma\eta\xi^2 - \tilde{p}^2}{\tilde{p}}, \frac{\sigma\eta\xi^2 - 2\tilde{p}\xi + \tilde{p}^2}{\xi - \tilde{p}}\right\}.$$

Notice that[18]

$$\frac{\bar{k}(l,\rho)}{l} < \frac{\bar{k}(l,\rho^*)}{l},\ \hat{\lambda}(\rho) > \hat{\lambda}(\rho^*),\ \text{and}\ \tilde{\lambda}_{\max}(l,\rho) < \tilde{\lambda}_{\max}(l,\rho^*). \tag{36}$$

It is clear from the arguments below (18) that $Z(\tilde{\lambda}_{\max}, \tilde{\lambda}^*_{\max}, l) < 0$ and from footnote 12 that if $\rho = \rho^*$, $\bar{k}(l,\rho)/l < \tilde{\kappa}_{\max}$ and $\tilde{\lambda}_{\max}(l,\rho) > \hat{\lambda}(\rho)$ hold for $l \in (l_0, l_1)$. So, as long as the difference between ρ and ρ^* is small, we have for $l \in (l_0, l_1)$,

$$\bar{k}(l,\rho^*)/l < \tilde{\kappa}_{\max}\ \text{and}\ \tilde{\lambda}_{\max}(l,\rho) > \hat{\lambda}(\rho) \tag{37}$$

and

$$Z(\tilde{\lambda}_{\max}, \hat{m}\tilde{\lambda}_{\max}, l) < 0, \tag{38}$$

where $\hat{m} \equiv \hat{\lambda}(\rho^*)/\hat{\lambda}(\rho) < 1$, and hence $\hat{m}\tilde{\lambda}_{\max} < \tilde{\lambda}^*_{\max}$. (36) and (37) imply

$$\tilde{\lambda}_{\min}(l,\rho) = \tilde{\lambda}_{\min}(l,\rho^*) = 0\ \text{and}\ \hat{\lambda}(\rho^*) < \hat{\lambda}(\rho) < \tilde{\lambda}_{\max}(l,\rho) < \tilde{\lambda}_{\max}(l,\rho^*)\ \text{for}\ l \in (l_0, l_1).$$

If the ray from the origin cuts twice the boundary of $A(l)$ on $\Lambda(l,\rho) \times \Lambda(l,\rho^*)$, then one of them is type (i) or type (ii) equilibrium with $J(0) < 0$, while the other is type (iii) or type (ii) equilibrium with $J(0) > 0$. The following Proposition shows the existence of such two intersections for some range of values of m.

Proposition 8 *Let the difference between ρ and ρ^* be such that (37) and (38) hold. For $l \in (l_0, l_1)$, there exists an open interval $M(l)$ such that for any $m \in M(l)$, $Z(\lambda, m\lambda, l) = 0$ has exactly two solutions for λ one of which corresponds to type (i) equilibrium and the other does type (iii) equilibrium.*

Proof. Consider $Z(\lambda, \lambda^*, l)$ along the ray $\lambda^* = \hat{m}\lambda$ which passes through $(\hat{\lambda}(\rho), \hat{\lambda}(\rho^*))$. Z is increasing in λ on $[0, \hat{\lambda}(\rho))$ and decreasing in λ on $(\hat{\lambda}(\rho), \infty)$. Note that for $l \in (l_0, l_1)$, $Z(0,0,l) < 0$ and $Z(\hat{\lambda}(\rho), \hat{\lambda}(\rho^*), l) > 0$ from (35) and $Z(\tilde{\lambda}_{\max}, \hat{m}\tilde{\lambda}_{\max}, l) < 0$ from (38). Therefore, along the ray $\lambda^* = \hat{m}\lambda$, Z changes its sign twice as λ increases from 0 to $\tilde{\lambda}_{\max}$, which implies that $\lambda^* = \hat{m}\lambda$ cuts twice the boundary of $A(l)$ and one intersection corresponds to type (i) equilibrium and the other does type (iii) equilibrium (see Figure 5). So, for each $l \in (l_0, l_1)$, we can find an open interval $M(l)$ that

[18] $\bar{k}(l,\rho) = [E(\tilde{p}, 0) - \tilde{w}l]/\rho$ is decreasing in ρ, $\hat{\lambda}(\rho)$ is increasing in ρ from footnote 11, and $\tilde{\lambda}_{\max}(l,\rho)$ is decreasing in ρ from Figure 3.

includes $\hat{m}$ such that for any $m \in M(l)$, $Z(\lambda, m\lambda, l) = 0$ has exactly two solutions for λ one of which satisfies $(\lambda, m\lambda) \in (0, \hat{\lambda}(\rho)) \times (0, \hat{\lambda}(\rho^*))$ and the other does $(\lambda, m\lambda) \in (\hat{\lambda}(\rho), \tilde{\lambda}_{\max}) \times (\hat{\lambda}(\rho^*), \tilde{\lambda}^*_{\max})$. ■

In the case $Z(\lambda, m\lambda, l) = 0$ has two solutions for λ, say λ^L and λ^H ($\lambda^L < \lambda^H$), the steady state with $(\lambda, \lambda^*) = (\lambda^L, m\lambda^L)$ Pareto dominates the other equilibrium, as it involves a higher level of utility for both countries. In the next section, we will show that the Pareto dominant steady state (type (i) equilibrium) that is unstable with $\rho = \rho^*$ can become stable if $\rho \neq \rho^*$.

4.2 Indeterminacy

Indeterminacy has been shown to arise in H-O models when markets are incomplete, as in Galor [12] for the case of overlapping generations of finitely lived consumers. In an H-O model with infinitely lived consumers, Nishimura and Shimomura [19] have shown that indeterminacy can arise when $\rho \neq \rho^*$ if preferences are quadratic and there is a negative income effect.[19] The possibility of indeterminacy arises with $\rho \neq \rho^*$ because factor price equalization does not give rise to a Pareto optimal allocation without international markets for lending. We conclude by showing that indeterminacy can arise if $\rho \neq \rho^*$, preferences are given by (31), and technologies take the following specification.[20]

Assumption 7: $\chi_i(w, r)$, $i = 1, 2$, are given by $a_i w + b_i r$, where a_i and b_i are constant and nonnegative with $a_1 b_2 - a_2 b_1 > 0$ and $b_2 < \theta^{-1}$.

Notice that $a_1 b_2 - a_2 b_1 > 0$ implies that good 1 is labor intensive (Assumption 1) and $b_2 < \theta^{-1}$ corresponds to Assumption 4.

The next Lemma, which is proven in Appendix, will be used in the proof of indeterminacy in Proposition 9.

Lemma 6 *Let Assumptions 5' and 7 hold. Then, we obtain (i)* $\lim_{\lambda \to 0} \tilde{z}_{1\lambda}(\lambda, l, \rho) < \infty$ *and* $\lim_{\lambda^* \to 0} \tilde{z}_{1\lambda}(\lambda^*, l, \rho^*) < \infty$; *(ii)* $\lim_{\lambda, \lambda^* \to 0} \Gamma = 0$.

Proposition 9 *Let the difference between ρ and ρ^* be such that (37) and (38) hold. If $l \in (l_0, l_1)$ is sufficiently close to l_0, then for any $m \in M(l)$, the dynamical system (equations (24)-(26)) has exactly two stationary solutions each of which is consistent with incomplete specialization and indeterminacy occurs around one of the steady state, while the other is saddle-point stable.*

Proof. From Proposition 8, for any $l \in (l_0, l_1)$, there are exactly two steady states with $m \in M(l)$. Let $(\lambda^L, \lambda^{L*})$ and $(\lambda^H, \lambda^{H*})$ denote the points in (λ, λ^*) space which correspond to the type

[19] Bond et al [8] obtain a continuum of equilibrium paths in a model with physical and human capital accumulation due to the fact that the two types of capital are perfect substitutes from the point of view of households. This illustrates the importance of having sufficient curvature in the problem to generate unique paths, as has been shown by Shannon and Zame [21].

[20] In the case of Leontief technologies, the unit cost functions become linear in w and r as in the following Assumption.

(i) and type (iii) equilibrium, respectively. Then, the steady state with $(\lambda^H, \lambda^{H*})$ is saddle-point stable. On the other hand, $J(0)$ is negative at the steady state with $(\lambda^L, \lambda^{L*})$. Let $J^L(\rho+\rho^*)$ be the value of $J(\rho+\rho^*)$ at the steady state with $\lambda = \lambda^L$. The result will be established by showing that $J^L(\rho+\rho^*) < 0$. From equation (28), we obtain $J^L(\rho+\rho^*)$ to be

$$J^L(\rho+\rho^*) = -\frac{(\rho+\rho^*)(\rho-\rho^*)\tilde{z}_1}{r'(\tilde{p})} + \Gamma\rho\rho^*(\rho+\rho^*) - \frac{\lambda}{r'(\tilde{p})}\left(\rho^2\tilde{z}_{1\lambda} + \rho^{*2}m\frac{H^*}{H}\tilde{z}^*_{1\lambda}\right),$$

where $r'(\tilde{p})$ is given by $-a_2/(a_1b_1 - a_2b_1)$ under Assumption 7. Note that for any m both λ^L and λ^{L*} $(= m\lambda^L)$ go to 0 as l goes to l_0. From Lemma 6, the last term in parentheses is bounded, so the last term will approach 0 as $\lambda, \lambda^* \to 0$. Also, we see that the second term will approach 0 as $\lambda, \lambda^* \to 0$. For the first term, we have $\lim_{l\to l_0} \tilde{z}_1(\lambda^L, l, \rho) < 0$, because $Z(0, 0, l_0) = H\tilde{z}_1(0, l_0, \rho) + H^*\tilde{z}_1(0, l_0, \rho^*) = 0$ and $\partial\tilde{z}_1/\partial\rho = r'(\tilde{p})\tilde{k}/\rho < 0$ together imply that $\tilde{z}_1(0, l_0, \rho) < 0$. Therefore, $J^L(\rho+\rho^*)$ is negative when l is sufficiently close to l_0. Thus, from Lemma 3, the characteristic equation has two roots with negative real parts at the steady state with $\lambda = \lambda^L$, which implies that indeterminacy occurs around the steady state. ■

The possibility of dynamic indeterminacy in a two country trade model when there are no markets for international lending and borrowing has been previously shown Shimomura [23] and Doi et al. [11] for the case of an endowment model of international trade in which one of the goods is durable and there is a negative income effect. Nishimura and Shimomura [19] discussed the possibility of dynamic indeterminacy in a two country H-O model, where they supposed $\rho \neq \rho^*$ and a specific utility function and Leontief technologies with $b_1 = 0$ and $a_2 = b_2 = 1$, and derived some conditions under which indeterminacy occurs around the steady state. However, their focus is mainly on the occurrence of indeterminacy and they did not considered the multiplicity of the steady states in our model. Indeed, one can verify that in their model, there is no possibility of the multiplicity because the negatively sloped region in their excess demand function is inconsistent with incomplete specialization under their assumed preferences and technologies.

Bond and Driskill [9] have shown that inferiority in consumption is not necessary to generate multiple steady states and indeterminacy in the model with durable consumption goods: indeterminacy can arise when both goods are normal as long as the exporting country has the higher marginal propensity to consume a good. In contrast, our results here show that inferiority in consumption is a necessary condition for the existence of multiple steady states and indeterminacy in the H-O model when factor price equalization holds. The difference between the two cases is due to the fact that with the factor price equalization property, the marginal utility of consumption in the two countries must be moving in the same direction along the optimal path. The H-O model of trade we examine here also requires that trade balance at each point in time. However, if there is factor price equalization along the optimal path, international lending and borrowing is redundant if the discount factors of the two countries are the same. As a result, a difference in discount factors between countries and inferiority in consumption are both necessary conditions for indeterminacy to occur.

5 Concluding Remarks

Our analysis have shown that the many of the results of the dynamic H-O model will extend to the case of non-homothetic preferences as long as both goods are normal and discount factors are the same between countries. In this case the steady state trade pattern will satisfy the static H-O theorem, a dynamic H-O theorem will hold, and the steady states will be saddle points. The main difference introduced by non-homotheticity when goods are normal and discount factors are equal is that the world capital stock will depend on the distribution of income across countries. Allowing discount factors to differ between countries (while keeping steady state rentals constant) introduces the possibility that the static and dynamic H-O theorems may fail to hold.

We have also provided an example to show that the results may differ dramatically if the labor intensive good is inferior. These differences include the possibility that there are multiple steady state equilibria, that the static H-O theorem is violated in the steady state, and that some steady state equilibria are Pareto dominated. If discount factors are the same across countries, steady state equilibria will be either saddle points or unstable equilibria. However, if discount factors differ across countries there is the possibility of local indeterminacy.

6 Appendix

6.1 *Proof of Lemma 1*

Totally differentiating equations (8) with respect to c_1, c_2, p and λ, we derive

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} dc_1 \\ dc_2 \end{bmatrix} = \begin{bmatrix} p \\ 1 \end{bmatrix} d\lambda + \begin{bmatrix} \lambda \\ 0 \end{bmatrix} dp.$$

Since the determinant of the coefficient matrix, $D = u_{11}u_{22} - u_{12}^2$, is positive at any point where $u_i(c_1, c_2) > 0$ (Assumption 2) and therefore invertible, we obtain

$$c_{1\lambda}(p,\lambda) \equiv \frac{\partial c_1}{\partial \lambda} = \frac{1}{D}(u_{22}p - u_{12}), \tag{39}$$

$$c_{2\lambda}(p,\lambda) \equiv \frac{\partial c_2}{\partial \lambda} = \frac{1}{D}(u_{11} - u_{12}p), \tag{40}$$

$$c_{1p}(p,\lambda) \equiv \frac{\partial c_1}{\partial p} = \frac{1}{D}\lambda u_{22} < 0, \tag{41}$$

$$c_{2p}(p,\lambda) \equiv \frac{\partial c_2}{\partial p} = -\frac{1}{D}\lambda u_{12}. \tag{42}$$

The results of Lemma 1 follow immediately from these comparative statics results.||

6.2 *Derivation of the Characteristic Equation (28)*

Expanding (27) yields

$$\det[xI-J]=x^3-\left\{\rho+\rho^*-\lambda r'\frac{\partial p}{\partial\lambda}-\left[z_1+\lambda c_{1\lambda}+(z_1^*+m\lambda c_{1\lambda}^*)\frac{H^*}{H}\right]\frac{\partial p}{\partial k}\right\}x^2$$

$$+\left\{\rho\rho^*-(\rho+\rho^*)\lambda r'\frac{\partial p}{\partial\lambda}-\left[\rho^*(z_1+\lambda c_{1\lambda})+\rho(z_1^*+m\lambda c_{1\lambda}^*)\frac{H^*}{H}+\left(E_\lambda+m\frac{H^*}{H}E_\lambda^*\right)\lambda r'\right]\frac{\partial p}{\partial k}\right\}x$$

$$+\lambda r'\left[\rho\rho^*\frac{\partial p}{\partial\lambda}+\left(\rho^*E_\lambda+\rho m\frac{H^*}{H}E_\lambda^*\right)\frac{\partial p}{\partial k}\right]. \tag{43}$$

Since $Hz_1+H^*z_1^*=0$ at the world trade equilibrium and $r'\frac{\partial p}{\partial\lambda}+(c_{1\lambda}+m\frac{H^*}{H}c_{1\lambda}^*)\frac{\partial p}{\partial k}=0$ from (23), the coefficient of x^2 becomes $-(\rho+\rho^*)$. Using the latter equation to substitute into the coefficient on x and the constant term yields

$$\rho\rho^*-\left[(\rho^*-\rho)z_1-\lambda(\rho c_{1\lambda}-r'E_\lambda)-\lambda m\frac{H^*}{H}(\rho^*c_{1\lambda}^*-r'E_\lambda^*)\right]\frac{\partial p}{\partial k}$$

$$\text{and}\quad-\lambda\left[\rho^*(\rho c_{1\lambda}-r'E_\lambda)+\rho m\frac{H^*}{H}(\rho^*c_{1\lambda}^*-r'E_\lambda^*)\right]\frac{\partial p}{\partial k},$$

respectively. Multiplying the result by $\Gamma\equiv\left(-r'\frac{\partial p}{\partial k}\right)^{-1}$ and using (19) yields (28) in the text.

6.3 *Proof of Lemma 4*

Let $\Upsilon\equiv\{(c_1,c_2)\in\mathbb{R}_+^2|0<\gamma c_1^\sigma c_2<\alpha$ and $0<\gamma c_1c_2^\eta<\beta\}$, which is the set of $(c_{1,}c_2)$ for which $u_i(c_1,c_2)>0$ for $i=1,2$. It is straightforward to show that (31) is strictly concave over the subset Υ when Assumption 5' is satisfied. The proof that the solutions to (8) are unique proceeds in two steps, which will only be sketched here. First, establish that any element of Υ has a unique representation as $(c_1,c_2)=\left(\left[\frac{(s\alpha)^\eta}{v\beta\gamma^{\eta-1}}\right]^{\frac{1}{\sigma\eta-1}},\left[\frac{(v\beta)^\sigma}{s\alpha\gamma^{\sigma-1}}\right]^{\frac{1}{\sigma\eta-1}}\right)$ for $0<s,\nu<1$. Second, substitute this representation into (8) and show that the resulting equations have unique solutions $s(p,\lambda),\nu(p,\lambda)\in(0,1)$ for any positive p and λ. Furthermore, these solutions have the properties that the respective function are decreasing in λ with $\lim_{\lambda\to0}s(p,\lambda)=1$, $\lim_{\lambda\to\infty}s(p,\lambda)=0$, $\lim_{\lambda\to0}v(p,\lambda)=1$, and $\lim_{\lambda\to\infty}v(p,\lambda)=0$. Based on these two results, we can conclude that for any positive p and λ, the system of equations (8) has an unique, interior, and positive solution, $(c_1(p,\lambda),c_2(p,\lambda))$, where

$$c_1(p,\lambda)=\left\{\frac{[\alpha s(p,\lambda)]^\eta}{\beta\gamma^{\eta-1}v(p,\lambda)}\right\}^{\frac{1}{\sigma\eta-1}}, \tag{44}$$

$$c_2(p,\lambda)=\left\{\frac{[\beta v(p,\lambda)]^\sigma}{\alpha\gamma^{\sigma-1}s(p,\lambda)}\right\}^{\frac{1}{\sigma\eta-1}}. \tag{45}$$

(i) From equations (44) and (45), it is clear that $\lim_{\lambda\to0}c_1(p,\lambda)=\bar{c}_1$ and $\lim_{\lambda\to0}c_2(p,\lambda)=\bar{c}_2$, since $\lim_{\lambda\to0}s(p,\lambda)=1$ and $\lim_{\lambda\to0}v(p,\lambda)=1$. On the other hand, from the first-order conditions (equations (8)), we see that $\lim_{\lambda\to\infty}c_i(p,\lambda)=0,\ i=1,2$.

(ii) Substituing the derivatives of (31) into (39) and (40) yields

$$c_{1\lambda}(\tilde{p},\lambda) = \frac{1}{D}[-\beta\eta c_2^{-\eta-1}\tilde{p} + \gamma];\ c_{2\lambda}(\tilde{p},\lambda) = \frac{1}{D}[-\alpha\sigma c_1^{-\sigma-1} + \gamma\tilde{p}], \tag{46}$$

$$\text{where } D(\tilde{p},\lambda) = \left[\frac{\sigma\eta}{s(\tilde{p},\lambda)v(\tilde{p},\lambda)} - 1\right]\gamma^2 > 0. \tag{47}$$

Since $\bar{c}_2/\eta\bar{c}_1 > \tilde{p} \Leftrightarrow \gamma > \beta\eta\left(\bar{c}_2\right)^{-\eta-1}\tilde{p}$, we have $c_{1\lambda}(\tilde{p},0) > 0$. Lemma 1 (ii) then implies $c_{2\lambda}(\tilde{p},0) < 0$. Indeed, one can verify that

$$\begin{aligned} &\gamma > \beta\eta\left(\bar{c}_2\right)^{-\eta-1}\tilde{p} \\ &\quad\Rightarrow \alpha\sigma\left(\bar{c}_1\right)^{-\sigma-1} > \gamma\tilde{p}. \end{aligned} \tag{48}$$

Since $c_{1\lambda}(\tilde{p},\lambda) > 0 \Leftrightarrow c_2 > (\tilde{p}\beta\eta/\gamma)^{1/(\eta+1)}$ and $\lim_{\lambda\to\infty} c_2(p,\lambda) = 0$, a sufficient condition for the existence of a λ^0 such that $c_{1\lambda}(\tilde{p},\lambda) > (<)\ 0$ for $\lambda < (>)\ \lambda^0$ is that $c_{2\lambda}(\tilde{p},\lambda) < 0$ for all λ. Suppose that $c_{2\lambda}(\tilde{p},\lambda') \geq 0$ holds for some λ'. We then have $c_{1\lambda}(\tilde{p},\lambda') < 0$ from Lemma 1 (ii). Since $c_{1\lambda}(\tilde{p},0) > 0$, the continuity of $c_{1\lambda}$ in λ ensures there is some $\lambda'' < \lambda'$ such that $c_{1\lambda}(\tilde{p},\lambda'') = 0$. We also have $c_1(\tilde{p},\lambda'') > c_1(\tilde{p},\lambda')$ because $c_{1\lambda} < 0$ on (λ'',λ'), which means $c_{2\lambda}(\tilde{p},\lambda'') > 0$ due to the fact that the numerator of $c_{2\lambda}$ in (46) is increasing in c_1. However, $c_{1\lambda}(\tilde{p},\lambda'') = 0$ and $c_{2\lambda}(\tilde{p},\lambda'') > 0$ contradicts Lemma 1 (ii). Therefore, $c_{2\lambda}(\tilde{p},\lambda) < 0$ for all λ.||

6.4 *Proof of Lemma 5*

In order to establish the result, we first prove the following:

Lemma A1: Suppose that $\xi > \tilde{p}$. If $\rho > -r'(\tilde{p})(\sigma\eta\xi^2 - \tilde{p}^2)/\tilde{p}$, then $\zeta_1(\rho)c_{1\lambda\lambda}(\tilde{p},\lambda) + c_{2\lambda\lambda}(\tilde{p},\lambda)$ is negative for $\lambda \leq \lambda^0$.

Proof. From (46), we obtain

$$\begin{aligned} c_{1\lambda\lambda}(\tilde{p},\lambda) &= \frac{1}{D}[\beta\eta(\eta+1)c_2^{-\eta-2}c_{2\lambda}\tilde{p} - c_{1\lambda}D_\lambda], \\ c_{2\lambda\lambda}(\tilde{p},\lambda) &= \frac{1}{D}[\alpha\sigma(\sigma+1)c_1^{-\sigma-2}c_{1\lambda} - c_{2\lambda}D_\lambda], \end{aligned}$$

where

$$D_\lambda \equiv \frac{\partial D}{\partial\lambda} = -\alpha\beta\sigma\eta c_1^{-\sigma-1}c_2^{-\eta-1}\left[(\sigma+1)\frac{c_{1\lambda}}{c_1} + (\eta+1)\frac{c_{2\lambda}}{c_2}\right]. \tag{49}$$

Using the definition of D in (47), D_λ is positive since $s_\lambda, v_\lambda < 0$ as established in the proof of Lemma 4. It then follows that $c_{1\lambda\lambda}$ is negative and $c_{2\lambda\lambda}$ is positive for $\lambda \in [0,\lambda^0]$, because $c_{1\lambda} \geq 0$ and $c_{2\lambda} < 0$ in this interval as established in Lemma 4. Notice that $c_1(\tilde{p},\lambda) \geq \bar{c}_1$ and $c_2(\tilde{p},\lambda) \leq \bar{c}_2$ hold for $\lambda \leq \lambda^0$ if $\xi > \tilde{p}$ holds. Therefore if inequality $\rho > -r'(\tilde{p})(\sigma\eta\xi^2 - \tilde{p}^2)/\tilde{p}$, which is identical to

$$\zeta_1(\rho)\tilde{p}\beta\eta\left(\bar{c}_2\right)^{-\eta-1} > \alpha\sigma\left(\bar{c}_1\right)^{-\sigma-1},$$

holds, then

$$\begin{aligned}\zeta_1(\rho)\tilde{p}\beta\eta c_2^{-\eta-1} &\geq \zeta_1(\rho)\tilde{p}\beta\eta\left(\bar{c}_2\right)^{-\eta-1}\\ &> \alpha\sigma\left(\bar{c}_1\right)^{-\sigma-1}\\ &\geq \alpha\sigma c_1^{-\sigma-1}\end{aligned}\tag{50}$$

for $\lambda \leq \lambda^0$. Based on the above, we obtain

$$\begin{aligned}&\zeta_1(\rho)c_{1\lambda\lambda}(\tilde{p},\lambda)+c_{2\lambda\lambda}(\hat{p},\lambda)\\ &=\frac{1}{D}\left[\zeta_1(\rho)\tilde{p}\beta\eta(\eta+1)c_2^{-\eta-2}c_{2\lambda}-\zeta_1(\rho)c_{1\lambda}D_\lambda+\alpha\sigma(\sigma+1)c_1^{-\sigma-2}c_{1\lambda}-c_{2\lambda}D_\lambda\right]\\ &<\frac{1}{D}\left\{\alpha\sigma c_1^{-\sigma-1}\left[(\sigma+1)\frac{c_{1\lambda}}{c_1}+(\eta+1)\frac{c_{2\lambda}}{c_2}\right]-[\zeta_1(\rho)c_{1\lambda}+c_{2\lambda}]D_\lambda\right\}\\ &=-\frac{D_\lambda}{D}\left[\frac{1}{\beta\eta c_2^{-\eta-1}}+\zeta_1(\rho)c_{1\lambda}+c_{2\lambda}\right]\\ &=-\frac{D_\lambda}{D}\left[\zeta_1(\rho)c_{1\lambda}+\frac{D+(-\alpha\sigma c_1^{-\sigma-1}+\gamma\tilde{p})\beta\eta c_2^{-\eta-1}}{D\beta\eta c_2^{-\eta-1}}\right]\\ &=-\frac{D_\lambda}{D}\left[\zeta_1(\rho)c_{1\lambda}+\frac{-\gamma^2+\gamma\tilde{p}\beta\eta c_2^{-\eta-1}}{D\beta\eta c_2^{-\eta-1}}\right]\\ &=-\frac{D_\lambda}{D}\left[\zeta_1(\rho)c_{1\lambda}+\frac{-\gamma c_{1\lambda}}{\beta\eta c_2^{-\eta-1}}\right]\\ &=-\frac{D_\lambda c_{1\lambda}}{D\beta\eta c_2^{-\eta-1}}\left[\zeta_1(\rho)\beta\eta c_2^{-\eta-1}-\gamma\right]\\ &\leq 0\end{aligned}$$

for $\lambda \leq \lambda^0$. Here the second inequality comes from (50), the third equality is due to (49), and the last inequality comes from (48) and (50). ■

From Lemma A1, we see that the excess demand function (32) is strictly concave in λ for $\lambda \in [0, \lambda^0]$.

Next, it is apparent from (46) that

$$\zeta_1(\rho)[\gamma-\beta\eta\left(\bar{c}_2\right)^{-\eta-1}\tilde{p}]>\alpha\sigma\left(\bar{c}_1\right)^{-\sigma-1}-\gamma\tilde{p}\ \Rightarrow\ \zeta_1(\rho)c_{1\lambda}(\tilde{p},0)+c_{2\lambda}(\tilde{p},0)>0.$$

One can easily verify that the former inequality is identical to inequality $\rho > -r'(\tilde{p})(\sigma\eta\xi^2 - 2\tilde{p}\xi + \tilde{p}^2)/(\xi - \tilde{p})$ under $\xi > \tilde{p}$.

Therefore, $\tilde{z}_{1\lambda\lambda} < 0$ for $\lambda \in [0, \lambda^0]$ and $\tilde{z}_{1\lambda}(0, l, \rho) > 0$ if Assumptions 5' and 6 hold. Since both $c_{1\lambda}$ and $c_{2\lambda}$ are negative for $\lambda > \lambda^0$, it is apparent from the continuity of $c_{i\lambda},\ i = 1, 2$, in λ that there is some $\hat{\lambda}(\rho) < \lambda^0$ such that

$$\tilde{z}_{1\lambda}(\lambda, l, \rho) = -\frac{r'(\tilde{p})\left[\zeta_1(\rho)c_{1\lambda}(\tilde{p}, \lambda) + c_{2\lambda}(\tilde{p}, \lambda)\right]}{\rho} \quad \begin{cases} > 0, & \text{if } \lambda < \hat{\lambda}(\rho), \\ < 0, & \text{if } \lambda > \hat{\lambda}(\rho), \end{cases} \tag{51}$$

where $\hat{\lambda}(\rho)$ is implicitly defined as the solution to $\tilde{z}_{1\lambda}(\lambda, l, \rho) = 0.||$

6.5 ***Proof of Lemma 6***

From (13), (19), and (39)-(41), we obtain

$$\begin{aligned} \tilde{z}_{1\lambda} &= c_{1\lambda} - \frac{r' E_\lambda}{\rho} = \frac{1}{u_{11}u_{22} - (u_{12})^2}\left(u_{22}\tilde{p} - u_{12} - r'\frac{u_{22}\tilde{p}^2 - 2u_{12}\tilde{p} + u_{11}}{\rho}\right), \\ z_{1p} &= c_{1p} - \frac{\partial y_1}{\partial p} = \frac{\lambda u_{22}}{u_{11}u_{22} - (u_{12})^2}. \end{aligned}$$

Notice that $\partial y_1/\partial p = 0$ under Assumption 7. Since $\lim_{\lambda\to 0} c_i(p, \lambda) = \bar{c}_i \in (0, \infty)$, $i = 1, 2$, we see

$$\lim_{\lambda\to 0} \tilde{z}_{1\lambda}(\lambda, l, \rho) < \infty \text{ and } \lim_{\lambda\to 0} z_{1p}(\tilde{p}, k, l, \lambda) = 0.$$

Finally, from (23), we have

$$\begin{aligned} \lim_{\lambda,\lambda^*\to 0} \Gamma &= \lim_{\lambda,\lambda^*\to 0}\left(-r'\frac{\partial p}{\partial k}\right)^{-1} \\ &= \lim_{\lambda,\lambda^*\to 0}\left[-\frac{Hz_{1p} + H^* z^*_{1p}}{H(r')^2}\right] \\ &= 0.|| \end{aligned}$$

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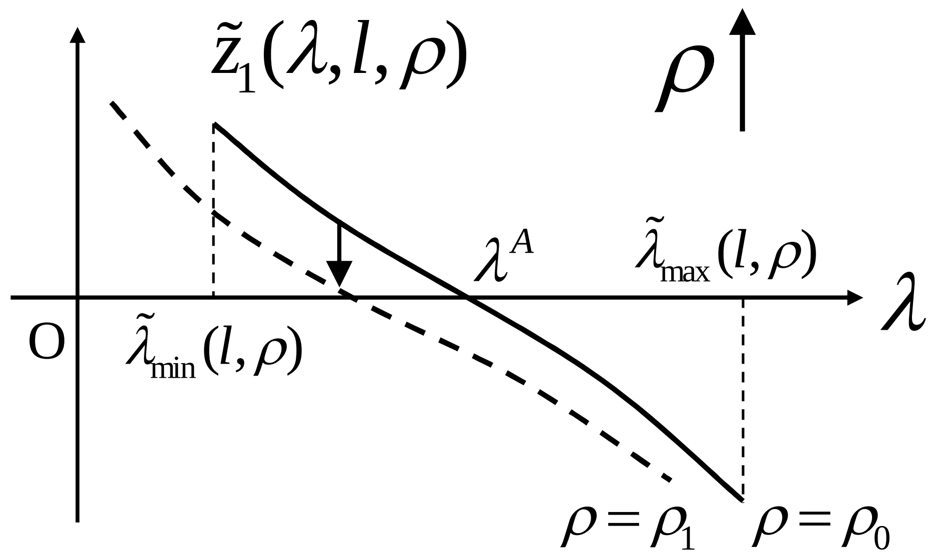
Figure 1



$\tilde{z}_1(\lambda, l, \rho)$
ρ
λ^A
$\tilde{\lambda}_{\max}(l, \rho)$
λ
O
$\tilde{\lambda}_{\min}(l, \rho)$
$\rho = \rho_1$
$\rho = \rho_0$

Figure 2

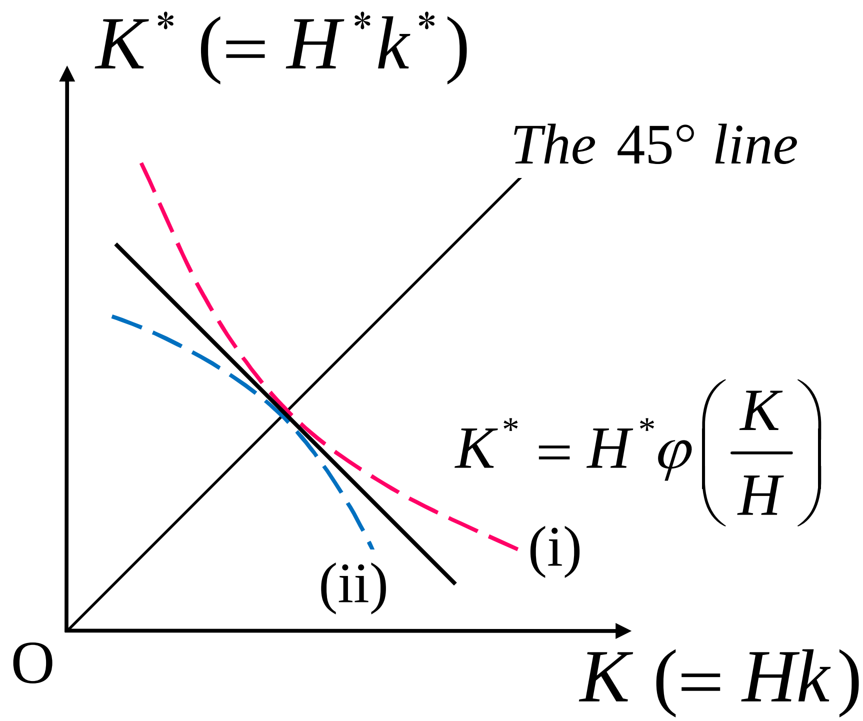


Figure 3

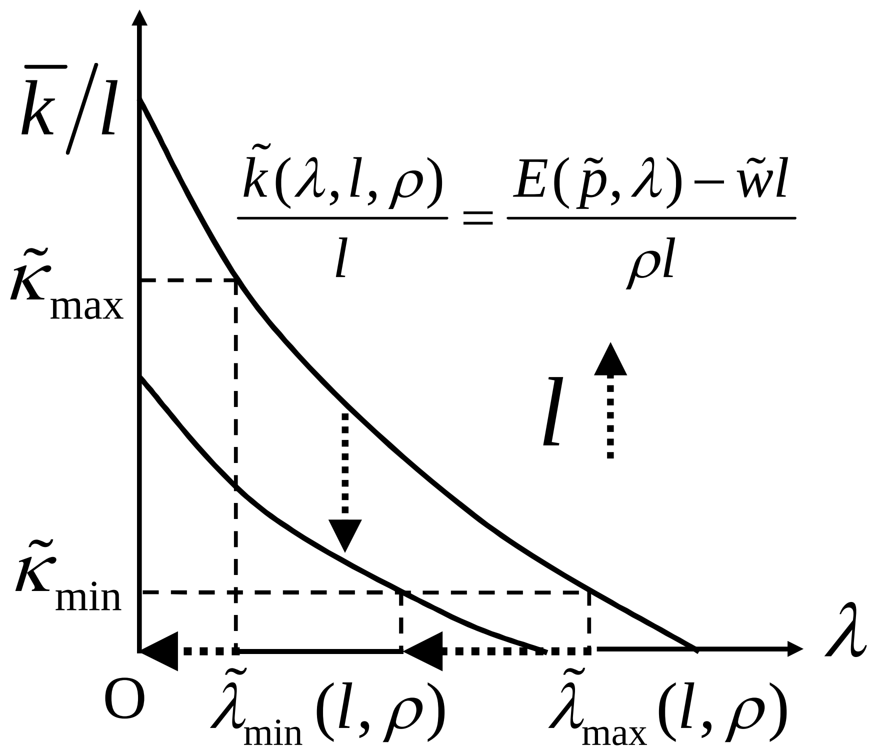
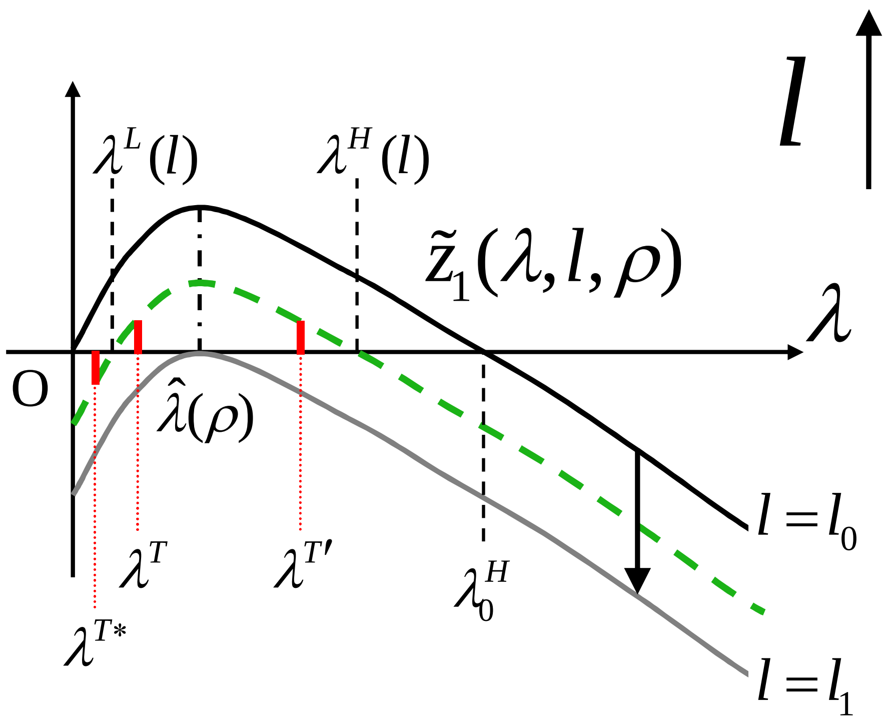
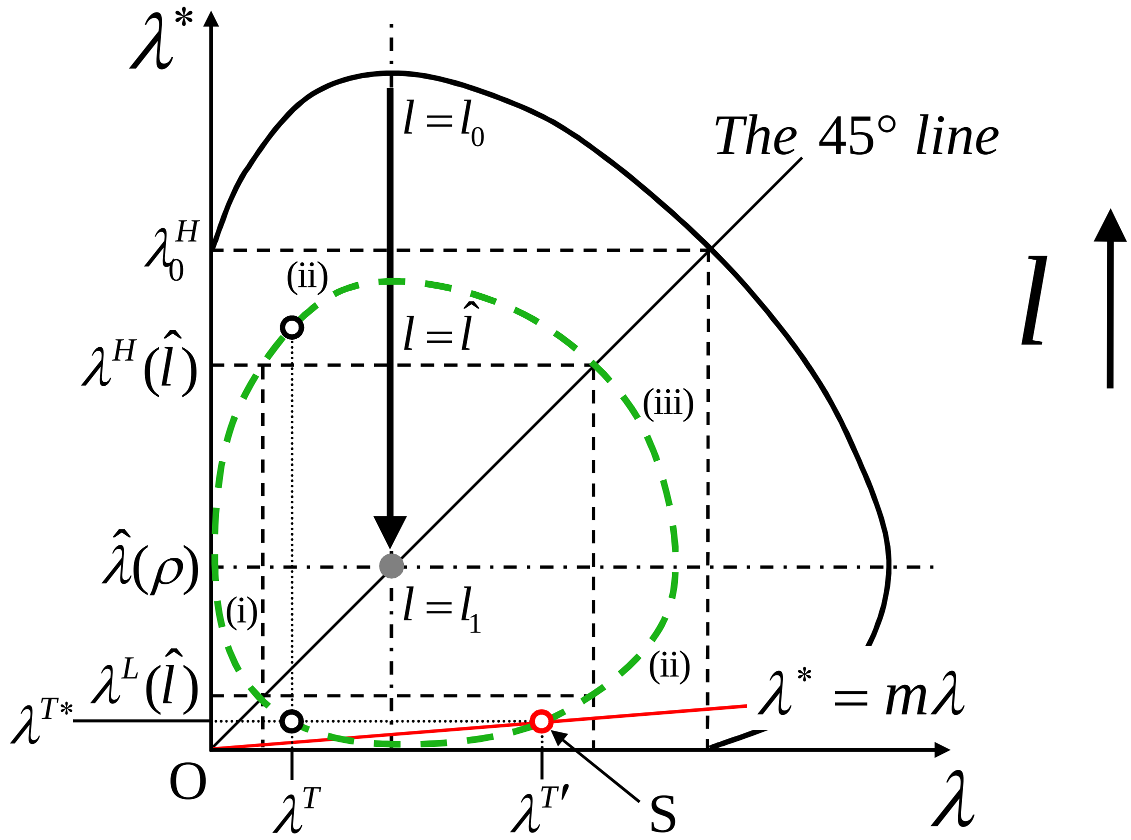

$\bar{k}/l$
$\dfrac{\tilde{k}(\lambda, l, \rho)}{l} = \dfrac{E(\tilde{p}, \lambda) - \tilde{w}l}{\rho l}$
$\tilde{\kappa}_{\max}$
l
$\tilde{\kappa}_{\min}$
λ
O
$\tilde{\lambda}_{\min}(l, \rho)$
$\tilde{\lambda}_{\max}(l, \rho)$

Figure 4



l
$\lambda^L(l)$
$\lambda^H(l)$
$\tilde{z}_1(\lambda, l, \rho)$
λ
O
$\hat{\lambda}(\rho)$
$l = l_0$
λ^T
$\lambda^{T'}$
λ_0^H
λ^{T*}
$l = l_1$

Figure 5



λ^*
$l = l_0$
The 45° line
λ_0^H
(ii)
$l = \hat{l}$
$\lambda^H(\hat{l})$
(iii)
l
$\hat{\lambda}(\rho)$
(i)
$l = l_1$
$\lambda^L(\hat{l})$
(ii)
$\lambda^* = m\lambda$
λ^{T*}
O
λ^T
$\lambda^{T\prime}$
S
λ