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“The Buy Price in Auctions with Discrete Type Distributions”

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The Buy Price in Auctions with Discrete Type Distributions *

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Abstract

This paper considers second-price, sealed-bid auctions with a buy price where bidders' types are discretely distributed. We characterize all equilibria, restricting our attention to equilibria where bidders whose types are less than a buy price bid their own valuations. Budish and Takeyama (2001) analyzed the two-bidder, two-type framework, and showed that if bidders are risk-averse, a seller can obtain a higher expected revenue from the auction with a certain buy price than from the auction without a buy price. We extend their revenue improvement result to the n-bidder, two-type framework. However, in case of three or more types, bidders' risk aversion is not a sufficient condition for the revenue improvement. Our example illustrates that even if bidders are risk-averse, a seller cannot always obtain a higher expected revenue from the auction with a buy price.

JEL classification: C72; D44

Key words: Auction; Buy price; Risk aversion

1 Introduction

We commonly observe that sellers set buy prices in Internet auctions. Since the winning bid is not above a buy price, it seems that the seller loses by determining an upper bound of the winning bid. Several papers, however, show that it may be reasonable for a seller to set a buy price. If a bidder wins by bidding a buy price, he certainly obtains some surplus. In contrast, if he wins by bidding the amount except a buy price, his surplus is random and depends on other bidders' reservation values. That is, a buy price plays a role of insurance for a risk-averse bidder. Therefore, by introducing a buy price, a seller can extract a risk premium from risk-averse bidders and then obtain a higher expected revenue.[1]

Budish and Takeyama (2001) first considered a second-price, sealed-bid auction with a buy price. They analyzed a simple model—the two-bidder, two-type framework. Their main result is that if bidders are risk-averse, a seller can obtain a higher expected revenue from the auction with a certain buy price. Hidvégi, Wang and Whinston (2006) and Reynolds and Wooders (2008) extended the analysis in two directions. One is that bidders' types are

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[1] For a similar reason, a seller can obtain a higher expected revenue from a first-price, sealed-bid auction than from a second-price, sealed-bid auction. See Maskin and Riley (1984) or Matthews (1987) for details.

continuously distributed. The other is that the auction is an open format. These two papers showed that if risk-averse bidders exhibit constant absolute risk aversion (CARA), a seller can obtain a higher expected revenue by properly setting a buy price. The above results indicate that bidders' risk-aversion is a sufficient condition for a revenue improvement.

This paper extends the analysis in a different direction. We consider a second-price, sealed-bid auction with a buy price where bidders' types are discretely distributed. Specifically, we analyze a general model—the n-bidder, m-type framework. In general, there are a lot of equilibria. To limit our attention to equilibria by reasonable strategies, we introduce the notion of *partial truth-telling*: a bidder whose type is less than a buy price bids his own valuation. Under this reasonable restriction, we characterize all equilibria. We show that there are only two kinds of equilibria where all bidders play partially truth-telling strategies. One is the symmetric equilibrium in which all bidders whose types are not less than a buy price actually bid it. The other is the asymmetric equilibrium where $n-1$ bidders play the strategy that any types above the buy price bid it and only one bidder plays a different strategy. For each class of equilibrium, we derive a necessary and sufficient condition for existence. By using this condition, we show that there always exists a buy price between the lowest valuation and the second lowest valuation under which we have a symmetric equilibrium. We also show that there is no asymmetric equilibrium in the two-bidder framework.

We analyze whether a seller can improve her payoffs by introducing a buy price. We consider also the case in which a seller is risk-averse. We show that if we can find a buy price under which the symmetric equilibrium exists between the highest valuation and the second highest valuation, then a seller can obtain a higher expected utility at the highest buy price under which we have the symmetric equilibrium unless both the seller and the bidders are risk-neutral. In any two-type framework, this condition always holds because of the above result on existence. Thus, we can obtain a generalization of the result of Budish and Takeyama (2001) in the two-type framework with any numbers of bidders. Our utility improvement results allow a seller's risk-aversion and do not depend on a CARA utility function unlike Hidvégi et al. (2006) and Reynolds and Wooders (2008).

It is not clear whether a seller obtains the highest expected utility from the symmetric equilibrium. We then consider the possibility that some asymmetric equilibrium gives a higher expected utility to the seller than the symmetric equilibrium does. However, we show that a seller cannot obtain a higher expected utility from any asymmetric equilibria than from the symmetric equilibrium.

If the condition for utility improvements does not hold, then we need to consider the auction with a buy price that is not greater than the second highest valuation. We show two examples. In the first example, the seller cannot improve her expected revenue by introducing such a buy price. In the second example, on the contrary, she can improve her expected revenue. The former concludes that even if bidders are risk-averse, a seller cannot always obtain a higher expected revenue from the auction with a buy price. In case of three or more types, bidders' risk aversion is not a sufficient condition for revenue improvements.

We introduce other results in related literature. Budish and Takeyama (2001) showed that if bidders are risk-averse, then a seller can obtain a higher expected revenue by properly setting a buy price from the second-price, sealed-bid auction with a buy price than from the first-price, sealed-bid auction without a buy price. In general, it is complicated to analyze the first-price, sealed-bid auction with three (or more) bidders where bidders' types are discretely distributed. Thus, we only make a comparison between the second-price, sealed-bid auctions with and without a buy price. In Internet auctions, two kinds of buy out prices are practically used. One is a buy price on Yahoo!. The other is a Buy It Now price on eBay.[2] Reynolds and Wooders (2008) compared English auctions with these two buy out

[2] In the case of Yahoo!, bidders can always bid a buy price throughout the auction. In the case of eBay, bidders can bid a Buy It Now price only before the bidding process starts.

prices. They showed that a seller can obtain a higher expected revenue from the auction with a certain buy price than from the auction with the same Buy It Now price if risk-averse bidders have CARA utility functions. Since we consider a static model, we cannot compare a buy price with a Buy It Now price. Some papers analyze auctions with a buy out price from a seller's point of view. Hidvégi et al. (2006) showed that a risk-averse seller can obtain a higher expected utility from the auction with a buy price. Mathews and Katzman (2006) obtained a similar utility improvement result in the auction with a Buy It Now price. We also show that a risk-averse seller can obtain a higher expected utility from the second-price, sealed-bid auction with a buy price.

The remainder of this paper is organized as follows. Section 2 describes the model. In Section 3, we characterize all equilibria. Section 4 examines whether a seller can improve her expected utility by introducing a buy price. And in Section 5, we conclude.

2 The model

We consider a second-price, sealed-bid auction with a buy price. Bidders' types are discretely distributed, and are drawn independently from an identical distribution. Bidders' valuations of an item depend only on their types. This auction consists of two stages: (i) a seller sets a buy price $B \in [0, +\infty)$, and (ii) an item is up for auction. We analyze mainly this auction given a buy price B and then argue which buy price B a seller should choose.

Let $N = \{1, \dots, n\}$ denote the set of bidders. The set of types for each bidder i is $T_i = \{v^1, \dots, v^m\}$ with $v^1 < \cdots < v^m$. We denote by f_μ the probability that a bidder's type is v^μ. We assume that for all μ, $f_\mu > 0$, and define $F_\mu = f_1 + \cdots + f_\mu$. Note that $F_m = 1$. In addition, let $F_0 = 0$. We assume that each bidder has a von-Neumann-Morgenstern utility function $U : \mathbb{R} \to \mathbb{R}$ with $U(0) = 0$, and that $U(\cdot)$ is strictly increasing and concave (possibly linear).

We suppose that bidders cannot bid above a buy price B. Thus, the set of actions for each bidder i is $A_i = [0, B]$. Bidder i's payoff function is $u_i : A \times T_i \to \mathbb{R}$, where $A = \times_{i=1}^n A_i$. Given $t_i \in T_i$ and $a \in A$, bidder i's utility is:

$$u_i(a; t_i) = \begin{cases} U(t_i - \max_{j \neq i} a_j) & \text{if } a_i \neq B \text{ and } a_i > \max_{j \neq i} a_j, \\ U(t_i - B) & \text{if } a_i = B \text{ and } a_i > \max_{j \neq i} a_j, \\ \frac{1}{M} U(t_i - a_i) & \text{if } a_i = \max_{j \neq i} a_j \text{ and } M = \mathfrak{card}\{j | a_j = a_i\}, \text{ and} \\ 0 & \text{if } a_i < \max_{j \neq i} a_j. \end{cases}$$

If no one bids a buy price B, then this auction is an ordinary second-price, sealed-bid auction. Thus, the highest bidder obtains the item and pays the second highest bid. If only one bidder bids a buy price B, then he immediately obtains the item but must pay it to the seller. If there are two or more bidders who submit the highest bid (it might be a buy price B.), then we adopt a tie-breaking rule that a winner is determined with equal probability.

A bidder i's strategy is $\sigma_i : T_i \to \Delta(A_i)$, where $\Delta(A_i)$ is the set of probability distributions over A_i.[3] A solution concept is Bayesian Nash equilibrium: a strategy profile $\sigma = (\sigma_i)_{n=1}^n$ is a Bayesian Nash equilibrium if for all i, all $t_i \in T_i$, and all $a_i' \in A_i$,

$$E[u_i(a; t_i) | \sigma_i, \sigma_{-i}, \rho(t_{-i})] \geq E[u_i(a_i', a_{-i}; t_i) | \sigma_{-i}, \rho(t_{-i})],$$

where σ_{-i} is the vector of the other n − 1 bidders' strategies, and $\rho(\cdot)$ is the probability distribution of the other n − 1 bidders' types.

[3] When $\sigma_i(\cdot)$ is a pure strategy, we often regard the range of $\sigma_i(\cdot)$ as A_i.

3 Characterization of equilibria

We consider the auction with a buy price $B \in (v^k, v^{k+1}]$ $(k = 1, \dots, m-1)$. In the auction, we have a lot of equilibria. To restrict our attention to equilibria by reasonable strategies, we propose the notion of partial truth-telling.

Definition 1. A strategy $\sigma_i(\cdot)$ is *partially truth-telling* if $\sigma_i(t_i) = t_i$ for all $t_i < B$.

When a bidder plays a partially truth-telling strategy, his type that is less than a buy price B bids his own valuation, which is a weakly dominant action. In this paper, we only consider equilibria where all bidders play partially truth-telling strategies.[4] Among partially truth-telling strategies, in particular, we pay much attention to the following one:

$$\sigma_i^*(t_i) = \begin{cases} B & \text{if } t_i \geq v^{k+1}, \\ t_i & \text{otherwise.} \end{cases}$$

The strategy $\sigma_i^*(\cdot)$ is reasonable, because it is weakly dominated for the bidder whose type is not less than a buy price B to bid an amount $b \neq B$.

The strategy $\sigma_i^*(\cdot)$ plays an important role. In fact, at least n $-$ 1 bidders play the strategy $\sigma_i^*(\cdot)$ in any equilibria by partially truth-telling strategies.

Proposition 1. *Any strategy profiles where at least two bidders play strategies except the strategy $\sigma_i^*(\cdot)$ do not become an equilibrium by partially truth-telling strategies.*

Proof. See Appendix.

By Proposition 1, it suffices to consider only two kinds of strategy profiles. One is the symmetric strategy profile where all bidders play the strategy $\sigma_i^*(\cdot)$. The other is asymmetric strategy profiles where only one bidder does not play the strategy $\sigma_i^*(\cdot)$, while all other bidders play it.

3.1 Symmetric equilibrium

In this subsection, we consider symmetric equilibrium. By Proposition 1, it suffices to consider the symmetric strategy profile $\sigma^* = (\sigma_i^*)_{i=1}^n$. The symmetric strategy profile σ^* is a Bayesian Nash equilibrium if and only if

$$\begin{aligned} &\left\{ \sum_{\nu=0}^{n-1} \frac{1}{n-\nu} \binom{n-1}{\nu} (1-F_k)^{n-1-\nu} (F_k)^{\nu} \right\} U(v^{\kappa} - B) \\ &\qquad \geq \sum_{\mu=1}^{k} \left\{ (F_{\mu})^{n-1} - (F_{\mu-1})^{n-1} \right\} U(v^{\kappa} - v^{\mu}) \end{aligned} \tag{1}$$

for all $\kappa = k+1, \dots, m$. The LHS of (1) is the expected payoff that a bidder's v^{κ}-type obtains by bidding a buy price B. ${}_{n-1}C_{\nu}\ (1-F_k)^{n-1-\nu}(F_k)^{\nu}$ is the probability that there are ν other bidders whose types are not greater than v^k. The winning probability is $1/\{n-\nu\}$ because the winner is determined among the bidders who bid a buy price B with equal probability. The RHS of (1) is the expected payoff that a bidder's v^{κ}-type obtains by bidding $b \in (v^k, B)$.

[4] If we analyze all equilibria, as Inami (2008) shows, we cannot accurately compare seller's expected revenues. Indeed, a seller can obtain both a higher expected revenue and a lower expected revenue from the equilibrium by weakly dominated strategies than from the equilibrium by partially truth-telling strategies. Similarly, in the auction without a buy price, a seller can obtain both a higher expected revenue and a lower expected revenue from the equilibrium by weakly dominated strategies than from the equilibrium by weakly dominant strategies. Thus, it would be reasonable to restrict our attention to equilibria by partially truth-telling strategies.

$(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}$ is the probability that the second highest bid among n $-$ 1 other bidders is v^μ. In this case, since this auction is a usual second-price, sealed-bid auction, the type wins and pays v^μ. If the type bid $b' \leq v^k$, then the type could not (surely) win against opponents' v^k-type because they bid v^k. In other words, the type cannot maximize the winning probability by bidding b'. Thus, the RHS of (1) is the maximum expected payoff by bidding the amount except a buy price B.

In fact, we do not need to consider all those inequalities.

Proposition 2. *The strategy profile σ^* is a Bayesian Nash equilibrium if and only if* (1) *holds for $\kappa = k+1$.*

Proof. The necessity part is straightforward. We only prove the sufficiency part.

Fix $\kappa \geq k+1$. For all $\mu \in \{1, \dots, k\}$,

$$U(v^\kappa - B) - U(v^{k+1} - B) \geq U(v^\kappa - v^\mu) - U(v^{k+1} - v^\mu),$$

where the inequality follows because $U(\cdot)$ is concave. In addition,

$$\begin{aligned}
\sum_{\nu=0}^{n-1} \frac{1}{n-\nu}\binom{n-1}{\nu}(1-F_k)^{n-1-\nu}(F_k)^\nu &= \frac{1-(F_k)^n}{n(1-F_k)} \\
&= \frac{1 + F_k + \cdots + (F_k)^{n-2} + (F_k)^{n-1}}{n} \\
&> (F_k)^{n-1} \\
&= \sum_{\mu=1}^{k} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left\{\sum_{\nu=0}^{n-1} \frac{1}{n-\nu}\binom{n-1}{\nu}(1-F_k)^{n-1-\nu}(F_k)^\nu\right\}\big\{U(v^\kappa - B) - U(v^{k+1} - B)\big\} \\
&\quad \geq \sum_{\mu=1}^{k} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\}\big\{U(v^\kappa - v^\mu) - U(v^{k+1} - v^\mu)\big\}.
\end{aligned}$$

Hence, if (1) holds for $\kappa = k+1$, then (1) also holds for $\kappa \geq k+2$. *Q.E.D.*

We consider when there exists a buy price $B \in (v^k, v^{k+1}]$ such that (1) for $\kappa = k+1$ holds. The following condition ensures the existence of such a buy price B:

$$\begin{aligned}
U(v^{k+1} - v^k) &> \frac{\sum_{\mu=1}^{k} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} U(v^{k+1} - v^\mu)}{\sum_{\nu=0}^{n-1} \frac{1}{n-\nu}\binom{n-1}{\nu}(1-F_k)^{n-1-\nu}(F_k)^\nu} \qquad (2)\\
&= \frac{n(1-F_k)\sum_{\mu=1}^{k} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} U(v^{k+1} - v^\mu)}{1-(F_k)^n}.
\end{aligned}$$

(2) can be interpreted from the view point of certainty equivalent. The RHS of (2) is the conditional expected utility given that a bidder's v^{k+1}-type wins in a usual second-price, sealed-bid auction without a buy price. Hence, (2) states that the certainty equivalent of the lottery corresponding to the RHS of (2) must be less than $v^{k+1} - v^k$.

If (2) holds, then we can also find the highest buy price B under which the symmetric strategy profile σ^* is a Bayesian Nash equilibrium. Consider (1) for $\kappa = k+1$:

$$U(v^{k+1} - B) \geq \frac{n(1-F_k)\sum_{\mu=1}^{k} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} U(v^{k+1} - v^\mu)}{1-(F_k)^n}. \qquad (3)$$

Arranging (3), we have

$$B \le v^{k+1} - U^{-1}\left(\frac{n(1-F_k)\sum_{\mu=1}^{k}\left\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\right\}U(v^{k+1}-v^\mu)}{1-(F_k)^n}\right).$$

Here let

$$B^*_{k+1} := v^{k+1} - U^{-1}\left(\frac{n(1-F_k)\sum_{\mu=1}^{k}\left\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\right\}U(v^{k+1}-v^\mu)}{1-(F_k)^n}\right).$$

Thus, when we consider the auction with a buy price $B \in (v^k, B^*_{k+1}]$, the symmetric strategy profile σ^* is a Bayesian Nash equilibrium. When we consider the auction with a buy price $B \in (B^*_{k+1}, v^{k+1}]$, the symmetric strategy profile σ^* is not an equilibrium. Thus, by Proposition 1, there is no symmetric equilibrium in the auction with a buy price $B \in (B^*_{k+1}, v^{k+1}]$.

Proposition 3. *Suppose that* (2) *holds. Then,*
(i) when we consider the auction with a buy price $B \in (v^k, B^*_{k+1}]$, the strategy profile σ^* is a unique symmetric equilibrium, and
(ii) when we consider the auction with a buy price $B \in (B^*_{k+1}, v^{k+1}]$, there is no symmetric equilibrium.

If (2) does not hold, then B^*_{k+1} is not greater than v^k. Thus, we cannot find a buy price $B \in (v^k, v^{k+1}]$ under which a symmetric equilibrium exists.

In fact, we can always find a buy price B under which a symmetric equilibrium exists between the lowest valuation and the second lowest valuation.

Proposition 4. (2) *for $k = 1$ always holds.*

Proof. We show that

$$U(v^2 - v^1) > \frac{n(1-F_1)(F_1)^{n-1}U(v^2-v^1)}{1-(F_1)^n}.$$

Since

$$1 > \frac{n(F_1)^{n-1}}{1+F_1+\cdots+(F_1)^{n-2}+(F_1)^{n-1}} = \frac{n(1-F_1)(F_1)^{n-1}}{1-(F_1)^n},$$

we immediately have the result. *Q.E.D.*

From the view point of certainty equivalent, the result in Proposition 4 is obvious. Since bidders obtain either the surplus $v^2 - v^1$ or the surplus 0 in the auction, the certainty equivalent of the lottery is clearly less than $v^2 - v^1$. By Proposition 4, we always have a symmetric equilibrium in the m-bidder, n-type framework.

There is no general relation between the intervals satisfying (2) and the intervals not satisfying (2). We show an example.

Example 1. *Consider a three-bidder, four-type framework. Let $U(x) = \sqrt{x}$, $f_1 = \frac{3}{10}$, $f_2 = \frac{1}{10}$, $f_3 = \frac{1}{10}$, $v^1 = 10$, $v^2 = 35.5$, $v^3 = 36$ and $v^4 = 40$.*

In this example, both (2) for $k = 1$ and (2) for $k = 3$ hold. On the contrary, (2) for $k = 2$ does not hold. (See Figure 1.) Thus, there is no monotonicity as to whether (2) holds.[5]

We consider the auction with a buy price B between other valuations, $(v^k, v^{k+1}]$ $(k \ne 1)$. In general, (2) does not always hold. However, fixing all parameters except the number of bidders, it depends on the number of bidders whether (2) for $k \ne 1$ holds.

[5]Strictly speaking, we have a problem that bidders' utility function $U(\cdot) = \sqrt{x}$ is not defined on $(-\infty, 0)$. To resolve this problem, we can modify the utility function $U(\cdot)$ as follows: at some point $\tilde{x}$ close to 0, $U(\cdot) = \sqrt{x}$ if $x \ge \tilde{x}$ and $U(\cdot) = x/\sqrt{\tilde{x}}$ if $x < \tilde{x}$. We can make the same modification in Example 3.

Figure 1: Example 1

$(v^k, v^{k+1}]$	The LHS of (2) for k	The RHS of (2) for k
$(v^3, v^4]$ $(k=3)$	2.00	1.31
$(v^2, v^3]$ $(k=2)$	0.71	0.77
$(v^1, v^2]$ $(k=1)$	5.05	0.98

Proposition 5. *Fix $U(\cdot)$, f_μ, and v^μ for all $\mu = 1, \cdots, k+1$. There exists n_0 such that for all $n \geq n_0$,*

$$U(v^{k+1} - v^k) > \frac{n(1-F_k)\sum_{\mu=1}^{k}\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\}U(v^{k+1} - v^\mu)}{1-(F_k)^n}.$$

Proof. Let

$$G_\mu(n) := \frac{n(F_\mu)^{n-1}}{1-(F_k)^n}$$

for all $\mu = 1, \ldots, k$. We have

$$\begin{aligned} & U(v^{k+1} - v^k) - \frac{n(1-F_k)\sum_{\mu=1}^{k}\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\}U(v^{k+1} - v^\mu)}{1-(F_k)^n} \\ &= U(v^{k+1} - v^k) - (1-F_k)\sum_{\mu=1}^{k} G_\mu(n)\{U(v^{k+1} - v^\mu) - U(v^{k+1} - v^{\mu+1})\}. \end{aligned} \tag{4}$$

For all μ, $(1-F_k)G_\mu(n) \to 0$ as $n \to 0$. Thus, there exists n_0 such that $(4) > 0$. Hence, we have the result. *Q.E.D.*

(2) for $k \neq 1$ holds with a large number of bidders. It is because if many bidders participate the auction, the second highest bid possibly decreases and then the certainty equivalent of the lottery corresponding to the RHS of (2) decreases. By Proposition 5, we also find that there exists n^* such that all intervals, $(v^k, v^{k+1}]$, satisfies (2). Furthermore, Proposition 5 has an important feature. In the Internet auctions, a seller usually faces many bidders and therefore (2) is more likely to hold.

3.2 Asymmetric equilibrium

In this subsection, we consider asymmetric equilibrium. By Proposition 1, it suffices to consider asymmetric strategy profiles where only one bidder, say, bidder 1, does not play the strategy $\sigma_i^*(\cdot)$ and all other bidders play it.

Consider the incentive of bidder 1. Since bidder 1 does not play the strategy $\sigma_1^*(\cdot)$, there exists $\kappa \geq k+1$ such that

$$\begin{aligned} & \sum_{\mu=1}^{k}\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\}U(v^\kappa - v^\mu) \\ & \qquad \geq \left\{\sum_{\nu=0}^{n-1}\frac{1}{n-\nu}\binom{n-1}{\nu}(1-F_k)^{n-1-\nu}(F_k)^\nu\right\}U(v^\kappa - B). \end{aligned} \tag{5}$$

The LHS of (5) is the maximum expected payoff that bidder 1's v^κ-type obtains by bidding the amount except a buy price B. The RHS of (5) is the expected payoff that he obtains by bidding a buy price B. If (5) does not hold, bidder 1's v^κ-type obtains a higher expected payoff by bidding a buy price B.

Suppose that such κ exists. By a similar argument to that of Proposition 2, we can find k^* such that (5) holds if $\kappa \le k^*$ and (5) does not hold if $\kappa \ge k^*+1$. Here, we fix $b \in (v^k, B)$ and then define the following strategy:

$$\overline{\sigma}_1(t_1; b) = \begin{cases} B & \text{if } t_1 \ge v^{k^*+1}, \\ b & \text{if } v^{k+1} \le t_1 \le v^{k^*}, \\ t_1 & \text{otherwise.} \end{cases}$$

We examine whether the asymmetric strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot; b), \sigma^*_{-1})$ is an equilibrium. First, consider the incentive of bidder 1. By the definition of k^*, for any b, bidder 1's strategy $\overline{\sigma}_1(\cdot; b)$ is a best response to other bidders' strategies $\sigma^*_{-1} = (\sigma^*_j)_{j\neq 1}$. Thus, it remains to consider the incentive of bidder j ($j = 2, \ldots, n$). To be an equilibrium, for all $\kappa \in \{k+1, \ldots, m\}$,

$$\begin{aligned} &\left\{\sum_{\nu=0}^{n-2} \binom{n-2}{\nu}\left(\frac{1-F_{k^*}}{n-\nu} + \frac{F_{k^*}}{n-1-\nu}\right)(1-F_k)^{n-2-\nu}(F_k)^\nu\right\}U(v^\kappa - B) \\ &\quad \ge \sum_{\mu=1}^{k}\left\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\right\}U(v^\kappa - v^\mu) + (F_{k^*} - F_k)(F_k)^{n-2}U(v^\kappa - b) \end{aligned} \tag{6}$$

must hold. The LHS of (6) is the expected payoff that bidder j's v^κ-type obtains by bidding a buy price B. The RHS of (6) is the maximum expected payoff that bidder j's v^κ-type obtains by bidding the amount except a buy price B. In this case, bidder j always wins against bidder 1 whose type is not greater than v^k. Furthermore, he wins against bidder 1 whose type is greater than v^k, but less than v^{k^*+1}.

From the same argument as that of Subsection 3.1, it suffices to consider (6) for $\kappa = k+1$.[6] Also, if (5) for $\kappa = k+1$ holds, then we can find k^*. Thus, it suffices to consider (5) for $\kappa = k+1$. Hence, both inequalities are also necessary conditions for equilibrium.

Proposition 6. *An asymmetric strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot; b), \sigma^*_{-1})$ is a Bayesian Nash equilibrium if and only if both* (5) *and* (6) *holds for $\kappa = k+1$.*

By Proposition 6, there is no asymmetric equilibrium by partially truth-telling strategies in the auction with a buy price $B \in (v^k, B^*_{k+1})$.

In general, given a buy price B, it is unclear whether there exists $b \in (v^k, B)$ such that the strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot; b), \sigma^*_{-1})$ is an equilibrium. The following inequality is a necessary and sufficient condition for the existence of such b:

$$\begin{aligned} &\left\{\sum_{\nu=0}^{n-2} \binom{n-2}{\nu}\left(\frac{1-F_{k^*}}{n-\nu} + \frac{F_{k^*}}{n-1-\nu}\right)(1-F_k)^{n-2-\nu}(F_k)^\nu\right\}U(v^{k+1} - B) \\ &\quad > \sum_{\mu=1}^{k}\left\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\right\}U(v^{k+1} - v^\mu) + (F_{k^*} - F_k)(F_k)^{n-2}U(v^{k+1} - B). \end{aligned} \tag{7}$$

[6] Let

$$\begin{aligned} I(x) := &\left\{\sum_{\nu=0}^{n-2} \binom{n-2}{\nu}\left(\frac{1-F_{k^*}}{n-\nu} + \frac{F_{k^*}}{n-1-\nu}\right)(1-F_k)^{n-2-\nu}(F_k)^\nu\right\}U(x - B) \\ &- \sum_{\mu=1}^{k}\left\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\right\}U(x - v^\mu) - (F_{k^*} - F_k)(F_k)^{n-2}U(x - b). \end{aligned}$$

Since we can show that $I(\cdot)$ is monotone increasing, it suffices to consider (6) for $\kappa = k+1$.

Since (7) holds with strict inequality, then we can find $b \in (v^k, B)$ such that (6) holds.

We have considered a specific asymmetric strategy profile. When we consider the auction with a buy price B under which (7) does not hold, other asymmetric strategy profiles might be equilibria. However, it is not the case. That is, whenever some asymmetric equilibrium exists, we can always find $b \in (v^k, B)$ such that the asymmetric strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot; b), \sigma^*_{-1})$ is an equilibrium.

Proposition 7. *Suppose that some asymmetric strategy profile $\sigma = (\sigma_1, \sigma^*_{-1})$ is an equilibrium. Then, there exists $b \in (v^k, B)$ such that the asymmetric strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot; b), \sigma^*_{-1})$ is also an equilibrium.*

Proof. See Appendix.

By Proposition 6 and Proposition 7, (7) is also a necessary and sufficient condition for the existence of asymmetric equilibrium.

However, we do not always have an asymmetric equilibrium by partially truth-telling strategies.

Proposition 8. *There is no asymmetric equilibrium by partially truth-telling strategies if there are only two bidders.*

Proof. See Appendix.

By Proposition 8, there is no asymmetric equilibrium by partially truth-telling strategies in the two-bidder framework regardless of the number of bidders' types. Thus, there is no asymmetric equilibrium by partially truth-telling strategies in the two-bidder, two-type framework of Budish and Takeyama (2001). Their restriction on a symmetric equilibrium does not lose generality.

4 Auction comparisons from a seller's point of view

In this section, we analyze whether a seller can improve her payoffs by introducing a buy price B if bidders are risk-averse. We take into account both bidders' risk attitude and a seller's risk attitude, and then compares the expected utilities. We assume that the seller has a von-Neumann-Morgenstern utility function $W : \mathbb{R}_+ \to \mathbb{R}_+$ with $W(0) = 0$, and assume that $W(\cdot)$ is strictly increasing and concave (possibly linear).

4.1 The comparison with the auction without a buy price B

First, we compare between the auctions with and without a buy price B. We consider the case in which a seller sets a buy price B between the highest valuation and the second highest valuation. We assume that (2) holds for $k = m-1$. Thus, there exists the symmetric equilibrium σ^* in the auction with a buy price $B \in (v^{m-1}, B^*_m]$. The seller's expected utility obtained from the symmetric equilibrium σ^* is:

$$\begin{aligned} R^B = &\sum_{\mu=1}^{m-1} \Big[nF_{m-1}\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} - (n-1)\big\{(F_\mu)^n - (F_{\mu-1})^n\big\}\Big] W(v^\mu) \\ &+ \big\{1 - (F_{m-1})^n\big\} W(B). \end{aligned} \tag{8}$$

Since R^B is increasing with respect to B, we evaluate (8) at $B = B^*_m$. We denote the expected utility by R^B_m.

Since the auction without a buy price B is a usual second-price, sealed-bid auction, the seller's expected utility is:

$$R^{NB} = \sum_{\mu=1}^{m} \Big[n\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} - (n-1)\big\{(F_\mu)^n - (F_{\mu-1})^n\big\} \Big] W(v^\mu). \tag{9}$$

If $R^B_m > R^{NB}$, then the seller can improve her utility by introducing a certain buy price B.

Theorem 1. *Consider a n-bidder, m-type framework. Suppose that* (2) *holds for* $k = m-1$. *Then,*

$$R^B_m \geq R^{NB},$$

where the equality holds if and only if both the seller and the bidders are risk-neutral.

Proof. See Appendix.

Even if bidders are risk-loving, we might have the symmetric equilibrium σ^*. However, by the same argument as that of the proof of Theorem 1, a seller obtains a strictly lower expected utility, with the result that she cannot improve her expected utility by introducing a buy price B.

In the two-type framework, by Proposition 4, (2) always holds for $k = 1$. We immediately have the following result.

Corollary 1. *Consider a n-bidder, two-type framework. Then,*

$$R^B_m \geq R^{NB},$$

where the equality holds if and only if both the seller and the bidders are risk-neutral.

Corollary 1 generalizes the result of Budish and Takeyama (2001) in the two-bidder, two-type framework with respect to the number of bidders and the seller's risk attitude.

4.2 The comparison with the auction with asymmetric equilibria

In this subsection, we investigate the possibility of further improvements. As long as we limit attention to a symmetric equilibrium, the expected utility R^B_m is the highest of all auctions with a buy price $B \in (v^{m-1}, v^m]$. We then take into account asymmetric equilibria and compare the expected utilities. The next subsection considers another possibility that a seller can obtain a higher expected utility from the auction with a buy price B that is not greater than the second highest valuation.

We continue to assume that (2) holds for $k = m-1$. Thus, by Proposition 6 , there is no asymmetric equilibrium by partially truth-telling strategies in the auction with a buy price $B \in (v^{m-1}, B^*_m)$. From the same argument as that of Subsection 3.2, there exists an asymmetric equilibrium if and only if

$$\begin{aligned} &\sum_{\nu=0}^{n-2} \left\{ \frac{1}{n-1-\nu} \binom{n-2}{\nu} (f_m)^{n-2-\nu} (F_{m-1})^{\nu} \right\} U(v^m - B) \\ &\quad > \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} U(v^m - v^\mu) + f_m (F_{m-1})^{n-2} U(v^m - B). \end{aligned} \tag{10}$$

Arranging (10), we have [7]

$$B < v^m - U^{-1}\left(\frac{(n-1)f_m \sum_{\mu=1}^{m-1}\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\}U(v^m - v^\mu)}{1-(F_{m-1})^{n-1} - (n-1)(f_m)^2(F_{m-1})^{n-2}}\right). \tag{11}$$

Here let

$$\overline{B_m^*} := v^m - U^{-1}\left(\frac{(n-1)f_m \sum_{\mu=1}^{m-1}\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\}U(v^m - v^\mu)}{1-(F_{m-1})^{n-1} - (n-1)(f_m)^2(F_{m-1})^{n-2}}\right).$$

Note that there is no asymmetric equilibrium in the auction with a buy price $B \in [\overline{B_m^*}, v^m]$.

From the above argument, there exists an asymmetric equilibrium in the auction with a buy price $B \in [B_m^*, \overline{B_m^*})$. When we compare seller's expected utilities, we need to consider two cases. One is the case in which a seller sets a buy price B_m^*. The other is the case in which she sets a buy price $B \in (B_m^*, \overline{B_m^*})$.

First, we consider the auction with the buy price B_m^* where there exist both a symmetric equilibrium and an asymmetric equilibrium. In this case, a seller can obtain a higher expected utility from the symmetric equilibrium. This is because in any asymmetric equilibria, one bidder whose type is not less than a buy price B does not bid the buy price B_m^* with probability 1.

Next, we consider the auction with a buy price $B \in (B_m^*, \overline{B_m^*})$. Since by definition, bidders cannot bid the buy price $\overline{B_m^*}$, and one bidder's v^m-type does not bid the buy price B with probability 1, we can derive an upper bound of the seller's expected utility with asymmetric equilibria:

$$\begin{aligned} R^{\overline{B}} = &\sum_{\mu=1}^{m-1}\Big[nF_{m-1}\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\} - (n-1)\{(F_\mu)^n - (F_{\mu-1})^n\}\Big]W(v^\mu) \\ &+ \sum_{\mu=1}^{m-1} f_m\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\}W(v^\mu) + \{1-(F_{m-1})^{n-1}\}W(\overline{B_m^*}). \end{aligned} \tag{12}$$

If $R_m^B > R^{\overline{B}}$, then a seller cannot improve her utility by taking account of asymmetric equilibria.

Proposition 9. *Consider a n-bidder, m-type framework. Suppose that* (2) *holds for* $k = m-1$ *and that* (10) *holds. Then,*

$$R_m^B > R^{\overline{B}}.$$

Proof. See Appendix.

If (10) does not hold, by Propositions 6 and 7, there is no asymmetric equilibrium by partially truth-telling strategies. Thus, a seller can obtain the highest expected utility from the auction with the buy price B_m^*.

4.3 Discussions

We have only analyzed the auction environment where a buy price B is set between the highest valuation and the second highest valuation, $(v^{m-1}, v^m]$. In this subsection, we

[7] Note that

$$\sum_{\nu=0}^{n-2}\frac{1}{n-1-\nu}\binom{n-2}{\nu}(f_m)^{n-2-\nu}(F_{m-1})^\nu - f_m(F_{m-1})^{n-2} = \frac{1-(F_{m-1})^{n-1}-(n-1)(f_m)^2(F_{m-1})^{n-2}}{(n-1)f_m}.$$

examine other auction environment in which a buy price B is not greater than the second highest valuation, v^{m-1}. We assume that a seller is risk-neutral and that bidders are risk-averse, and study the two-bidder, three-type framework. Since we consider the two-bidder framework, by Proposition 8, we need not care about asymmetric equilibria by partially truth-telling strategies.

First, we consider the auction with a buy price $B \in (v^2, v^3]$. The symmetric equilibrium exists if and only if (2) for $k = 2$ holds:

$$U(v^3 - v^2) \geq \frac{2f_1 U(v^3 - v^1)}{1 + f_1 - f_2}. \tag{13}$$

If (13) does not hold, by Theorem 1, a seller cannot improve her revenue by introducing a buy price $B \in (v^2, v^3]$.

Next, we consider the auction with a buy price $B \in (v^1, v^2]$. Since, by Proposition 4, (2) for $k = 1$ always holds, it suffices to compare seller's expected revenues in the same way of Subsection 4.1. To obtain a higher revenue from the auction with the buy price B_2^*, which is not greater than the second highest valuation v^2, than from the auction without a buy price,

$$\begin{aligned} &U\left(\frac{-(f_3)^2 v^3 + \{1 - 2f_2 + 2f_1 f_2 - (f_1)^2 + (f_2)^2\} v^2 - 2f_1(1 - f_1)v^1}{1 - (f_1)^2}\right) \\ &\quad > \frac{2f_1}{1 + f_1} U(v^2 - v^1) \end{aligned} \tag{14}$$

must hold.

Example 2. *Let* $U(x) = 1 - e^{-\frac{1}{2}x}$, $f_1 = \frac{7}{10}$, $f_2 = \frac{1}{10}$, $v^1 = 0.25$, $v^2 = 0.50$, *and* $v^3 = 0.55$.

We have the followings:

$$\begin{aligned} &\text{the LHS of (13)} \simeq 0.025 < 0.12 \simeq \text{the RHS of (13), and} \\ &\text{the LHS of (14)} \simeq 0.0960 < 0.0968 \simeq \text{the RHS of (14).} \end{aligned}$$

In Example 2, a seller cannot obtain a higher expected revenue from the auction with a buy price B even if bidders are risk-averse. Thus, Theorem 1 does not extend to a general case. In other words, bidders' risk-aversion is not a sufficient condition for revenue improvements.

Example 3. *Let* $U(x) = \sqrt{x}$, $f_1 = \frac{7}{10}$, $f_2 = \frac{1}{10}$, $v^1 = 0.25$, $v^2 = 0.50$, *and* $v^3 = 0.75$.

We have the followings:

$$\begin{aligned} &\text{the LHS of (13)} = 0.50 < 0.62 \simeq \text{the RHS of (13), and} \\ &\text{the LHS of (14)} \simeq 0.43 > 0.41 \simeq \text{the RHS of (14).} \end{aligned}$$

In Example 3, there does not exist a symmetric equilibrium in the auction with a buy price $B \in (v^2, v^3]$. In the auction with the buy price B_2^*, the event that one bidder is a v^2-type and the other bidder is a v^3-type occurs with positive probability. In this case, the v^2-type can win the auction with positive probability. And this results in an inefficient allocation. That is, the item can be allocated to the bidder who does not have the highest valuation. On the contrary, a seller can extract a risk premium from the v^2-type. Thus, depending on the parameter settings, a seller can obtain a higher expected revenue from the auction with the buy price B_2^* than from the auction without a buy price B as Example 3.

5 Conclusion

We have considered a second-price, sealed-bid auction with a buy price. To restrict our attention to equilibria by reasonable strategies, we have introduced the notion of partial truth-telling, and then characterized all equilibria by partially truth-telling strategies. We have shown that if there exists a buy price between the highest valuation and the second highest valuation under which we have a symmetric equilibrium, and bidders are risk-averse, a seller can obtain a higher expected revenue from the auction with a certain buy price. This improvement result extends to the case in which a seller is risk-averse. Also, in the two-type framework with any number of bidders, a seller can always improve her utility by introducing a certain buy price. On the other hand, we have shown an example that a seller cannot obtain a higher expected revenue from the auction with a buy price even if bidders are risk-averse. In case of three or more types, bidders' risk aversion is not a sufficient condition for revenue improvements.

We do not derive an optimal buy price except the case in which the number of bidders' types is two. In the three or more type framework, we must consider not only the auction with a buy price between the highest valuation and the second highest valuation, but also the auction with a buy price that is not greater than the second highest valuation. To find an optimal auction, we need to compare seller's expected revenues with a buy price on different intervals. Additionally, in case of three or more bidders, we necessarily take account of a seller's expected revenue from an asymmetric equilibrium. It is left for future research.

Appendix

Proof of Proposition 1

We assume, by way of contradiction, that the strategy profile $\hat{\sigma}$ where some two bidders, say, bidder i and bidder j, do not play the strategy $\sigma_i^*(\cdot)$ is an equilibrium by partially truth-telling strategies. For each bidder i and bidder j, there exist types $t_i \geq B$ and $t_j \geq B$ such that $\hat{\sigma}_i(t_i) \neq B$ and $\hat{\sigma}_j(t_j) \neq B$. Here, we denote the infimum of the support of $\hat{\sigma}_i(t_i)$ and $\hat{\sigma}_j(t_j)$ by $\underline{b}_i$ and $\underline{b}_j$, respectively. Without loss of generality, we assume that $\underline{b}_i \geq \underline{b}_j$.

In the equilibrium, by the definition of $\underline{b}_j$, it is optimal for the t_j-type to bid $\underline{b}_j$ or $\underline{b}_j + \varepsilon$ for sufficiently small ε. When the highest bid is not a buy price B, this auction is a usual second-price, sealed-bid auction. Thus, the winning probability of the t_j-type must be maximized among the bids except a buy price B.

Suppose that the t_j-type bids $b_j \in (\underline{b}_i, B)$. Since the event that bidder i is a t_i-type, bidder j is a t_j-type, and n $-$ 2 other bidders are types less than a buy price B occurs with positive probability, the t_j-type can increase his winning probability by bidding b_j rather than $\underline{b}_j$ or $\underline{b}_j + \varepsilon$ for sufficiently small ε, which contradicts the assumption. This completes the proof.

Proof of Proposition 7

Suppose that some asymmetric strategy profile $\sigma = (\sigma_1(\cdot), \sigma_{-1}^*)$ is an equilibrium. Then, there must exist k^* and therefore

$$\begin{aligned}&\sum_{\mu=1}^{k} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} U(v^{k+1} - v^\mu) \\ &\quad \geq \left\{ \sum_{\nu=0}^{n-1} \frac{1}{n-\nu} \binom{n-1}{\nu} (1-F_k)^{n-1-\nu} (F_k)^\nu \right\} U(v^{k+1} - B).\end{aligned} \tag{15}$$

must hold.

Here let $\tilde{F}$ be the probability that bidder 1 does not bid a buy price B. Then, $\tilde{F} \in (F_k, F_{k^*}]$. Since bidder j ($j = 2, \ldots, n$) plays the equilibrium strategy $\sigma_j^*(\cdot)$, we have

$$\left\{ \sum_{\nu=0}^{n-2} \binom{n-2}{\nu} \left(\frac{1-\tilde{F}}{n-\nu} + \frac{\tilde{F}}{n-1-\nu} \right) (1-F_k)^{n-2-\nu} (F_k)^\nu \right\} U(v^{k+1} - B) \geq \pi_j(\varepsilon),$$

where $\pi_j(\varepsilon)$ is the expected payoff obtained by bidding the amount $B - \varepsilon$ for $\varepsilon > 0$. Since

$$\lim_{\varepsilon \to 0} \pi_j(\varepsilon) > \sum_{\mu=1}^{k} \left\{ (F_\mu)^{n-1} - (F_{\mu-1})^{n-1} \right\} U(v^{k+1} - v^\mu) + (\tilde{F} - F_k)(F_k)^{n-2} U(v^{k+1} - B),$$

we have

$$\begin{aligned} &\left\{ \sum_{\nu=0}^{n-2} \binom{n-2}{\nu} \left(\frac{1-\tilde{F}}{n-\nu} + \frac{\tilde{F}}{n-1-\nu} \right) (1-F_k)^{n-2-\nu} (F_k)^\nu \right\} U(v^{k+1} - B) \\ &\quad > \sum_{\mu=1}^{k} \left\{ (F_\mu)^{n-1} - (F_{\mu-1})^{n-1} \right\} U(v^{k+1} - v^\mu) + (\tilde{F} - F_k)(F_k)^{n-2} U(v^{k+1} - B). \end{aligned}$$

Here let

$$\begin{aligned} J(x) :=& \left\{ \sum_{\nu=0}^{n-2} \binom{n-2}{\nu} \left(\frac{1-x}{n-\nu} + \frac{x}{n-1-\nu} \right) (1-F_k)^{n-2-\nu} (F_k)^\nu \right\} U(v^{k+1} - B) \\ &\quad - \sum_{\mu=1}^{k} \left\{ (F_\mu)^{n-1} - (F_{\mu-1})^{n-1} \right\} U(v^{k+1} - v^\mu) - (x - F_k)(F_k)^{n-2} U(v^{k+1} - B). \end{aligned}$$

$J(\cdot)$ is linear with respect to x. By assumption, $J(\tilde{F}) > 0$. In addition, $J(F_k) \leq 0$. This is because if $J(F_k) > 0$, then (1) holds for $\kappa = k+1$, which contradicts the assumption. Thus, $J(\cdot)$ is increasing. This implies that $J(F_{k^*}) > 0$. That is, we have

$$\begin{aligned} &\left\{ \sum_{\nu=0}^{n-2} \binom{n-2}{\nu} \left(\frac{1-F_{k^*}}{n-\nu} + \frac{F_{k^*}}{n-1-\nu} \right) (1-F_k)^{n-2-\nu} (F_k)^\nu \right\} U(v^{k+1} - B) \\ &\quad > \sum_{\mu=1}^{k} \left\{ (F_\mu)^{n-1} - (F_{\mu-1})^{n-1} \right\} U(v^{k+1} - v^\mu) + (F_{k^*} - F_k)(F_k)^{n-2} U(v^{k+1} - B). \end{aligned} \tag{16}$$

Since both (15) and (16) hold, by Proposition 6, there exists $b \in (v^k, B)$ such that the asymmetric strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot; b), \sigma_{-1}^*)$ is an equilibrium. This completes the proof.

Proof of Proposition 8

Suppose, by way of contradiction, that an asymmetric equilibrium exists. Then, by Propositions 6 and 7,

$$\begin{aligned} &\sum_{\mu=1}^{k} \left\{ (F_\mu) - (F_{\mu-1}) \right\} U(v^{k+1} - v^\mu) \\ &\quad \geq \frac{1}{2}(1 - F_k) U(v^{k+1} - B) + (F_k) U(v^{k+1} - B), \end{aligned} \tag{17}$$

$$\begin{aligned} &\frac{1}{2}(1 - F_{k^*}) U(v^{k+1} - B) + (F_{k^*}) U(v^{k+1} - B) \\ &\quad > \sum_{\mu=1}^{k} \left\{ (F_\mu) - (F_{\mu-1}) \right\} U(v^{k+1} - v^\mu) + (F_{k^*} - F_k) U(v^{k+1} - B). \end{aligned} \tag{18}$$

Hence,

$$
\begin{aligned}
&\frac{1}{2}(1-F_{k^*})U(v^{k+1}-B)+(F_{k^*})U(v^{k+1}-B)\\
&\quad > \sum_{\mu=1}^{k}\big\{(F_\mu)-(F_{\mu-1})\big\}U(v^{k+1}-v^\mu)+(F_{k^*}-F_k)U(v^{k+1}-B) \qquad (19)\\
&\quad \geq \frac{1}{2}(1-F_k)U(v^{k+1}-B)+(F_{k^*})U(v^{k+1}-B).
\end{aligned}
$$

Rearranging (19), we have

$$F_k > F_{k^*}. \qquad (20)$$

However, (20) does not hold because $k^* > k$ and $f_\mu > 0$ for all μ. Thus, both (17) and (18) do not hold simultaneously, which contradicts the assumption. This completes the proof.

Proof of Theorem 1

At first, we recall that

$$B_m^* = v^m - U^{-1}\left(\frac{nf_m\sum_{\mu=1}^{m-1}\big\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\big\}U(v^m-v^\mu)}{1-(F_{m-1})^n}\right).$$

If (8) > (9), then the seller can obtain a higher expected utility from the auction with the buy price B_m^*.

$$
\begin{aligned}
R_m^B - R^{NB} &= \sum_{\mu=1}^{m-1}\Big[nF_{m-1}\big\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\big\}-(n-1)\big\{(F_\mu)^n-(F_{\mu-1})^n\big\}\Big]W(v^\mu)\\
&\quad + \{1-(F_{m-1})^n\}W(B_m^*)\\
&\quad - \sum_{\mu=1}^{m}\Big[n\big\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\big\}-(n-1)\big\{(F_\mu)^n-(F_{\mu-1})^n\big\}\Big]W(v^\mu)\\
&= -nf_m\sum_{\mu=1}^{m-1}\big\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\big\}W(v^\mu)-\big\{1-nf_m(F_{m-1})^{n-1}-(F_{m-1})^n\big\}W(v^m)\\
&\quad + \big\{1-(F_{m-1})^n\big\}W(B_m^*)\\
&= \big\{1-(F_{m-1})^n\big\}\Bigg[-\frac{nf_m\sum_{\mu=1}^{m-1}\big\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\big\}W(v^\mu)}{1-(F_{m-1})^n}\\
&\quad - \frac{\big\{1-nf_m(F_{m-1})^{n-1}-(F_{m-1})^n\big\}}{1-(F_{m-1})^n}W(v^m)+W(B_m^*)\Bigg]\\
&\geq \big\{1-(F_{m-1})^n\big\}\Bigg[-W\Bigg(\frac{nf_m\sum_{\mu=1}^{m-1}\big\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\big\}v^\mu}{1-(F_{m-1})^n}\\
&\quad + \frac{\big\{1-nf_m(F_{m-1})^{n-1}-(F_{m-1})^n\big\}v^m}{1-(F_{m-1})^n}\Bigg)+W(B_m^*)\Bigg] \qquad (21)\\
&\geq 0,
\end{aligned}
$$

where (21) follows because $W(\cdot)$ is concave and the last inequality follows because $W(\cdot)$ is monotone. Indeed,

$$
\begin{aligned}
& B_m^* - \frac{nf_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} v^\mu}{1-(F_{m-1})^n} - \frac{\big\{1 - nf_m (F_{m-1})^{n-1} - (F_{m-1})^n\big\} v^m}{1-(F_{m-1})^n} \\
= & \frac{nf_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} (v^m - v^\mu)}{1-(F_{m-1})^n} \\
& - U^{-1}\left(\frac{nf_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} U(v^m - v^\mu)}{1-(F_{m-1})^n} \right) \\
\geq & \ 0, \qquad\qquad (22)
\end{aligned}
$$

where the inequality follows because for all μ ($\mu = 1, \dots, m-1$), $0 < nf_m\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\}/\big\{1-(F_{m-1})^n\big\} < 1$, $nf_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\}/\big\{1-(F_{m-1})^n\big\} < 1$, and $U^{-1}(\cdot)$ is convex. Especially, (21) holds with equality if and only if a seller is risk-neutral. And (22) holds with equality if and only if bidders are risk-neutral. Hence, $R_m^B \geq R^{NB}$ holds with equality if and only if both the seller and the bidders are risk-neutral. This completes the proof.

Proof of Proposition 9

At first, we recall that

$$
\overline{B_m^*} = v^m - U^{-1}\left(\frac{(n-1)f_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} U(v^m - v^\mu)}{1 - (F_{m-1})^{n-1} - (n-1)(f_m)^2 (F_{m-1})^{n-2}} \right).
$$

For notational simplicity, we rewrite B_m^* and $\overline{B_m^*}$ as:

$$
\begin{aligned}
B_m^* &= v^m - U^{-1}\left(\frac{C}{C'}\right) \text{ and} \\
\overline{B_m^*} &= v^m - U^{-1}\left(\frac{C}{C''}\right),
\end{aligned}
$$

where $C = \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} U(v^m - v^\mu)$, $C' = \{1-(F_{m-1})^n\}/\{nf_m\}$, and $C'' = \{1 - (F_{m-1})^{n-1} - (n-1)(f_m)^2 (F_{m-1})^{n-2}\}/\{(n-1)f_m\}$.

If (8) > (12), then the seller obtains a higher expected utility from the auction with the buy price B_m^*. Thus, the seller cannot obtain further improvements from the auction with a buy price $B \in (B_m^*, \overline{B_m^*})$.

$$\begin{aligned}
R_m^B - R^{\overline{B}} = &\sum_{\mu=1}^{m-1} \Big[nF_{m-1}\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} - (n-1)\big\{(F_\mu)^n - (F_{\mu-1})^n\big\}\Big] W(v^\mu) \\
&+ \big\{1 - (F_{m-1})^n\big\} W(B_m^*) \\
&- \sum_{\mu=1}^{m-1} \Big[nF_{m-1}\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} - (n-1)\big\{(F_\mu)^n - (F_{\mu-1})^n\big\}\Big] W(v^\mu) \\
&- f_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} W(v^\mu) - \big\{1 - (F_{m-1})^{n-1}\big\} W(\overline{B_m^*}) \\
= &\big\{1 - (F_{m-1})^n\big\} W(B_m^*) - f_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} W(v^\mu) \\
&- \big\{1 - (F_{m-1})^{n-1}\big\} W(\overline{B_m^*}) \\
= &W(B_m^*) - \Big[(F_{m-1})^n W(B_m^*) + f_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} W(v^\mu) \\
&+ \big\{1 - (F_{m-1})^{n-1}\big\} W(\overline{B_m^*}) \Big] \\
\geq &W(B_m^*) - W\Big((F_{m-1})^n B_m^* + f_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} v^\mu \\
&+ \big\{1 - (F_{m-1})^{n-1}\big\} \overline{B_m^*} \Big) \\
> &\, 0,
\end{aligned}$$

where the first inequality follows because $W(\cdot)$ is concave and the second inequality follows because $W(\cdot)$ is monotone. Indeed,

$$
\begin{aligned}
& B_m^* - \left[(F_{m-1})^n B_m^* + f_m \sum_{\mu=1}^{m-1} \big\{ (F_\mu)^{n-1} - (F_{\mu-1})^{n-1} \big\} v^\mu + \big\{ 1 - (F_{m-1})^{n-1} \big\} \overline{B_m^*} \right] \\
&= -\big\{ 1 - (F_{m-1})^n \big\} U^{-1}\left(\frac{C}{C'} \right) + f_m \sum_{\mu=1}^{m-1} \big\{ (F_\mu)^{n-1} - (F_{\mu-1})^{n-1} \big\} (v^m - v^\mu) \\
&\quad + \big\{ 1 - (F_{m-1})^{n-1} \big\} U^{-1}\left(\frac{C}{C''} \right) \\
&\geq -\big\{ 1 - (F_{m-1})^n \big\} U^{-1}\left(\frac{C}{C'} \right) + f_m (F_{m-1})^{n-1} U^{-1}\left(\frac{C}{(F_{m-1})^{n-1}} \right) \\
&\quad + \big\{ 1 - (F_{m-1})^{n-1} \big\} U^{-1}\left(\frac{C}{C''} \right) \\
&= \big\{ 1 - (F_{m-1})^n \big\} \left[-U^{-1}\left(\frac{C}{C'} \right) + \frac{f_m (F_{m-1})^{n-1}}{\big\{ 1 - (F_{m-1})^n \big\}} U^{-1}\left(\frac{C}{(F_{m-1})^{n-1}} \right) \right. \\
&\quad \left. + \frac{\big\{ 1 - (F_{m-1})^{n-1} \big\}}{\big\{ 1 - (F_{m-1})^n \big\}} U^{-1}\left(\frac{C}{C''} \right) \right] \\
&\geq \big\{ 1 - (F_{m-1})^n \big\} \left[-U^{-1}\left(\frac{C}{C'} \right) \right. \\
&\quad \left. + U^{-1}\left(\frac{f_m C}{\big\{ 1 - (F_{m-1})^n \big\}} + \frac{\big\{ 1 - (F_{m-1})^{n-1} \big\} C}{\big\{ 1 - (F_{m-1})^n \big\} C''} \right) \right] \\
&> 0,
\end{aligned}
$$

where the first two inequalities follow because $U^{-1}(\cdot)$ is convex and the last inequality follows because $U^{-1}(\cdot)$ is monotone. Indeed,

$$
\begin{aligned}
& \frac{f_m C}{\{1 - (F_{m-1})^n\}} + \frac{\{1 - (F_{m-1})^{n-1}\} C}{\{1 - (F_{m-1})^n\} C''} - \frac{C}{C'} \\
&= \frac{\big\{ 1 - (F_{m-1})^{n-1} - (n-1) f_m C'' \big\} C}{\big\{ 1 - (F_{m-1})^n \big\} C''} \\
&= \frac{(n-1)(f_m)^2 (F_{m-1})^{n-2} C}{\big\{ 1 - (F_{m-1})^n \big\} C''} \\
&> 0.
\end{aligned}
$$

This completes the proof.

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