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"Coextrema Additive Operators"

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Coextrema Additive Operators[*]

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Abstract

This paper proposes a class of weak additivity concepts for an operator on the set of real valued functions on a finite state space Ω, which include additivity and comonotonic additivity as extreme cases. Let $\mathcal{E} \subseteq 2^{\Omega}$ be a collection of subsets of Ω. Two functions x and y on Ω are $\mathcal{E}$-coextrema if, for each $E \in \mathcal{E}$, the set of minimizers of x restricted on E and that of y have a common element, and the set of maximizers of x restricted on E and that of y have a common element as well. An operator I on the set of functions on Ω is $\mathcal{E}$-coextrema additive if $I(x+y) = I(x)+I(y)$ whenever x and y are $\mathcal{E}$-coextrema. The main result characterizes homogeneous $\mathcal{E}$-coextrema additive operators.

JEL classification: C71, D81, D90.

Keywords: Choquet integral; comonotonicity; non-additive probabilities; capacities.

1 Introduction

The purpose of this paper is to characterize operators on the set of real valued functions on a finite set which is *coextrema additive*: let Ω be a finite set and let $\mathcal{E} \subseteq 2^{\Omega}$ be a collection of subsets of Ω. Two functions x and y on Ω are said to be $\mathcal{E}$-coextrema if, for each $E \in \mathcal{E}$, the set of minimizers of function x restricted on E and that of function y have a common element, and the set of maximizers of x restricted on E and that of y have a common element as well. An operator I on the set of functions on Ω is $\mathcal{E}$-coextrema additive if $I(x + y) = I(x) + I(y)$ whenever x and y are $\mathcal{E}$-coextrema. Note that if two functions are comonotonic, then they are $\mathcal{E}$-extrema, *a fortiori.*

The main result shows that a homogeneous coextrema additive operator I can be represented as $I(x) = \sum_{E \in \mathcal{E}} \{\lambda_E \max_{\omega \in E} x(\omega) + \mu_E \min_{\omega \in E} x(\omega)\}$, where λ_E and μ_E are unique constants,

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when the collection $\mathcal{E}$ satisfies a certain regularity condition. This expression can also be written as the Choquet integral with respect to a certain non-additive (signed) measure. Therefore, a homogeneous coextrema additive operator corresponds to a special class of the Choquet integral, which is expressed as a weighted sum of "optimistic evaluation" $\max_{\omega \in E} x(\omega)$ and "pessimistic evaluation" $\min_{\omega \in E} x(\omega)$. For the case where $I(1) = 1$, we have $\sum_{E \in \mathcal{E}} (\lambda_E + \mu_E) = 1$, and then these weights can be interpreted as beliefs on events in $E \in \mathcal{E}$ if these are non-negative numbers.

As a corollary, our result shows that for the special case where $\mathcal{E}$ consists of singletons and the whole set Ω, a homogeneous $\mathcal{E}$-coextrema operator is exactly the Choquet integral of a NEO-additive capacity, which is axiomatized by Chateaunuff, Eichberger, and Grant (2002). Thus, our result provides a natural, and important generalization of the NEO-additive capacity result. Eichberger, Kelsey, and Schipper (2006) applied a NEO-additive capacity model to the Bertrand and Cournot competition models to study combined effects of optimism and pessimism in economic environments.

While in the NEO-additive capacity, optimism and pessimism are about the whole states of the world, our model can accommodate more delicate combinations of optimism and pessimism measured in a family of events. Thus our $\mathcal{E}$-coextrema additivity model provides a rich framework for analyzing effects optimism and pessimism in economic problems.

Kajii, Kojima, and Ui (2007) considered the class of cominimum additive operators, and each cominimum additive operator is shown to be a weighted sum of minimums. The class of comaximum operators is defined and characterized similarly. However, the class of coextrema additive operators is *not* the intersection of the two, and the characterization result reported in this paper cannot be done by adopting these results. In fact, the reader will see that the issue of characterization is far more technically involved.

Ghirardato, Maccheroni, and Marinacci (2004) axiomatized the following class of operators called the α-MEU functional: $I(x) = \alpha \min_{q \in C} \int x dq + (1 - \alpha) \max_{q \in C} \int x dq$ where C is a convex set of additive measures. It can be readily verified that the NEO-additive capacity model is a special class of the α-MEU functional, and so $\mathcal{E}$-coextrema additive operators are also α-MEU functionals, when $\mathcal{E}$ consists of singletons and the whole set Ω. But for general $\mathcal{E}$, there is no direct connection as far as we can tell.

The organization of this paper is as follows. After a summary of basic concepts and preliminary results in Section 2, a formal definition of the coextrema operator is given in Section 3. Section 3 also contains some discussions on the operator, including potential applications to economics and social sciences. The main result is stated in Section 4, and a proof is provided in Section 5.

2 The model and preliminary results

Let Ω be a finite set, whose generic element is denoted by ω. Denote by $\mathcal{F}$ the collection of all non-empty subsets of Ω, and by $\mathcal{F}_1$ the collection of singleton subsets of Ω. A typical interpretation is that Ω is the set of the states of the world and a subset $E \subseteq \Omega$ is an event.

We shall fix a collection $\mathcal{E} \subseteq \mathcal{F}$, $\mathcal{E} \neq \emptyset$, throughout the analysis. Write $\sigma(\mathcal{E})$ for the algebra of Ω generated by $\mathcal{E}$, i.e., the smallest σ-algebra containing each element of $\mathcal{E}$. Let $\Pi(\mathcal{E}) \subseteq \mathcal{F}$ be the collection of minimal elements of $\sigma(\mathcal{E})$, which constitutes a well defined *partition* of Ω, since Ω is a finite set. A generic element of partition $\Pi(\mathcal{E})$ will be denoted by S. For each $F \in \mathcal{F}$, let $\kappa(F) \in \sigma(\mathcal{E})$ denote the minimal $\sigma(\mathcal{E})$-measurable set containing F; that is, $\kappa(F) := \cap\{E \in \sigma(\mathcal{E}) : F \subseteq E\}$.

Remark 2.1 Note that every element of $\Pi(E)$ belongs to $\sigma(\mathcal{E})$ and that any element $E \in \sigma(\mathcal{E})$ is the union of some elements of $\Pi(E)$. So in particular, for every $E \in E$ and every $S \in \Pi(E)$, either $S \subseteq E$ or $S \subseteq E^c$ holds. By construction, $\kappa(F) = \cup\{S \in \Pi(\mathcal{E}) : S \cap F \neq \emptyset\}$, i.e., $\kappa(F)$ is the union of elements in partition $\Pi(\mathcal{E})$ intersecting F. It is readily verified that if $E \in \sigma(E)$, then $\kappa(E \cap F) = E \cap \kappa(F)$ holds for any $F \in \mathcal{F}$, and so in particular $\kappa(E) = E$.

Example 2.1 Let $\Omega = \{1, 2, \cdots, 8\}$ and $\mathcal{E} = \{E_1, E_2, E_3, E_4\}$ where $E_1 = \{1, 2, 3, 4\}$, $E_2 = \{3, 4, 5, 6\}$, $E_3 = \{1, 2, 5, 6\}$, $E_4 = \{5, 6, 7, 8\}$. Then, $\Pi(\mathcal{E}) = \{S_1, \ldots, S_4\}$, where $S_1 = \{1, 2\}$, $S_2 = \{3, 4\}$, $S_3 = \{5, 6\}$, $S_4 = \{7, 8\}$. In this case, $E_1 = S_1 \cup S_2$, $E_2 = S_2 \cup S_3$, $E_3 = S_1 \cup S_3$, $E_1 = S_3 \cup S_4$. For instance, for $R = \{1, 3, 5, 7\}$, we have $\kappa(R) = \Omega$, because every $S \in \Pi(\mathcal{E})$ intersects R.

A set function $v : 2^\Omega \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *game* or a *non-additive signed measure*. Since each game is identified with a point in $\mathbb{R}^\mathcal{F}$, we denote by $\mathbb{R}^\mathcal{F}$ the set of all games. For a game $v \in \mathbb{R}^\mathcal{F}$, we use the following definitions:

- v is *non-negative* if $v(E) \geq 0$ for all $E \in 2^\Omega$.
- v is *monotone* if $E \subseteq F$ implies $v(E) \leq v(F)$ for all $E, F \in 2^\Omega$. A monotone game is non-negative.
- v is *additive* if $v(E \cup F) = v(E) + v(F)$ for for all $E, F \in 2^\Omega$ with $E \cap F = \emptyset$, which is equivalent to $v(E) + v(F) = v(E \cup F) + v(E \cap F)$ for all $E, F \in 2^\Omega$.
- v is *convex* (or *supermodular*) if $v(E) + v(F) \leq v(E \cup F) + v(E \cap F)$ for all $E, F \in 2^\Omega$.
- v is *normalized* if $v(\Omega) = 1$.
- v is a *non-additive measure* if it is monotone. A normalized non-additive measure is called a *capacity*.

- v is a *measure* if it is non-negative and additive. A normalized measure is called a *probability measure*.
- The *conjugate* of v, denoted by v', is defined as $v'(E) = v(\Omega) - v(\Omega \backslash E)$ for all $E \in 2^{\Omega}$. Note that $(v')' = v$ and $(v + w)' = v' + w'$ for $v, w \in \mathbb{R}^{\mathcal{F}}$.

For $T \in \mathcal{F}$, let $u_T \in \mathbb{R}^{\mathcal{F}}$ be the *unanimity game* on T defined by the rule: $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. Let w_T be the conjugate of u_T. Then $w_T(S) = 1$ if $T \cap S \neq \emptyset$ and $w_T(S) = 0$ otherwise. Note that when $T = \{\omega\}$, i.e., T is a singleton set, $u_T = w_T$ and they are additive. The following result is well known as the Möbius inversion in discrete and combinatorial mathematics (cf. Shapley, 1953).

Lemma 2.1 *The collection $\{u_T\}_{T \in \mathcal{F}}$ is a linear base for $\mathbb{R}^{\mathcal{F}}$, so is the collection $\{w_T\}_{T \in \mathcal{F}}$. The unique collection of coefficients $\{\beta_T\}_{T \in \mathcal{F}}$ satisfying $v = \sum_{T \in \mathcal{F}} \beta_T u_T$ is given by $\beta_T = \sum_{E \subseteq T, E \neq \emptyset} (-1)^{|T|-|E|} v(E)$.*

By convention, we shall omit the empty set in the summation indexed by subsets of Ω. By the definition of u_T, we have $v(E) = \sum_{T \subseteq E} \beta_T$ for all $E \in \mathcal{F}$. The collection of coefficients $\{\beta_T\}_{T \in \mathcal{F}}$ is referred to as the Möbius transform of v. If $v = \sum_{T \in \mathcal{F}} \beta_T u_T$, then the conjugate v' is given by $v' = \sum_{T \in \mathcal{F}} \beta_T w_T$. Using the formula in Lemma 2.1, by direct computation, one can show that for each $E \in \mathcal{F}$:

$$w_E = \sum_{T \subseteq E} (-1)^{|T|-1} u_T. \tag{1}$$

Remark 2.2 If $v = \sum_{T \in \mathcal{F}} \beta_T u_T$, the game v is additive if and only if $\beta_T = 0$ unless $|T| = 1$. Obviously, $\sum_{\omega \in \Omega} \beta_{\{\omega\}} u_{\{\omega\}}$ is an additive game. So, we can also write $v = p + \sum_{T \in \mathcal{F}, |T| > 1} \beta_T u_T$ where p is an additive game.

By convention, a function $x : \Omega \to \mathbb{R}$ is identified with an element of $\mathbb{R}^{\Omega}$, and we denote by 1_E the indicator function of event $E \in \mathcal{F}$. For a function $x \in \mathbb{R}^{\Omega}$, and an event E, we write $\min_E x := \min_{\omega \in E} x(\omega)$ and $\arg\min_E x := \arg\min_{\omega \in E} x(\omega)$. Similarly, we write $\max_E x := \max_{\omega \in E} x(\omega)$ and $\arg\max_E x := \arg\max_{\omega \in E} x(\omega)$.

Definition 2.1 For $x \in \mathbb{R}^{\Omega}$ and $v \in \mathbb{R}^{\mathcal{F}}$, the *Choquet integral* of x with respect to v is defined as

$$\int x dv = \int_{\underline{x}}^{\bar{x}} v(x \geq \alpha) d\alpha + \underline{x} v(\Omega), \tag{2}$$

where $\bar{x} = \max_{\Omega} x$, $\underline{x} = \min_{\Omega} x$, and $v(x \geq \alpha) = v(\{\omega \in \Omega : x(\omega) \geq \alpha\})$.

By definition, $\int 1_E dv = v(E)$. A direct computation reveals that, for any two sets E and F in $\mathcal{F}$,

$$\int (1_E + 1_F) \, dv = v(E \cup F) + v(E \cap F). \tag{3}$$

Then for each event T, we see from (2) that $\int xdu_T = \min_T x$ and $\int xdw_T = \max_T x$. Also it can be readily verified that the Choquet integral is additive in games. Recall that for a game v, there is a unique set of coefficients $\{\beta_T : T \in \mathcal{F}\}$ such that $v = \sum_T \beta_T u_T$ by Lemma 2.1. Using additivity, therefore, we have $\int xdv = \sum_T \beta_T \min_T x$, as is pointed out in Gilboa and Schmeidler (1994).

Note that the additivity implies the following property: for any $T \in \mathcal{F}$ and real numbers λ and μ, $\int xd\left(\lambda w_T + \mu u_T\right) = \int xd\left(\lambda w_T\right) + \int xd\left(\mu u_T\right) = \lambda \max_T x + \mu \min_T x$, and so

$$\int xd\left(\sum_{E\in\mathcal{F}'} \lambda_E w_E + \mu_E u_E\right) = \sum_{E\in\mathcal{F}'} \{\lambda_E \max_E x + \mu_E \min_E x\}, \tag{4}$$

for any collection of events $\mathcal{F}' \subseteq \mathcal{F}$ and collections of real numbers $\{\lambda_E : E \in \mathcal{F}'\}$ and $\{\mu_E : E \in \mathcal{F}'\}$.

Definition 2.2 Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. Two functions $x, y \in \mathbb{R}^\Omega$ are said to be $\mathcal{E}$*-cominimum*, provided $\arg\min_E x \cap \arg\min_E y \neq \emptyset$ for all $E \in \mathcal{E}$. Two functions $x, y \in \mathbb{R}^\Omega$ are said to be $\mathcal{E}$*-comaximum*, provided $\arg\max_E x \cap \arg\max_E y \neq \emptyset$ for all $E \in \mathcal{E}$.

Remark 2.3 Clearly, x and y are $\mathcal{E}$-cominimum, if and only if $-x$ and $-y$ are $\mathcal{E}$-comaximum. Also, the $\mathcal{E}$-cominimum and the $\mathcal{E}$-comaximum relations are invariant of adding a constant. In particular, if two indicator functions 1_A and 1_B are $\mathcal{E}$-cominimum, $1_{\Omega\backslash A}$ $(=1 - 1_A)$ and $1_{\Omega\backslash B}$ $(=1 - 1_B)$ are $\mathcal{E}$-comaximum, and vice versa.

A function $I : \mathbb{R}^\Omega \to \mathbb{R}$ is referred to as an operator.

Definition 2.3 An operator I is said to be homogeneous if $\mathrm{I}(\alpha x) = \alpha I(x)$ for any $\alpha > 0$.

Kajii, Kojima, and Ui (2007) studied $\mathcal{E}$-cominimum and $\mathcal{E}$-comaximum operators defined as follows:

Definition 2.4 An operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ is $\mathcal{E}$-cominimum (resp. comaximum) additive provided $I(x + y) = I(x) + I(y)$ whenever x and y are $\mathcal{E}$-cominimum (resp. comaximum).

A pair of functions x and y are said to be *comonotonic* if $\left(x(\omega) - x(\omega')\right)\left(y(\omega) - y(\omega')\right) \geq 0$ for any $\omega, \omega' \in \Omega$. Notice that if $\mathcal{E} = \mathcal{F}$, a pair of functions x and y are comonotonic if and only if they are $\mathcal{E}$-cominimum, as well as $\mathcal{E}$-comaximum. So when $\mathcal{E} = \mathcal{F}$, the $\mathcal{E}$-cominimum additivity, as well as the $\mathcal{E}$-comaximum additivity, is equivalent to the comonotonic additivity which Schmeidler (1986) characterized. Then in general both the $\mathcal{E}$-cominimum and the $\mathcal{E}$-comaximum additivity imply the comonotonic additivity. Therefore, the following can be obtained from Schmeidler's theorem in a straightforward manner.[1]

[1] Schmeidler (1986) assumes monotonicity instead of homogeneity of the operator, but the method of his proof can be adopted for this result with little modification.

Theorem 2.1 *If an operator* $I : \mathbb{R}_{+}^{\Omega} \to \mathbb{R}$ *is homogenous and satisfies* $\mathcal{E}$*-cominimum additivity (or* $\mathcal{E}$*-comaximum additivity), then there exists a unique game* $v \in \mathbb{R}^{\mathcal{F}}$ *such that* $I(x) = \int xdv$ *for all* $x \in \mathbb{R}^{\Omega}$*. Moreover, game* v *is defined by the rule* $v(E) = I(1_E)$*.*

We say that a game v is $\mathcal{E}$-cominimum additive (resp. $\mathcal{E}$-comaximum additive) if the operator $I(x) := \int xdv$ is $\mathcal{E}$-cominimum additive (resp. $\mathcal{E}$-comaximum additive). Since $\mathcal{E}$-cominimum additivity as well as $\mathcal{E}$-comaximum additivity implies comonotonic additivity, Theorem 2.1 assures that this is a consistent terminology.

Obviously, the properties of $\mathcal{E}$-cominimum additive or $\mathcal{E}$-comaximum additive operators depend on the structure of the family $\mathcal{E}$.

Definition 2.5 Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. An event $T \in \mathcal{F}$ is $\mathcal{E}$-complete provided, for any two distinct points ω_1 and ω_2 in T, there is $E \in \mathcal{E}$ such that $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. The collection of all $\mathcal{E}$-complete events is called the $\mathcal{E}$-complete collection and denoted by $\Upsilon(\mathcal{E})$. A collection $\mathcal{E}$ is said to be complete if $\mathcal{E} = \Upsilon(\mathcal{E})$.

Note that a singleton set is automatically $\mathcal{E}$-complete, so is any $E \in \mathcal{E}$. For each T, consider the graph where the set of vertices is T and the set of edges consists of the pairs of vertices $\{\omega_1, \omega_2\}$ with $\{\omega_1, \omega_2\} \subseteq E \subseteq T$ for some $E \in \mathcal{E}$. This graph is a complete graph if and only if T is $\mathcal{E}$ -complete.

Remark 2.4 For $E, E' \in \mathcal{E}$, $E \cup E'$ is not necessarily $\mathcal{E}$-complete. However, by definition, for any $T \in \Upsilon(\mathcal{E})$ with $|T| > 1$, T coincides with the union of sets in $\mathcal{E}$ which are included in T, thus T is the union of (partition) elements in $\Pi(\mathcal{E})$ which are included in T. In particular, T must contain at least one element of $\Pi(\mathcal{E})$.

It can be shown that for any $\mathcal{E} \subseteq \mathcal{F}$, $\Upsilon(\mathcal{E})$ is complete, i.e., $\Upsilon(\mathcal{E}) = \Upsilon(\Upsilon(\mathcal{E}))$. See Kajii, Kojima, and Ui (2007) for further discussions on this concept, as well as for the proofs of the results shown in the rest of this section.

Example 2.2 In Example 2.1, $S = S_1 \cup S_2 \cup S_3 = \{1, 2, 3, 4, 5, 6\}$ is $\mathcal{E}$-complete, but $S_2 \cup S_3 \cup S_4 = \{3, 4, 5, 6, 7, 8\}$ is not $\mathcal{E}$-complete since there is no $E \in \mathcal{E}$ with $\{3, 7\} \subseteq E \subseteq S_2 \cup S_3 \cup S_4$.

The completeness plays a crucial role in our analysis, as is indicated in the next result:

Lemma 2.2 *Two functions* x *and* y *are* $\mathcal{E}$*-cominimum (resp.* $\mathcal{E}$*-comaximum) if and only if they are* $\Upsilon(\mathcal{E})$*-cominimum (resp.* $\Upsilon(\mathcal{E})$*-comaximum).*

The idea of "cominimum" can be stated in terms of sets by looking at the indicator functions. Say that a pair of sets A and B is an $\mathcal{E}$-decomposition pair if for any $E \in \mathcal{E}$, $E \subseteq A \cup B$ implies that $E \subseteq A$ or $E \subseteq B$ or both. Then the following can be shown:

Lemma 2.3 *Two indicator functions 1_A and 1_B are $\mathcal{E}$-cominimum if and only if the pair of sets A and B constitutes an $\mathcal{E}$-decomposition pair.*

Remark 2.5 From Lemma 2.3 and Remark 2.3, we see that two indicator functions 1_A and 1_B are $\mathcal{E}$-comaximum if and only if for any $E \in \mathcal{E}$, $E \subseteq \Omega \backslash (A \cap B)$ implies that $E \subseteq \Omega \backslash A$ or $E \subseteq \Omega \backslash B$ or both.

Finally, a characterization of cominimum additive and comaximum additive operators is given below.

Theorem 2.2 *Let $v \in \mathbb{R}^{\mathcal{F}}$ be a game, and let $I(x) = \int x dv$. Write $v = \sum_{T \in \mathcal{F}} \beta_T u_T = \sum_{T \in \mathcal{F}} \eta_T w_T$. Then,*
(1) the following three statements are equivalent: (i) *operator I is $\mathcal{E}$-cominimum additive;* (ii) $v(A) + v(B) = v(A \cup B) + v(A \cap B)$ *for any $\mathcal{E}$-decomposition pair A and B;* (iii) $\beta_T = 0$ *for any $T \notin \Upsilon(\mathcal{E})$, and*
(2) the following three statements are equivalent: (i) *operator I is $\mathcal{E}$-comaximum additive;* (ii)*;* $v(A^c) + v(B^c) = v(A^c \cup B^c) + v(A^c \cap B^c)$ *for any $\mathcal{E}$-decomposition pair A and B;* (iii) $\eta_T = 0$ *for any $T \notin \Upsilon(\mathcal{E})$.*

3 Coextrema additive operators

In this paper we study pairs of functions which share *both* a minimizer and a maximizer for events in a given collection $\mathcal{E}$, which is fixed throughout.

Definition 3.1 Two functions $x, y \in \mathbb{R}^{\Omega}$ are said to be $\mathcal{E}$-*coextrema*, provided they are both $\mathcal{E}$-*cominimum and $\mathcal{E}$-comaximum;* that is, $\arg\min_E x \cap \arg\min_E y \neq \emptyset$ and $\arg\max_E x \cap \arg\max_E y \neq \emptyset$ for all $E \in \mathcal{E}$.

Analogous to the cases of cominimum and comaximum functions, the notion of $\mathcal{E}$-coextrema functions induces the following additivity property of an operator $I : \mathbb{R}^{\Omega} \to \mathbb{R}$.

Definition 3.2 An operator $I : \mathbb{R}^{\Omega} \to \mathbb{R}$ is $\mathcal{E}$-coextrema additive provided $I(x+y) = I(x)+I(y)$ whenever x and y are $\mathcal{E}$-coextrema.

The completion $\Upsilon(\mathcal{E})$ plays an important role here again: the following is an immediate consequence of the definition and Lemma 2.2.

Lemma 3.1 Two functions x and y are $\mathcal{E}$-coextrema if and only if they are $\Upsilon(\mathcal{E})$-coextrema.

By definition, the $\mathcal{E}$-coextrema additivity implies the comonotonic additivity. So by Theorem 2.1, we obtain the following result.

Lemma 3.2 If an operator $I : \mathbb{R}^\Omega \to R$ is homogeneous and $\mathcal{E}$-coextrema additive for some $\mathcal{E} \subseteq \mathcal{F}$, then there exists a unique game v such that $I(x) = \int xdv$ for any $x \in \mathbb{R}^\Omega$. Moreover, v is defined by the rule $v(E) = I(1_E)$.

Thus the following definition is justified:

Definition 3.3 A game v is said to be $\mathcal{E}$-coextrema additive provided $\int(x+y)dv = \int xdv + \int ydv$ whenever x and y are $\mathcal{E}$-coextrema.

Our goal is to establish that a game v is $\mathcal{E}$-coextrema additive if and only if v can be expressed in the form

$$v = \sum_{E \in \Upsilon(\mathcal{E})} \{\lambda_E w_E + \mu_E u_E\}. \tag{5}$$

Note that from (4), this is equivalent to say that the original operator I can be written as

$$I(x) = \sum_{E \in \Upsilon(\mathcal{E})} \{\lambda_E \max_E x(\omega) + \mu_E \min_E x(\omega)\}. \tag{6}$$

In addition, if $\mathcal{E}$ is complete, i.e., $\mathcal{E} = \Upsilon(\mathcal{E})$, we have the expression written in Introduction.

Remark 3.1 Note that by definition $u_{\{\omega\}} = w_{\{\omega\}}$, and they are the probability measure δ_ω which assigns probability one to $\{\omega\}$. Since $\Upsilon(E)$ contains all the singleton subsets of Ω, the (5) has a trivial redundancy for E with $|E| = 1$. Taking this into account, (5) can be written as:

$$v = p + \sum_{E \in \Upsilon(\mathcal{E}) \setminus \mathcal{F}_1} \{\lambda_E w_E + \mu_E u_E\}, \tag{7}$$

where p is an additive measure given by $p := \sum_{\omega \in \Omega}(\lambda_{\{\omega\}} + \mu_{\{\omega\}}\}\delta_\omega$. Similarly, (6) can be written as

$$I(x) = \int xdp + \sum_{E \in \Upsilon(\mathcal{E}) \setminus \mathcal{F}_1} \{\lambda_E \max_E x + \mu_E \min_E x\}. \tag{8}$$

We will also show that these expressions are unique under some conditions.

As we mentioned before, a leading case for our set up is to interpret Ω as the set of states describing uncertainty and function x as a random variable over Ω. Then the class of operators which can be written as in (8) with underlying capacity of the form (7) has a natural interpretation that the value of x is the sum of its expected value $\int xdp$ and a weighted average of the most optimistic outcome and the most pessimistic outcomes on events in $\Upsilon(\mathcal{E})$. That is, $I(x)$ is the expectation biased by optimism and pessimism conditional on various events in $\Upsilon(\mathcal{E})$.

Alternatively, interpret Ω as a collection of individuals (i.e., a society), and $x(\omega)$ as the wealth allocated to individual ω. Then $\int xdp$ can be seen as the (weighted) average income of the society, and $\max_E x$ and $\min_E x$ correspond to the wealthiest and the poorest in group E, respectively.

In particular, when p is the uniform distribution and $\lambda_E = -1$ and $\mu_E = 1$, then the problem of maximizing (8) subject to $\int xdp$ being held constant means that that of reducing the sum of wealth differences in various groups in $\Upsilon(\mathcal{E})$.

An interesting special subclass of (8) is the class of NEO-additive capacities obtained by Chateaunuff, Eichberger, and Grant (2002): a NEO-additive capacity is a capacity of the form $v = (1 - \lambda - \mu)\, q + \lambda w_\Omega + \mu u_\Omega$, i.e., $\mathcal{E} = \{\Omega\}$ in (8) and $I(1_\Omega) = 1$.[2] More generally, let $\mathcal{E}$ be a partition of Ω, and write $\mathcal{E} = \{E_1, ..., E_K\}$. Then (8) is essentially $v = p + \sum \lambda_k w_{E_k} + \mu_k u_{E_k}$, where p is an additive game. Not only this is a generalization of the NEO-additive capacity, but also it is a generalization of the E-capacities of Eichberger and Kelsey (1999), which correspond to the case where $\lambda_k = 0$ for all k.

4 Main characterization result

One direction of the characterization can be readily established, as is shown below.

Lemma 4.1 *Let $v = \sum_{E \in \Upsilon(\mathcal{E})} \{\lambda_E w_E + \mu_E u_E\}$. Then v is $\mathcal{E}$-coextrema additive.*

Proof. Let x and y be $\mathcal{E}$-coextrema functions. Then by Lemma 3.1, x and y are $\Upsilon(\mathcal{E})$-coextrema. For every $E \in \Upsilon(\mathcal{E})$, let $\bar{\omega} \in \arg\max_E x \cap \arg\max_E y$ and $\underline{\omega} \in \arg\min_E x \cap \arg\min_E y$. Then, $\max_E(x+y) = (x+y)(\bar{\omega}) = x(\bar{\omega}) + y(\bar{\omega}) = \max_E x + \max_E y$, and $\min_E(x+y) = (x+y)(\underline{\omega}) = x(\underline{\omega}) + y(\underline{\omega}) = \min_E x + \min_E y$.

Using these relations, since the Choquet integral is additive in games (see (4)), we have

$$\begin{aligned}
\int (x+y) dv &= \int (x+y) d[\sum_{E \in \Upsilon(\mathcal{E})} \{\lambda_E w_E + \mu_E u_E\}], \\
&= \sum_{E \in \Upsilon(\mathcal{E})} \{\lambda_E \max_E(x+y) + \mu_E \min_E(x+y)\}, \\
&= \sum_{E \in \Upsilon(\mathcal{E})} \{\lambda_E (\max_E x + \max_E y) + \mu_E (\min_E x + \min_E y)\}, \\
&= \sum_{E \in \Upsilon(\mathcal{E})} \{\lambda_E \max_E x + \mu_E \min_E x\} + \sum_{E \in \Upsilon(\mathcal{E})} \{\lambda_E \max_E y + \mu_E \min_E y\}, \\
&= \int x dv + \int y dv,
\end{aligned}$$

which completes the proof. ■

The other direction is far more complicated. Observe first that since both $\{u_T : T \in \mathcal{F}\}$ and $\{w_T : T \in \mathcal{F}\}$ constitute linear bases, if the collection of events $\Upsilon(\mathcal{E})$ contains a sufficient variety of events, not only coextrema additive games but also many other games can be expressed as in

[2]When $\lambda = 0$, i.e., there is no part for optimism, this type of capacity is also referred to as an ε-contamination. See Kajii, Kojima, and Ui (2007) for more discussions.

(5) or (6). In other words, for these expressions to be interesting, it is important to establish the uniqueness, and one can easily expect that the collection $\Upsilon(\mathcal{E})$ should not contain too many elements for this purpose. On the other hand, $\mathcal{E}$ must be rich enough relative to Ω as the following example shows.

Example 4.1 Let $|\Omega| \geq 3$ and $\mathcal{E} = \{\{1,2,3\}\}$. Then $\Upsilon(\mathcal{E})\backslash\mathcal{F}_1 = \mathcal{E}$. Notice that in general when $|E| = 3$, if x and y are coextrema on E, then x and y are automatically comonotonic on E. So any non-game v of the form $v = \sum_{T\subseteq\{1,2,3\}} \beta_T u_T$ is $\mathcal{E}$-coextrema additive, in particular $u_{\{1,2\}}$ is $\mathcal{E}$-coextrema additive. But it can be shown that $u_{\{1,2\}}$ cannot be written in the form (5).

To exclude cases like Example 4.1, we need to guarantee that $\Upsilon(\mathcal{E})$ does not contain too many elements. The key condition formally stated below roughly says that the elements of $\mathcal{E}$, as well as their intersections, are not too small, i.e., the collection $\mathcal{E}$ are "coarse" enough:

Coarseness Condition $|E| \geq 4$ for every $E \in \mathcal{E}$ and $|S| \geq 2$ for every $S \in \Pi(\mathcal{E})$.

The Coarseness Condition is satisfied in Example 2.1, but it is violated in Example 4.1.

Remark 4.1 Obviously, if $\mathcal{E}$ is coarse, it contains no singleton set. However, as far as the representation result stated below is concerned, singletons are inessential since $\Upsilon(\mathcal{E})$ automatically contains all the singletons anyway. Put it differently, we could state the condition by first excluding singletons from $\mathcal{E}$ and then construct the relevant field and partition.

We are now ready to state the main result of this paper.

Theorem 4.1 *Let $\mathcal{E}$ be a collection of events which satisfies the coarseness condition. Let v be a game. Then the following two conditions are equivalent:*
(i) v is $\mathcal{E}$-coextrema additive; (ii) there exist an additive game p and two sets of real numbers, $\{\lambda_E : E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1\}$ and $\{\mu_E : E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1\}$, such that

$$v = p + \sum_{E\in\Upsilon(\mathcal{E})\backslash\mathcal{F}_1} \{\lambda_E w_E + \mu_E u_E\}. \tag{9}$$

Moreover, (9) is unique; that is, if $v = p' + \sum_{E\in\Upsilon(\mathcal{E})\backslash\mathcal{F}_1}\{\lambda'_E w_E + \mu'_E u_E\}$ where p' is additive, then $p = p'$, and $\lambda'_E = \lambda_E$ and $\mu'_E = \mu_E$ hold for every $E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1$.

We shall prove this result in the next section, but we note here that the coarseness condition is indispensable for Theorem 4.1. Recall that in Example 4.1 the coarseness condition is violated and there is a coextrema additive game which cannot be expressed in the form (9). The next example is also instructive for this point.

Example 4.2 *Let* $\Omega = \{1,2,3,4\}$, $\mathcal{E} = \{\{1,2,3\},\{1,2,4\},\{3,4\}\}$. In this case, it is $\Upsilon(\mathcal{E})\backslash\mathcal{F}_1 = \mathcal{E}\cup\{\Omega\}$. But if x and y are $\mathcal{E}$-coextrema, then it is comonotonic on both $\{1,2,3\}$ and $\{1,2,4\}$, and hence it is comonotonic on Ω. So any non-additive measure v is $\mathcal{E}$-coextrema additive.

Let us conclude this section with a couple of applications of Theorem 4.1. The first concerns a characterization of the generalized NEO-additive, E-capacities outlined before. Let $\mathcal{E}$ be a partition of Ω, and write $\mathcal{E} = \{E_1, ..., E_K\}$ as before. It can be readily verified that $\Upsilon(\mathcal{E}) = \mathcal{E} \cup \mathcal{F}_1$. Trivially, $\Pi(\mathcal{E}) = \mathcal{E}$. So if $|E_k| \geq 4$ for every $k = 1, ..., K$, by Theorem 4.1, $\mathcal{E}$ satisfies the coarseness condition and then v is $\mathcal{E}$-coextrema additive if and only if v can be written as $v = p + \sum \lambda_k w_{E_k} + \mu_k u_{E_k}$, where p is an additive game.

The second is a generalization of the variation averse operator proposed in Gilboa (1989). Let $T > 1$ and $M \geq 2$ be integers and set $\Omega = \{(m, t) : m = 1, ..., 2M, t = 1, ..., T\}$. The intended interpretation is that t is the time and at each time t there are m states representing some uncertainty. Let $\mathcal{E}$ be the collection of all sets of the following forms: $\{(m, t) : m = 1, ..., 2M\}$; $\{(m, t) : m = 1, ..., M\} \cup \{(m, t+1) : m = 1, ..., M\}$; and $\{(m, t) : m = M+1, ..., 2M\} \cup \{(m, t+1) : m = M+1, ..., 2M\}$. It can be readily verified that $\Upsilon(\mathcal{E}) = \mathcal{E} \cup \mathcal{F}_1$, and every set in $\Pi(\mathcal{E})$ contains M points. So the coarseness condition is met, and by Theorem 4.1, an $\mathcal{E}$-extrema additive capacity has the form in 8). Arguing analogously as in Kajii, Kojima, and Ui (2007), the coefficients for the $\mathcal{E}$-events of the form $\{(m, t) : m = 1, ..., 2M\}$ represent measurements of optimism and pessimism about the uncertainty, whereas the coefficients for the $\mathcal{E}$-events of the other forms represent measurements of (conditional) degrees of variation loving and variation aversion.

5 The proof

This section is devoted to the proof of Theorem 4.1. Since Lemma 3.2 has already shown that (ii) implies (i), it suffices to establish the other direction. The proof consists of several steps: basically, starting with an $\mathcal{E}$-coextrema game v, we shall first show that a restriction of v is $\mathcal{E}$-comaximum. Then we show that this construction is invariant of the way the restriction is chosen as long as a certain condition is satisfied, which then implies the existence of a well-defined $\mathcal{E}$-comaximum additive game v_1. We then show that the game $v_2 := v - v_1$ is $\mathcal{E}$-cominimum additive. Theorem 2.2 can be applied to v_1 and v_2 to obtain the desired expression.

Let v be an $\mathcal{E}$-coextrema additive game with $v = \sum_{T \in \mathcal{F}} \beta_T u_T$. For any $R \in \mathcal{F}$, let $v_{|R}$ be the game defined by the rule $v_{|R}(E) = v(E \cap R)$ for all $E \in \mathcal{F}$, i.e., $v_{|R} = \sum_{T \subseteq R} \beta_T u_T$. Define $\mathcal{E}_{\cap R} = \{E \cap R \,|\, E \in \mathcal{E}, E \cap R \neq \emptyset\}$, which is the collection of intersections of elements of $\mathcal{E}$ and R, and also define $\mathcal{E}_{\subseteq R} = \{E \,|\, E \in \mathcal{E}, E \subseteq R\}$, which is the collection of elements of $\mathcal{E}$ contained in R. Note that $\mathcal{E}_{\subseteq R} \subseteq \mathcal{E}_{\cap R}$.

To construct the desired $\mathcal{E}$-comaximum additive game v_1, we first observe the following property.

Lemma 5.1 *Let v be $\mathcal{E}$-coextrema additive. Let $R \in \mathcal{F}$ be such that $\mathcal{E}_{\subseteq R} = \emptyset$ and $\mathcal{E}_{\cap R} \neq \emptyset$. Then, $v_{|R}$ is $\mathcal{E}_{\cap R}$-comaximum additive.*

Proof. Let 1_S and 1_T be $\mathcal{E}_{\cap R}$-comaximum. It is enough to show that $v_{|R}(S\cup T)+v_{|R}(S\cap T) = v_{|R}(S)+v_{|R}(T)$, which is rewritten as $v((S\cap R)\cup(T\cap R))+v((S\cap R)\cap(T\cap R)) = v(S\cap R)+v(T\cap R)$. Therefore, it suffices to show that $1_{S\cap R}$ and $1_{T\cap R}$ are $\mathcal{E}$-coextrema because v is $\mathcal{E}$-coextrema additive.

Fix any $E \in \mathcal{E}$. Since $\mathcal{E}_{\subseteq R} = \emptyset$, either $E\cap R = \emptyset$, or $E\cap R \neq \emptyset$ and $E\backslash R \neq \emptyset$. If $E\cap R = \emptyset$, then $1_{S\cap R}$ and $1_{T\cap R}$ are 0 on E and thus have a common minimizer and maximizer on E. If $E\cap R \neq \emptyset$ and $E\backslash R \neq \emptyset$, then $1_{S\cap R}$ and $1_{T\cap R}$ have a common maximizer in $E\cap R \subseteq E$ since 1_S and 1_T are $\mathcal{E}_{\cap R}$-comaximum, and $1_{S\cap R}$ and $1_{T\cap R}$ have a common minimizer in $E\backslash R \subseteq E$ since $1_{S\cap R}$ and $1_{T\cap R}$ are 0 on R^c. Therefore, $1_{S\cap R}$ and $1_{T\cap R}$ are $\mathcal{E}$-coextrema. ■

By this lemma and Theorem 2.2, $v_{|R}$ has a unique expression

$$v_{|R} = \sum_{\omega\in R} \nu^R_{\{\omega\}} w_{\{\omega\}} + \sum_{E'\in\Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1} \nu^R_{E'} w_{E'}. \tag{10}$$

To obtain the desired game v_1 which will constitute a part of the expression (9), we want the second part of the right hand side of (10) in the following form:$\sum_{E\in\Upsilon(\mathcal{E})\backslash\mathcal{F}_1} \nu^R_{E\cap R} w_{E\cap R}$. Since each $E' \in \mathcal{E}_{\cap R}\backslash\mathcal{F}_1$ is written as $E' = E\cap R$ for some $E \in \mathcal{E}$, one way to proceed is to associate each E' with the corresponding E. Of course, this procedure is not well defined in general, since there may be many such E for candidates. So our next step is to find a condition on the set R so that this procedure in fact unambiguously works. It turns out that the following property is suitable for this purpose.

Definition 5.1 A set $R \in \mathcal{F}$ is a *representation* of $\mathcal{E}$ if $\mathcal{E}_{\subseteq R} = \emptyset$, $\kappa(R) = \Omega$, and $|R\cap E| \geq 2$ for all $E \in \mathcal{E}$. Moreover we say that $R \in \mathcal{F}$ is a *minimal representation* of $\mathcal{E}$ if R is a representation of $E \in \mathcal{E}$ and any proper subset of R is not a representation.

In Example 2.1, the set R is a representation for $\mathcal{E}$. Another example follows below.

Example 5.1 Let $\Omega = \{1,2,3,4,5,6\}$, and set $\mathcal{E} = \{\{1,2,3,4\},\{3,4,5,6\}\}$. Then $\Pi(\mathcal{E}) = \{\{1,2\},\{3,4\},\{5,6\}\}$. $R = \{3,4\}$ is not a representation, since $\kappa(R) = \{3,4\} \neq \Omega$. $R = \{2,3,4,6\}$ is a representation but not minimal. $R = \{2,4,6\}$ is a minimal representation.

Lemma 5.2 *When $\mathcal{E}$ is coarse, if $T \in \mathcal{F}$ satisfies $\mathcal{E}_{\cap T} = \emptyset$, then there is a representation R such that $T \subseteq R$.*

Proof. Construct R by the following procedure: first set $R = T$ and then for each $S \in \Pi(\mathcal{E})$; if $S \in \mathcal{E}$ and $|T\cap S| \leq 1$, then add a point or two to R from $S\backslash T$ (recall that $|S| \geq 4$ if $S \in \mathcal{E}$ by the coarseness) so that two points from S are contained in R; if $S \in \mathcal{E}$ and $|T\cap S| \geq 2$, do nothing; if $S \notin \mathcal{E}$ and $T\cap S = \emptyset$, then add a point to R (note $S\backslash R \neq \emptyset$ by the coarseness); if $S \notin \mathcal{E}$ and $T\cap S \neq \emptyset$, do nothing. Then by construction, $\kappa(R) = \Omega$, and $|R\cap E| \geq 2$ for all

$E \in \mathcal{E}$. Notice also that for any $E \in \mathcal{E}$, there is some point which is not added to R, so $\mathcal{E}_{\subseteq R} = \emptyset$ follows. ■

Note that if R is a representation of $\mathcal{E}$, $\kappa(R) = \Omega$ holds by definition and so every $S \in \Pi(\mathcal{E})$ must necessarily intersect R. Roughly speaking, a representation is obtained by choosing some representative elements from each S in $\Pi(\mathcal{E})$ when $\mathcal{E}$ is coarse. Formally, we have the following result:

Lemma 5.3 *If $\mathcal{E}$ is coarse, there exists a minimal representation, which can be constructed by the following rule: for each $S \in \Pi(\mathcal{E})$, choose two distinct elements from S if $S \in \mathcal{E}$, and one element if $S \notin \mathcal{E}$, and set R to be the set of chosen elements. Moreover, every minimal representation can be constructed in this way, and so in particular minimal representations contain exactly the same number of points.*

Proof. The set R constructed as above is well defined since by the coarseness condition every S has at least two elements. We claim that R is a representation. For all $E \in \mathcal{E}$, there is $S \in \Pi(\mathcal{E})$ with $S \subseteq E$. If $S = E$, R contains exactly two points belonging to E. If $S \subsetneq E$, then there is another $S' \neq S$ with $S' \subseteq E$ because E is the union of some elements in $\Pi(\mathcal{E})$. Since R contains one element of S and S', it contains at least two points belonging to E. Therefore, $|R \cap E| \geq 2$ for all $E \in \mathcal{E}$. Also, every $S \in \Pi(\mathcal{E})$ intersects with R and so $\kappa(R) = \Omega$. Finally, notice that $E \subseteq R$ is possible only if $E \in \Pi(\mathcal{E})$. But by the coarseness condition, $|E| \geq 4$ and so this case cannot occur in the construction, thus $\mathcal{E}_{\subseteq R} = \emptyset$.

Next we claim that R is minimal. Let R' be a proper subset of R and pick any $\omega \in R \backslash R'$. Let $S \in \Pi(\mathcal{E})$ be the set where ω is chosen from. If $S \in \mathcal{E}$, then R contains exactly two elements of S by construction. Then $|R' \cap S| = 1$, and so R' is not a representation. If $S \notin \mathcal{E}$, then ω is the only one element from S. Then $S \cap R' = \emptyset$ which implies $\kappa(R') \subseteq \Omega \backslash S$, and so R' is not a representation.

Finally, let R be a minimal representation. Then $S \cap R \neq \emptyset$ for every $S \in \Pi(\mathcal{E})$ so R contains at least one point from each S. If $S \in \Pi(\mathcal{E})$ and $S \in \mathcal{E}$, then $|R \cap S| \geq 2$ so at least two points from such S must be contained in R. Let R' the collection of all these points in the intersections, which is a minimum representation as we have shown above. Since $R' \subseteq R$, we conclude $R' = R$, which completes the proof. ■

Example 5.2 In Example 2.1, none of elements in $\Pi(\mathcal{E})$ belongs to $\mathcal{E}$. So to obtain a minimal representation one can choose exactly one point from each $S \in \Pi(\mathcal{E})$. For instance, $R = \{1, 3, 5, 7\}$ is a minimal representation.

When R constitutes a representation of $\mathcal{E}$, we can associate each $E \in \Upsilon(\mathcal{E}_{\cap R}) \backslash \mathcal{F}_1$ to some *unique* element in $\Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$, as is shown in the next result.

Lemma 5.4 *Assume that $\mathcal{E}$ is coarse, and let $R \in \mathcal{F}$ be a representation of $\mathcal{E}$. Then $\kappa(F) \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1$ for any $F \in \Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$. Conversely, if $E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1$, then $E \cap R$ is a unique element of $\Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$ such that $\kappa(E \cap R) = E$. In short, given R, the restriction of κ, denoted by κ_R, constitutes a bijection between $\Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$ and $\Upsilon(\mathcal{E})\backslash\mathcal{F}_1$ by the rule $\kappa_R(F) = \kappa(F)$ for all $F \in \Upsilon(\mathcal{E}_{\subseteq R})\backslash\mathcal{F}_1$, and $\kappa_R^{-1}(E) = E \cap R$ for all $E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1$.*

Proof. Note that $\mathcal{E}$ and $\mathcal{E}_{\cap R}$ contain no singleton since $\mathcal{E}$ is coarse and R is a representation of $\mathcal{E}$. Also note that from the basic property of κ and $\kappa(R) = \Omega$ by the definition of representation, we have for each $E \in \mathcal{E}$, $E \cap R \in \mathcal{E}_{\cap R}$ and $\kappa(E \cap R) = \kappa(E) \cap \kappa(R) = \kappa(E) \cap \Omega = \kappa(E) = E$.

We first show that $\kappa(F) \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1$ for all $F \in \Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$. Fix any $F \in \Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$. Choose two distinct points $\omega_1, \omega_2 \in \kappa(F)$ arbitrarily, and we shall show that there is an $E \in \mathcal{E}$ such that $\{\omega_1, \omega_2\} \subseteq E \subseteq \kappa(F)$. By the construction of $\kappa(F)$, there are S_1, $S_2 \in \Pi(\mathcal{E})$ (possibly $S_1 = S_2$) such that $\omega_1 \in S_1$, $\omega_2 \in S_2$, and both $S_1 \cap F$ and $S_2 \cap F$ are non-empty. Suppose first that $S_1 \neq S_2$. Then we can select two distinct points $\omega_1' \in S_1 \cap F$ and $\omega_2' \in S_2 \cap F$. Since $F \in \Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$, there exists $F' \in \mathcal{E}_{\cap R}$ such that $\omega_1', \omega_2' \in F' \subseteq F$ by the definition of completeness. By the definition of $\mathcal{E}_{\cap R}$, there is $E \in \mathcal{E}$ with $F' = E \cap R$. Using the property of κ (see Remark 2.1), and the definition of a representation, $\kappa(F') = E \cap \kappa(R) = E$ and $\kappa(F') \subseteq \kappa(F)$. So we have $\{\omega_1, \omega_2\} \subseteq F' \subseteq \kappa(F') = E \subseteq \kappa(F)$, as we wanted. Suppose then $S_1 = S_2$ $(= \hat{S})$. Recall that $F \in \Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$ implies that F is the union of some elements in $\mathcal{E}_{\cap R}$. Since $\hat{S} \in \Pi(\mathcal{E})$, this means that there is at least one $E \in \mathcal{E}$ such that $\hat{S} \subseteq E$ and $E \cap R \subseteq F$. Then again by the definition of representation, $E = \kappa(E \cap R) \subseteq \kappa(F)$, and so this E has the desired property.

Next, we show that the restriction κ_R is a map from $\Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$ onto $\Upsilon(\mathcal{E})\backslash\mathcal{F}_1$. Fix any $E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1$. Since $E \in \sigma(\mathcal{E})$, $\kappa_R(E \cap R) = \kappa(E) \cap \kappa(R) = \kappa(E) \cap \Omega = \kappa(E) = E$; that is, $E \cap R$ is in the inverse image of κ_R. Thus, it is enough to show that $E \cap R \in \Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$. By the definition of completeness, there exist $E_1, \ldots, E_K \in \mathcal{E}$ such that $E = \bigcup_{k=1}^K E_k$ and that for any pair of points $\omega, \omega' \in E$, $\omega, \omega' \in E_k$ holds for some k. So in particular, for any distinct points $\omega, \omega' \in E \cap R \subseteq E$, there exists k with $\omega, \omega' \in E_k$ and thus $\omega, \omega' \in E_k \cap R \in \mathcal{E}_{\cap R}$ since $\omega, \omega' \in R$. Therefore, κ_R is onto.

Finally we show that κ_R is one to one, i.e., $\kappa_R(F) = E$ occurs for $F \in \Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$ only if $F = E \cap R$. Note that $F \in \Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$ implies that there exist $E_1, \ldots, E_K \in \mathcal{E}$ such that $F = \bigcup_{k=1}^K (E_k \cap R) = (\bigcup_{k=1}^K E_k) \cap R$. Since R is a representation, R must intersect any $\Pi(\mathcal{E})$ -component of E_k for all k, and so $\kappa(F) = \kappa((\bigcup_{k=1}^K E_k) \cap R) = \bigcup_{k=1}^K E_k$. So $\kappa_R(F) = E$ implies $\bigcup_{k=1}^K E_k = E$ and so $F = E \cap R$ must hold. This completes the proof. ■

By Lemma 5.4, if R be a representation of $\mathcal{E}$, then, by rewriting (10), we have

$$v_{|R} = \sum_{\omega \in R} \lambda_{\{\omega\}}^R w_{\{\omega\}} + \sum_{E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1} \lambda_E^R w_{E \cap R} \tag{11}$$

where $\lambda_E^R = \nu_{E \cap R}^R$ for each $E \in \Upsilon(\mathcal{E})$. By construction, the coefficients $\{\lambda_E^R : E \in \Upsilon(\mathcal{E})\}$ are uniquely determined with respect to a representation R except for singletons. It turns out that these do not depend upon the choice of representation R, which we shall demonstrate in the following in a few lemmas. Let R, $R' \in \mathcal{F}$ be representations of $\mathcal{E}$, and so there are corresponding expressions of the form (11). We write $R \doteq R'$ if $\lambda^{R}_{\{\omega\}} = \lambda^{R'}_{\{\omega\}}$ for all $\omega \in R \cap R'$ and $\lambda^{R}_{E} = \lambda^{R'}_{E}$ for all $E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1$. Note that the first part holds vacuously if $R \cap R' = \emptyset$.

Lemma 5.5 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Let $R, R', R'' \in \mathcal{F}$ be representations of $\mathcal{E}$. Suppose that $R \doteq R'$ and $R' \doteq R''$. Then, $R \doteq R''$ holds if $R \cap R'' \subseteq R'$.*

Proof. By definition, $\lambda^{R}_{\{\omega\}} = \lambda^{R''}_{\{\omega\}}$ for all $\omega \in R \cap R' \cap R''$ $(= R \cap R'')$ and $\lambda^{R}_{E} = \lambda^{R''}_{E}$ for all $E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1$. ■

Lemma 5.6 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Let $R, R' \in \mathcal{F}$ be representations of $\mathcal{E}$. Then $R \doteq R'$ holds if $R \cap R'$ is a representation. In particular, if $R \subseteq R'$, $R \doteq R'$ holds.*

Proof. Set $R^* = R \cap R'$. Note that by construction, for all $T \in \mathcal{F}$, $v_{|R^*}(T) = v_{|R}(T \cap R^*) = v_{|R'}(T \cap R^*)$. Using (11) on the other hand, we have

$$\begin{aligned} v_{|R}(T \cap R^*) &= \sum_{\omega \in R} \lambda^{R}_{\{\omega\}} w_{\{\omega\}}(T \cap R^*) + \sum_{E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1} \lambda^{R}_{E} w_{E \cap R}(T \cap R^*) \\ &= \sum_{\omega \in R^*} \lambda^{R}_{\{\omega\}} w_{\{\omega\}}(T) + \sum_{E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1} \lambda^{R}_{E} w_{E \cap R^*}(T) \end{aligned}$$

and

$$v_{|R'}(T \cap R^*) = \sum_{\omega \in R^*} \lambda^{R'}_{\{\omega\}} w_{\{\omega\}}(T) + \sum_{E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1} \lambda^{R'}_{E} w_{E \cap R^*}(T).$$

Thus, for all $T \in \mathcal{F}$,

$$\sum_{\omega \in R^*} \lambda^{R}_{\{\omega\}} w_{\{\omega\}}(T) + \sum_{E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1} \lambda^{R}_{E} w_{E \cap R^*}(T) = \sum_{\omega \in R^*} \lambda^{R'}_{\{\omega\}} w_{\{\omega\}}(T) + \sum_{E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1} \lambda^{R'}_{E} w_{E \cap R^*}(T).$$

Since R^* is also a representation by assumption, by Lemma 5.4, $\Upsilon(\mathcal{E}_{|R^*})\backslash\mathcal{F}_1$ and $\Upsilon(\mathcal{E})\backslash\mathcal{F}_1$ are isomorphic. Since $\{w_T : T \in \mathcal{F}\}$ are linearly independent, this means that the games in $\{w_{\{\omega\}}\}_{\omega \in R^*} \cup \{w_{E \cap R^*}\}_{E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1}$ are linearly independent. Therefore, the respective coefficients on the both sides of the above equation must coincide each other, which completes the proof. ■

Lemma 5.7 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Let $R, R' \in \mathcal{F}$ be minimal representations of $\mathcal{E}$. Then $R \doteq R'$.*

Proof. If $R = R'$, then obviously $R \doteq R'$, and so let $R \neq R'$. By Lemma 5.3, $|R| = |R'|$ and so there is $\omega' \in R' \backslash R$. Let $S \in \Pi(\mathcal{E})$ be the unique element with $\omega' \in S$. Recall that a representation intersects every elements of $\Pi(\mathcal{E})$, and hence we can pick an $\omega \in R \cap S$. By construction $\omega \neq \omega'$. Set $R^1 = (R \backslash \{\omega\}) \cup \{\omega'\}$, i.e., R^1 is obtained by substituting ω with ω' both of which belong to S. So R^1 is also a minimal representation by Lemma 5.3.

We shall show that $R \doteq R^1$. For this, consider first $\hat{R} = R \cup \{\omega'\}$. Notice that $\hat{R}$ is a representation; since $R \subseteq \hat{R}$ and R is a representation, it is clear that $\kappa(\hat{R}) = \Omega$, and $\left|\hat{R} \cap E\right| \geq 2$ for all $E \in \mathcal{E}$. Since $\mathcal{E}$ is coarse and R is minimal, for all $E \in \mathcal{E}$, we have $|E \backslash R| \geq 2$ and so $\left|E \backslash \hat{R}\right| \geq 1$. Hence $\mathcal{E}_{\subseteq \hat{R}} = \emptyset$, which proves that $\hat{R}$ is a representation. By construction, both $R \cap \hat{R} = R$ and $R^1 \cap \hat{R} = R^1$ are representations, so by Lemma 5.6, $R \doteq \hat{R}$ and $\hat{R} \doteq R^1$. Note that $R \cap R^1 \subseteq \hat{R}$, which implies that $R \doteq R^1$ by Lemma 5.5.

Recall that both R and R' are finite and they can be obtained by the method described in Lemma 5.3, so repeating the argument above, i.e., replacing one ω in R with another $\omega' \in R' \backslash R$, we can construct a sequence of minimal representations R^0 $(= R)$ R^1, R^2, $\cdots$, $R^k = R'$ such that $R^{m-1} \doteq R^m$ for each $m = 1, .., k$. By definition, $\lambda_E^{R^{m-1}} = \lambda_E^{R^m}$ holds for all $E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$ for every $m = 1, ..., k$, hence $\lambda_E^R = \lambda_E^{R'}$ holds for all $E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$. For any $\omega \in R \cap R'$, since such ω is never replaced along the sequence above, we have $\lambda_{\{\omega\}}^{R^{m-1}} = \lambda_{\{\omega\}}^{R^m}$ for every $m = 1, ..., k$, and hence $\lambda_{\{\omega\}}^R = \lambda_{\{\omega\}}^{R'}$. Therefore, we conclude that $R \doteq R'$. ■

Lemma 5.8 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Let $R, R' \in \mathcal{F}$ be representations of $\mathcal{E}$. Then $R \doteq R'$.*

Proof. Choose any two minimal representations Γ and Γ' such that $\Gamma \subseteq R$, $\Gamma' \subseteq R'$, and $\Gamma \cap \Gamma' \subseteq R \cap R'$. Notice that by Lemma 5.3 such minimal representations always exist and can be constructed as follows: for any $\omega \in R \cap R'$, then select this ω from $S \in \Pi(\mathcal{E})$ which contains ω. Now by Lemma 5.6, $R \doteq \Gamma$ and $\Gamma' \doteq R'$ hold. Also, by Lemma 5.7, $\Gamma \doteq \Gamma'$ holds. These imply that $\lambda_E^R = \lambda_E^{R'}$ for all $E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$ and that $\lambda_{\{\omega\}}^R = \lambda_{\{\omega\}}^{R'}$ for all $\omega \in \Gamma \cap \Gamma'$. Since the choice of $\Gamma \cap \Gamma' \subseteq R \cap R'$ is arbitrary as is pointed out above, we must have $\lambda_{\{\omega\}}^R = \lambda_{\{\omega\}}^{R'}$ for all $\omega \in R \cap R'$. Therefore, we conclude that $R \doteq R'$. ■

Since there is a representation containing any $\omega \in \Omega$, Lemma 5.8 implies that there exists a unique collection of constants $\{\lambda_E\}_{E \in \Upsilon(\mathcal{E})}$ such that, for *any* representation R of $\mathcal{E}$, $v_{|R} = \sum_{\omega \in R} \lambda_{\{\omega\}} w_{\{\omega\}} + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \lambda_E w_{E \cap R}$. Using this collection, define two games v_1 and v_2 by the following rule:

$$v_1 = \sum_{E \in \Upsilon(\mathcal{E})} \lambda_E w_E \text{ and } v_2 = v - v_1. \tag{12}$$

By Theorem 2.2, v_1 is $\mathcal{E}$-comaximum additive. To show that v_2 is $\mathcal{E}$-cominimum additive, we use the following property of v_2.

Lemma 5.9 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Then for any $T \in \mathcal{F}$,*

$$v_2(T) = v_2(\bigcup_{E \in \mathcal{E}_{\subseteq T}} E). \tag{13}$$

Proof. Case 1: $\mathcal{E}_{\subseteq T} = \emptyset$, i.e., no element in $\mathcal{E}$ is contained in T. Then, $v_2(\bigcup_{E\in\mathcal{E}_{\subseteq T}} E) = v(\emptyset) - v_1(\emptyset) = 0$, so we need to show that $v_2(T) = v(T) - v_1(T) = 0$. Note that there exists a representation R of $\mathcal{E}$ such that $T \subseteq R$ (see Lemma 5.2). Then, $v(T) = v(T \cap R) = v_{|R}(T) = \sum_{E\in\Upsilon(\mathcal{E}),\ E\cap T\neq\emptyset} \lambda_E = v_1(T)$, as claimed.

Case 2: $\mathcal{E}_{\subseteq T} \neq \emptyset$. Let $E^* = \bigcup_{E\in\mathcal{E}_{\subseteq T}} E$ and $T^* = T\backslash E^*$. We want to show that $v_2(T) = v_2(E^*)$. By construction, $E^* \in \sigma(\mathcal{E})$ is the union of some elements in $\Pi(\mathcal{E})$, choose one point from each of these elements and let A be the collection of these points. Note that $\kappa(A) = E^*$, and that $\mathcal{E}_{\subseteq A} = \mathcal{E}_{\subseteq T^*\cup A} = \emptyset$ follows from the coarseness. Thus, Case 1 applies to A and $T^* \cup A$, and we have

$$v_2(A) = v_2(T^* \cup A) = 0. \tag{14}$$

Now we claim that 1_{E^*} and $1_{T^*\cup A}$ are $\mathcal{E}$-coextrema. Note first that $E^* \cap (T^* \cup A) = A$ by construction. To see that they are $\mathcal{E}$-comaximum, recall Remark 2.5, and pick $F \in \mathcal{E}$ with $F \subseteq \Omega\backslash A$. Then $F \cap E^* = \emptyset$ must follow, since both F and E^* are in $\sigma(\mathcal{E})$ and so for any $S \in \Pi(\mathcal{E})$ with $S \subseteq F$, $A \cap S \neq \emptyset$ would hold if $S \subseteq E^*$. Then $F \subseteq \Omega\backslash E^*$ as desired. To see that they are $\mathcal{E}$-cominimum as well, notice that if $F \in \mathcal{E}$ and $F \subseteq E^* \cup (T^* \cup A) = T$, then $F \subseteq E^*$ by construction. Thus E^* and $(T^* \cup A)$ are an $\mathcal{E}$-decomposition pair, and so apply Lemma 2.3.

By the coextrema additivity of v, $v(E^* \cup (T^* \cup A)) + v(E^* \cap (T^* \cup A)) = v(E^*) + v(T^* \cup A)$, which can be re-written as

$$v(E^* \cup T^*) + v(A) = v(E^*) + v(T^* \cup A). \tag{15}$$

On the other hand, since 1_{E^*} and $1_{T^*\cup A}$ are $\mathcal{E}$-comaximum and v_1 is $\mathcal{E}$-comaximum additive,

$$v_1(E^* \cup T^*) + v_1(A) = v_1(E^*) + v_1(T^* \cup A). \tag{16}$$

Subtracting (16) from (15), and using the definition of v_2, and the fact $T = E^* \cup T^*$, we have

$$v_2(T) + v_2(A) = v_2(E^*) + v_2(T^* \cup A).$$

Applying (14) here, we obtain the desired equation. ■

Now we are ready to show that v_2 is $\mathcal{E}$-cominimum additive.

Lemma 5.10 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Then, v_2 is $\mathcal{E}$-cominimum additive and thus it has a unique expression*

$$v_2 = \sum_{E\in\Upsilon(\mathcal{E})} \mu_E u_E.$$

Proof. Let 1_A and 1_B be $\mathcal{E}$-cominimum, i.e., A and B constitute an $\mathcal{E}$-decomposition pair by Lemma 2.3. We need to show that $v_2(A \cup B) + v_2(A \cap B) = v_2(A) + v_2(B)$.

Note that for each $S \in \Pi(\mathcal{E})$ such that there is an $E \in \mathcal{E}$ with $S \subseteq E \subseteq A \cup B$, if $S \nsubseteq A \cap B$, then either $S \cap (B \backslash A) \neq \emptyset$ or $S \cap (A \backslash B) \neq \emptyset$, but not both; if both hold then A and B would not be an $\mathcal{E}$-decomposition pair.

For each $S \in \Pi(\mathcal{E})$ with $S \nsubseteq A \cap B$, choose a point ω_S from $S \cap (A \backslash B)$ if $S \cap (A \backslash B) \neq \emptyset$, or from $S \cap (B \backslash A)$ if $S \cap (B \backslash A) \neq \emptyset$. Let Ω^* be the set of chosen points. Finally, set $A^* = A \cup (B \backslash \Omega^*)$ and $B^* = B \cup (A \backslash \Omega^*)$. Notice that $A^* \cup B^* = A \cup B$ by construction.

We claim that if $E \in \mathcal{E}$ satisfies $E \subseteq A^*$, then $E \subseteq A$. Indeed, suppose that there is a point $\omega \in E \cap (A^* \backslash A)$. Since $E \in \mathcal{E}$, we can find (a unique) $S \in \Pi(\mathcal{E})$ with $\omega \in S \subseteq E$, and $\omega \in S \cap (B \backslash A)$. By the construction of Ω^*, this means that $S \cap ((B \backslash A) \cap \Omega^*) \neq \emptyset$ so $E \cap ((B \backslash A) \cap \Omega^*) \neq \emptyset$, which is impossible since $E \subseteq A^* = A \cup (B \backslash \Omega^*)$.

Similarly, if $E \in \mathcal{E}$ satisfies $E \subseteq B^*$, then $E \subseteq B$. To sum up, the collections of $\mathcal{E}$-elements contained in A^*, B^*, $A^* \cup B^*$ and $A^* \cap B^*$ coincide with those of A, B, $A \cup B$ and $A \cap B$, respectively. Therefore, by Lemma 5.9, we are done if $v_2(A^* \cup B^*) + v_2(A^* \cap B^*) = v_2(A^*) + v_2(B^*)$. For this, it suffices to show that 1_{A^*} and 1_{B^*} are $\mathcal{E}$-coextrema. Indeed, since v is $\mathcal{E}$-coextrema additive, we have $v(A^* \cup B^*) + v(A^* \cap B^*) = v(A^*) + v(B^*)$, and since v_1 is $\mathcal{E}$-comaximum additive, we have $v_1(A^* \cup B^*) + v_1(A^* \cap B^*) = v_1(A^*) + v_1(B^*)$. Since $v_2 = v - v_1$, the desired equation is established from these two equations.

To see 1_{A^*} and 1_{B^*} are $\mathcal{E}$-cominimum, notice that A and B constitutes a decomposition pair by assumption, and so do A^* and B^*; if $E \subseteq A^* \cup B^*$ with $E \in \mathcal{E}$, then $E \subseteq A \cup B$, which implies $E \subseteq A$ or $E \subseteq B$ and hence $E \subseteq A^*$ or $E \subseteq B^*$ as we have shown above. Thus 1_{A^*} and 1_{B^*} are $\mathcal{E}$-cominimum by Lemma 2.3.

It remains to show that 1_{A^*} and 1_{B^*} are $\mathcal{E}$-comaximum. Pick any $E \in \mathcal{E}$ with $E \subseteq \Omega \backslash (A^* \cap B^*)$. We need to show that $E \subseteq \Omega \backslash A^*$ or $E \subseteq \Omega \backslash B^*$ or both (see Remark 2.5). Suppose $E \cap (A^* \cup B^*) \neq \emptyset$ or else the implication holds trivially, and so it suffices to show that $E \cap (A^* \backslash B^*) = \emptyset$ or $E \cap (B^* \backslash A^*) = \emptyset$. If neither of these holds, then pick $\omega_A \in E \cap (A^* \backslash B^*)$ and $\omega_B \in E \cap (B^* \backslash A^*)$. Note that $A^* \backslash B^* = A \backslash B^*$ and $B^* \backslash A^* = B \backslash A^*$ holds, and thus ω_A and ω_B must belong to Ω^* by the construction of A^* and B^*. Since $E \in \mathcal{E}$, there must be $S_A \in \Pi(\mathcal{E})$ and $S_B \in \Pi(\mathcal{E})$ and $E_A \in \mathcal{E}$ and $E_B \in \mathcal{E}$ such that $\omega_A \in S_A \subseteq E_A \cap E \subseteq A \cup B$ and $\omega_B \in S_B \subseteq E_B \cap E \subseteq A \cup B$. But then, by the coarseness, both $S_A \cap B^*$ and $S_B \cap A^*$ are non-empty, which implies $E \cap (A^* \cap B^*) \neq \emptyset$, a contradiction. This completes the proof. ■

Since v_1 is $\mathcal{E}$-comaximum additive and v_2 is $\mathcal{E}$-cominimum additive, we have the desired expression $v = v_1 + v_2 = \sum_{E \in \Upsilon(\mathcal{E})} \lambda_E w_E + \sum_{E \in \Upsilon(\mathcal{E})} \mu_E u_E = p + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} (\lambda_E w_E + \mu_E u_E)$ where $p = \sum_{\omega \in \Omega} p_{\{\omega\}} u_{\{\omega\}}$ and $p_{\{\omega\}} = \lambda_{\{\omega\}} + \mu_{\{\omega\}}$. It remains to show that this is a unique representation.

Lemma 5.11 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Then, the expression $v = \sum_{\omega\in\Omega} p_{\{\omega\}} u_{\{\omega\}} + \sum_{E\in\Upsilon(\mathcal{E})\setminus\mathcal{F}_1} (\lambda_E w_E + \mu_E u_E)$ is unique; that is, if $v = \sum_{\omega\in\Omega} p'_{\{\omega\}} u_{\{\omega\}} + \sum_{E\in\Upsilon(\mathcal{E})\setminus\mathcal{F}_1} (\lambda'_E w_E + \mu'_E u_E)$ then $p_{\{\omega\}} = p'_{\{\omega\}}$ for all $\omega \in \Omega$, $\lambda_E = \lambda'_E$, and $\mu_E = \mu'_E$ for all $E \in \Upsilon(\mathcal{E})\setminus\mathcal{F}_1$.*

Proof. Let $R \in \mathcal{F}$ be a representation of $\mathcal{E}$. Then,

$$v_{|R} = \sum_{\omega\in R} p_{\{\omega\}} u_{\{\omega\}} + \sum_{E\in\Upsilon(\mathcal{E})\setminus\mathcal{F}_1} \lambda_E w_{E\cap R} = \sum_{\omega\in R} p'_{\{\omega\}} u_{\{\omega\}} + \sum_{E\in\Upsilon(\mathcal{E})\setminus\mathcal{F}_1} \lambda'_E w_{E\cap R}.$$

By Lemma 5.4, $\Upsilon(\mathcal{E}_{\cap R})\setminus\mathcal{F}_1$ and $\Upsilon(\mathcal{E})\setminus\mathcal{F}_1$ are isomorphic and thus $\{w_{\{\omega\}}\}_{\omega\in R} \cup \{w_{E\cap R}\}_{E\in\Upsilon(\mathcal{E})\setminus\mathcal{F}_1}$ are linearly independent. Therefore, $p_{\{\omega\}} = p'_{\{\omega\}}$ for all $\omega \in R$ and $\lambda_E = \lambda'_E$ for all $E \in \Upsilon(\mathcal{E})\setminus\mathcal{F}_1$. Since the choice of R was arbitrary, $p_{\{\omega\}} = p'_{\{\omega\}}$ for all $\omega \in \Omega$. The linear independence also guarantees that the expression $v - \sum_{\omega\in\Omega} p_{\{\omega\}} w_{\{\omega\}} - \sum_{E\in\Upsilon(\mathcal{E})\setminus\mathcal{F}_1} \lambda_E w_E = \sum_{E\in\Upsilon(\mathcal{E})\setminus\mathcal{F}_1} \mu_E u_E$ must also be unique. ■

The proof of Theorem 4.1 is now complete.

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