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Discussion Paper No. 656


"A Double-Track Auction for Substitutes and Complements "


Ning Sun and Zaifu Yang



July 2008

KYOTO UNIVERSITY

KYOTO, JAPAN

# A Double-Track Auction for Substitutes and Complements[1]

Ning Sun[2] and Zaifu Yang[3]

This revision: June 16, 2008

**Abstract:** We propose a new tâtonnement process called a double-track auction for efficiently allocating multiple heterogeneous indivisible items in two distinct sets $S_1$ and $S_2$ to many buyers who view items in the same set as substitutes but items across the two sets as complements. The auctioneer initially announces sufficiently low prices for items in one set, say $S_1$, but sufficiently high prices for items in the other set $S_2$. In each round, the buyers respond by reporting their demands at the current prices and the auctioneer adjusts prices upwards for items in $S_1$ but downwards for items in $S_2$ based on buyers' reported demands until the market is clear. Unlike any existing auction, this auction is a blend of a multi-item ascending auction and a multi-item descending auction. We prove that the auction finds an efficient allocation and its market-clearing prices in finitely many rounds. Based on the auction we also establish a dynamic, efficient and strategy-proof mechanism.

**Keywords:** Market design, dynamic auction, tâtonnement process, gross substitutes and complements, Walrasian equilibrium, incentives.

[1]We are deeply grateful to Atsushi Kajii and Larry Samuelson for many insightful comments and constructive suggestions and to three anonymous referees for very useful comments. We also wish to thank Tommy Andersson, Vince Crawford, Jean-Jacques Herings, Mamoru Kaneko, Gerard van der Laan, Bo Larsson, Rudolf Muller, Michael Ostrovsky, Lars-Gunnar Svensson, Dolf Talman, Stef Tijs, Walter Trockel, and participants in the seminars at Bielefeld, Chinese Academy of Sciences, Hitotsubashi, Kyoto, Lund, Maastricht, Shanghai, Shanghai Caida, Shanghai Jiaotong, Tilburg, Tsukuba, Xidian, Yokohama Kokudai, as well as at the 2007 Japanese Economic Association Annual Spring Meeting and the 2007 North American Summer Meeting of the Econometric Society for helpful comments. We are, however, solely responsible for any remaining errors. Ning Sun would like to thank Institute for Economic Research, Kyoto University for its financial support and hospitality. Zaifu Yang gratefully acknowledges financial support by the Ministry of Education, Science and Technology of Japan, and by the Netherlands Organization for Scientific Research (NWO), and by CentER for Economic Research, Tilburg University.

[2]N. Sun, School of Economics, Shanghai University of Finance and Economics, Shanghai, China; `nsun@mail.shufe.edu.cn`

[3]Z. Yang, Faculty of Business Administration, Yokohama National University, Yokohama 240-8501, Japan; `yang@ynu.ac.jp`

# 1 Introduction

Tâtonnement processes or auctions are fundamental instruments for discovering market-clearing prices and efficient allocations. The study of such processes provides one way of addressing the question of price formation and has long been a major issue of economic research. In 1874 Leon Walras formulated a first tâtonnement process-a type of auction. Samuelson (1941), Arrow and Hurwicz (1958), were among the first to study the convergence of certain tâtonnement processes. They proved that such processes converge globally to an equilibrium for any economy with divisible goods when the goods are substitutable. This study then generated great hope that such processes might also work for a larger class of economies with divisible goods. But Scarf (1960) soon dashed such hopes by showing that when goods exhibit complementarity, such processes can oscillate and will never tend towards equilibrium. Later it was Scarf (1973) who developed a remarkable process that can find an equilibrium in any reasonable economy with *divisible goods*.

The current paper explores adjustment processes for markets with *indivisible goods*.[4] To motivate it, let us review the related literature. In a seminal paper, Kelso and Crawford (1982) developed an auction-like process that allows each firm to hire several workers.[5] They showed that their process efficiently allocates workers with competitive salaries to firms, provided that every firm views all the workers as substitutes. This condition is called *gross substitutes* (GS) and has been widely used, adapted and extended in auction, matching, and equilibrium models.[6] Gul and Stacchetti (2000) devised an elegant ascending auction that finds a Walrasian equilibrium in finitely many steps when all goods are substitutes. While their analysis is mathematically sophisticated and quite demanding, Ausubel (2006) significantly simplified the analysis by developing a simpler and more elegant dynamic auction. Based on his auction, he also proposed a novel dynamic strategy-proof procedure yielding a Vickrey-Clarke-Groves outcome. As in Kelso and Crawford (1982), Milgrom (2000) proposed a less information demanding auction for finding an approximate equilibrium but converging to an equilibrium in the limit. However, all these processes were designed and work only for substitutes. It is widely recognized[7] that complementarities

[4]For a related but different problem, Scarf (1986) introduced a theory of testing optimality of production plans in the presence of indivisibility.

[5]Special but well-studied models typically assume that every consumer demands at most one item or every person needs only one opposite sex partner. See Gale and Shapley (1962), Shapley and Scarf (1974), Crawford and Knoer (1981), Demange, Gale and Sotomayor (1986) among others.

[6]See Roth and Sotomayor (1990), Bikhchandani and Mamer (1997), Laan, Talman and Yang (1997), Ma (1998), Bevia, Quinzii and Silva (1999), Gul and Stacchetti (1999), Fujishige and Yang (2003), Milgrom (2004), Crawford (2005), Hatfield and Milgrom (2005), and Ostrovsky (2007) among others.

[7]The current state of the art is well documented in Milgrom (2000), Jehiel and Moldovanu (2003), Klemperer (2004) and Maskin (2005). We quote from Milgrom (2000, p. 258): "The problem of bidding for complements has inspired continuing research both to clarify the scope of the problem and to devise

pose a challenge for designing dynamic mechanisms for discovering market-clearing prices and efficient allocations.

This paper aims to show that certain typical patterns of complementarity together with substitutability can be handled by a new dynamic auction design. More specifically, we study a market model where a seller wishes to sell two distinct sets $S_1$ and $S_2$ of several heterogeneous items to a number of buyers. The buyers view items in the same set as substitutes but items across the two sets as complements. This condition is called *gross substitutes and complements* (GSC), generalizing the GS condition. Many typical situations fit this general description, stretching from the sale of computers and software packages to consumers, to the allocation of workers and machines to firms, take-off and landing slots to airliners, etc.[8] In our earlier analysis (Sun and Yang (2006B)), we showed that if all agents in an exchange economy have GSC preferences, the economy has a Walrasian equilibrium. But the method is non-constructive, and so in particular, the important issue of how to find the equilibrium prices and allocation is not dealt with. The existing auctions, however, are hindered by the exposure problem and cannot handle this situation. In contrast, in this paper we propose a new tâtonnement process –a double-track auction that can discover a Walrasian equilibrium. The auction proceeds as follows. The auctioneer initially calls out sufficiently low prices for items in one set, say $S_1$, but sufficiently high prices for items in the other set $S_2$ so that all items in $S_1$ are over-demanded but those in $S_2$ are under-demanded. In each round, buyers are asked to report their demands at the current prices. Based on buyers' reported demands, the auctioneer adjusts prices upwards for those over-demanded items in $S_1$ but downwards for those under-demanded items in $S_2$ until the market is clear. In finitely many rounds the auction pinpoints an efficient allocation and its market-clearing prices. Unlike traditional tâtonnement processes that typically adjust prices continuously, the auction process adjusts prices only in integer or fixed quantities.

The proposed auction circumvents the exposure problem confronting the existing auctions and differs markedly from them in that it adjusts simultaneously prices of items in $S_1$ and $S_2$ respectively in opposite directions,[9] whereas the existing auctions typically adjust all prices simultaneously only in one direction (either ascending or descending). When

practical auction designs that overcome the exposure problem." The so-called *exposure problem* refers to a phenomenon concerning an ascending auction that at the earlier stages of the auction, all items were over-demanded, but as the prices are going up, some or all items may be exposed to the possibility that no bidder wants to demand them anymore, because complementary items have become too expensive. As a result, the ascending auction will get stuck in disequilibrium.

[8]Ostrovsky (2007) independently proposed a similar condition for a supply chain model where prices of goods are fixed and a non-Walrasian equilibrium (weak core) solution is used. See also Shapley (1962), Samuelson (1974), Rassenti, Smith and Bulfin (1982), Krishna (2002) and Milgrom (2007).

[9]This double-track idea can be used elsewhere such as to extend Kelso-Crawford's job-matching model by permitting complementarities among employees; see Sun and Yang (2006A).

all items are substitutes (i.e., either $S_1 = \emptyset$ or $S_2 = \emptyset$), the proposed auction coincides with Ausubel's (2006) auction and is similar to Gul and Stacchetti (2000). In general the proposed auction deals with the circumstances including complements that go beyond the existing models with substitutes. Another attractive feature of the proposed auction is that it only requires the buyers to report their demands at several price vectors along a finite path rather than their entire values over all possible bundles so that their privacy can be protected. This is important, because businessmen generally do not like to reveal their values or costs. Based upon the proposed auction, we also establish a dynamic, efficient and strategy-proof mechanism for the environments with complements. So there is no benefit to any buyer from acting strategically rather than bidding truthfully in this mechanism. To design the new auction, it is crucial to introduce a new characterization of the GSC condition called *generalized single improvement* (GSI), generalizing the *single improvement* (SI) property of Gul and Stacchetti (1999). GSI plays the same important role in our auction design as SI does in Ausubel (2006), Gul and Stacchetti (2000).

This paper proceeds as follows. Section 2 introduces the market model. Section 3 presents the double-track auction and discusses its basis, its properties and its convergence. Section 4 introduces a dynamic, efficient and incentive compatible procedure based on the double-track auction. Section 5 concludes.

# 2 The Market Model

An auctioneer (or seller) wishes to sell a set $N = \{\beta_1, \beta_2, \cdots, \beta_n\}$ of $n$ indivisible items to a finite group $I$ of buyers (or bidders). The items may be heterogeneous and can be divided into two sets $S_1$ and $S_2$ (i.e., $N = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$). For instance, one can think of $S_1$ as computers and of $S_2$ as software packages. Items in the same set can also be heterogeneous. Every buyer $i$ has a utility function $u^i : 2^N \rightarrow \mathbb{R}$ specifying his valuation $u^i(B)$ (in units of money) on each bundle $B$ with $u^i(\emptyset) = 0$, where $2^N$ denotes the family of all bundles of items. It is standard to assume that $u^i$ is weakly increasing, every buyer can pay up to his value and has quasi-linear utilities in money, and the seller values every bundle at zero. Note, however, that weak monotonicity can be dropped; see Sun and Yang (2006A-B).

A price vector $p = (p_1, \cdots, p_n) \in \mathbb{R}^n$ specifies a price $p_h$ for each item $\beta_h \in N$. Buyer $i$'s demand correspondence $D^i(p)$, the net utility function $v^i(A, p)$, and the indirect utility function $V^i(p)$, are defined respectively by

$$\begin{array}{rcl} D^i(p) & = & \arg\max_{A \subseteq N}\{u^i(A) - \sum_{\beta_h \in A} p_h\}, \\ v^i(A, p) & = & u^i(A) - \sum_{\beta_h \in A} p_h, \text{ and} \\ V^i(p) & = & \max_{A \subseteq N}\{u^i(A) - \sum_{\beta_h \in A} p_h\}. \end{array} \quad (2.1)$$

It is known that for any utility function $u^i : 2^N \to \mathbb{R}$, the indirect utility function $V^i$ is a decreasing, continuous and convex function.

An allocation of items in $N$ is a *partition* $\pi = (\pi(i), i \in I)$ of items among all buyers in $I$, i.e., $\pi(i) \cap \pi(j) = \emptyset$ for all $i \neq j$ and $\cup_{i \in I} \pi(i) = N$. Note that $\pi(i) = \emptyset$ is allowed. At allocation $\pi$, buyer $i$ receives bundle $\pi(i)$. An allocation $\pi$ is *efficient* if $\sum_{i \in I} u^i(\pi(i)) \geq \sum_{i \in I} u^i(\rho(i))$ for every allocation $\rho$. Given an efficient allocation $\pi$, let $R(N) = \sum_{i \in I} u^i(\pi(i))$. We call $R(N)$ *the market value* of the items which is the same for all efficient allocations.

**Definition 2.1** *A Walrasian equilibrium $(p, \pi)$ consists of a price vector $p \in \mathbb{R}^n_+$ and an allocation $\pi$ such that $\pi(i) \in D^i(p)$ for every $i \in I$.*

It is well-known that every equilibrium allocation is efficient, but an equilibrium may not always exist. To ensure the existence of an equilibrium, we need to impose some conditions on the model. The most important one is called *gross substitutes and complements condition*, which is defined below.[10]

**Definition 2.2** *The utility function $u^i$ of buyer $i$ satisfies the* gross substitutes and complements (GSC) *condition if for any price vector $p \in \mathbb{R}^n$, any item $\beta_k \in S_j$ for $j = 1$ or $2$, any $\delta \geq 0$, and any $A \in D^i(p)$, there exists $B \in D^i(p + \delta e(k))$ such that $[A \cap S_j] \setminus \{\beta_k\} \subseteq B$ and $[A^c \cap S_j^c] \subseteq B^c$.*

GSC says that buyer $i$ views items in each set $S_j$ as substitutes, but items across the two sets $S_1$ and $S_2$ as complements, in the sense that if the buyer wants to demand a bundle $A$ at prices $p$ and if now the price of some item $\beta_k \in S_j$ is increased, then he would still want to demand the items both in $A$ and in $S_j$ whose prices did not rise, but he would not want to demand any item in another set $S_j^c$ which was not in his choice set $A$ at prices $p$. In particular, when either $S_1 = \emptyset$ or $S_2 = \emptyset$, GSC reduces to the gross substitutes (GS) condition of Kelso and Crawford (1982). GS excludes complements and requires that all the items be substitutes. This case has been studied extensively in the literature; see e.g., Kelso and Crawford (1982), Gul and Stacchetti (1999, 2000), Milgrom (2000), and Ausubel (2006). Now we state the assumptions for the current model:

(A1) *Integer private values*: Every buyer $i$'s utility function $u^i : 2^N \to \mathbb{Z}_+$ takes integer values and is his private information.

[10] The following piece of notation is used throughout the paper. For any positive integer $k \leq n$, $e(k)$ denotes the $k$th unit vector in $\mathbb{R}^n$. Let $\mathbb{Z}^n$ stand for the integer lattice in $\mathbb{R}^n$ and $0$ the $n$-vector of 0's. For any subset $A$ of $N$, let $e(A) = \sum_{\beta_k \in A} e(k)$. When $A = \{\beta_k\}$, we also write $e(A)$ as $e(k)$. For any subset $A$ of $N$, let $A^c$ denote its complement, i.e., $A^c = N \setminus A$. For any vector $p \in \mathbb{R}^n$ and any set $A \in 2^N$, let $p(A) = \sum_{\beta_k \in A} p_k e(k)$. So we have $p(N) = p$ for any $p \in \mathbb{R}^n$. For any finite set $A$, $\sharp(A)$ denotes the number of elements in $A$. For any set $D \subseteq \mathbb{R}^n$, $\mathrm{co}(D)$ denotes its convex hull.

(A2) *Gross substitutes and complements*: Every buyer $i$'s utility function $u^i$ satisfies the GSC condition with respect to the two sets $S_1$ and $S_2$.

(A3) *The auctioneer's knowledge*: The auctioneer knows some integer value $U^*$ greater than any buyer's possible maximum value.

The essence and difficulty of designing a mechanism for locating a Walrasian equilibrium in this market lie in the facts that every buyer's valuation of any bundle of goods is private information and is therefore unobservable to the auctioneer (A1); and that there are multiple indivisible substitutes and complements for sale (A2). We point out that (A3) is merely a technical assumption used only in Theorem 3.9.

# 3 The Double-Track Adjustment Process

## 3.1 The Basis

This subsection provides the basis on which the double-track procedure will be established for finding a Walrasian equilibrium in the market described in the previous section. We begin with a new characterization of the GSC condition.

**Definition 3.1** *The utility function $u^i$ of buyer $i$ has the* generalized single improvement (GSI) *property if for any price vector $p \in \mathbb{R}^n$ and any bundle $A \notin D^i(p)$, there exists a bundle $B \in 2^N$ such that $v^i(A,p) < v^i(B,p)$ and $B$ satisfies exactly one of the following conditions*:
(i): *$A \cap S_j = B \cap S_j$, and $\sharp[(A \setminus B) \cap S_j^c] \leq 1$ and $\sharp[(B \setminus A) \cap S_j^c] \leq 1$ for either $j = 1$, or $j = 2$;*
(ii): *either $B \subseteq A$ and $\sharp[(A \setminus B) \cap S_1] = \sharp[(A \setminus B) \cap S_2] = 1$, or $A \subseteq B$ and $\sharp[(B \setminus A) \cap S_1] = \sharp[(B \setminus A) \cap S_2] = 1$.*

GSI says that for buyer $i$, every suboptimal bundle $A$ at prices $p$ can be strictly improved by either adding an item to it, or removing an item from it, or doing both in either set $A \cap S_j$. The bundle $A$ can be also strictly improved by adding simultaneously one item from each set $S_j$ to it, or removing simultaneously one item from each set $A \cap S_j$, $j = 1, 2$. We call bundle $B$ *a GSI improvement of* $A$. When either $S_1$ or $S_2$ is empty, GSI coincides with the single improvement (SI) property of Gul and Stacchetti (1999) which in turn is equivalent to the GS condition. The GSI property plays a crucial role both in proving several of our main results and in our auction design. We now state the following theorem whose proof together with those of Theorems 3.3, 3.5 and 3.10 and Lemmas 3.4, 3.6 and 3.8 is deferred to the Appendix.

**Theorem 3.2** *Conditions GSC and GSI are equivalent.*

Let $p, q \in \mathbb{R}^n$ be any vectors. With respect to the order $(S_1, S_2)$, we define their generalized *meet* $s = (s_1, \cdots, s_n) = p \wedge_g q$ *and join* $t = (t_1, \cdots, t_n) = p \vee_g q$ by

$$\begin{array}{llll} s_k = \min\{p_k, q_k\}, & \beta_k \in S_1, & s_k = \max\{p_k, q_k\}, & \beta_k \in S_2; \\ t_k = \max\{p_k, q_k\}, & \beta_k \in S_1, & t_k = \min\{p_k, q_k\}, & \beta_k \in S_2. \end{array}$$

Note that the two operations are different from the standard meet and join operations. A subset $W$ of $\mathbb{R}^n$ is called *a generalized lattice* if $p \wedge_g q$, $p \vee_g q \in W$ for any $p, q \in W$. A generalized lattice is a rotated standard lattice. Given a generalized lattice $W$, we say a function $f : W \to \mathbb{R}$ is *a generalized submodular function* if $f(p \wedge_g q) + f(p \vee_g q) \leq f(p) + f(q)$ for all $p, q \in W$. A useful characterization of the generalized submodular function is given in Lemma 3 in the Appendix. Ausubel and Milgrom (2002, Theorem 10) showed that items are substitutes for a buyer if and only if his indirect utility function is submodular. Our next theorem generalizes their result from GS to GSC preferences and will be used to establish Theorem 3.5 below.

**Theorem 3.3** *A utility function $u^i$ satisfies the GSC condition if and only if the indirect utility function $V^i$ is a generalized submodular function.*

For the market model, define the Lyapunov function $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{L}(p) = \sum_{\beta_h \in N} p_h + \sum_{i \in I} V^i(p) \tag{3.2}$$

where $V^i$ is the indirect utility function of buyer $i \in I$. This type of function is well-known in the literature for economies with divisible goods (see e.g., Arrow and Hahn (1971) and Varian (1981)) but was only recently explored ingeniously by Ausubel (2005, 2006) in the context of indivisible goods. His Proposition 1 in both papers shows that if an equilibrium exists, then the set of equilibrium price vectors coincides with the set of minimizers of the Lyapunov function. The following Lemma 3.4 strengthens this result by providing a necessary and sufficient condition for the existence of an equilibrium.

**Lemma 3.4** *For the market model, $p^* \in \mathbb{R}^n$ is a Walrasian equilibrium price vector if and only if it is a minimizer of the Lyapunov function $\mathcal{L}$ defined by* (3.2) *with its value $\mathcal{L}(p^*)$ equal to the market value $R(N)$.*

Given a subset $W$ of $\mathbb{R}^n$, we define a new order on $W \times W$ with respect to the order $(S_1, S_2)$ as follows: for any $p, q \in W$, $p \leq_g q$ if and only if $p(S_1) \leq q(S_1)$ and $p(S_2) \geq q(S_2)$. A point $p^* \in W$ is called *a smallest element* if $p^* \leq_g q$ for every $q \in W$. Similarly, a point $q^* \in W$ is called *a largest element* if $q^* \geq_g p$ for every $p \in W$. It is easy to verify that a compact generalized lattice has a unique smallest (largest) element in it. A set

$D \subseteq \mathbb{R}^n$ is *integrally convex* if $D = \text{co}(D)$ and $x \in D$ implies $x \in \text{co}(D \cap N(x))$, where $N(x) = \{z \in \mathbb{Z}^n \mid ||z - x||_\infty < 1\}$ and $|| \cdot ||_\infty$ means the maximum norm, i.e., every point $x \in D$ can be represented as a convex combination of integral points in $N(x) \cap D$. Favati and Tardella (1990) originally introduced this concept for discrete subsets of $\mathbb{Z}^n$. The following theorem will be used to prove the convergence of the double-track procedure.

**Theorem 3.5** *Assume that the market model satisfies Assumptions* (A1) *and* (A2)*. Then* (i) *the Lyapunov function* $\mathcal{L}$ *defined by* (3.2) *is a continuous, convex and generalized submodular function*;
(ii) *the set of Walrasian equilibrium price vectors in the model forms a nonempty, compact, integrally convex and generalized lattice, implying that all its vertices including both its smallest and largest equilibrium price vectors, denoted by* $\underline{p}$ *and* $\bar{p}$ *respectively, are integer vectors.*

The theorem asserts that (i) the Lyapunov function is a well-behaved function meaning that a local mimimum is also a global mimimum; and (ii) the set of Walrasian equilibrium price vectors possesses an elegant geometry: (1) the set is an integral polyhedron, i.e., all vertices including $\underline{p}$ and $\bar{p}$ are integer vectors; and (2) the intersection of the set with any unit hypercube $\{x\} + [0, 1]^n$ for $x \in \mathbb{Z}^n$ is integrally convex and thereby all of its vertices are integer vectors.

## 3.2 An Illustration

The existing auctions typically adjust all prices simultaneously in one direction, are either ascending or descending, and generally do not work in the environments with complements. It is helpful to use a simple example to illustrate how an ascending (or descending) auction might be plagued by the exposure problem and how the new auction proposed in this paper overcomes the problem and succeeds in finding a Walrasian equilibrium. Consider now a market where a seller wishes to sell two volumes $A$ and $B$ of a book to two buyers. *Each buyer knows his values privately and the seller knows only that all values are below* 6. Buyers' values are given in the Table 1, and the seller values every bundle at zero. Observe that every buyer views $A$ and $B$ as complements.

Table 1: Buyers' values over items.

| | $\emptyset$ | $A$ | $B$ | $AB$ |
|---|---|---|---|---|
| Buyer 1 | 0 | 2 | 2 | 5 |
| Buyer 2 | 0 | 2 | 2 | 5 |

**The ascending auction:** In an ascending auction, the seller initially announces a low price vector of $p(0) = (p_A(0), p_B(0)) = (0, 0)$ so that every buyer demands both $A$ and $B$.

Buyers respond by reporting their demand sets at $p(0)$: $D^1(p(0)) = D^2(p(0)) = \{AB\}$. According to the reported demand sets, the seller subsequently adjusts the price vector $p(0)$ to the next one $p(1) = p(0) + \delta(0) = (1, 1)$ by increasing the price of every good by 1, because both goods are over-demanded at $p(0)$. The seller faces a similar situation at $p(1)$ and $p(2)$. The auction ends up with the price vector $p(3) = (3, 3)$ at which no bidder wants to demand the items anymore, and thus gets stuck in disequilibrium. We summarize the entire process in the Table 2. The reader can also verify that starting with a high price vector $p(0) = (p_A(0), p_B(0)) = (q, q)$ for any integer $q \geq 6$ so that no buyer demands any item, a descending auction will terminate with the price vector $\bar{p} = (2, 2)$ at which both buyers demand both items, and thus get stuck in disequilibrium, too. We remind the reader that prices in auction processes are adjusted in integer or fixed quantities.

Table 2: The data created by the ascending auction for the example.

| Price vector | Buyer 1 | Buyer 2 | Price variation |
|---|---|---|---|
| $p(0) = (0, 0)$ | $\{AB\}$ | $\{AB\}$ | $\delta(0) = (1, 1)$ |
| $p(1) = (1, 1)$ | $\{AB\}$ | $\{AB\}$ | $\delta(1) = (1, 1)$ |
| $p(2) = (2, 2)$ | $\{AB\}$ | $\{AB\}$ | $\delta(2) = (1, 1)$ |
| $p(3) = (3, 3)$ | $\{\emptyset\}$ | $\{\emptyset\}$ | $\delta(3) = (0, 0)$ |

**The double-track auction:** Unlike the previous two cases, in the current double-track auction, the seller initially announces a price vector of $p(0) = (p_A(0), p_B(0)) = (0, 6)$ (a low price for item $A$ but a high price for item $B$) so that every buyer demands only item $A$ and not item $B$. Buyers respond by reporting their demand sets at $p(0)$: $D^1(p(0)) = D^2(p(0)) = \{A\}$. Using the reported demands, the seller subsequently adjusts the price vector $p(0)$ to the next one $p(1) = p(0) + \delta(0) = (1, 5)$ by increasing the price of $A$ by 1 but decreasing the price of $B$ by 1, because $A$ is over-demanded but $B$ is under-demanded at $p(0)$. At $p(1)$, the seller faces a similar situation. An interesting moment occurs when $p(1)$ advances to $p(2) = (2, 4)$ at which $B$ is clearly still under-demanded, but $A$ can be seen as either over-demanded or balanced. According to the rule of the double-track auction (to be discussed soon in detail), the seller treats $A$ as balanced and so she adjusts $p(2)$ to $p(3) = (2, 3)$ by decreasing the price of $B$ by 1 and holding the price of $A$ constant. At $p(3)$, the market reaches an equilibrium in which the seller can assign items $A$ and $B$ to buyer 1 and asks him to pay 5, while buyer 2 gets nothing and pays nothing. We can summarize the entire process in the Table 3. Observe that in this process, the seller increases the price of item $A$ (since it is over-demanded) but decreases the price of item $B$ (since it is under-demanded) until the market is clear. So to a large extent, this double-track auction is also similar to the classical Walrasian tâtonnement process.

Table 3: The data created by the double-track auction for the example.

| Price vector | Buyer 1 | Buyer 2 | Price variation |
|---|---|---|---|
| $p(0) = (0, 6)$ | $\{A\}$ | $\{A\}$ | $\delta(0) = (1, -1)$ |
| $p(1) = (1, 5)$ | $\{A\}$ | $\{A\}$ | $\delta(1) = (1, -1)$ |
| $p(2) = (2, 4)$ | $\{\emptyset, A\}$ | $\{\emptyset, A\}$ | $\delta(2) = (0, -1)$ |
| $p(3) = (2, 3)$ | $\{\emptyset, A, AB\}$ | $\{\emptyset, A, AB\}$ | $\delta(3) = (0, 0)$ |

## 3.3 The Formal Procedure

We are now ready to give a formal description of the double-track adjustment process.[11] This process can be seen as an extension of Ausubel (2006) from GS to GSC preferences environments and thus from the standard order $\leq$ to the new order $\leq_g$.[12] More specifically, when either $S_1 = \emptyset$ or $S_2 = \emptyset$ (i.e., all items are substitutes to the buyers), this new process coincides exactly with Ausubel's. The new order $\leq_g$ differs from the standard order $\leq$ used by the existing auctions in that the new process adjusts prices of items in one set upwards but at the same time adjusts prices of items in the other set downwards. Therefore, we define an $n$-dimensional cube for price adjustment by

$$\square = \{\delta \in \mathbb{R}^n \mid 0 \leq \delta_k \leq 1, \forall \beta_k \in S_1, \ -1 \leq \delta_l \leq 0, \forall \beta_l \in S_2\}.$$

For any buyer $i \in I$, any price vector $p \in \mathbb{Z}^n$ and any price variation $\delta \in \square$, choose

$$\tilde{S}^i \in \arg\min_{S \in D^i(p)} \{\sum_{\beta_h \in S} \delta_h\}. \tag{3.3}$$

The next lemma asserts that for any buyer $i$, any $p \in \mathbb{Z}^n$ and any $\delta \in \square$, his optimal bundle $\tilde{S}^i$ in (3.3) chosen from $D^i(p)$ remains constant for all price vectors on the line segment from $p$ to $p + \delta$. This property is crucial for the auctioneer to adjust the current price vector to the next one and is a consequence of the GSI property.

**Lemma 3.6** *If Assumptions* (A1) *and* (A2) *hold for the market model, then for any $i \in I$, any $p \in \mathbb{Z}^n$ and any $\delta \in \square$, the solution $\tilde{S}^i$ of Formula* (3.3) *satisfies $\tilde{S}^i \in D^i(p + \lambda\delta)$ and the Lyapunov function $\mathcal{L}(p + \lambda\delta)$ is linear in $\lambda$, for any parameter $\lambda \geq 0$ such that $0 \leq \lambda\delta_k \leq 1$ for every $\beta_k \in S_1$ and $-1 \leq \lambda\delta_l \leq 0$ for every $\beta_l \in S_2$.*

Given a current price vector $p(t) \in \mathbb{Z}^n$, the auctioneer first asks every buyer $i$ to report his demand $D^i(p(t))$. Then she uses every buyer's reported demand $D^i(p(t))$ to determine the next price vector $p(t + 1)$. The underlying rationale for the auctioneer is to choose a

[11]We refer to Yang (1999) for various adjustment processes for finding Walrasian equilibria, Nash equilibria and their refinements in the continuous models.

[12]This process can be also viewed as a direct generalization of Gul and Stacchetti (2000) from GS to GSC environments. We adopt here the Lyapunov function approach instead of matroid theory used by Gul and Stacchetti, because the former is more familiar in economics and much simpler than the latter.

direction $\delta \in \square$ so as to reduce the value of the Lyapunov function $\mathcal{L}$ as large as possible. To achieve this, she needs to solve the following problem

$$\max_{\delta\in\square}\{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\} \tag{3.4}$$

Note that the above formula involves every buyer's valuation of every bundle of goods, so it uses private information. Apparently, it is impossible for the auctioneer to know such information unless the buyers tell her. Fortunately, she can fully infer the difference between $\mathcal{L}(p(t))$ and $\mathcal{L}(p(t)+\delta)$ just from the reported demands $D^i(p(t))$ and the price variation $\delta$. To see this, we know from the definition of the Lyapunov function that for any given $p(t) \in \mathbb{Z}^n$ and $\delta \in \square$, the difference is given by

$$\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta) = \sum_{i\in I}(V^i(p(t)) - V^i(p(t)+\delta)) - \sum_{\beta_h\in N}\delta_h \tag{3.5}$$

Although, at prices $p(t)$, each buyer $i$ may have many optimal choices, his indirect utility $V^i(p(t))$ at $p(t)$ is unique since every optimal choice gives him the same indirect utility. Lemma 3.6 tells us that some $\tilde{S}^i$ of his optimal choices remains unchanged when prices vary from $p(t)$ to $p(t)+\delta$. It is immediately clear that his indirect utility $V^i(p(t)+\delta)$ at prices $p(t)+\delta$ equals $V^i(p(t)) - \sum_{\beta_h\in\tilde{S}^i}\delta_h$. Now we obtain the change in indirect utility for buyer $i$ when prices move from $p(t)$ to $p(t)+\delta$. This change is unique and is given by

$$V^i(p(t)) - V^i(p(t)+\delta) = \min_{S\in D^i(p(t))}\sum_{\beta_h\in S}\delta_h = \sum_{\beta_h\in\tilde{S}^i}\delta_h \tag{3.6}$$

where $\tilde{S}^i$ is a solution given by (3.3) for buyer $i$ with respect to price vector $p(t)$ and the variation $\delta$. Consequently, the equation (3.5) becomes the following simple formula whose right side involves only price variation $\delta$ and optimal choices at $p(t)$:

$$\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta) = \sum_{i\in I}\Big(\min_{S\in D^i(p(t))}\sum_{\beta_h\in S}\delta_h\Big) - \sum_{\beta_h\in N}\delta_h = \sum_{i\in I}\sum_{\beta_h\in\tilde{S}^i}\delta_h - \sum_{\beta_h\in N}\delta_h \tag{3.7}$$

The next result shows that the set of solutions to Problem (3.4) is a generalized lattice and both its smallest and largest elements are integral, resembling Theorem 3.5 and following also from the generalized submodularity of the Lyapunov function.

**Lemma 3.7** *If Assumptions* (A1) *and* (A2) *hold for the market model, then the set of solutions to Problem* (3.4) *is a nonempty, integrally convex and generalized lattice and both its smallest and largest elements are integer vectors.*

Given the current price vector $p(t)$, the next price vector $p(t+1)$ is given by $p(t+1) = p(t)+\delta(t)$, where $\delta(t)$ is the unique smallest element as described in the above lemma. Since $\delta(t)$ is an integer vector, this implies that the auctioneer does not need to search everywhere in the cube $\square$ for achieving a maximal decrease in the value of the Lyapunov function. It suffices to search only the vertices (i.e., the integer vectors) of the cube $\square$ and doing so will lead to the same maximal value decrease of the Lyapunov function. Let $\Delta = \square \cap \mathbb{Z}^n$. By (3.7), the decision Problem (3.4) of the seller boils down to computing the unique smallest solution $\delta(t)$ (in the order $\leq_g$) of the optimization problem:

$$\max_{\delta \in \Delta} \Big\{ \sum_{i \in I} \Big( \min_{S \in D^i(p(t))} \sum_{\beta_h \in S} \delta_h \Big) - \sum_{\beta_h \in N} \delta_h \Big\} \tag{3.8}$$

The max-min in the above formula has a meaningful and interesting interpretation: when the prices are adjusted from $p(t)$ to $p(t+1) = p(t) + \delta(t)$, all buyers try to minimize their losses in indirect utility whereas the seller strives for the highest gain. Nevertheless, the entire computation for (3.8) is carried out solely by the seller according to buyers' reported demands $D^i(p(t))$. The computation of (3.8) is fairly simple because the seller can easily calculate the value $(\min_{S \in D^i(p(t))} \sum_{\beta_h \in S} \delta_h)$ for each given $\delta \in \Delta$ and buyer $i$. Now we summarize the adjustment process as follows.

**The dynamic double-track (DDT) auction**

Step 1: The auctioneer announces an initial price vector $p(0) \in \mathbb{Z}^n_+$ with $p(0) \leq_g \underline{p}$. Let $t := 0$ and go to Step 2.

Step 2: After the announcement of $p(t)$, the auctioneer asks every buyer $i$ to report his demand $D^i(p(t))$. Then according to (3.8) and reported demands $D^i(p(t))$, the auctioneer computes the unique smallest element $\delta(t)$ (in the order $\leq_g$) and obtains the next price vector $p(t+1) := p(t) + \delta(t)$. If $p(t+1) = p(t)$, then the auction stops. Otherwise, let $t := t + 1$ and return to Step 2.

First, observe that this auction simultaneously adjusts prices upwards for items in $S_1$ and downwards for items in $S_2$. So on the side of $S_1$, the auction runs like an English auction, while on the other side of $S_2$, it does like a Dutch auction. But the auction does not run two sides independently. Second, the auction rules adhere to the Wilson doctrine (Wilson (1987)) in the sense that they are simple, transparent and detail-free to the bidders. Third, to ensure $p(0) \leq_g \underline{p}$, the auctioneer just needs to set the initial prices of items in $S_1$ so low and those of items in $S_2$ so high that all items in $S_1$ are over-demanded but all items in $S_2$ are under-demanded. This can be easily done because every buyer's utility function $u^i$ is weakly increasing with $u^i(\emptyset) = 0$ and is bounded above from $U^*$ given in Assumption (A3). For instance, the auctioneer can simply take $p(0) = (p_1(0), \cdots, p_n(0))$ by setting $p_k(0) = 0$ for any $\beta_k \in S_1$ and $p_k(0) = U^*$ for any $\beta_k \in S_2$. Note that the choice of initial prices of the items can have an effect on the speed of the auction's convergence.

Observe from the proof of the following Lemma 3.8 in the Appendix that Lemma 3.8 (i) and (ii) are independent of the choice of $p(0)$.

**Lemma 3.8** *Under Assumptions* (A1) *and* (A2), *the DDT auction has the following properties*:

(i) $p(t) \le_g \underline{p}$ *implies* $p(t+1) \le_g \underline{p}$.

(ii) $p(t+1) = p(t)$ *implies* $p(t) \ge_g \underline{p}$.

We are ready to establish the following convergence theorem for the DDT auction.

**Theorem 3.9** *For the market model under Assumptions* (A1)–(A3), *the DDT auction converges to the smallest equilibrium price vector* $\underline{p}$, *in a finite number of rounds.*

Proof: Recall that by Theorem 3.5 (ii), the market model has not only a nonempty set of equilibrium price vectors but also a unique smallest equilibrium price vector $\underline{p}$. Let $\{p(t), t = 0, 1, \cdots\}$ be the sequence of price vectors generated by the auction. Note that $p(t+1) = p(t) + \delta(t)$, $\delta(t) \in \square \cap \mathbb{Z}^n$ for $t = 0, 1, \cdots$, and that $p(t) \le_g p(t+1)$ for $t = 0, 1, \cdots$, and all $p(t)$ are integer vectors. Step 1 of the auction implies that $p(0) \le_g \underline{p}$ and Lemma 3.8 (i) implies that $p(t) \le_g \underline{p}$ for all $t$. Since $\delta(t)$ is an integer vector for any $t$ and the sequence $\{p(t), t = 0, 1, \cdots\}$ is bounded above from $\underline{p}$, the sequence must be finite. This means that $p(t^*) = p(t^* + 1)$ for some $t^*$, i.e., the sequence can be written as $\{p(t), t = 0, 1, \cdots, t^*\}$. Note that $p(t) \ne p(t+1)$ and $\delta(t) \ne 0$ for any $t = 0, 1, \cdots, t^* - 1$. By Lemma 3.8 (ii), $p(t^*) \ge_g \underline{p}$. Because of $p(t^*) \le_g \underline{p}$, it is clear $p(t^*) = \underline{p}$. This shows that the auction indeed terminates with the smallest equilibrium price vector $\underline{p}$, in a finite number of rounds. □

The DDT auction has the drawback that it converges to an equilibrium price vector only if $p(0) \le_g \underline{p}$. To overcome this shortcoming, we propose the following modified DDT auction which can start from any integer price vector and still converges to an equilibrium price vector. Analogous to the discrete set $\Delta$, define the discrete set $\Delta^* = -\Delta$. Through $\Delta^*$, we lower prices of items in $S_1$ but raise prices of items in $S_2$.

**The global dynamic double-track (GDDT) auction**

Step 1: Choose any initial price vector $p(0) \in \mathbb{Z}^n_+$. Let $t := 0$ and go to Step 2.

Step 2: The auctioneer asks every buyer $i$ to report his demand $D^i(p(t))$ at $p(t)$. Then based on reported demands $D^i(p(t))$, the auctioneer computes the unique smallest element $\delta(t)$ (in the order $\le_g$) according to (3.8). If $\delta(t) = 0$, go to Step 3. Otherwise, set the next price vector $p(t+1) := p(t) + \delta(t)$ and $t := t + 1$. Return to Step 2.

Step 3: The auctioneer asks every buyer $i$ to report his demand $D^i(p(t))$ at $p(t)$. Then based on reported demands $D^i(p(t))$, the auctioneer computes the unique largest element $\delta(t)$ (in the order $\le_g$) from the following problem of type (3.8):

$$\max_{\delta \in \Delta^*} \Big\{ \sum_{i \in I} \Big( \min_{S \in D^i(p(t))} \sum_{\beta_h \in S} \delta_h \Big) - \sum_{\beta_h \in N} \delta_h \Big\}, \tag{3.9}$$

If $\delta(t) = 0$, then the auction stops. Otherwise, set the next price vector $p(t+1) := p(t) + \delta(t)$ and $t := t + 1$. Return to Step 3.

First, observe that Step 2 of the GDDT auction is the same as Step 2 of the DDT auction and Step 3 of the GDDT auction is also the same as Step 2 of the DDT auction except that in Step 3 we switch the role of $S_1$ and $S_2$ by moving from $\Delta$ to $\Delta^*$. Second, the GDDT auction terminates in Step 3 and never goes from Step 3 to Step 2. Third, because the order $\leq_g$ is defined in the specified order of $(S_1, S_2)$, the auctioneer computes the unique largest element $\delta(t)$ in Step 3 (which is equivalent to the unique smallest element if we redefine the order $\leq_g$ in the order of $(S_2, S_1)$). Note that Theorem 3.10 dispenses with Assumption (A3).

**Theorem 3.10** *For the market model under Assumptions* (A1) *and* (A2)*, starting with any integer price vector, the GDDT auction converges to an equilibrium price vector in a finite number of rounds.*

# 4 The Dynamic Strategy-Proof Procedure

We now address the strategic issue such as When confronting an auction, is honesty the best policy for every bidder? More specifically, does sincere bidding constitute a Nash equilibrium (or its variants) of the auction game? If it is the case, the auction is said to be *strategy-proof*. The (sealed-bid) Vickrey-Clarke-Groves (VCG) auction is strategy-proof. The dynamic auction of Ausubel (2006) not only possesses this important strategy-proof property but also offers advantages of informational efficiency, transparency and privacy preservation. The auction of Demange, Gale and Sotomayor (1986) also has the same properties but applies to a less general model in which every buyer can demand only one item. The outcome yielded by a dynamic strategy-proof auction often coincides with the VCG outcome. According to Gul and Staachetti (1999, 2000), the VCG outcome typically lies outside the set of Walrasian equilibria in the sense that the VCG payment is generally below the Walrasian equilibrium payment. Ausubel and Milgrom (2002) further observed that in the presence of complementarity, the VCG outcome may lie outside the core.

Built upon the proposed GDDT auction, we will develop a dynamic strategy-proof auction for the current more general environment with both complements and substitutes, thus extending Ausubel's auction for substitutes. We use the following notation. Let $\mathcal{M}$ denote the market with the set $I$ of bidders and the set $N$ of items, and for each bidder $i \in I$, let $\mathcal{M}_{-i}$ denote the market $\mathcal{M}$ *without bidder* $i$. Let $I_{-i} = I \setminus \{i\}$ for every $i \in I$ and for convenience also let $\mathcal{M}_{-0} = \mathcal{M}$, $I_{-0} = I$, $M = I \cup \{0\}$, and $M_{-i} = M \setminus \{i\}$ for $i \in M$. Furthermore, let $\mathcal{U}$ denote the family of all utility functions $u : 2^N \to \mathbb{Z}_+$ satisfying Assumptions (A1) and (A2).

We now introduce the following PGDDT auction mechanism in which every bidder acts strategically and may not behave as a price-taker. The mechanism runs the GDDT auction for all markets $\mathcal{M}_{-m}$ ($m \in M$) simultaneously in parallel and in coordination. The GDDT auction works for every market $\mathcal{M}_{-m}$ exactly as described in Section 3 but needs the following modifications: Consider any market $\mathcal{M}_{-m}$. At $t \in \mathbb{Z}_+$ and $p^{-m}(t) \in \mathbb{Z}^n_+$, every bidder $i \in I_{-m}$ reports a choice set $C^i_{-m}(t) \subseteq 2^N$ (which need not be his demand set $D^i(p^{-m}(t))$) and the problem (3.8) or (3.9) becomes the next one for $\Delta$ or $\Delta^*$ respectively,

$$\max_{\delta \in \Delta(\text{ or } \Delta^*)} \Big\{ \sum_{i \in I_{-m}} \Big( \min_{S \in C^i_{-m}(t)} \sum_{\beta_h \in S} \delta_h \Big) - \sum_{\beta_h \in N} \delta_h \Big\} \tag{4.10}$$

If the auctioneer finds a unique smallest (largest) solution $\sigma^{-m}(t)$ of (4.10) for $\Delta$ ($\Delta^*$) in the order $\leq_g$, she obtains the next price vector $p^{-m}(t+1) = p^{-m}(t) + \delta^{-m}(t)$ whenever $\delta^{-m}(t) \neq 0$. We say the GDDT auction *finds an allocation* $\pi^{-m}$ in $\mathcal{M}_{-m}$ if $\delta^{-m}(t) = 0$ for $\Delta^*$ (i.e., in Step 3 of the auction) and $\pi^{-m}(i) \in C^i_{-m}(t)$ for all $i \in I_{-m}$. The GDDT auction needs to go back to Step 2 from Step 3 if $\delta^{-m}(t) = 0$ for $\Delta^*$ but it finds no allocation $\pi^{-m}$ in $\mathcal{M}_{-m}$ such that $\pi^{-m}(i) \in C^i_{-m}(t)$ for all $i \in I_{-m}$—this modification is meant to tolerate minor mistakes or manipulations committed by bidders. The GDDT auction *detects serious manipulation* if it finds no unique smallest (largest) solution $\delta^{-m}(t)$ of (4.10) for $\Delta$ ($\Delta^*$), or if $p^{-m}_h(t+1) < 0$ for some $\beta_h \in N$, or if it never finds an allocation in $\mathcal{M}_{-m}$ in which case the auction is said to *stop at time* $\infty$. Now we have

**The parallel global dynamic double-track (PGDDT) auction**[13]

Step 1: Run the GDDT auction simultaneously in parallel for every market $\mathcal{M}_{-m}$ ($m \in M$) by starting with a common initial price vector $p^{-m}(0) = p(0) \in \mathbb{Z}^n_+$. At $t \in \mathbb{Z}_+$ and $p^{-m}(t) \in \mathbb{Z}^n$, every bidder $i \in I_{-m}$ reports a choice $C^i_{-m}(t) \subseteq 2^N$ and the auctioneer finds the next price vector $p^{-m}(t+1) = p^{-m}(t) + \delta^{-m}(t)$. If the GDDT auction detects serious manipulations in any market, go to Step 3. Otherwise, the GDDT auction continues until it finds an allocation $\pi^{-m}$ in every market $\mathcal{M}_{-m}$ ($m \in M$) at $p^{-m}(T^{-m}) \in \mathbb{Z}^n_+$, and $T^{-m} \in \mathbb{Z}_+$. Go to Step 2.

Step 2: In this case all markets are clear. For every bidder $i \in I$, every $m \in M_{-i}$ and every $t = 0, 1, \cdots, T^{-m} - 1$, let $\Delta^{-m}_i(t)$ denote the "indirect utility change" of bidder $i$ in $I_{-m}$ when prices move from $p^{-m}(t)$ to $p^{-m}(t+1)$, where

$$\Delta^{-m}_i(t) = \min_{S \in C^i_{-m}(t)} \sum_{\beta_h \in S} \delta^{-m}_h(t) \tag{4.11}$$

[13]We can also use the DDT auction to construct a similar parallel auction.

Every bidder $i \in I$ is assigned the bundle $\pi^{-0}(i)$ of the allocation $\pi^{-0}$ found in the market $\mathcal{M}_{-0} = \mathcal{M}$ and required to pay $q_i$ and then the auction stops, where

$$q_i = \sum_{j \in I_{-i}} \Big( \sum_{t=0}^{T^{-0}-1} \Delta_j^{-0}(t) - \sum_{t=0}^{T^{-i}-1} \Delta_j^{-i}(t) \Big) + \sum_{\beta_h \in N} p_h^{-i}(T^{-i}) - \sum_{\beta_h \in N \setminus \pi^{-0}(i)} p_h^{-0}(T^{-0}) \quad (4.12)$$

Step 3: In this case every bidder receives no item but is assigned a payoff of $-\infty$. The auction stops.

The payment $q_i$ of bidder $i$ has an intuitive interpretation: $q_i$ is equal to the accumulation of "indirect utility changes" of his opponents $l \in I_{-i}$ along the path from $p^{-i}(T^{-i})$ to $p(0)$ (in the market $\mathcal{M}_{-i}$) and the path from $p(0)$ to $p^{-0}(T^{-0})$ (in the market $\mathcal{M}$ ) by subtracting $\sum_{\beta_h \in N \setminus \pi^{-0}(i)} p_h^{-0}$–the equilibrium payments by bidder $i$'s opponents in the market $\mathcal{M}$, and adding $\sum_{\beta_h \in N} p_h^{-i}(T^{-i})$–the equilibrium payments by bidder $i$'s opponents in the market $\mathcal{M}_{-i}$.

It is simple but important to observe that the PGDDT auction tolerates minor mistakes or manipulations committed by bidders and allows them to correct so that for any time $t^* \in \mathbb{Z}_+$, no matter what has happended before $t^*$, as long as from $t^*$ on every bidder $i$ bids according to his GSC utility function $u^i$, the auction will find a Walrasian equilibrium in every market in finitely many rounds and thus terminates in Step 2, because the GDDT auction converges to a Walrasian equilibrium from any integer price vector.

To study the incentive properties of the PGDDT auction mechanism, we will formulate this auction as an extensive-form dynamic game of incomplete information in which bidders are players. Prior to the start of the (auction) game, nature reveals to every player $i \in I$ only his own utility function $u^i \in \mathcal{U}$ of private information and a joint probability distribution $F(\cdot)$ from which the profile $\{u^i\}_{i \in I}$ is drawn. Let $H_i^t$ be the part of the information (or history) of play that player $i$ has observed just before he submits his choice sets at time $t \in \mathbb{Z}_+$. A natural and sensible specification is that $H_i^t$ comprises the complete set of all observable price vectors and all players' choice sets, i.e.,

$$H_i^t = \{p^{-m}(t), p^{-m}(s), C_{-m}^j(s) \mid m \in M, j \in I, 0 \leq s < t, m \neq j\}$$

Note that $H_i^t = H_j^t$ for all $i, j \in I$, namely, all bidders share a common history just like in an English auction. Let $T^*$ be the time when the PGDDT auction stops at Steps 2 or 3. If the auction has found an allocation in any $\mathcal{M}_{-m}$, for consistency and convenience, we define $C_{-m}^i(t) = C_{-m}^i(T^{-m})$ and $p^{-m}(t) = p^{-m}(T^{-m})$ for any $i \in I_{-m}$ and any $t \in \mathbb{Z}_+$ between $T^{-m}$ and $T^*$. After any history $H_i^t$ and at any time $t \in \mathbb{Z}_+$, each player $i$ updates his posterior beliefs $\mu_i(\cdot \mid t, H_i^t, u^i)$ over opponents' utility functions; see also Ausubel (2006). We stress that even after the auction is finished, player $i$ may not know his opponents' utility functions precisely.

A (dynamic) *strategy* $\sigma_i$ of player $i(i \in I)$ is a set-valued function $\{(t, m, H_i^t, u^i) \mid t \in \mathbb{Z}_+, m \in M_{-i}, u^i \in \mathcal{U}\} \to 2^N$, which tells him to bid $\sigma_i(t, m, H_i^t, u^i) \subseteq 2^N$ for every market $\mathcal{M}_{-m}(m \in M_{-i})$ at each time $t \in \mathbb{Z}_+$ when he observes $H_i^t$. Let $\Sigma_i$ denote player $i's$ strategy space of all such strategies $\sigma_i$. We say that $\sigma_i$ is *a sincerely bidding strategy* for player $i$ if he always reports his demand set $D^i(p^{-m}(t))$ as defined by (2.1)with respect to his true utility function $u^i$ for any $t \in \mathbb{Z}_+$, $m \in M_{-i}$ and $p^{-m}(t) \in \mathbb{Z}^n$, i.e.,

$$\sigma_i(t, m, H_i^t, u^i) = C_{-m}^i(t) = D^i(p^{-m}(t)) = \arg\max_{A \subseteq N}\{u^i(A) - \sum_{\beta_h \in A} p_h^{-m}(t)\}$$

Clearly, the strategy space $\Sigma_i$ of player $i$ contains sincere bidding strategies and also various other strategies.

Given the auction rules, the outcome of this auction game depends entirely upon the realization of utility functions and the strategies the bidders take. When every bidder $i \in I$ takes a strategy $\sigma_i$ and the PGDDT auction terminates in Step 2, then bidder $i \in I$ receives bundle $\pi^{-0}(i)$ and pays $q_i$ given by (4.12). When every bidder $i \in I$ takes a strategy $\sigma_i$ and the PGDDT auction stops in Step 3, every bidder gets nothing but a payoff of $-\infty$. In summary, every player $i's$ payoff function $W_i(\cdot, \cdot)$ is given by

$$W_i\Big(\{\sigma_j\}_{j \in I}, \{u^j\}_{j \in I}\Big) = \begin{cases} u^i(\pi^{-0}(i)) - q_i & \text{if the auction stops in Step 2,} \\ -\infty & \text{if the auction stops in Step 3.} \end{cases}$$

For auction games of incomplete information, the ex post equilibrium was used by Crémer and McLean (1985) for a sealed-bid auction (see also Krishna (2002)) and the ex post perfect equilibrium by Ausubel (2006) for a dynamic auction. Stronger than Bayesian equilibrium or perfect Bayesian equilibrium, these notions of equilibrium have a number of additional desirable properties, i.e., they are not only robust against any regret but also independent of any probability distribution. Following Ausubel (2006), the $\sharp(I)$-tuple $\{\sigma_i\}_{i \in I}$ is an ex post perfect equilibrium if for any time $t \in \mathbb{Z}_+$, any history profile $\{H_i^t\}_{i \in I}$, and any realization $\{u^i\}_{i \in I}$ of profile of utility functions of private information, the continuation strategy $\sigma_i(\cdot \mid t, H_i^t, u^i)$ of every player $i \in I$ (i.e., $\sigma_i(s, m, H_i^s \mid t, H_i^t, u^i) \subseteq 2^N$ for all $s \geq t$, $m \in M_{-i}$ and $H_i^s$) constitutes his best response against the continuation strategies $\{\sigma_j(\cdot \mid t, H_j^t, u^j)\}_{j \in I_{-i}}$ of player $i$'s opponents of the game even if the realization $\{u^i\}_{i \in I}$ becomes common knowledge.

Before presenting our next result, we briefly review the VCG auction for the marekt $\mathcal{M}$. In this auction every bidder $i \in I$ reports his utility function $u^i$ to the auctioneer. Then she computes an efficient allocation $\pi$ with respect to all bidders' reported $u^i$ and assigns bundle $\pi(i)$ to bidder $i$ and charges him a payment of $q_i^* = u^i(\pi(i)) - R(N) + R_{-i}(N)$, where $R(N)$ and $R_{-i}(N)$ are the market values of the markets $\mathcal{M}$ and $\mathcal{M}_{-i}$ based on $u^i$ $(i \in I)$, respectively. Bidder $i$'s VCG payoff equals $R(N) - R_{-i}(N)$.

**Theorem 4.1** *Suppose that the market $\mathcal{M}$ satisfies Assumptions* (A1) *and* (A2).
(i) *When every bidder bids sincerely, the PGDDT auction converges to a Walrasian equilibrium and yields a Vickrey-Clarke-Groves outcome for the market $\mathcal{M}$ in a finite number of rounds.*
(ii) *Sincere bidding is an ex post perfect equilibrium in the PGDDT auction.*

Proof: We first prove (i). By the argument in Section 3, we see that when every bidder $i$ bids sincerely according to his true GSC utility function $u^i$ , the auction terminates at Step 2 and finds a Walrasian equilibrium $(p^{-m}(T^{-m}), \pi^{-m})$ in every market $\mathcal{M}_{-m}$, $m \in M$. By the rules, every bidder $i$ receives bundle $\pi^{-0}(i)$ and pays $q_i$ of (4.12). It follows from (3.6) that

$$\Delta_i^{-m}(t) = \min_{S \in C^i_{-m}(t)} \sum_{\beta_h \in S} \delta_h^{-m}(t) = V^i(p^{-m}(t)) - V^i(p^{-m}(t+1))$$

for all $i \in I$ and $m \in M$ $(i \neq m)$, where $C^i_{-m}(t) = D^i(p^{-m}(t))$ and $V^i$ is bidder $i$'s indirect utility function based on $u^i$. Using these equations, we will show that $q_i$ coincides with the VCG payment $q_i^* = u^i(\pi^{-0}(i)) - R(N) + R_{-i}(N)$, where $R(N) = \sum_{j\in I} u^j(\pi^{-0}(j))$ and $R_{-i}(N) = \sum_{j \in I_{-i}} u^j(\pi^{-i}(j))$. Observe that payment $q_i$ of (4.12) satisfies

$$\begin{aligned}
q_i &= \textstyle\sum_{j\in I_{-i}} \Big( \sum_{t=0}^{T^{-0}-1} (V^j(p^{-0}(t)) - V^j(p^{-0}(t+1))) \\
&\qquad - \textstyle\sum_{t=0}^{T^{-i}-1} (V^j(p^{-i}(t)) - V^j(p^{-i}(t+1))) \Big) \\
&\quad + \textstyle\sum_{\beta_h \in N} p_h^{-i}(T^{-i}) - \sum_{\beta_h \in N \setminus \pi^{-0}(i)} p_h^{-0}(T^{-0}) \\
&= \textstyle\sum_{j\in I_{-i}} \Big( (V^j(p^{-0}(0)) - V^j(p^{-0}(T^{-0}))) - (V^j(p^{-i}(0)) - V^j(p^{-i}(T^{-0}))) \Big) \\
&\qquad + \textstyle\sum_{\beta_h \in N} p_h^{-i}(T^{-i}) - \sum_{\beta_h \in N \setminus \pi^{-0}(i)} p_h^{-0}(T^{-0}) \\
&= \Big( \textstyle\sum_{j\in I_{-i}} V^j(p^{-i}(T^{-0})) + \sum_{\beta_h \in N} p_h^{-i}(T^{-i}) \Big) \\
&\qquad - \Big( \textstyle\sum_{j\in I_{-i}} V^j(p^{-0}(T^{-0})) + \sum_{\beta_h \in N \setminus \pi^{-0}(i)} p_h^{-0}(T^{-0}) \Big) \\
&= \textstyle\sum_{j\in I_{-i}} u^j(\pi^{-i}(j)) - \sum_{j\in I_{-i}} u^j(\pi^{-0}(j)) \\
&= u^i(\pi^{-0}(i)) - R(N) + R_{-i}(N) \\
&= q_i^*.
\end{aligned}$$

Bidder $i's$ payoff $u^i(\pi^{-0}(i)) - q_i$ equals his VCG payoff $R(N) - R_{-i}(N)$.

Now we prove (ii). Consider any time $t^* \in \mathbb{Z}_+$, any history profile $\{H_j^{t^*}\}_{j\in I}$ (which may be on or off the equilibrium path), and any realization $\{u^j\}_{j\in I}$ of profile of utility functions in $\mathcal{U}^I$ of private information.[14] Take any player $i \in I$. Suppose that in the continuation game from time $t^*$ on, every opponent $j (j \in I_{-i})$ of player $i$ bids sincerely at any $t \in \mathbb{Z}_+ (t \geq t^*)$ and any $\mathcal{M}_{-m}$ $(m \in M_{-j})$, namely,

$$\sigma_j(t, m, H_j^t, u^j) = C^j_{-m}(t) = D^j(p^{-m}(t)) = \arg\max_{A \subseteq N} \{u^j(A) - \sum_{\beta_h \in A} p_h^{-m}(t)\}$$

[14]In this case, the outcome of the game depends on the histories $H_j^{t^*}$ and the strategies that all bidders will take in the continuation game starting from $t^*$. Bidders cannot change histories but can influence the path of the future from $t^*$ on.

Clearly, in this continuation game from time $t^*$, when all opponents of player $i$ choose sincere bidding strategies, because of the payoff of $-\infty$, bidder $i$ strictly prefers a strategy which results in the auction terminating at Step 2, to any other strategies which result in the auction stopping at Step 3. Therefore, it sufficient to compare the sincere bidding strategy with any other strategies which also result in the auction finishing at Step 2. Suppose that $\sigma_i'(\cdot \mid t^*, H_i^{t^*}, u^i)$ ($\sigma_i'$ in short) is such a continuation strategy of player $i$ resulting in an allocation $\rho$ for $\mathcal{M}$, and that bidder $i$'s (continuation) sincere bidding strategy results in an allocation $\pi$ for $\mathcal{M}$. Without any loss of generality, we assume that by the time $t^*$, the auction for the markets $\mathcal{M}$ and $\mathcal{M}_{-i}$ has not yet finished, i.e., $t^* < T^{-0}$ and $t^* < T^{-i}$. When player $i$ chooses the strategy $\sigma_i'$, his payment $q_i'$ given by (4.12) is

$$\begin{aligned}
q_i' &= \textstyle\sum_{j\in I_{-i}}\Big(\sum_{t=0}^{t^*-1}\Delta_j^{-0}(t)+\sum_{t=t^*}^{T^{-0}-1}[V^j(p^{-0}(t))-V^j(p^{-0}(t+1))]\\
&\quad\textstyle -\sum_{t=0}^{t^*-1}\Delta_j^{-i}(t)-\sum_{t=t^*}^{T^{-i}-1}[V^j(p^{-i}(t))-V^j(p^{-i}(t+1))]\Big)\\
&\quad\textstyle +\sum_{\beta_h\in N}p_h^{-i}(T^{-i})-\sum_{\beta_h\in N\setminus\rho(i)}p_h^{-0}(T^{-0})\\
&= \textstyle\sum_{j\in I_{-i}}\Big(\sum_{t=0}^{t^*-1}[\Delta_j^{-0}(t)-\Delta_j^{-i}(t)]+V^j(p^{-0}(t^*))+V^j(p^{-i}(T^{-i}))-V^j(p^{-i}(t^*))\Big)\\
&\quad\textstyle +\sum_{\beta_h\in N}p_h^{-i}(T^{-i})\\
&\quad\textstyle -\Big(\sum_{j\in I_{-i}}V^j(p^{-0}(T^{-0}))+\sum_{\beta_h\in N\setminus\rho(i)}p_h^{-0}(T^{-0})\Big)\\
&= \textstyle \mathit{constant}-\sum_{j\in I_{-i}}u^j(\rho(j)),
\end{aligned}$$

where $V^j$ is bidder $j$'s indirect utility function based on $u^j$ and *constant* is given by

$$\begin{aligned}
\mathit{constant} &= \textstyle\sum_{j\in I_{-i}}\Big(\sum_{t=0}^{t^*-1}[\Delta_j^{-0}(t)-\Delta_j^{-i}(t)]\Big)\\
&\quad\textstyle +\sum_{j\in I_{-i}}\Big(V^j(p^{-0}(t^*))+V^j(p^{-i}(T^{-i}))-V^j(p^{-i}(t^*))\Big)+\sum_{\beta_h\in N}p_h^{-i}(T^{-i})
\end{aligned}$$

Observe that *constant* is totally determined by the history profile $\{H_j^{t^*}\}_{j\in I}$ and the market $\mathcal{M}_{-i}$ without bidder $i$, and does not depend on player $i$'s strategy $\sigma_i'$, (and that $\Delta_j^{-0}(t)$ and $\Delta_j^{-i}(t)$ for $t < t^*$ cannot be expressed by $V^j$, because player $j$ may not have bid according to $u^j$ before $t^*$). Analogously we can show that when bidder $i$ uses the (continuation) sincere bidding strategy, his payment $\tilde{q}_i$ will be $\tilde{q}_i = \mathit{constant} - \sum_{j\in I_{-i}} u^j(\pi(j))$, where *constant* is the same as the previous one. Furthermore, we know from the argument in Section 3 that (in the continuation game) when bidders bid sincerely according to their utility functions $u^i$, $i \in I$, the resulted allocation $\pi$ must be efficient for $\mathcal{M}$. This implies that

$$u^i(\pi(i)) + \sum_{j\in I_{-i}} u^j(\pi(j)) \geq u^i(\rho(i)) + \sum_{j\in I_{-i}} u^j(\rho(j)).$$

Consequently, for bidder $i's$ payoff $\tilde{W}_i$ with the sincere bidding strategy and his payoff $W_i'$ with the strategy $\sigma_i'$, we have

$$\begin{aligned}
\tilde{W}_i &= \textstyle u^i(\pi(i))-\tilde{q}_i = u^i(\pi(i))-(\mathit{constant}-\sum_{j\in I_{-i}}u^j(\pi(j)))\\
&= \textstyle u^i(\pi(i))+\sum_{j\in I_{-i}}u^j(\pi(j))-\mathit{constant}\\
&\geq \textstyle u^i(\rho(i))+\sum_{j\in I_{-i}}u^j(\rho(j))-\mathit{constant} = u^i(\rho(i))-q_i'\\
&= W_i'.
\end{aligned}$$

This shows that sincere bidding is an ex post perfect equilibrium. □

The current dynamic procedure yields the same outcome as that of the VCG auction, but offers several advantages over the VCG auction: First, it utilizes information from every buyer efficiently and judiciously in that it only requires him to report his demand sets on a number of price vectors, whereas the VCG auction is sealed-bid and requires every buyer to report his entire values. In reality, businessmen generally do not like to reveal their values even if truth-telling may be theoretically a dominant strategy; see e.g., Rothkopf (2007). Second, the current procedure gives a simple and transparent way of computing efficient allocations, equilibrium prices and VCG payments using observable information, whereas the VCG auction tells only a way of computing VCG payments assuming that all buyers' values and efficient allocations are already given.

While both the current dynamic procedure and Ausubel's (2006) compute a Walrasian equilibrium in every market $\mathcal{M}_{-m}$ ($m \in M$) somehow like the VCG auction that needs to compute every market $\mathcal{M}_{-m}$ value $R_{-m}(N)$, the current procedure and analysis differ from Ausubel's in several aspects: First, the current procedure applies to the environment with both complements and substitutes, while Ausubel's applies to the environment with substitutes. Second, his procedure and payment rule are not symmetric, whereas the current procedure and payment rule are symmetric and simpler. Third, Ausubel's analysis on the VCG outcome focuses on economies with divisible goods and relies on calculus and Theorem 1 of Krishna and Maenner (2001) but he mentioned that his analysis can be analogously done for his model with indivisible goods under the GS condition, whereas the current analysis is quite different from his and in fact very elementary and simple.

# 5 Concluding Remarks

We conclude with a short summary highlighting the main contributions of the current paper and pointing out some open question. We proposed the (G)DDT auction that finds Walrasian equilibria and tells us about Walrasian equilibria in the circumstances containing complements that move beyond those we could handle before. The essential feature of the proposed dynamic auction is that it adjusts the prices of items in one set upwards but those of items in the other set downwards. Based upon the GDDT auction, we also introduced a dynamic, efficient and strategy-proof mechanism for the same environments. The GSI property plays a crucial role in establishing these procedures.

The current model is of private value. For models with interdependent values, we refer to Milgrom and Weber (1982), Cremér and McLean (1985), and more recently to Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), Krishna (2002), Perry and Reny (2002, 2005), and Ausubel (2004). It is of considerable interest but also significantly more difficult to extend the current model to the interdependent value framework.

# Appendix

The following lemma gives a different formulation of the GSC condition, saying that instead of adjusting the price of a single item, one can actually simultaneously increase the prices of several items in one set $S_j$ and decrease the prices of several items in another set $S_j^c$. The original definition of GSC has the advantage of simplicity and is easy to use in checking whether a utility function has the GSC property or not, whereas this alternative shows the rich properties of GSC and is very useful in mechanism design and proving theorems.

**Lemma 1** A utility function $u^i : 2^N \to \mathbb{R}$ satisfies the GSC condition if and only if for any price vectors $p, q \in \mathbb{R}^n$ with $q_k \geq p_k$ for all $\beta_k \in S_j$ for $j = 1$ or $2$ and $q_l \leq p_l$ for all $\beta_l \in S_j^c$, and for any bundle $A \in D^i(p)$, there exists a bundle $B \in D^i(q)$ such that

$$\{\beta_k \mid \beta_k \in A \cap S_j \text{ and } q_k = p_k\} \subseteq B \text{ and } \{\beta_l \mid \beta_l \in A^c \cap S_j^c \text{ and } q_l = p_l\} \subseteq B^c.$$

Proof: "Sufficiency" is obvious. Let us prove "Necessity". First, recall that Lemma 1 of Sun and Yang (2006B) says a utility function $u^i : 2^N \to \mathbb{R}$ satisfies the GSC condition if and only if for any $p \in \mathbb{R}^n$, any $\beta_k \in S_j$ for $j = 1$ or $2$, any $\delta \geq 0$, and any $A \in D^i(p)$, there exists $B \in D^i(p - \delta e(k))$ such that $[A^c \cap S_j] \setminus \{\beta_k\} \subseteq B^c$ and $[A \cap S_j^c] \subseteq B$.

For any $p \in \mathbb{R}^n$ and any $A \in D^i(p)$, we consider the following three basic cases and the other cases can be proved in an analogously recursive way.

*Case* (i), $\tilde{p} = p + \delta_k e(k) + \delta_{k'} e(k')$, where the two different objects $\beta_k$ and $\beta_{k'}$ are both in $S_j$ and $\delta_k > 0, \delta_{k'} > 0$. By the definition of the GSC condition, there exists $B' \in D^i(p + \delta_k e(k))$ such that $[A \cap S_j] \setminus \{\beta_k\} \subseteq B'$ and $[A^c \cap S_j^c] \subseteq B'^c$. Since $\tilde{p} = (p + \delta_k e(k)) + \delta_{k'} e(k')$, for $B' \in D^i(p + \delta_k e(k))$, there is $B \in D^i(\tilde{p})$ such that $[B' \cap S_j] \setminus \{\beta_{k'}\} \subseteq B$ and $[B'^c \cap S_j^c] \subseteq B^c$. Thus we have $[A \cap S_j] \setminus \{\beta_k, \beta_{k'}\} \subseteq B$ and $[A^c \cap S_j^c] \subseteq B^c$, namely,

$$\{\beta_x \mid \beta_x \in A \cap S_j \text{ and } \tilde{p}_x = p_x\} \subseteq B \text{ and } \{\beta_y \mid \beta_y \in A^c \cap S_j^c \text{ and } \tilde{p}_y = p_y\} \subseteq B^c.$$

*Case* (ii), $\tilde{p} = p - \delta_l e(l) - \delta_{l'} e(l')$, where the two different objects $\beta_l$ and $\beta_{l'}$ are both in $S_j^c$ and $\delta_l > 0, \delta_{l'} > 0$. It follows from the above equivalent formulation of the GSC condition that there exists $B' \in D^i(p - \delta_l e(l))$ such that $[A^c \cap S_j^c] \setminus \{\beta_l\} \subseteq B'^c$ and $[A \cap S_j] \subseteq B'$. Since $\tilde{p} = (p - \delta_l e(l)) - \delta_{l'} e(l')$, for $B' \in D^i(p - \delta_l e(l))$ there is $B \in D^i(\tilde{p})$ such that $[B'^c \cap S_j^c] \setminus \{\beta_{l'}\} \subseteq B^c$ and $[B' \cap S_j] \subseteq B$. Thus we obtain that $[A^c \cap S_j^c] \setminus \{\beta_l, \beta_{l'}\} \subseteq B^c$ and $[A \cap S_j] \subseteq B$, namely,

$$\{\beta_x \mid \beta_x \in A \cap S_j \text{ and } \tilde{p}_x = p_x\} \subseteq B \text{ and } \{\beta_y \mid \beta_y \in A^c \cap S_j^c \text{ and } \tilde{p}_y = p_y\} \subseteq B^c.$$

*Case* (iii), $\tilde{p} = p + \delta_k e(k) - \delta_l e(l)$, where $\beta_k \in S_j$, $\beta_l \in S_j^c$, and $\delta_k > 0, \delta_l > 0$. By the definition of the GSC condition, there exists $B' \in D^i(p+\delta_k e(k))$ such that $[A \cap S_j] \setminus \{\beta_k\} \subseteq B'$ and $[A^c \cap S_j^c] \subseteq B'^c$. Note that $\tilde{p} = (p+\delta_k e(k)) - \delta_l e(l)$. Then, it follows from the above equivalent formulation of the GSC condition that for $B' \in D^i(p+\delta_k e(k))$ there is $B \in D^i(\tilde{p})$ such that $[B'^c \cap S_j^c] \setminus \{\beta_l\} \subseteq B^c$ and $[B' \cap S_j] \subseteq B$. So we have $[A^c \cap S_j^c] \setminus \{\beta_l\} \subseteq B^c$ and $[A \cap S_j] \setminus \{\beta_k\} \subseteq B$, namely,

$$\{\beta_x \mid \beta_x \in A \cap S_j \text{ and } \tilde{p}_x = p_x\} \subseteq B \text{ and } \{\beta_y \mid \beta_y \in A^c \cap S_j^c \text{ and } \tilde{p}_y = p_y\} \subseteq B^c.$$

$\square$

To prove Theorem 3.2, we need to introduce an auxiliary lemma.

**Lemma 2** A utility function $u^i : 2^N \to \mathbb{R}$ satisfies the GSI condition, if and only if, for any price vector $p \in \mathbb{R}^n$ and any set $A \notin D^i(p)$, there exists another set $B(\neq A)$ satisfying one of the conditions (i) and (ii) of Definition 3.1 and $v^i(A, p) \leq v^i(B, p)$.

**Proof** "Necessity" is obvious. Let us prove "Sufficiency". Suppose that for any price vector $p \in \mathbb{R}^n$ and any set $A \notin D^i(p)$, there exists a set $B(\neq A)$ satisfying one of the conditions (i) and (ii) of Definition 3.1 and $v^i(A, p) \leq v^i(B, p)$. Continuity of the indirect utility function $V^i(\cdot)$ and the net utility function $v^i(A, \cdot)$ both in the price vector implies that there is a sufficiently small $\epsilon > 0$ such that $V^i(q) > v^i(A, q)$ for $q = p + \epsilon\, e(A^c) - \epsilon\, e(A)$. That is, $A \notin D^i(q)$. Thus, for the price vector $q$, there exists a set $B(\neq A)$ satisfying one of the conditions (i) and (ii) of Definition 3.1 and $v^i(A, q) \leq v^i(B, q)$. This leads to $v^i(B, p) - v^i(A, p) = v^i(B, q) - v^i(A, q) + [\sharp(A \setminus B) + \sharp(B \setminus A)]\epsilon > v^i(B, q) - v^i(A, q) \geq 0$. $\square$

**Proof of Theorem 3.2** We first prove that GSC implies GSI. By Lemma 2, it is sufficient to show that for any price vector $p \in \mathbb{R}^n$ and any set $A \notin D^i(p)$, there exists another set $B(\neq A)$ satisfying one of the conditions (i) and (ii) of Definition 3.1 and $v^i(A, p) \leq v^i(B, p)$.

First, observe that since the utility obtained by consuming any bundle of items is finite, regardless of the prices of other items the buyer $i$ will never demand item $\beta_k$ when its price is too high but will always demand it when its price is very low (may be quite negative). Formally, for the given price vector $p$ there exists a large real number $M^*$ such that for any price vector $q \in \mathbb{R}^n$, any $T \in D^i(q)$ and any $\beta_k \in N$, $q_k \geq p_k + M^*$ implies $\beta_k \notin T$, and $q_k \leq p_k - M^*$ implies $\beta_k \in T$.

Now choose any set $C \in D^i(p)$. Since $A \notin D^i(p)$, we clearly have $v^i(C, p) = V^i(p) > v^i(A, p)$ and thus $C \neq A$. There are two possibilities. Case (1) $C \setminus A \neq \emptyset$, and Case (2) $C \setminus A = \emptyset$ and $A \setminus C \neq \emptyset$. Define $\hat{p} = p + M^* e(A^c \cap C^c)$. Then, we still have $C \in D^i(\hat{p})$ and $v^i(C, \hat{p}) = V^i(\hat{p}) = V^i(p) > v^i(A, p) = v^i(A, \hat{p})$, and consequently $A \notin D^i(\hat{p})$. It might be helpful to draw figures when considering the following cases with respect to various sets.

In Case (1), i.e., $C \setminus A \neq \emptyset$, choose an item $\beta_k \in C \setminus A$ and assume $\beta_k \in S_j$ for some $j = 1$ or 2. Let $\breve{p} = \hat{p} + M^*e([C \setminus (A \cup \{\beta_k\})] \cap S_j) - M^*e(A \cap S_j^c)$. Note that when $\hat{p}$ changes to $\breve{p}$, the price of item $\beta_k$ does not change. Then, with regard to $C \in D^i(\hat{p})$ and $\beta_k \in C \cap S_j$, it follows from the GSC condition and Lemma 1 that there exists a set $\bar{C} \in D^i(\breve{p})$ such that $\beta_k \in \bar{C}$. Clearly, $\{\beta_k\} \subseteq (\bar{C} \setminus A) \cap S_j$. Meanwhile, observe that $\breve{p} = p + M^*e(A^c \setminus [(C \cap S_j^c) \cup \{\beta_k\}]) - M^*e(A \cap S_j^c)$. Then, by the definition of $M^*$ and the construction of $\breve{p}$, we have $(\bar{C} \setminus A) \cap S_j \subseteq \{\beta_k\}$ and $A \cap S_j^c \subseteq \bar{C}$. In summary, it yields $(\bar{C} \setminus A) \cap S_j = \{\beta_k\}$ and $A \cap S_j^c \subseteq \bar{C}$.

In Subcase (1-1) in which $(\bar{C} \setminus A) \cap S_j^c \neq \emptyset$, select an item $\beta_h \in (\bar{C} \setminus A) \cap S_j^c$. Let $\tilde{p} = \breve{p} + M^*e([C \setminus (A \cup \{\beta_h\})] \cap S_j^c) - M^*e(A \cap S_j)$. Note that when $\breve{p}$ changes to $\tilde{p}$, the price of item $\beta_h$ does not change. Then, with regard to $\bar{C} \in D^i(\breve{p})$ and $\beta_h \in \bar{C} \cap S_j^c$, it follows from the GSC condition and Lemma 1 that there exists a bundle $B \in D^i(\tilde{p})$ such that $\beta_h \in B$. Observe that $\tilde{p} = p + M^*e(A^c \setminus \{\beta_k, \beta_h\}) - M^*e(A)$. Then the definition of $M^*$ and the construction of $\tilde{p}$ imply that $A \subseteq B$, and $B \setminus A \subseteq \{\beta_k, \beta_h\}$. Thus we have $A \setminus B = \emptyset$, and $B \setminus A = \{\beta_h, \beta_k\}$ or $\{\beta_h\}$. Namely, the set $B$ satisfies the condition (i) or (ii) of Definition 3.1.

In Subcase (1-2) in which $(\bar{C} \setminus A) \cap S_j^c = \emptyset$ and $(A \setminus \bar{C}) \cap S_j \neq \emptyset$, choose an item $\beta_h \in (A \setminus \bar{C}) \cap S_j$. Let $\tilde{p} = \breve{p} + M^*e((C \setminus A) \cap S_j^c) - M^*e((A \cap S_j) \setminus \{\beta_h\}))$. Note that when $\breve{p}$ changes to $\tilde{p}$, the price of item $\beta_h$ does not change. Then, with regard to $\bar{C} \in D^i(\breve{p})$ and $\beta_h \in S_j \setminus \bar{C}$, it follows from the GSC condition and Lemma 1 that there exists a set $B \in D^i(\tilde{p})$ such that $\beta_h \notin B$. Next, observe that $\tilde{p} = p + M^*e(A^c \setminus \{\beta_k\}) - M^*e(A \setminus \{\beta_h\})$. Then the definition of $M^*$ and the construction of $\tilde{p}$ imply that $A \setminus B \subseteq \{\beta_h\}$ and $B \setminus A \subseteq \{\beta_k\}$. Therefore we have $A \setminus B = \{\beta_h\}$, and $B \setminus A = \{\beta_k\}$ or $\emptyset$. This shows that the set $B$ satisfies the condition (i) or (ii) of Definition 3.1.

In Subcase (1-3) in which $\bar{C} = A \cup \{\beta_k\}$, let $\tilde{p} = \breve{p}$ and $B = \bar{C}$. Then, $B$ satisfies the condition (i) of Definition 3.1.

In Case (2), i.e., $C \subseteq A$ and $A \setminus C \neq \emptyset$, choose an item $\beta_k \in A \setminus C$ and assume $\beta_k \in S_j$ for some $j = 1$ or 2. Let $\breve{p} = \hat{p} - M^*e((A \setminus \{\beta_k\}) \cap S_j)$. Note that when $\hat{p}$ changes to $\breve{p}$, the price of item $\beta_k$ does not change. Then, with regard to $C \in D^i(\hat{p})$ and $\beta_k \in S_j \setminus C$, it follows from the GSC condition and Lemma 1 that there exists a set $\bar{C} \in D^i(\breve{p})$ such that $\beta_k \notin \bar{C}$. Meanwhile, note that $\breve{p} = p + M^*e(A^c) - M^*e((A \cap S_j) \setminus \{\beta_k\})$. Then, by the definition of $M^*$ and the construction of $\breve{p}$, we have $\bar{C} \subseteq A$ and $(A \setminus \bar{C}) \cap S_j \subseteq \{\beta_k\}$. Consequently, it leads to $\bar{C} \subseteq A$ and $(A \setminus \bar{C}) \cap S_j = \{\beta_k\}$.

In Subcase (2-1) in which $(A \setminus \bar{C}) \cap S_j^c \neq \emptyset$, choose an item $\beta_h \in (A \setminus \bar{C}) \cap S_j^c$. Let $\tilde{p} = \breve{p} - M^*e((A \setminus \{\beta_h\}) \cap S_j^c)$. Note that when $\breve{p}$ changes to $\tilde{p}$, the price of item $\beta_h$ does not change. Then, with regard to $\bar{C} \in D^i(\breve{p})$ and $\beta_h \in S_j^c \setminus \bar{C}$, it follows from the GSC condition and Lemma 1 that there exists a bundle $B \in D^i(\tilde{p})$ such that $\beta_h \notin B$. Next, note that

$\tilde{p} = p + M^*e(A^c) - M^*e(A \setminus \{\beta_k, \beta_h\})$. Then the definition of $M^*$ and the construction of $\tilde{p}$ imply that $B \subseteq A$ and $A \setminus B \subseteq \{\beta_h, \beta_k\}$. Therefore, we have $B \subseteq A$, and $A \setminus B = \{\beta_h, \beta_k\}$ or $\{\beta_h\}$. Thus the set $B$ satisfies the condition (i) or (ii) of Definition 3.1.

In Subcase (2-2) in which $\bar{C} = A \setminus \{\beta_k\}$, let $\tilde{p} = \check{p}$ and $B = \bar{C}$. Then, $B$ satisfies the condition (i) of Definition 3.1.

By summing up all above cases, we conclude that there always exist a price vector $\tilde{p}$ and a set $B \in D(\tilde{p})$ satisfying one of the condition (i) and (ii) of Definition 3.1. By the construction of $B$, we see $B \neq A$ in each case. Then, it follows from $B \in D^i(\tilde{p})$ that $v^i(B, \tilde{p}) = V^i(\tilde{p}) \geq v^i(A, \tilde{p})$. Furthermore, by the construction of $\tilde{p}$, we see $\tilde{p}([A \setminus B] \cup [B \setminus A]) = p([A \setminus B] \cup [B \setminus A])$. As a result, we have $v^i(B, p) - v^i(A, p) = v^i(B, \tilde{p}) - v^i(A, \tilde{p}) \geq 0$. In this way we proved that GSC implies GSI.

It remains to show that GSI implies GSC. Choose any price vector $p \in \mathbb{R}^n$, $\beta_k \in S_j$ for some $j = 1$ or 2, $\delta \geq 0$, and $A \in D^i(p)$. It is clear that if $\beta_k \notin A$, then $A \in D^i(p + \delta e(k))$. If we choose $B = A$, then the GSC condition is immediately satisfied. Now we assume that $\beta_k \in A$. Let $\delta^* = V^i(p) - V^i(p + \delta e(k))$. Then we have $0 \leq \delta^* \leq \delta$, $A \in D^i(p + \epsilon e(k))$ and $V^i(p + \epsilon e(k)) = V^i(p) - \epsilon$ for all $\epsilon \in [0, \delta^*]$. We need to consider two separate cases. First, if $\delta^* = \delta$, then we have $A \in D^i(p + \delta e(k))$ and we can choose $B = A$. Clearly, the GSC condition is satisfied. In the rest, we deal with the case of $\delta^* < \delta$. In this case we have $V^i(p + \epsilon e(k)) = V^i(p + \delta^* e(k))$ and $A \notin D^i(p + \epsilon e(k))$ for all $\epsilon > \delta^*$. In particular, we have $A \notin D^i(p + \delta e(k))$. Now let $\{\delta_\nu\}$ be any sequence of positive real numbers which converges to 0. Since $A \notin D^i(p + (\delta^* + \delta_\nu)e(k))$, it follows from the GSI condition that there exists a GSI improvement set $B_\nu$ of $A$ such that $v^i(B_\nu, p + (\delta^* + \delta_\nu)e(k)) > v^i(A, p + (\delta^* + \delta_\nu)e(k))$. Notice that $\beta_k$ does not belong to any such GSI improvement set $B_\nu$. Suppose that this statement is false. Then for some $\nu$ we would have $v^i(B_\nu, p + \delta^* e(k)) - \delta_\nu = v^i(B_\nu, p + (\delta^* + \delta_\nu)e(k)) > v^i(A, p + (\delta^* + \delta_\nu)e(k)) = v^i(A, p + \delta^* e(k)) - \delta_\nu$. This leads to $v^i(B_\nu, p + \delta^* e(k)) > v^i(A, p + \delta^* e(k)) = V^i(p + \delta^* e(k))$, yielding a contradiction. Meanwhile, since the number of sets $B_\nu$ is finite, without loss of generality we can assume that there exists a positive integer $\nu^*$ such that $B_\nu = B$ for all $\nu \geq \nu^*$. Then by the continuity of net utility function $v^i(B, \cdot)$, we have $v^i(B, p + \delta^* e(k)) = v^i(A, p + \delta^* e(k)) = V^i(p + \delta^* e(k)) = V^i(p + \delta e(k))$. In addition, since $\beta_k \notin B$, we have $v^i(B, p + \delta e(k)) = v^i(B, p + \delta^* e(k)) = V^i(p + \delta e(k))$. This implies $B \in D^i(p + \delta e(k))$. Furthermore, since $B$ is a GSI improvement set of $A$ and $\beta_k \notin B$, it satisfies either
(i): $A \cap S_j^c = B \cap S_j^c$, and $(A \setminus B) \cap S_j = \{\beta_k\}$ and $\sharp[(B \setminus A) \cap S_j] \leq 1$; or
(ii): $B \subseteq A$, and $(A \setminus B) \cap S_j = \{\beta_k\}$ and $\sharp[(A \setminus B) \cap S_j^c] = 1$.
This concludes that $[A \cap S_j] \setminus \{\beta_k\} \subseteq B$ and $A^c \cap S_j^c \subseteq B^c$ and thus the GSC condition is satisfied. $\square$

The next lemma extends a well-known property of a submodular function to its new generalization, saying that (i) the marginal utility of an additional item decreases if the bundle of items to which it is added gets smaller in its complement set (gets larger in its same set for (ii)). Property (i) is new but (ii) is familiar.

**Lemma 3** A function $f$ is a generalized submodular function if and only if
(i): for any $x \in \mathbb{R}^n$, any $\beta_k \in S_j$, any $\beta_l \in S_j^c$, any $\delta_k > 0$ and $\delta_l > 0$,

$$f(x + \delta_k e(k) - \delta_l e(l)) - f(x - \delta_l e(l)) \leq f(x + \delta_k e(k)) - f(x); \text{ and}$$

(ii): for any $x \in \mathbb{R}^n$, any distinct $\beta_k, \beta_l \in S_j$, any $\delta_k > 0$ and $\delta_l > 0$,

$$f(x + \delta_k e(k) + \delta_l e(l)) - f(x + \delta_l e(l)) \leq f(x + \delta_k e(k)) - f(x).$$

Proof: Suppose that $f$ is a generalized submodular function. In the case of (i), let $p = x + \delta_k e(k)$ and $q = x - \delta_l e(l)$. Then $p \wedge_g q = x$ and $p \vee_g q = x + \delta_k e(k) - \delta_l e(l)$. Clearly (i)'s conclusion holds. It is also easy to check the case of (ii).

Suppose that both (i) and (ii) hold. Take any $p, q \in \mathbb{R}^n$. With respect to $S_1$ and $S_2$, let

$$J_{S_1} = \{j \mid p_j > q_j \text{ and } \beta_j \in S_1\}$$

$$K_{S_1} = \{k \mid p_k < q_k \text{ and } \beta_k \in S_1\}$$

$$J_{S_2} = \{j \mid p_j > q_j \text{ and } \beta_j \in S_2\}$$

$$K_{S_2} = \{k \mid p_k < q_k \text{ and } \beta_k \in S_2\}.$$

We consider the most general case, namely, all the above four sets are nonempty. So there exists a nonnegative vector $\delta = (\delta_1, \cdots, \delta_n) \gg 0$, such that $p_j = q_j + \delta_j$ for all $j \in J_{S_1} \cup J_{S_2}$ and $p_j = q_j - \delta_j$ for all $j \in K_{S_1} \cup K_{S_2}$. Let $J_{S_1} = \{h_1, \cdots, h_s\}$, $K_{S_1} = \{i_1, \cdots, i_t\}$,

$J_{S_2} = \{j_1, \cdots, j_u\}$, and $K_{S_2} = \{k_1, \cdots, k_v\}$. Then we have

$$
\begin{array}{rl}
& f(p) - f(p \wedge_g q) \\
= & f(p) - f\Big(p - \sum_{l=1}^{s} \delta_{h_l} e(h_l) + \sum_{l=1}^{v} \delta_{k_l} e(k_l)\Big) \\
= & \sum_{l=1}^{s} \Big[ f\Big(p - \sum_{r=1}^{l-1} \delta_{h_r} e(h_r)) - f\Big(p - \sum_{r=1}^{l} \delta_{h_r} e(h_r)\Big)\Big] \\
& + \sum_{l=1}^{v} \Big[ f\Big(p - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l-1} \delta_{k_r} e(k_r)\Big) \\
& \quad - f\Big(p - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l} \delta_{k_r} e(k_r)\Big)\Big] \\
\geq & \sum_{l=1}^{s} \Big[ f\Big(p - \sum_{r=1}^{l-1} \delta_{h_r} e(h_r) + \delta_{i_1} e(i_1)\Big) - f\Big(p - \sum_{r=1}^{l} \delta_{h_r} e(h_r) + \delta_{i_1} e(i_1)\Big)\Big] \\
& + \sum_{l=1}^{v} \Big[ f\Big(p - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l-1} \delta_{k_r} e(k_r) + \delta_{i_1} e(i_1)\Big) \\
& \quad - f\Big(p - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l} \delta_{k_r} e(k_r)) + \delta_{i_1} e(i_1)\Big)\Big] \\
\vdots & \\
\geq & \sum_{l=1}^{s} \Big[ f\Big(p - \sum_{r=1}^{l-1} \delta_{h_r} e(h_r) + \sum_{r=1}^{t} \delta_{i_r} e(i_r)\Big) \\
& \quad - f\Big(p - \sum_{r=1}^{l} \delta_{h_r} e(h_r) + \sum_{r=1}^{t} \delta_{i_r} e(i_r)\Big)\Big] \\
& + \sum_{l=1}^{v} \Big[ f\Big(p - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l-1} \delta_{k_r} e(k_r) + \sum_{r=1}^{t} \delta_{i_r} e(i_r)\Big) \\
& \quad - f\Big(p - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l} \delta_{k_r} e(k_r) + \sum_{r=1}^{t} \delta_{i_r} e(i_r)\Big)\Big] \\
\geq & \sum_{l=1}^{s} \Big[ f\Big(p - \sum_{r=1}^{l-1} \delta_{h_r} e(h_r) + \sum_{r=1}^{t} \delta_{i_r} e(i_r) - \delta_{j_1} e(j_1)\Big) \\
& \quad - f\Big(p - \sum_{r=1}^{l} \delta_{h_r} e(h_r) + \sum_{r=1}^{t} \delta_{i_r} e(i_r) - \delta_{j_1} e(j_1)\Big)\Big] \\
& + \sum_{l=1}^{v} \Big[ f\Big(p - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l-1} \delta_{k_r} e(k_r) + \sum_{r=1}^{t} \delta_{i_r} e(i_r) - \delta_{j_1} e(j_1)\Big) \\
& \quad - f\Big(p - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l} \delta_{k_r} e(k_r) + \sum_{r=1}^{t} \delta_{i_r} e(i_r) - \delta_{j_1} e(j_1)\Big)\Big] \\
\vdots & \\
\geq & \sum_{l=1}^{s} \Big[ f\Big(p - \sum_{r=1}^{l-1} \delta_{h_r} e(h_r) + \sum_{r=1}^{t} \delta_{i_r} e(i_r) - \sum_{r=1}^{u} \delta_{j_r} e(j_r)\Big) \\
& \quad - f\Big(p - \sum_{r=1}^{l} \delta_{h_r} e(h_r) + \sum_{r=1}^{t} \delta_{i_r} e(i_r) - \sum_{r=1}^{u} \delta_{j_r} e(j_r)\Big)\Big] \\
& + \sum_{l=1}^{v} \Big[ f\Big(p - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l-1} \delta_{k_r} e(k_r) + \sum_{r=1}^{t} \delta_{i_r} e(i_r) - \sum_{r=1}^{u} \delta_{j_r} e(j_r)\Big) \\
& \quad - f\Big(p - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l} \delta_{k_r} e(k_r) + \sum_{r=1}^{t} \delta_{i_r} e(i_r) - \sum_{r=1}^{u} \delta_{j_r} e(j_r)\Big)\Big] \\
= & \sum_{l=1}^{s} \Big[ f\Big(p \vee_g q - \sum_{r=1}^{l-1} \delta_{h_r} e(h_r)\Big) - f\Big(p \vee_g q - \sum_{r=1}^{l} \delta_{h_r} e(h_r)\Big)\Big] \\
& + \sum_{l=1}^{v} \Big[ f\Big(p \vee_g q - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l-1} \delta_{k_r} e(k_r)\Big) \\
& \quad - f\Big(p \vee_g q - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l} \delta_{k_r} e(k_r)\Big)\Big] \\
= & f(p \vee_g q) - f(q)
\end{array}
$$

Therefore we have $f(p \wedge_g q) + f(p \vee_g q) \leq f(p) + f(q)$. In the above derivation, the first two inequalities follow from case (ii) and the last two follow from case (i). □

**Proof of Theorem 3.3** *Necessity*: Choose any two distinct items $\beta_k, \beta_l \in N$, any $p \in \mathbb{R}^n$, any $\delta_k > 0$, and any $\delta_l > 0$. If $V^i(p) - V^i(p + \delta_k e(k)) = 0$, the monotonicity of $V^i(\cdot)$ implies that $V^i(p + \delta_l e(l) + \delta_k e(k)) - V^i(p + \delta_l e(l)) \leq 0 = V^i(p + \delta_k e(k)) - V^i(p)$ and $V^i(p - \delta_l e(l) + \delta_k e(k)) - V^i(p - \delta_l e(l)) \leq 0 = V^i(p + \delta_k e(k)) - V^i(p)$. We can now assume that $V^i(p) - V^i(p + \delta_k e(k)) = \epsilon_k > 0$. Then it follows that $0 < \epsilon_k \leq \delta_k$,

$V^i(p + \epsilon_k e(k)) = V^i(p + \delta_k e(k)) = V^i(p) - \epsilon_k$, and there are a bundle $A \in D^i(p)$ and a bundle $B \in D^i(p + \epsilon_k e(k))$ (for example, $B = A$) with $\beta_k \in A \cap B$. We need to consider the following two situations.

Case 1: $\beta_l$ and $\beta_k$ are in the same set $S_j$. With regard to $A \in D^i(p)$ and $B \in D^i(p + \epsilon_k e(k))$, it follows from the GSC condition and $\beta_k \in A \cap B$ that there are two bundles $C \in D^i(p + \delta_l e(l))$ with $\beta_k \in C$ and $D \in D^i(p + \delta_l e(l) + \epsilon_k e(k))$ with $\beta_k \in D$. As a result, we have

$$\begin{array}{rl} & V^i(p + \delta_l e(l) + \delta_k e(k)) - V^i(p + \delta_l e(l)) \\ \leq & V^i(p + \delta_l e(l) + \epsilon_k e(k)) - V^i(p + \delta_l e(l)) \\ = & v^i(D, p + \delta_l e(l) + \epsilon_k e(k)) - V^i(p + \delta_l e(l)) \\ = & v^i(D, p + \delta_l e(l)) - \epsilon_k - V^i(p + \delta_l e(l)) \\ \leq & -\epsilon_k = V^i(p + \delta_k e(k)) - V^i(p). \end{array}$$

Case 2: $\beta_l$ and $\beta_k$ are not in the same set $S_j$. With regard to $A \in D^i(p)$ and $B \in D^i(p+\epsilon_k e(k))$, it follows from the GSC condition, Lemma 1 and $\beta_k \in A \cap B$ that there are two bundles $C \in D^i(p - \delta_l e(l))$ with $\beta_k \in C$ and $D \in D^i(p - \delta_l e(l) + \epsilon_k e(k))$ with $\beta_k \in D$, which leads to

$$\begin{array}{rl} & V^i(p - \delta_l e(l) + \delta_k e(k)) - V^i(p - \delta_l e(l)) \\ \leq & V^i(p - \delta_l e(l) + \epsilon_k e(k)) - V^i(p - \delta_l e(l)) \\ = & v^i(D, p - \delta_l e(l) + \epsilon_k e(k)) - V^i(p - \delta_l e(l)) \\ = & v^i(D, p - \delta_l e(l)) - \epsilon_k - V^i(p - \delta_l e(l)) \\ \leq & -\epsilon_k = V^i(p + \delta_k e(k)) - V^i(p). \end{array}$$

In summary, we see through Lemma 3 that $V^i$ is a generalized submodular function.

*Sufficiency*: Suppose to the contrary that there are some $p \in \mathbb{R}^n$, $\beta_k \in S_j$, $\delta_k > 0$, and $A \in D^i(p)$ such that for every $B \in D^i(p+\delta_k e(k))$ we have $[A \cap S_j] \setminus \{\beta_k\} \not\subseteq B$ or $A^c \cap S_j^c \not\subseteq B^c$. Let $\epsilon_k = V^i(p) - V^i(p+\delta_k e(k))$. Clearly, $0 \leq \epsilon_k \leq \delta_k$, $V^i(p+\epsilon_k e(k)) = V^i(p+\delta_k e(k))$, and $A \in D^i(p+\epsilon_k e(k))$. Since $A \notin D^i(p+\delta_k e(k))$, it holds that $D^i(p+\epsilon_k e(k)) \neq D^i(p+\delta_k e(k))$ and $\epsilon_k < \delta_k$. Let $q = p+\epsilon_k e(k)$ and $\theta_k = \delta_k - \epsilon_k > 0$. Then $V^i(q) = V^i(q+\theta_k e(k))$. Observe that $A \in D^i(q)$ and $B \notin D^i(q + \theta_k e(k))$ for every bundle $B$ satisfying $[A \cap S_j] \setminus \{\beta_k\} \subseteq B$ and $A^c \cap S_j^c \subseteq B^c$. This means that $V^i(q + \theta_k e(k)) > v^i(B, q + \theta_k e(k))$ for every bundle $B$ satisfying $[A \cap S_j] \setminus \{\beta_k\} \subseteq B$ and $A^c \cap S_j^c \subseteq B^c$. Furthermore, the continuity of $V^i(\cdot)$ and $v^i(B, \cdot)$ implies that there exists a sufficiently small positive number $\theta$ so that

$$\begin{array}{rl} & V^i\Big(q + \theta_k e(k) - \theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)\Big) \\ > & v^i\Big(B, q + \theta_k e(k) - \theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)\Big) \end{array}$$

for every bundle $B$ satisfying $[A \cap S_j] \setminus \{\beta_k\} \subseteq B$ and $A^c \cap S_j^c \subseteq B^c$. This means that if $B \in D^i\Big(q + \theta_k e(k) - \theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)\Big)$, then $[A \cap S_j] \setminus \{\beta_k\} \not\subseteq B$ or

$A^c \cap S_j^c \not\subseteq B^c$. Then choosing $B \in D^i\Big(q + \theta_k e(k) - \theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)\Big)$ yields

$$\begin{array}{rl} & V^i\Big(q + \theta_k e(k) - \theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)\Big) \\ = & v^i(B, q + \theta_k e(k) - \theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)) \\ = & v^i(B, q + \theta_k e(k) + \bar{p}) \\ = & v^i(B, q + \theta_k e(k)) - \sum_{\beta_k \in B} \bar{p}_k \\ = & v^i(B, q + \theta_k e(k)) + \sharp(B \cap ([A \cap S_j] \setminus \{\beta_k\}))\theta - \sharp(B \cap (A^c \cap S_j^c))\theta \\ < & v^i(B, q + \theta_k e(k)) + \sharp([A \cap S_j] \setminus \{\beta_k\})\theta \\ \leq & V^i(q + \theta_k e(k)) + \sharp([A \cap S_j] \setminus \{\beta_k\})\theta, \end{array}$$

where $\bar{p} = -\theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)$. Therefore we have

$$\begin{array}{rl} & V^i\Big(q - \theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)\Big) \\ = & V^i(q) + \sharp([A \cap S_j] \setminus \{\beta_k\})\theta \\ = & V^i(q + \theta_k e(k)) + \sharp([A \cap S_j] \setminus \{\beta_k\})\theta \\ > & V^i\Big(q + \theta_k e(k) - \theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)\Big). \end{array}$$

Let $x = q$ and $y = q + \theta_k e(k) - \theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)$. Then the above inequality leads to

$$\begin{array}{rcl} V^i(x \wedge_g y) + V^i(x \vee_g y) & = & V^i\Big(q - \theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)\Big) + V^i(q + \theta_k e(k)) \\ & > & V^i(q) + V^i\Big(q + \theta_k e(k) - \theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)\Big) \\ & = & V^i(x) + V^i(y), \end{array}$$

contradicting the hypothesis that $V^i$ is a generalized submodular function. □

**Proof of Lemma 3.4** Suppose that $p^*$ is an equilibrium price vector. Then we know from Gul and Stacchetti (1999, Lemma 6) that for any efficient allocation $\pi^*$, $(p^*, \pi^*)$ constitutes an equilibrium. Clearly, $\sum_{i \in I} u^i(\pi^*(i)) = R(N)$ the market value of the objects. Furthermore, we have $\mathcal{L}(p^*) = \sum_{i \in I} V^i(p^*) + \sum_{\beta_h \in N} p_h^* = \sum_{i \in I}(u^i(\pi^*(i)) - \sum_{\beta_h \in \pi^*(i)} p_h^*) + \sum_{\beta_h \in N} p_h^* = R(N)$. Note that for any $p \in \mathbb{R}^n$ and $i \in I$, $V^i(p) \geq u^i(\pi^*(i)) - \sum_{\beta_h \in \pi^*(i)} p_h$. Thus for any $p \in \mathbb{R}^n$, we have

$$\mathcal{L}(p) = \sum_{i \in I} V^i(p) + \sum_{\beta_h \in N} p_h \geq \sum_{i \in I} u^i(\pi^*(i)) = R(N) = \mathcal{L}(p^*).$$

Hence, $\mathcal{L}(p^*) = \min_{p \in \mathbf{R}^n} \mathcal{L}(p)$, i.e., $p^*$ is a minimizer of the function $\mathcal{L}$ with $\mathcal{L}(p^*) = R(N)$.

Suppose that $\hat{p}$ is a minimizer of $\mathcal{L}$ with its value $\mathcal{L}(\hat{p}) = R(N)$. Let $\rho$ be any efficient allocation of the model. We will show that $(\hat{p}, \rho)$ is an equilibrium. Clearly, we have $V^i(\hat{p}) \geq u^i(\rho(i)) - \sum_{\beta_h \in \rho(i)} \hat{p}_h$ for every $i \in I$. We need to show that $V^i(\hat{p}) = u^i(\rho(i)) - \sum_{\beta_h \in \rho(i)} \hat{p}_h$ for every $i \in I$. Suppose to the contrary that $V^j(\hat{p}) > u^j(\rho(j)) - \sum_{\beta_h \in \rho(j)} \hat{p}_h$ for some bidder $j$. Adding the previous inequalities over all bidders leads to $\mathcal{L}(\hat{p}) > R(N)$. This contradicts the hypothesis that $\hat{p}$ is a minimizer of $\mathcal{L}$ with $\mathcal{L}(\hat{p}) = R(N)$. Thus $(\hat{p}, \rho)$ must be an equilibrium. □

**Proof of Theorem 3.5** By Theorem 3.1 of Sun and Yang (2006B) the model has an equilibrium. Then by Lemma 3.4 the set of equilibria is equal to the set of minimizers of the Lyapunov function $\mathcal{L}$. Let

$$\Lambda = \arg\min\{\mathcal{L}(p) \mid p \in \mathbb{R}^n\}.$$

It follows from Theorem 3.3 and the remark after Formula (2.1) that the Lyapunov function $\mathcal{L}$ is a continuous, convex and generalized submodular function. Now we prove statement (ii). We first show that $\Lambda$ is a generalized lattice. Take any $p, q \in \Lambda$. So we have $\mathcal{L}(p) = \mathcal{L}(q) = R(N)$, where $R(N)$ is the market value. Clearly, $R(N) \le \mathcal{L}(p \wedge_g q) \le \mathcal{L}(p) + \mathcal{L}(q) - \mathcal{L}(p \vee_g q) \le 2R(N) - R(N) = R(N)$. This shows that $\mathcal{L}(p \wedge_g q) = \mathcal{L}(p \vee_g q) = R(N)$ and $p \wedge_g q, p \vee_g q \in \Lambda$. So the set $\Lambda$ is a nonempty, convex and generalized lattice. Clearly, $\Lambda$ is also compact.

Next, we prove that $\Lambda$ is also an integrally convex set. Suppose the statement is false. Define

$$A = \{p \in \Lambda \mid p \notin \mathrm{co}(\Lambda \cap N(p))\},$$

where $N(p) = \{z \in \mathbb{Z}^n \mid \|z - p\|_\infty < 1\}$. Then $A$ is a nonempty subset of $\Lambda$. Observe that $p \in \mathrm{co}(\Lambda \cap N(p))$ for every $p \in \Lambda \cap \mathbb{Z}^n$ because $N(p) = \{p\}$ for every $p \in \mathbb{Z}^n$. And so, $A \cap \mathbb{Z}^n = \emptyset$. Let $p^* \in A$ be a vector that has at least as many integral coordinates as any other vector in $A$ has. Thus, the number of integral coordinates of $p^*$ is the largest among all vectors in $A$. Since $\mathrm{co}(N(p^*))$ is a hypercube, it is a generalized lattice. Let $q^*$ be the generalized smallest element of $\mathrm{co}(N(p^*))$. Obviously, $q^* \in \mathbb{Z}^n$, $q^* \neq p^*$, and $q^*_h = p^*_h$ whenever $p^*_h$ is an integer. Let $\delta^* = p^* - q^*$. Clearly, $\delta^*_h = 0$ whenever $p^*_h$ is an integer. Then, $\delta^* \in \square$ (defined before Lemma 3.6), $\delta^* \notin \mathbb{Z}^n$, and $0 < \|\delta^*\|_\infty < 1$. Define $\bar{\lambda} = 1/\|\delta^*\|_\infty > 1$. By Lemma 3.6[15] we know that $\mathcal{L}(q^* + \lambda\delta^*)$ is linear in $\lambda$ on the interval $[0, \bar{\lambda}]$. Recall that $p^*$ is a minimizer of the Lyapunov function $\mathcal{L}$. Thus, if $q^* \notin \Lambda$, i.e., $\mathcal{L}(q^*) > \mathcal{L}(p^*) = \mathcal{L}(q^* + \delta^*)$, then $\mathcal{L}(p^*) > \mathcal{L}(q^* + \bar{\lambda}\delta^*)$, yielding a contradiction. We now consider the case where $q^* \in \Lambda$, i.e., $\mathcal{L}(q^*) = \mathcal{L}(p^*) = \mathcal{L}(q^* + \delta^*)$. Then, it follows from the linearity of $\mathcal{L}$ in $\lambda$ that $\mathcal{L}(p^*) = \mathcal{L}(q^* + \bar{\lambda}\delta^*)$. That is, $q^* + \bar{\lambda}\delta^* \in \Lambda$. By the construction of $\bar{\lambda}$, $q^* + \bar{\lambda}\delta^*$ has more integral coordinates than $p^*$. Therefore, by the choice of $p^*$, we see that $q^* + \bar{\lambda}\delta^* \in \Lambda \setminus A$. That is, $q^* + \bar{\lambda}\delta^* \in \mathrm{co}(\Lambda \cap N(q^* + \bar{\lambda}\delta^*))$. Moreover, observe that $N(q^* + \bar{\lambda}\delta^*) \subseteq N(p^*)$, $q^* \in \mathrm{co}(\Lambda \cap N(p^*))$, and $p^*$ is a convex combination of $q^*$ and $q^* + \bar{\lambda}\delta^*$. As a result, we have $p^* \in \mathrm{co}(\Lambda \cap N(p^*))$, contradicting the hypothesis that $p^* \in A$.

Finally, by definition, we know that every vertex of an integrally convex set is an integral vector and thus every vertex of $\Lambda$ must be integral as well. So all the vertices of $\Lambda$, including the generalized smallest and largest equilibrium price vectors $\underline{p}$ and $\bar{p}$, are integral vectors.

[15]Note tha Lemma 3.6 and its proof are independent of the current theorem and its proof.

Furthermore, since the set $\Lambda$ is bounded, $\Lambda$ has a finite number of vertices. Clearly, $\Lambda$ is an integral polyhedron. $\square$

We extend and modify the arguments of Propositions 2 and 5 of Ausubel (2006) under the GS condition to prove the following two lemmas under the GSC condition.

**Proof of Lemma 3.6** Assume by way of contradiction that there exists $\lambda > 0$ such that $0 \leq \lambda\delta_k \leq 1$ for any $\beta_k \in S_1$ and $-1 \leq \lambda\delta_l \leq 0$ for any $\beta_l \in S_2$ but $\tilde{S}^i \notin D^i(p + \lambda\delta)$. By the GSI property, for $\tilde{S}^i$ there exists a GSI improvement bundle $A$ with $v^i(A, p+\lambda\delta) > v^i(\tilde{S}^i, p + \lambda\delta)$. By the construction of $\tilde{S}^i$, we see that $v^i(\tilde{S}^i, p + \lambda\delta) \geq v^i(C, p + \lambda\delta)$ for all $C \in D^i(p)$, and hence $A \notin D^i(p)$. Then it follows from Assumption (A1) and $p \in \mathbb{Z}^n$ that

$$v^i(A, p) \leq v^i(\tilde{S}^i, p) - 1.$$

On the other hand, since $0 \leq \lambda\delta_k \leq 1$ for any $\beta_k \in S_1$ and $-1 \leq \lambda\delta_l \leq 0$ for any $\beta_l \in S_2$, and $A$ is a GSI improvement bundle of $\tilde{S}^i$, we must have $|\sum_{\beta_h \in \tilde{S}^i} \lambda\delta_h - \sum_{\beta_h \in A} \lambda\delta_h| \leq 1$ and thus

$$\sum_{\beta_h \in \tilde{S}^i} \lambda\delta_h - 1 \leq \sum_{\beta_h \in A} \lambda\delta_h.$$

The two inequalities imply that $v^i(A, p + \lambda\delta) \leq v^i(\tilde{S}^i, p + \lambda\delta)$, yielding a contradiction.

We now prove that the Lyapunov function $\mathcal{L}(p + \lambda\delta)$ is linear in $\lambda$ for any $\lambda > 0$ such that $0 \leq \lambda\delta_k \leq 1$ for any $\beta_k \in S_1$ and $-1 \leq \lambda\delta_l \leq 0$ for any $\beta_l \in S_2$. By the first part of the lemma, for any such $\lambda$ and any bidder $i \in I$ we know $\tilde{S}^i \in D^i(p + \lambda\delta)$, which immediately yields

$$\mathcal{L}(p + \lambda\delta) = \mathcal{L}(p) + \lambda(\sum_{\beta_h \in N} \delta_h - \sum_{i \in I} \sum_{\beta_h \in \tilde{S}^i} \delta_h).$$

This shows that $\mathcal{L}(p + \lambda\delta)$ is indeed linear in $\lambda$ on the interval. $\square$

**Proof of Lemma 3.8** (i) Suppose to the contrary that in the DDT auction process there exists a price vector $p(t)$ such that $p(t) \leq_g \underline{p}$ but $p(t + 1) \not\leq_g \underline{p}$. Then, we have $p(t) \wedge_g \underline{p} = p(t)$ but

$$p(t) \leq_g (p(t + 1) \wedge_g \underline{p}) \leq_g p(t + 1) \quad \text{and} \quad (p(t + 1) \wedge_g \underline{p}) \neq p(t + 1) \qquad (*)$$

On the other hand, recall from Lemma 3.4 that, since $\underline{p}$ is the smallest equilibrium price vector in the order of $\leq_g$, it minimizes $\mathcal{L}(\cdot)$ and so $\mathcal{L}(\underline{p}) \leq \mathcal{L}(p(t + 1) \vee_g \underline{p})$. Since $\mathcal{L}(\cdot)$ is a generalized submodular function by Theorem 3.5 (i), we have $\mathcal{L}(p(t + 1) \vee_g \underline{p}) + \mathcal{L}(p(t + 1) \wedge_g \underline{p}) \leq \mathcal{L}(p(t + 1)) + \mathcal{L}(\underline{p})$. Adding the previous inequalities leads to $\mathcal{L}(p(t + 1) \wedge_g \underline{p}) \leq \mathcal{L}(p(t+1))$. By the construction of $p(t+1)$, this implies that $\mathcal{L}(p(t+1) \wedge_g \underline{p}) = \mathcal{L}(p(t+1))$ and so $p(t + 1) \leq_g (p(t + 1) \wedge_g \underline{p})$, contradicting inequality $(*)$.

(ii) Suppose to the contrary that there exists a price vector $p(t)$ such that $p(t+1) = p(t)$ but $p(t) \not\geq_g \underline{p}$. Then $p(t) \wedge_g \underline{p}$ is less than $\underline{p}$ in at least one component in the order of $\leq_g$. Since $\underline{p}$ is the smallest equilibrium price vector in the order of $\leq_g$, we know that $p(t) \wedge_g \underline{p}$ is not an equilibrium price vector of the market model. Applying Lemma 3.4, this implies that $\mathcal{L}(\underline{p}) < \mathcal{L}(p(t) \wedge_g \underline{p})$. Since $\mathcal{L}(\cdot)$ is a generalized submodular function, we also have that $\mathcal{L}(p(t) \vee_g \underline{p}) + \mathcal{L}(p(t) \wedge_g \underline{p}) \leq \mathcal{L}(p(t)) + \mathcal{L}(\underline{p})$. Adding the previous inequalities implies that $\mathcal{L}(p(t) \vee_g \underline{p}) < \mathcal{L}(p(t))$. Since $(p(t) \vee_g \underline{p}) \geq_g p(t)$ and $(p(t) \vee_g \underline{p}) \neq p(t)$, there exists $p'$, a strict convex combination of $p(t)$ and $p(t) \vee_g \underline{p}$, such that $p' \in p(t) + \square$ and $\mathcal{L}(p') < \mathcal{L}(p(t))$ due to the convexity of $\mathcal{L}(\cdot)$ by Theorem 3.5 (i) and the previous strict inequality. By Lemma 3.7, we know that $\mathcal{L}(p(t) + \delta(t)) < \mathcal{L}(p(t))$, and hence $p(t+1) \neq p(t)$, contradicting the hypothesis. $\square$

**Proof of Theorem 3.10** By Theorem 3.5 (ii) the market has a Walrasian equilibrium and by Lemma 3.4 the Lyapunov function $\mathcal{L}(\cdot)$ attains its mimimum value at any equilibrium price vector and is bounded from below. Since the prices and utility functions take only integer values, the Lyapunov function is an integer valued function and it lowers by a positive integer value in each round of the GDDT auction. This guarantees that the auction terminates in finitely many rounds, i.e., $\delta(t^*) = 0$ in Step 3 for some $t^* \in \mathbb{Z}_+$.

Let $p(0), p(1), \cdots, p(t^*)$ be the generated finite sequence of price vectors. Let $\bar{t} \in \mathbb{Z}_+$ be the time when the GDDT auction finds $\delta(\bar{t}) = 0$ at Step 2. We claim that $\mathcal{L}(p) \geq \mathcal{L}(p(\bar{t}))$ for all $p \geq_g p(\bar{t})$. Suppose to the contrary that there exists some $p \geq_g p(\bar{t})$ such that $\mathcal{L}(p) < \mathcal{L}(p(\bar{t}))$. By the convexity of $\mathcal{L}(\cdot)$ via Theorem 3.5 (i), there is a strict convex combination $p'$ of $p$ and $p(\bar{t})$ such that $p' \in p(\bar{t}) + \square$ and $\mathcal{L}(p') < \mathcal{L}(p(\bar{t}))$. By Lemma 3.7 we know that $\mathcal{L}(p(\bar{t}) + \delta(\bar{t})) < \mathcal{L}(p(\bar{t}))$, and so $\delta(\bar{t}) \neq 0$ in Step 2 of the GDDT auction, yielding a contradiction. Therefore, we have $\mathcal{L}(p \vee_g p(\bar{t})) \geq \mathcal{L}(p(\bar{t}))$ for all $p \in \mathbb{R}^n$, because $p \vee_g p(\bar{t}) \geq_g p(\bar{t})$ for all $p \in \mathbb{R}^n$. We will further show that $\mathcal{L}(p \vee_g p(t)) \geq \mathcal{L}(p(t))$ for all $t = \bar{t}+1, \bar{t}+2, \cdots, t^*$ and $p \in \mathbb{R}^n$. By induction, it sufficies to prove the case of $t = \bar{t}+1$. Notice that $p(\bar{t}+1) = p(\bar{t}) + \delta(\bar{t})$, where $\delta(\bar{t}) \in \Delta^*$ is determined in Step 3 of the GDDT auction. Assume by way of contradiction that there is some $p \in \mathbb{R}^n$ such that $\mathcal{L}(p \vee_g p(\bar{t}+1)) < \mathcal{L}(p(\bar{t}+1))$. Then if we start the GDDT auction from $p(\bar{t}+1)$, we can by the same previous argument find a $\delta(\neq 0) \in \Delta$ in Step 2 such that $\mathcal{L}(p(\bar{t}+1) + \delta) < \mathcal{L}(p(\bar{t}+1))$. Since $\mathcal{L}(\cdot)$ is a generalized submodular function, we have $\mathcal{L}(p(\bar{t}) \vee_g (p(\bar{t}+1) + \delta)) + \mathcal{L}(p(\bar{t}) \wedge_g (p(\bar{t}+1) + \delta)) \leq \mathcal{L}(p(\bar{t}) + \mathcal{L}(p(\bar{t}+1) + \delta)$. Recall that $\mathcal{L}(p(\bar{t}) \vee_g (p(\bar{t}+1)+\delta)) \geq \mathcal{L}(p(\bar{t}))$. It follows that $\mathcal{L}(p(\bar{t}) \wedge_g (p(\bar{t}+1)+\delta)) \leq \mathcal{L}(p(\bar{t}+1)+\delta) < \mathcal{L}(p(\bar{t}+1))$. Observe that $\delta' = 0 \wedge_g (\delta(\bar{t}) + \delta) \in \Delta^*$ and $p(\bar{t}) \wedge_g (p(\bar{t}+1) + \delta) = p(\bar{t}) + \delta'$. This yields $\mathcal{L}(p(\bar{t}) + \delta') < \mathcal{L}(p(\bar{t}) + \delta(\bar{t}))$ and so $\delta' \neq \delta(\bar{t})$, contradicting the definition of $\delta(\bar{t}) \in \Delta^*$ by which $\mathcal{L}(p(\bar{t}) + \delta(\bar{t})) = \min_{\delta \in \Delta^*} \mathcal{L}(p(\bar{t}) + \delta)$.

By the symmetry between Step 2 and Step 3, as above we can also show that $\mathcal{L}(p \wedge_g$

$p(t^*)) \geq \mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$. We proved above that $\mathcal{L}(p \vee_g p(t^*)) \geq \mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$. Since $\mathcal{L}(\cdot)$ is a generalized submodular function, we have $\mathcal{L}(p) + \mathcal{L}(p(t^*)) \geq \mathcal{L}(p \vee_g p(t^*)) + \mathcal{L}(p \wedge_g p(t^*)) \geq 2\mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$. This shows that $\mathcal{L}(p(t^*)) \leq \mathcal{L}(p)$ holds for all $p \in \mathbb{R}^n$ and by Lemma 3.4, $p(t^*)$ is an equilibrium price vector. □

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