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Efficient Risk-Sharing Rules with Heterogeneous Risk Attitudes and Background Risks

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[5]The exposition of this paper follows that of Hara and Kuzmics (2006), but some of the results of this paper have also been obtained in a series of papers by Huang (2002a, 2002b, 2004, 2005, 2006). Other results by Huang, such as multiplicative background risks and implications on demands and prices for options, are not included here.

Abstract

In an exchange economy in which there is a complete set of markets for macroeconomic risks but no market for idiosyncratic risks, we consider how the efficient risk-sharing rules for the macroeconomic risk are affected by the heterogeneity in the consumers' risk attitudes and idiosyncratic risks. We provide sufficient conditions under which an idiosyncratic risk increases cautiousness (the derivative of the reciprocal of the absolute risk aversion), the determinant of the curvatures of the efficient risk-sharing rules. While the curvature of the risk-sharing rules at high consumption levels are governed by the consumers' risk attitudes, the curvature at low consumption levels depend not only on the risk attitudes but also on the lower tail distributions of the idiosyncratic risks.

JEL Classification Codes: D51, D58, D81, G11, G12, G13.

Keywords: Efficient risk-sharing rules, relative risk aversion, absolute risk tolerance, Inada condition, idiosyncratic risks, background risks, incomplete markets.

1 Introduction

The characterization of efficient risk allocation in complete asset markets has been well studied in the literature. The most celebrated result is the *mutual fund theorem*: Assume that all consumers have expected utility functions. Define the (absolute) *risk tolerance* as the reciprocal of the Arrow-Pratt measure of absolute risk aversion, and call its first derivative (absolute) *cautiousness*. Assume also that they have the same probabilistic belief over all states of the world. It is then easy to establish that at every efficient allocation, each consumer's realized consumption levels can be written as a strictly increasing function of realized aggregate consumption levels. This function is called an efficient *risk-sharing rule*. The theorem then asserts that if all consumers have a constant, common cautiousness, then their risk-sharing rules are linear (or, to be more precise, affine). The more general case, in which the cautiousness is not constant or common, was investigated in the companion paper, Hara, Huang, and Kuzmics (2006), henceforth HHK, and the references therein. A benchmark result for this case (Proposition 3 of HHK) is that the curvature of the risk-sharing rule and the level of cautiousness have a one-to-one correspondence. More specifically, the curvature is a linear function of the difference between the individual consumer's cautiousness and the representative consumer's counterpart.

The characterization of efficient risk allocation in incomplete asset markets is much more difficult. The reason is that the risk-sharing rules depend not only on the individual consumers' risk aversion but also on the risks that they cannot hedge due to the incompleteness of asset markets. Franke, Stapleton, and Subrahmanyam (1998) (henceforth FSS) exploited a nicely tractable approach to this difficult problem.[1] They considered an environment in which two disjoint types of risks are present. The first one is the macroeconomic risks, which would affect all consumers' consumption levels and for which there is a complete set of markets. Hence, via asset transactions, they can attain, subject to the budget constraint, any consumption pattern as long as it is a function of the aggregate endowments of the economy. The second type of risks are idiosyncratic risks. These are risks affecting the individual consumers' initial endowments, which are independently distributed from the macroeconomic risks, and for which there is no asset market at all. Each individual consumers must therefore bear all of his own idiosyncratic risk.

Mathematically, if an individual consumer with an expected utility function v shares the risk ζ of the macroeconomic risk and owns the idiosyncratic risk ξ, then his expected utility equals $E(v(\zeta + \xi))$. By the law of iterated expectation, this can be rewritten $E(E(v(\zeta + \xi) \mid \zeta))$. Here, by the assumption of stochastic independence between the macroeconomic and idiosyncratic risks, if we define an *induced* utility function u by $u(x) = E(v(x + \xi))$ for every deterministic consumption level x, then the expected utility can be written as $E(u(\zeta))$. To characterize efficient risk allocation in this incomplete market setting, therefore, it is sufficient

[1] Prior to this, for example, Weil (1992) also used the same approach although his model was for the equity premium and risk-free rate puzzles and imposed ex-ante homogeneity assumptions on consumers' risk attitudes and initial endowments.

to apply the results on efficient risk allocations in complete markets in which the *original* utility function v has been replaced by the induced utility function u. It is for this reason that the idiosyncratic risks ξ can synonymously be called *background risks.* It is also at the heart of the FSS approach. They identified a case in which the curvature of a risk-sharing rule can be unambiguously characterized (Theorem 3): if all consumers' original utility functions u exhibit common constant cautiousness and if some consumers have background risks but others do not, then the sharing rule for any consumer without background risk is a concave function of aggregate consumption.

In this paper, we extend their analysis on efficient risk-sharing rules in three ways. We do not restrict attention to original utility functions being in the class of constant cautiousness (or, equivalently, hyperbolic absolute risk aversion, HARA for short), and we incorporate two types of heterogeneity. One is with regards to the consumers' original utility function and the other is with regards to the distributions of background risks.

Proposition 3 of HHK identifies cautiousness as the determinant of the curvature of risk-sharing rules. In our first result (Theorem 1) in this paper, we find sufficient conditions on the original utility function under which the cautiousness is increased by the presence of a background risk at any level of consumption. These conditions are met not only by HARA utility functions, which therefore implies that Theorem 3 of FSS can be derived directly from this result, but also by other, more general utility functions. More importantly in the present context, it also points to the directions along which Theorem 3 of FSS cannot be generalized. Specifically, we show that even in an economy of two consumers who have the same original utility function and of which one has a riskier background risk than the other in the sense of second-order stochastic dominance, if the cautiousness of the common original function is strictly decreasing or if the two background risks are both nonzero, then the consumer with the less risky background risk may well have a convex, rather than concave, risk-sharing rule.

While no general result in the spirit of Theorem 3 of FSS can be obtained, we can say more about the behavior of cautiousness and of risk-sharing rules for the cases of very low and very high realizations of aggregate consumption. Our first result there is that at high aggregate consumption levels, the effect of the background risks on the curvature of the risk-sharing rules is almost negligible (Proposition 2) and, as identified in Proposition 3 of HHK, the curvature is determined solely by the cautiousness of the original utility functions (Proposition 3). The second result shows that for the curvature at low consumption levels, the lower tail distribution of the idiosyncratic risk is a key factor (Theorem 2). If it puts a strictly positive probability on the minimum levels that these risks can attain (as in the case of a discrete random variable taking finitely many values), then the curvature is again determined solely by the cautiousness of the original utility functions. On the other hand, if the cumulative distribution function of the idiosyncratic risk of a consumer is continuous from the right at the minimum level (as in the case of a continuous random variable), then the curvature depends intricately on the degree of non-zero coefficients of the Taylor series expansion of the cumulative distribution function. In particular, the curvature no longer has a one-to-one correspondence with the cautiousness of the original utility functions (Proposition 4).

Although our result for high aggregate consumption levels are intuitive, the results for low aggregate consumption levels are not quite so. The driving force behind them is the Inada condition of the original utility functions. The condition is often considered merely as a technical assumption to guarantee the interior consumptions, but it in fact turns out to be an important property to capture the asymptotic cautiousness of the induced utility functions as well.

The curvature of the risk-sharing rules do indeed matter to the prediction of equilibrium prices and allocations. The mutual fund theorem clarifies when the risk-sharing rules are linear and how the (constant) slopes are related to the individual consumers' risk attitudes. Whenever the theorem fails and the risk-sharing rules have nonzero curvatures, one way for consumers to attain their optimal consumption patterns (implied by their risk-sharing rules) is to buy or sell options written on the aggregate consumption. As can be seen from Leland (1980), Brennan and Solanki (1981), and Proposition 5 of Huang (2005), the curvature measures how much options a consumer should buy or sell in proportion to the consumer's share in the aggregate consumption. Moreover, as can easily be derived from equation (3) of Lemma 1 of HHK (which is due to Wilson (1968)), the cautiousness and relative risk aversion of the representative consumer are the weighted averages of the individual counterparts where the weights are the slopes of the risk-sharing rules. Nonzero curvatures in the risk-sharing rules are, therefore, responsible for the variation of the representative consumer's risk attitudes as aggregate consumption levels vary. This has important implications on asset pricing, as explained in HHK. Hara (2006) explored implications in a continuous-time model via Ito's Lemma.

This paper is organized as follows. The formal model and preliminary results are presented in the next section. The effect of the presence of background risks on induced utility functions for all consumption levels is investigated in Section 3 in which we also investigate the robustness, by means of examples, of Theorem 3 of FSS by relaxing any one of the main assumptions made in that theorem. We then show in Section 4 that the asymptotic behavior of the cautiousness of the induced utility function for high consumption levels is the same as that for the original utility function. Implications of this result on the risk-sharing rules are also given. Section 5 investigates the asymptotic cautiousness for low consumption levels, and presents the most intricate result in this paper, on the influence of the lower tail distribution of the background risk on cautiousness. Again, implications of these results on the efficient risk-sharing rules are explored. Section 6 concludes, suggesting a future direction of research.

2 Model

There are I consumers, $i \in \{1, \dots, I\}$. Consumer i has a von-Neumann Morgenstern (also known as Bernoulli) utility function $v_i : (\underline{c}_i, \infty) \to \boldsymbol{R}$, where $\underline{c}_i > -\infty$. Note that the domain is assumed to be bounded from below but not from above. We assume that v_i is infinitely many times differentiable and satisfies $v_i'(x_i) > 0$ and $v_i''(x_i) < 0$ for every $x_i > \underline{c}_i$. As we alluded to in the introduction, we impose the Inada condition, $v_i'(x_i) \to \infty$ as $x_i \to \underline{c}_i$ and $v_i'(x_i) \to 0$ as $x_i \to \infty$. This condition allows us to apply results in HHK. We also assume, to simplify the exposition, that the limit $\lim_{x_i \to \underline{c}_i} s_i(x_i)$ exists and equals zero, and that the limits $\lim_{x_i \to \infty} s_i'(x_i)$ and $\lim_{x_i \to \underline{c}_i} s_i'(x_i)$ exist and are finite. These assumptions are satisfied by utility functions exhibiting strictly decreasing linear risk tolerance (or, equivalently, strictly increasing hyperbolic absolute risk aversion).

Each consumer faces two sources of risks. The first one is the risk about the macroeconomic risk, which can be completely shared among consumers. This risk is described by a probability measure space $(\Omega, \mathscr{F}, P)$, for which the expectation operator is given by E. The second one is the idiosyncratic component, which is the risk each consumer must bear without hedging. This risk is described by a probability measure space $(\Theta, \mathscr{G}, Q)$, for which the expectation operator is denoted by E^Q. The probability measure space describing the entire risk of the economy is therefore the product one, $(\Omega \times \Theta, \mathscr{F} \otimes \mathscr{G}, P \otimes Q)$, so that the two probability measures P and Q are stochastically independent. The corresponding expectation operator is denoted by $E^{P \otimes Q}$.[2]

We assume that each consumer i owns endowments $\xi_i : \Theta \to \boldsymbol{R}$ on the idiosyncratic component. The cumulative distribution function of ξ_i is denoted by $G_i : \boldsymbol{R} \to [0, 1]$, with $G_i(z_i) = Q\left(\{\theta \in \Theta \mid \xi_i(\theta) \leq z_i\}\right)$ for every $z_i \in \boldsymbol{R}$. For simplicity, we use the following assumptions throughout the paper. First, the support of the distribution of G_i is bounded, that is, there are two numbers $\underline{e}_i$ and $\overline{e}_i$ such that $G_i\left(\underline{e}_i\right) = 0$ and $G_i\left(\overline{e}_i\right) = 1$. Second, ξ_i has zero mean, that is, $\int_{\underline{e}_i}^{\overline{e}_i} y_i \, dG_i(y_i) = 0$. The first assumption guarantees that all the expected values that we consider in the subsequent analysis are well defined and Leibnitz's rule is applicable, so that the order of integration and differentiation for smooth functions can be swapped. The second is a normalization and implies that $\underline{e}_i \leq 0$ and $\overline{e}_i \geq 0$. Note also that while the idiosyncratic risks are stochastically independent of the macroeconomic risks, the ξ_i may well be correlated with one another.

If $\zeta_i : \Omega \to \boldsymbol{R}$ is consumer i's share of the aggregate marketed endowment, his final consumption is $\zeta_i + \xi_i$, from he obtains the expected utility level $E^{P \otimes Q}\left(v_i\left(\zeta_i + \xi_i\right)\right)$. By Fubini's Theorem, this expected utility equals

$$E\left(E^Q\left(v_i\left(\zeta_i + \xi_i\right)\right)\right). \tag{1}$$

Hence, if we define his *induced utility function*

$$u_i\left(x_i\right) = E^Q\left(v_i\left(x_i + \xi_i\right)\right),$$

then the expected utility level (1) equals $E\left(u_i\left(\zeta_i\right)\right)$. Hence identifying properties of the efficient

[2]Although we assume throughout the paper that there is a single probability measure Q on Θ, we could easily accommodate the case in which consumers have differing probabilities on Θ.

allocations of the aggregate endowment ζ with respect to the original utility functions v_i in the presence of the idiosyncratic risks ξ_i is equivalent to identifying those with respect to the induced utility functions u_i without any idiosyncratic risk. Since the idiosyncratic risks ξ_i have been put in the background of the induced utility function u_i, these risk shall henceforth be referred to as *background risks* as well.

In this reformulation, the realized consumption level, inclusive of the realized background risk, must of course be in the domain $(\underline{c}_i, \infty)$ almost surely. To guarantee this, we concentrate on the consumption levels $x_i > \underline{c}_i - \underline{e}_i$. Denote $\underline{d}_i = \underline{c}_i - \underline{e}_i$, then the domain of the induced utility function u_i is $(\underline{d}_i, \infty)$.

Define the consumer's original (absolute) risk tolerance, $s_i : (\underline{c}_i, \infty) \to \boldsymbol{R}_{++}$, by

$$s_i(x_i) = -\frac{v_i'(x_i)}{v_i''(x_i)}.$$

This is just the reciprocal of the consumer's Arrow-Pratt coefficient of absolute risk aversion $a_i(x_i) = -v_i''(x_i)/v_i'(x_i)$. The (absolute) risk tolerance of the corresponding induced utility function u_i shall be denoted by $t_i : (\underline{c}_i, \infty) \to \boldsymbol{R}_{++}$. By Leibnitz's rule,

$$t_i(x_i) = -\frac{E^Q(v_i'(x_i + \xi_i))}{E^Q(v_i''(x_i + \xi_i))}.$$

Following the terminology coined by Wilson (1968) the derivative of risk tolerance shall be called (absolute) *cautiousness*. The consumer's original cautiousness is therefore given by $s_i'(x_i)$, while the consumer's induced (absolute) cautiousness is given by $t_i'(x_i)$ for x_i in the respective domain.

Denote by $\psi_i(x_i)$ the prudence of v_i of Kimball (1990):

$$\psi_i(x_i) = -\frac{v_i'''(x_i)}{v_i''(x_i)}.$$

Also denote by φ_i the prudence of the induced utility function u_i, then we have

$$\varphi_i(x_i) = -\frac{E^Q(v_i'''(x_i + \xi_i))}{E^Q(v_i''(x_i + \xi_i))}.$$

The following relationship among the risk tolerance, prudence, and cautiousness is easy to prove and yet useful.

Lemma 1 *1. For every $x_i > \underline{c}_i$, $s_i'(x_i) = s_i(x_i)\psi_i(x_i) - 1$.*

2. For every $x_i > \underline{d}_i$, $t_i'(x_i) = t_i(x_i)\varphi_i(x_i) - 1$.

In HHK we analyzed efficient risk sharing rules for consumers who were heterogeneous with respect to their risk-attitudes, but who did not face these idiosyncratic background risks. To characterize efficient allocations in the present context, all we need is to find implications of the background risks ξ_i on the induced utility functions u_i. This is the task of this paper.

The following is a review of HHK. Let $\underline{d} = \sum \underline{d}_i$. An infinitely differentiable function $f : (\underline{d}, \infty) \to (\underline{d}_1, \infty) \times \cdots \times (\underline{d}_I, \infty)$ is an *efficient risk-sharing rule* with respect to the induced utility functions u_i if there exists a vector of strictly positive utility weights, $(\lambda_1, \ldots, \lambda_I)$, such that for every stochastic aggregate endowment ζ of the economy, the social welfare function $\sum \lambda_i E(u_i(\zeta_i))$ is maximized under the resource constraint $\sum \zeta_i = \zeta$ at $(\zeta_1, \ldots, \zeta_I) = (f_1(\zeta), \ldots, f_I(\zeta))$. This means that the allocation $(f_1(\zeta), \ldots, f_I(\zeta))$ is a Pareto-efficient allocation of the macroeconomic ζ with respect to the induced utility functions $(u_1, \ldots, u_I)$. This is equivalent to saying that $(f_1(\zeta) + \xi_1, \ldots, f_I(\zeta) + \xi_I)$ is a constrained efficient allocation of $\zeta + \sum \xi_i$ with respect to the original utility functions $(v_1, \ldots, v_I)$ subject to the constraint that each consumer i must consume his own idiosyncratic risk ξ_i. Corresponding to this social welfare maximization problem, we can define an expected utility function $u : (\underline{d}, \infty) \to \boldsymbol{R}$ of the representative consumer so that $u(x) = \sum \lambda_i u_i(f_i(x))$ and hence $E(u(\zeta)) = \sum \lambda_i E(u_i(f_i(\zeta)))$. Denote by $t : (\underline{d}, \infty) \to \boldsymbol{R}$ the risk tolerance of u.

The following lemma was more or less established in HHK.

Lemma 2 *Assume that u_i satisfies the Inada condition for every i.*

1. *For every i and $x \in \left(\underline{d}, \overline{d}\right)$,*

$$\frac{f_i''(x)}{f_i'(x)} = \frac{1}{t(x)} \left(t_i' \left(f_i(x) \right) - t'(x) \right). \tag{2}$$

2. *Suppose that $\lim_{x_i \to \infty} t_i'(x_i)$ exists (and may be ∞) for every i. Define $\overline{I}$ as the set of consumers i such that $\lim_{x_i \to \infty} t_i'(x_i) \geq \lim_{x_j \to \infty} t_j'(x_j)$ for every j. Then $\lim_{x \to \infty} t'(x)$ exists and equals $\lim_{x_i \to \infty} t_i'(x_i)$ for every $i \in \overline{I}$. Moreover, both $\lim_{x \to \infty} \sum_{i \in \overline{I}} f_i(x)/x$ and $\lim_{x \to \infty} \sum_{i \in \overline{I}} f_i'(x)$ exist and equal 1. Furthermore, $f_i''(x) < 0$ for every $i \notin \overline{I}$ and every sufficiently large $x > \underline{d}$.*

3. *Suppose that $\lim_{x_i \to \underline{d}_i} t_i'(x_i)$ exists (and may be ∞ or $-\infty$) for every i. Define $\underline{I}$ as the set of consumers i such that $\lim_{x_i \to \underline{c}_i} t_i'(x_i) \leq \lim_{x_j \to \underline{c}_j} t_j'(x_j)$ for every j. Then $\lim_{x \to \underline{d}} t'(x)$ exists and equals $\lim_{x_i \to \underline{d}_i} t_i'(x_i)$ for every $i \in \underline{I}$. Moreover, both $\lim_{x \to \underline{d}} \sum_{i \in \underline{I}} \left(f_i(x) - \underline{d}_i \right) / \left(x - \underline{d} \right)$ and $\lim_{x \to \underline{d}} \sum_{i \in \underline{I}} f_i'(x)$ exist and equal 1. Furthermore, $f_i''(x) > 0$ for every $i \notin \underline{I}$ and every sufficiently small $x > \underline{d}$.*

Proof of Lemma 2 Part 1 is nothing but Proposition 3 of HHK. The first statement of part 2 is part 2 of Proposition 10 of HHK. The second statement is part 1 of the same proposition. The third statement follows from part 1 of this lemma, the first statement of this part, and the fact, proved in that paper, that $f_i(x) \to \infty$ as $x \to \infty$. The third part follows analogously. ///

3 Cautiousness and Risk-Sharing Rules for All Consumption Levels

3.1 Cautiousness

Gollier and Pratt (1996, Propositions 2 and 3) gave sufficient conditions under which, if ξ_i has a positive variance, then $t_i(x_i) < s_i(x_i)$, that is, the background risk makes the consumer less risk tolerant (more risk averse). They called utility functions having this property *risk vulnerable.* The following result provides sufficient conditions under which $t_i'(x_i) \geq s_i'(x_i)$.

Theorem 1 *If $s_i'(x_i) \geq 0$, $s_i''(x_i) \leq 0$, and $s_i'''(x_i) \geq 0$ for every $x_i > \underline{c}_i$, then $t_i'(x_i) \geq s_i'(x_i)$ for every $x_i > \underline{d}_i$. The inequality is strict if, in addition, $s_i'(x_i) \neq 0$ for every $x_i > \underline{c}_i$.*

This theorem says that at any given consumption level $x_i > \underline{d}_i$, the cautiousness $t_i'(x_i)$ of the induced utility function u_i is not exceeded by the cautiousness $s_i'(x_i)$ of the original utility function v_i if the cautiousness s_i' is a non-negative, non-increasing, and convex function of consumption levels. The first condition sign is nothing but non-increasing absolute risk aversion (DARA). The second sign condition is that the risk tolerance s_i be concave, which implies that the absolute risk aversion a_i is convex.[3]

Proof of Theorem 1 Let $x_i > \underline{d}_i$. By Lemma 1 and direct calculation,

$$
\begin{aligned}
t_i'(x_i) &= \frac{E^Q(v_i'(x_i+\xi_i))\,E^Q(v_i'''(x_i+\xi_i))}{(E^Q(v_i''(x_i+\xi_i)))^2} - 1 \\
&= E^Q\left(\frac{v_i'(x_i+\xi_i)\,v_i'''(x_i+\xi_i)}{(v_i''(x_i+\xi_i))^2}\frac{(v_i''(x_i+\xi_i))^2}{v_i'(x_i+\xi_i)}\right)\frac{E^Q(v_i'(x_i+\xi_i))}{(E^Q(v_i''(x_i+\xi_i)))^2} - 1 \\
&= E^Q\left((s_i'(x_i+\xi_i)+1)\frac{(v_i''(x_i+\xi_i))^2}{v_i'(x_i+\xi_i)}\right) \\
&\quad \times E^Q\left(\frac{v_i'(x_i+\xi_i)}{-v_i''(x_i+\xi_i)}\frac{-v_i''(x_i+\xi_i)}{E^Q(-v_i''(x_i+\xi_i))}\right)\frac{1}{E^Q(-v_i''(x_i+\xi_i))} - 1.
\end{aligned}
$$

[3]The converse, however, does not hold. Even when the absolute risk aversion is convex, the absolute risk tolerance may not be concave. An undesirable implication of concave absolute risk tolerance, which is not implied by convex absolute risk aversion, is increasing relative risk aversion: Let $\underline{c}_i = 0$, then, by the Inada condition, $s_i(x_i) \to 0$ as $x_i \to 0$. Thus the concavity of s_i implies that its elasticity is not greater than one; and it is strictly less than one beyond any point at which s_i'' is strictly negative. But it can be shown that the elasticity is strictly less than one if and only if the first derivative of the relative risk aversion is strictly positive.

By applying Jensen's inequality to the hyperbolic function and noting that the function $z_i \mapsto -v_i''(x_i+z_i)/E^Q(-v_i''(x_i+\xi_i))$ has the property of a Radon-Nikodym derivative, we obtain

$$\begin{aligned}
&E^Q\left(\frac{v_i'(x_i+\xi_i)}{-v_i''(x_i+\xi_i)}\frac{-v_i''(x_i+\xi_i)}{E^Q(-v_i''(x_i+\xi_i))}\right)\\
\geq\ &\frac{1}{E^Q\left(\dfrac{-v_i''(x_i+\xi_i)}{v_i'(x_i+\xi_i)}\dfrac{-v_i''(x_i+\xi_i)}{E^Q(-v_i''(x_i+\xi_i))}\right)}\\
=\ &\frac{E^Q(-v_i''(x_i+\xi_i))}{E^Q\left(\dfrac{(v_i''(x_i+\xi_i))^2}{v_i'(x_i+\xi_i)}\right)},
\end{aligned}$$

where the weak inequality $\geq$ holds as a strict inequality unless $s_i'(x_i+\xi_i)=0$. Thus

$$t_i'(x_i)\geq E^Q\left(\left(s_i'(x_i+\xi_i)+1\right)\frac{(v_i''(x_i+\xi_i))^2}{v_i'(x_i+\xi_i)}\right)\frac{1}{E^Q\left(\dfrac{(v_i''(x_i+\xi_i))^2}{v_i'(x_i+\xi_i)}\right)}-1.$$

Since $-v_i''(x_i+z_i)/v_i'(x_i+z_i)$ and $-v_i''(x_i+z_i)$ are non-increasing functions of z_i, so is their product $(v_i''(x_i+z_i))^2/v_i'(x_i+z_i)$. Since $s_i''\leq 0$, $s_i'(x_i+z_i)$ is also a non-increasing function of z_i. Thus

$$E^Q\left(\left(s_i'(x_i+\xi_i)+1\right)\frac{(v_i''(x_i+\xi_i))^2}{v_i'(x_i+\xi_i)}\right)\geq E^Q\left(\left(s_i'(x_i+\xi_i)+1\right)\right)E^Q\left(\frac{(v_i''(x_i+\xi_i))^2}{v_i'(x_i+\xi_i)}\right).$$

Hence

$$t_i'(x_i)\geq E^Q\left(s_i'(x_i+\xi_i)+1\right)-1=E^Q\left(s_i'(x_i+\xi_i)\right)$$

Finally, since $s_i'''(x_i+z_i)\geq 0$, $s_i'(x_i+z_i)$ is a convex function of z_i. Thus, by Jensen's inequality, $E^Q(s_i'(x_i+\xi_i))\geq s_i'(x_i)$, which completes the proof. ///

The sign restrictions on the first two derivatives of s_i are sufficient for the original utility function v_i to be risk-vulnerable in the sense of Gollier and Pratt (1996). The sign restrictions on the first three derivatives of s_i also imply that the prudence of the induced utility function exceeds the prudence of the original utility function, that is, $\varphi_i(x_i)\geq\psi_i(x_i)$ for all $x_i>\underline{d}_i$. Indeed, Theorem 1 established that $t_i'(x_i)\geq s_i'(x_i)$, which is equivalent to $\varphi_i(x_i)t_i(x_i)\geq\psi_i(x_i)s_i(x_i)$ by Lemma 1. Risk-vulnerability means that $t_i(x_i)\leq s_i(x_i)$. These two inequalities together imply that $\varphi_i(x_i)\geq\psi_i(x_i)$.

To obtain $t_i'(x_i)>s_i'(x_i)$ locally, it is, of course, sufficient that the three conditions are satisfied for the range of the background risk only, that is, for all z_i such that $x_i+\underline{e}_i\leq z_i\leq x_i+\overline{e}_i$. However, if we assume the three conditions to hold globally, the assumption of DARA (s_i' being always nonnegative) becomes redundant. Indeed, suppose then that there were to exist an $x_0>\underline{c}_i$ such that $s_i'(x_0)<0$. Let $\varepsilon\in(0,-s_i'(x_0))$. Then, by the assumption that $s_i''(x_i)\leq 0$ for all $x_i\geq\underline{c}_i$, $s_i'(x_i)\leq-\varepsilon<0$ for all $x_i\geq x_0$. But then there must be an $\bar{x}_i$ such that $s_i(x_i) < 0$ for all $x_i > \bar{x}_i$. We thus arrive at a contradiction.

The conditions for Theorem 1 satisfied by every HARA utility function, with the second and third derivatives being always zero.

Assumption 1 There exist a $\tau_i \in \boldsymbol{R}$ and a $\gamma_i \in \boldsymbol{R}_{++}$ such that $s_i(x_i) = \tau_i + \gamma_i x_i$ for every $x_i > \underline{c}_i$.

Corollary 1 *Under Assumption 1, if* $\mathrm{Var}(\xi_i) > 0$*, then* $t'_i(x_i) > s'_i(y_i)$ *for every* $x_i > \underline{d}_i$ *and every* $y_i > \underline{d}_i$

Corollary 1 shows that if the cautiousness s'_i of the original utility function v_i is constant, then we have $t'_i(x_i) > s'_i(y_i)$ regardless of the choice of $x_i > \underline{d}_i$ and $y_i > \underline{d}_i$. Although Theorem 1 still holds if the cautiousness is strictly decreasing, we then lose such unambiguous ranking in cautiousness between the original and the induced utility functions. Suppose, for instance, that the original utility function is given by $v_i(x_i) = \ln(1 - \exp(-x_i))$. It is easy to check that $v'_i(x_i) > 0$ and $v''_i(x_i) < 0$ for every $x_i > 0$, and that the Inada condition is satisfied. Then the risk tolerance is $s_i(x_i) = 1 - \exp(-x_i)$ and the cautiousness is $s'_i(x_i) = \exp(-x_i)$, which is exponentially decreasing. Moreover, $s''_i(x_i) < 0$ and $s''_i(x_i) > 0$, and hence s_i satisfies the conditions of Theorem 1. Consider two consumers. The first consumer has a background risk, which takes values 1 or -1 with probability 1/2 each, while the second consumer has no background risk. Figure 1 illustrates that both the original cautiousness s'_i (solid curve) as well as the induced cautiousness t'_i (dashed curve) are strictly decreasing. While the induced cautiousness is higher than the original cautiousness for every given level of consumption, the induced cautiousness at sufficiently high levels of consumption is in fact smaller than the original cautiousness at sufficiently low levels of consumption.

While Theorem 1 compares the cautiousness of the original utility function and the induced utility function, it has nothing to say on the comparison of the cautiousness of two induced utility functions, for which one background risk is riskier than the other in the second order stochastic dominance relation. To see this, consider the following example. Assume that both consumers' original utility functions are $v_i(x_i) = -x_i^{-3}/3$, exhibiting constant relative risk aversion 4. The first consumer has a background risk that takes values 1 or -1 with probability 1/2 each (variance 1), while the second consumer has a background risk that takes values 1, 0, and -1 with probability 1/3 each (variance 2/3). Note that the second consumer's background risk second-order stochastically dominates the first consumer's background risk.

The cautiousness t_i for the induced utility functions u_i are depicted in Figure 2. It shows that cautiousness for each consumer increases up to a point, after which it decreases; and that neither is uniformly higher than the other.

3.2 Risk-Sharing Rules

The main qualitative result (Theorem 3) of FSS can be immediately derived from Corollary 1.

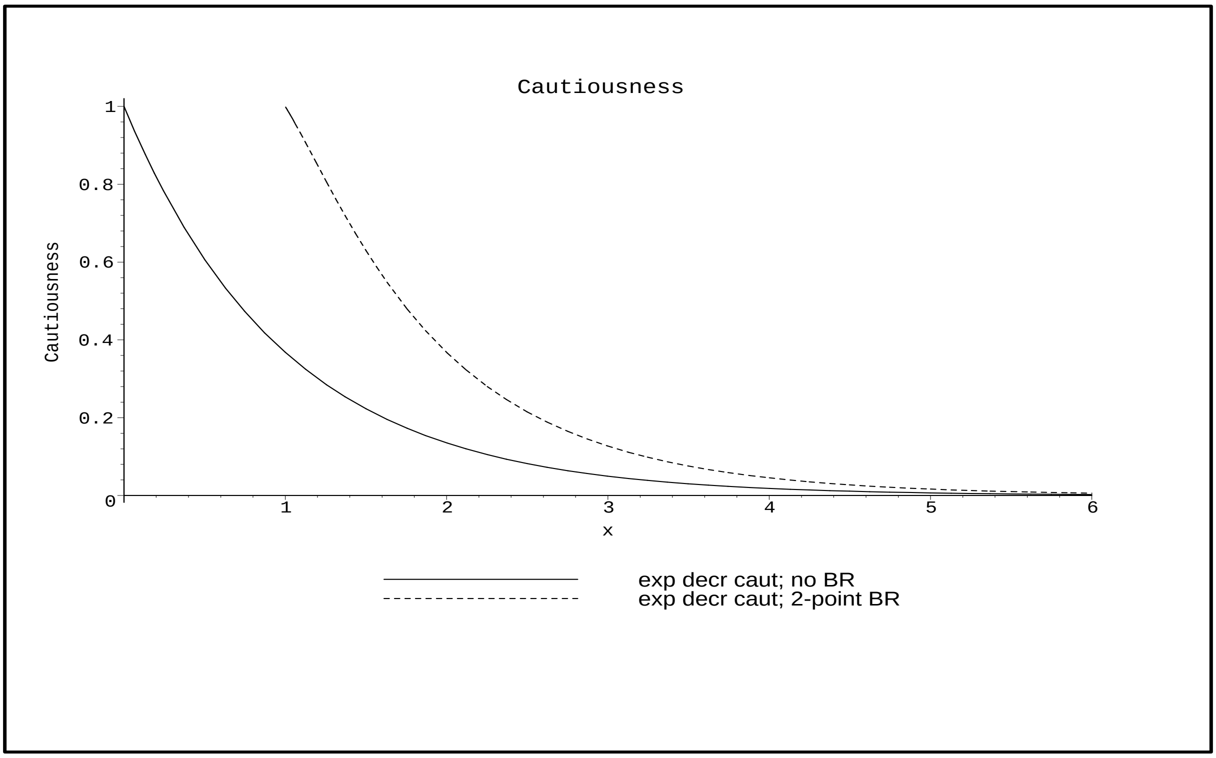

Figure 1: Cautiousness of a utility function exhibiting exponentially decreasing cautiousness without background risk, and the same utility function with a two-point (probability 1/2 on −1 and 1 each) background risk.

Proposition 1 (Theorem 3 of FSS) *Under Assumption 1, suppose in addition that* $\gamma_1 = \cdots = \gamma_I$*, there is a consumer* i *such that* $\mathrm{Var}(\xi_i) = 0$*, and there is another consumer* j *such that* $\mathrm{Var}(\xi_j) > 0$*. Then, for every consumer* i *with* $\mathrm{Var}(\xi_i) = 0$*, we have* $f_i''(x) < 0$ *for every* x*.*

Proof of Proposition 1 If $\mathrm{Var}(\xi_i) = 0$, then, by part 1 of Lemma 2 and Corollary 1, $f_i''(x)/f_i'(x) \leq f_j''(x)/f_j'(x)$ for every j, and strict inequality for some j, as $\mathrm{Var}(\xi_j) > 0$ for some j. We cannot have $f_i''(x) \geq 0$ for any x, because, if we had, then $f_j''(x) \geq 0$ for every j, with strict inequality for some j, which would contradict $\sum_j f_j''(x) = 0$. Hence $f_i''(x) < 0$ for every x. ///

Two crucial assumptions of Proposition 1 (Theorem 3 in FSS) are that the original utility functions exhibit constant cautiousness (which is, in addition, assumed to be common across consumers) and that there exists at least one consumer who is not exposed to any background risk. In this subsection, we show, by using the two examples of the preceding subsection, that if we relax either of these two assumptions, then the proposition loses its validity. This is true even when all consumers in the economy have the same original utility function.[4] In these examples,

[4]It is easy to see that if consumers have different original utility functions then generally whether a consumer has a concave or convex risk-sharing rule depends also on the now different levels of original cautiousness and not solely on the background risk.

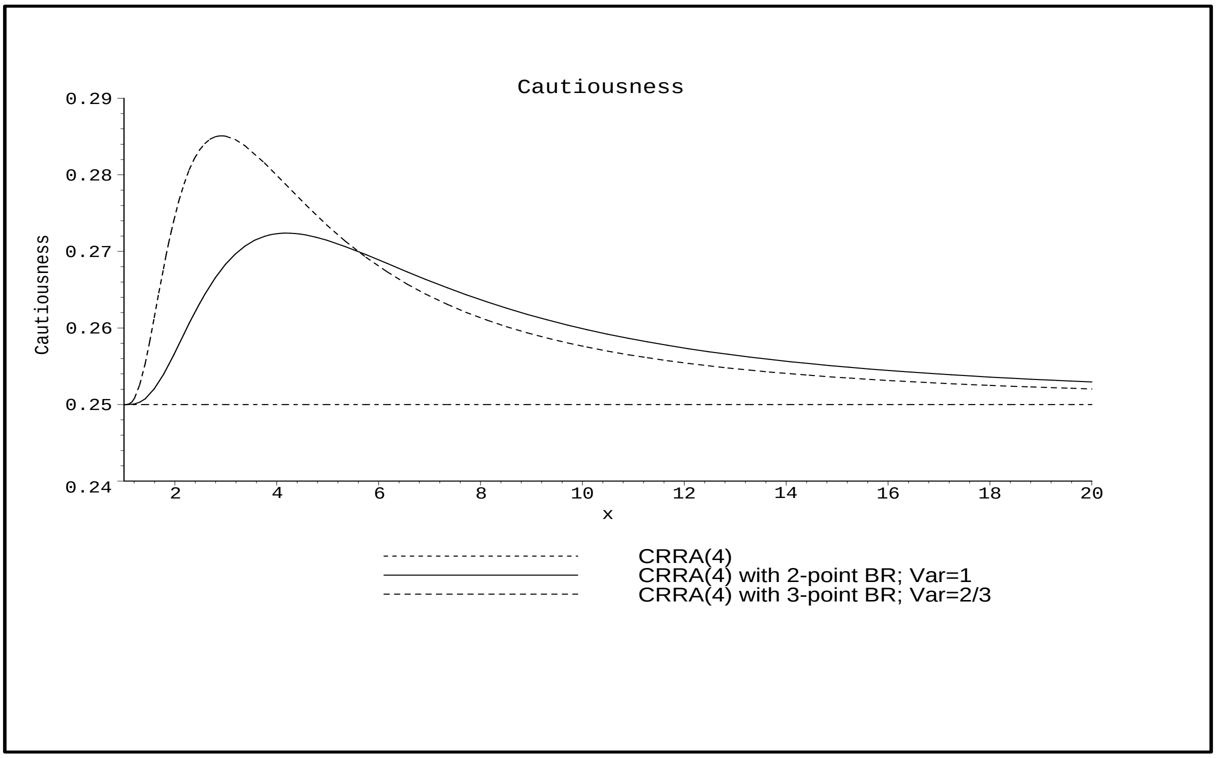

Figure 2: Cautiousness of a utility function exhibiting constant relative risk aversion 4 without background risk, the same utility function with a two-point (probability 1/2 on −1 and 1 each) background risk, and the same utility function with 3-point background risk (probability 1/3 on −1, 0, and 1 each).

the values of the utility weights λ_i become crucial for the determination of the curvature of sharing rules.[5] This is in stark contrast with our results for high and low consumption levels in sections 4 and 5, Theorem 3 of FSS, all the results in HHK, and the mutual fund theorem, which hold for all specifications of utility weights and thus for all efficient risk-sharing rules.

Let us now present our first class of examples. We assume that both consumers have the same original utility function $v_i(x_i) = \ln(1 - \exp(-x_i))$ as discussed above. As before (see Figure 1) the first consumer has a background risk, which takes values 1 or −1 with probability 1/2 each, while the second consumer has no background risk. Since $t_2(x_2) = s_2(x_2)$, $t_1'(x_i) > t_2'(x_i)$ for every $x_i > 0$ by Theorem 1. This means that in the presence of the background risk, the first consumer is more cautious than the second whenever they enjoy the same consumption level.

According to (2), $f_1''(x) > 0 > f_2''(x)$, that is, the risk-sharing rule for the first consumer is locally convex and that for the second consumer is locally concave, if and only if $t_1'(f_1(x)) > t_2'(f_2(x))$, that is, the first consumer is more cautious than the second, when their consumption levels are as implied by the sharing rules f_i. Recall here that the sharing rules do depend on

[5]If the efficient risk-sharing rule corresponds to an equilibrium allocation, then the utility weights depend, in turn, on initial endowments.

the utility weights λ_i and hence the consumers do not necessarily enjoy the same consumption level. We are thus led to consider three cases, $\lambda_1/\lambda_2 \in \{1, 2, 3\}$. The results are demonstrated in Figures 3 and 4. When $\lambda_1/\lambda_2 = 1$, the risk-sharing rule of the first consumer, who has a background risk, is uniformly convex; when $\lambda_1/\lambda_2 = 3$, it is uniformly concave; and when $\lambda_1/\lambda_2 = 2$, it is convex up to a unique inflection point, beyond which it is concave.

This class of examples illustrates a weakness of Proposition 1 (Theorem 3 of FSS). Since all consumers are assumed to have the same constant cautiousness, any consumer with no background risk has a lower cautiousness than any consumer with a background risk, regardless of their individual consumptions levels, and hence regardless of the utility weights λ_i and aggregate consumption levels x. On the other hand, even if a consumer with no background risk has a lower cautiousness than a consumer with a background risk whenever they enjoy the same consumption level, the former may in fact be higher than the latter when their consumption levels are different. The difference can indeed arise, and this accounts for our finding that the curvature of the sharing rules depends quite sensitively on the choice of utility weights λ_i and aggregate consumption level x.

Let us now move on to the second class of examples. We assume that both consumers' original utility functions are $v_i(x_i) = -x_i^{-3}/3$, exhibiting constant relative risk aversion 4. The first consumer has a background risk that takes values 1 or -1 with probability 1/2 each (variance 1), while the second consumer has a background risk that takes values 1, 0, and -1 with probability 1/3 each (variance 2/3). This implies that the second consumer's background risk second-order stochastically dominates the first consumer's background risk. This is the example with a comparison between the original and induced cautiousness given in Figure 2, which shows that an SOSD ranking of consumers' background risks does not in general lead to a ranking of the consumers' levels of cautiousness. As in our first class of examples, the curvatures of the risk-sharing rules depend on the weights λ_i. We considered three cases, $\lambda_1/\lambda_2 \in \{1/3, 1, 3\}$. The results are demonstrated in Figures 5 and 6. When $\lambda_1/\lambda_2 = 1/3$ and when $\lambda_1/\lambda_2 = 1$, the risk-sharing rule of the first consumer, who has a larger background risk, is concave up to a unique inflection point after which it is convex; when $\lambda_1/\lambda_2 = 3$, it is uniformly concave.

Since Proposition 1 (Theorem 3 of FSS) is concerned only with the extreme case where at least one consumer has no background risk, it would be natural to hope to extend the proposition to the case where two consumers' background risks are comparable according to second-order stochastic dominance. This class of examples, however, dashes such a hope. The underlying reasoning is that even if we start with an original utility function exhibiting constant cautiousness, the presence of one background risk makes the cautiousness increase up to some point, after which it decreases. This non-monotonicity in cautiousness makes Theorem 1 inapplicable to predict the change in cautiousness when a mean-preserving spread is added. The result is then that the curvature of the risk-sharing rules depends sensitively on the utility weights λ_i and the aggregate consumption level x.

While these examples demonstrate that no broad generalizations of Proposition 1 are possible, one can obtain more insight into the behavior of the induced cautiousness as well as that of the risk-sharing rules for the limits when aggregate consumption tends to its lowest and its highest values, respectively. This is done in the next two sections.

4 Cautiousness and Risk-Sharing Rules for High Consumption Levels

The following proposition identifies the asymptotic behavior of the cautiousness $t_i'(x_i)$ as $x_i \to \infty$.

Proposition 2 *If* $\lim_{x_i\to\infty} s_i'(x_i) > 0$, *then* $\lim_{x_i\to\infty} t_i'(x_i)$ *exists and equals* $\lim_{x_i\to\infty} s_i'(x_i)$, *and* $u_i'(x_i) \to 0$ *as* $x_i \to \infty$.

Proof of Proposition 2 Let $x_i > \underline{d}_i$ and H_i be the probability measure defined in the proof of Theorem 1. Then, by the mean value theorem, there exist a $z_i^1 \in [\underline{e}_i, \overline{e}_i]$ and a $z_i^2 \in [\underline{e}_i, \overline{e}_i]$ such that $s_i(x_i + z_i^1) = E^{Q_i}(s_i(x_i + \xi_i)) = t_i(x_i)$ and $\psi_i(x_i + z_i^2) = E^{Q_i}(\psi_i(x_i + \xi_i)) = \varphi_i(x_i)$. Hence, by Lemma 1,

$$\begin{aligned} t_i'(x_i) &= t_i(x_i)\varphi_i(x_i) - 1 \\ &= s_i\left(x_i + z_i^1\right)\psi_i\left(x_i + z_i^2\right) - 1 \\ &= \frac{s_i\left(x_i + z_i^1\right)}{s_i\left(x_i + z_i^2\right)}\left(s_i'\left(x_i + z_i^2\right) + 1\right) - 1. \end{aligned}$$

Write $\gamma_i = \lim_{x_i\to\infty} s_i'(x_i)$. By assumption, $s_i'\left(x_i + z_i^2\right) + 1 \to \gamma_i + 1$ as $x_i \to \infty$. Moreover, $s_i(x_i) \to \infty$ as $x_i \to \infty$ and $|s_i(x_i + z_i^1) - s_i(x_i + z_i^2)| \leq 2\gamma_i(\overline{e}_i - \underline{e}_i)$ for every sufficiently large x_i. Hence $\dfrac{s_i\left(x_i + z_i^1\right)}{s_i\left(x_i + z_i^2\right)} \to 1$ as $x_i \to \infty$. Thus $t_i'(x_i) \to \gamma_i$.

The upper side of the Inada condition, $u_i'(x_i) \to 0$ as $x_i \to \infty$, follows from $u_i'(x_i) \leq v_i'(x_i + \underline{e}_i)$ and $v_i'(x_i + \underline{e}_i) \to 0$ as $x_i \to \infty$. ///

Proposition 2 shows that its asymptotic behavior of the cautiousness of the induced utility function, as the consumption level tends unboundedly large, is the same as the cautiousness of the original utility function.

The implications of Proposition 2 on the risk-sharing rules are given in the following proposition.

Proposition 3 *Define* $\overline{I}$ *as the set of consumers* i *such that* $\lim_{x_i\to\infty} s_i'(x_i) \geq \lim_{x_j\to\infty} s_j'(x_j)$ *for every* j. *Then, for every efficient risk-sharing rule* f *and the corresponding representative consumer's cautiousness* t', $\lim_{x\to\infty} t'(x)$ *exists and equals* $\lim_{x_i\to\infty} s_i'(x_i)$ *for every* $i \in \overline{I}$, *both* $\lim_{x\to\infty} \sum_{i\in\overline{I}} f_i(x)/x$ *and* $\lim_{x\to\infty} \sum_{i\in\overline{I}} f_i'(x)$ *exist and equal 1, and* $f_i''(x) < 0$ *for every* $i \notin \overline{I}$ *and every sufficiently large* $x > \underline{d}$.

This proposition can be proved by Proposition 2 and part 2 of Lemma 2. The following proposition is the main result, which is concerned with low consumption levels.

5 Cautiousness and Risk-Sharing Rules for Low Consumption Levels

In this section we present the second main result of this paper, which is concerned with the asymptotic behavior of cautiousness as the consumption levels tend to the minimum level $\underline{d}_i$. We will see that if the cumulative distribution function G_i is polynomial on the right side of its minimum value $\underline{e}_i$, then the asymptotic cautiousness, $\lim_{x_i \to \underline{d}_i} t_i'(x_i)$, depends on the *rate* of convergence of the distribution function as $x_i \to \underline{e}_i$ from right, as measured by the minimum degree of non-zero coefficients of the polynomial. By saying that v_i satisfies Assumption 1 on $\left(\underline{c}_i, \underline{c}_i + \overline{\delta}\right]$ with $\overline{\delta} > 0$, we mean that there exist a $\tau_i \in \boldsymbol{R}$ and a $\gamma_i \in \boldsymbol{R}_{++}$ such that $s_i(x_i) = \tau_i + \gamma_i x_i$ for every $x_i \in \left(\underline{c}_i, \underline{c}_i + \overline{\delta}\right]$.

Theorem 2 *Suppose that there exist a $\overline{\delta} > 0$ such that v_i satisfies Assumption 1 on $\left(\underline{c}_i, \underline{c}_i + \overline{\delta}\right]$, and that there exists a sequence $(\kappa_0, \kappa_1, \kappa_2, \dots)$ of real numbers such that $\kappa_n = 0$ for every sufficiently large n and*

$$G_i(z_i) = \sum_{n=0}^{\infty} \kappa_n \frac{(z_i - \underline{e}_i)^n}{n!} \tag{3}$$

for every $z_i \in \left[\underline{e}_i, \underline{e}_i + \overline{\delta}\right]$. Let N_i be the nonnegative integer such that $\kappa_{N_i} \neq 0$ and $\kappa_n = 0$ for every $n < N_i$. If $N_i \gamma_i < 1$, then $u_i'(x_i) \to \infty$ as $x_i \to \underline{d}_i$, and $\lim_{x_i \to \underline{d}_i} t_i'(x_i)$ exists and equals

$$\frac{\gamma_i}{1 - N_i \gamma_i}.$$

The first condition of this theorem says that the original utility function v_i is a HARA-utility function on some lower tail interval of its domain $(\underline{c}_i, \infty)$. In Remark 2 after the proof of this theorem, we will explain how to dispense with this assumption.

The second condition says that the cumulative distribution function G_i is a polynomial on the right side of $\underline{e}_i$. The nonnegative integer N_i designates the minimum degree of nonzero coefficients of the polynomial. Since $G_i(z_i) > 0$ for every $z_i \in \left(\underline{e}_i, \underline{e}_i + \overline{\delta}\right]$, there must be such an N_i, and it must satisfy $\kappa_{N_i} > 0$. In Remark 1 after the proof of the theorem, we will explain why it is difficult to weaken this assumption to the assumption that G_i is merely real analytic, so that $\kappa_n = 0$ for possibly infinitely many n. If $\kappa_0 > 0$, then $N_i = 0$, G_i is not continuous from left at $\underline{e}_i$, and κ_0 equals the probability that the background risk takes $\underline{e}_i$. On the other hand, if $\kappa_0 = 0$, then $N_i \geq 1$, G_i is continuous from left at $\underline{e}_i$, and N_i equals the minimum degree of differentiation for which the right derivative of G_i at $\underline{e}_i$ is nonzero. Note that this condition accommodates the case in which the background risk is a discrete random variable with only finitely many possible realizations: We can take $\overline{\delta}$ to be smaller than the difference between the smallest and the second smallest realizations and $\kappa_n = 0$ for every $n \geq 0$.

The third condition, $N_i\gamma_i < 1$, is a joint assumption on the original utility function v_i and the cumulative distribution function G_i. This imposes no restriction on v_i if $N_i = 0$, that is, there is a strictly positive probability on the realization on $\underline{e}_i$. Otherwise, it requires the cautiousness γ_i be smaller than the reciprocal of the minimum degree of nonzero coefficients of the polynomial for G_i. Without this condition, the induced utility function u_i may not satisfy the lower side of the Inada condition. Note that these conditions do not impose any restriction on the variance of the idiosyncratic risks or on the value of κ_{N_i}, except that it is strictly positive. What this means, for the case in which G_i is a uniform distribution and $\kappa_n = 0$ for every $n \neq 1$, is that the asymptotic cautiousness does not depend on the length of the support.

The conclusion of the theorem is that we need to multiply $1 - N_i\gamma_i$ to the asymptotic cautiousness of the original utility function v_i to obtain the asymptotic cautiousness of the induced utility function u_i. The latter is therefore larger. Also, the thinner the lower tail distribution of G_i, the larger the multiplier. If $\underline{c}_i = 0$, then γ_i equals the reciprocal of the relative risk aversion. If $0 < N_i\gamma_i < 1$, then the relative risk aversion is greater than N_i in the limit as the consumption level is close to zero.

The Inada condition, $u_i'(x_i) \to \infty$ as $x_i \to \underline{d}_i$, is also guaranteed by Theorem 2. In fact, it is guaranteed by the conditions of Proposition 3 of Huang (2002a), which are weaker than the conditions of this theorem. However, we are not sure if his conditions guarantee that the limit of its cautiousness t_i' exists as the consumption level converges to the minimum subsistence level $\underline{d}_i$. Since the existence of this limit is needed to characterize the curvature of the risk-sharing rules near the minimum subsistence level, we opt for the stronger conditions.

The proof of the theorem is relegated to the appendix. We now turn to its implications on the risk sharing rules.

Proposition 4 *Suppose that the same set of assumptions as in Theorem 2 is met for every consumer i. Define $\underline{I}$ as the set of consumers i such that*

$$\frac{\gamma_i}{1 - N_i\gamma_i} \leq \frac{\gamma_j}{1 - N_j\gamma_j}$$

for every j. Then, for every efficient risk-sharing rule f and the corresponding representative consumer's cautiousness t', both $\lim_{x\to\underline{d}} t'(x)$ exists and equals $\gamma_i/(1 - N_i\gamma_i)$ for every $i \in \underline{I}$, $\lim_{x\to\underline{d}} \sum_{i\in\underline{I}} (f_i(x) - \underline{d}_i) / (x - \underline{d})$ and $\lim_{x\to\underline{d}} \sum_{i\in\underline{I}} f_i'(x)$ exist and equal 1, and $f_i''(x) > 0$ for every $i \notin \underline{I}$ and every sufficiently small $x > \underline{d}$.

This can be proved by Theorem 2 and part 3 of Lemma 2.

A special case of interest is where the original utility functions exhibit constant relative risk aversion. In this case, Propositions 3 and 4 can be merged into a simple proposition.

Corollary 2 *Assume that for every i:*

1. *v_i exhibit constant relative risk aversion $\beta_i > 0$.*

2. *There exist a* $\overline{\delta}^i > 0$ *and a sequence* $(\kappa_0^i, \kappa_1^i, \kappa_2^i, \dots)$ *of real numbers such that* $\kappa_n^i = 0$ *for every sufficiently large* n *and*

$$G_i(z_i) = \sum_{n=0}^{\infty} \kappa_n^i \frac{(z_i - \underline{e}_i)^n}{n!}$$

for every $z_i \in \left[\underline{e}_i, \underline{e}_i + \overline{\delta}^i\right]$. *Denote by* N_i *the nonnegative integer such that* $\kappa_{N_i}^i \neq 0$ *and* $\kappa_n^i = 0$ *for every* $n < N_i$.

3. $\beta_i > N_i$.

Define $\overline{I}$ *as the set of consumers* i *such that* $\beta_i \leq \beta_j$ *for every* $j \in \{1, \dots, I\}$, *and* $\underline{I}$ *as the set of consumers* i *such that* $\beta_i - N_i \geq \beta_j - N_j$ *for every* $j \in \{1, \dots, I\}$. *Then, for every efficient risk-sharing rule* f,

1. *Both* $\lim_{x \to \infty} \sum_{i \in \overline{I}} f_i(x)/x$ *and* $\lim_{x \to \infty} \sum_{i \in \overline{I}} f_i'(x)$ *exist and equal 1, and* $f_i''(x) < 0$ *for every* $i \notin \overline{I}$ *and every sufficiently large* $x > -\sum_{i=1}^{I} \underline{e}_i$.

2. *Both* $\lim_{x \to \underline{d}} \sum_{i \in \underline{I}} (f_i(x) - \underline{d}_i) / (x - \underline{d})$ *and* $\lim_{x \to \underline{d}} \sum_{i \in \underline{I}} f_i'(x)$ *exist and equal 1, and* $f_i''(x) > 0$ *for every* $i \notin \underline{I}$ *and every sufficiently small* $x > -\sum_{i=1}^{I} \underline{e}_i$.

The first conclusion tells us that the risk sharing rule is concave over some region of high consumption levels unless the consumer is the least risk averse one. The second conclusion is quite illustrative. The relevant number for the asymptotic curvature for low consumption levels is $\beta_i - N_i$, the constant relative risk aversion minus the minimum degree of differentiation with a nonzero derivative of the cumulative distribution function, the latter of which measures how thin the lower tail distribution is. It tells us that the risk-sharing rule is likely to be convex the smaller the number $\beta_i - N_i$ is, that is, the less risk averse he is or the thinner the lower tail distribution is.

The corollary can be proved by $s_i'(x_i) = 1/\beta_i$ for every $x_i > \underline{c}_i$ and

$$\frac{\lim_{x_i \to \underline{c}_i} s_i'(x_i)}{1 - N_i \lim_{x_i \to \underline{c}_i} s_i'(x_i)} = \frac{1}{\beta_i - N_i}.$$

Let us now provide two examples on the curvature of risk-sharing rules. For both examples, suppose that there are two consumers $i = 1, 2$. Each consumer i's original utility function v_i exhibits constant relative risk aversion $\beta_i > 1$. The second consumer has no background risk, so that $N_2 = 0$.

Suppose first that the first consumer's background risk follows a uniform distribution. Then $N_1 = 1$. According to Corollary 2, if, moreover, $0 < \beta_1 - \beta_2 < 1$, then, over some range of high consumption levels, the risk sharing rule is concave for the first consumer and convex for the second (because $\beta_1 > \beta_2$); but over some range of low consumption levels, it is convex for the first consumer and concave for the second (because $\beta_1 - N_1 < \beta_2 - N_2$). Hence there must at least one inflection point for each consumer's risk sharing rule, and neither of them is convex or concave over the entire region $(\underline{d}, \infty)$ of consumption levels. In particular, the risk-sharing rule for the second consumer is concave up to the smallest inflection point, and convex beyond the largest inflection point.[6] According to part 3 of Theorem 17 of HHK, this is something you can never observe in complete markets in which every consumer exhibits constant cautiousness, however many consumers there are, however many states there are, and whatever the probability distribution of the aggregate endowment is. In this sense, this risk-sharing rule is characteristic of incomplete markets.

Suppose second that the first consumer's background risk may take only finitely many values. Then $N_1 = 0$. According to Corollary 2 and Theorem 1, if, moreover, $\beta_1 - \beta_2$ is positive but sufficiently small, then over some range of high consumption levels and also over some range of low consumption levels, the risk sharing rule is concave for the first consumer and convex for the second; but over some range of intermediate consumption levels, it is convex for the first consumer and concave for the second. Hence there must at least two inflection point for each consumer's risk sharing rule, and, again, neither of them is convex or concave over the entire region $(\underline{d}, \infty)$ of consumption levels. Again, this is something you can never observe in complete markets in which every consumer exhibits constant cautiousness. These risk-sharing rules are a sign of incomplete markets.

6 Conclusion

In a model of a static exchange economy under uncertainty, we have investigated how the cautiousness (the derivative of the reciprocal of the Arrow-Pratt measure of absolute risk aversion) for macroeconomic risks is affected by the presence of idiosyncratic (background) risks. We gave sufficient conditions (Theorem 1) on the original utility function under which the cautiousness, at any given level of consumption, is higher in the presence of a background risk than in its absence. We also showed that while the cautiousness over a range of high consumption levels is left almost unaffected (Proposition 2) by its presence, the cautiousness over a range of low consumption levels is affected by the lower tail distribution of the idiosyncratic risks (Theorem 2).

We have also explored some implications of these results on the curvature of risk-sharing rules. While FSS concluded that the riskier the background risk, the more likely it is for the consumer's risk-sharing rule to be convex, our results indicate that such a claim cannot be easily established beyond the case they looked into. Even in an economy consisting of two consumers with the same original utility function and SOSD-rankable background risks, if the cautiousness of the original utility function is strictly decreasing (rather than constant), or if

[6] There may well be only one inflection point, in which case the largest and smallest inflection points coincide.

both consumers do indeed have background risks, then the consumer with a riskier background risk may well have an everywhere concave risk-sharing rule, depending on the utility weights to be used in the social welfare function. Moreover, regardless of the choice of these utility weights, the impact of the background risks on the curvature of the risk-sharing rule for high consumption levels is almost negligible (Proposition 3), and the lower tail distributions, rather than the variance, of the background risks matter for the curvature of the risk-sharing rules for low consumption levels (Proposition 4).

An interesting direction of future research is to extend the present analysis to a dynamic setting. Extending a static analysis to a dynamic setting tends to be more difficult when the markets are incomplete than when they are complete, because the possibility of dynamic asset trading may diminish the relevance of market incompleteness for risk sharing, as exemplified by Levine and Zame (2002). On the other hand, as subsequently shown by Kubler and Schmedders (2001), the relevance of market incompleteness depends subtly on both the persistence of endowment shocks and the time-discount rates. When assessing the impact of background risks on sharing rules, we will have to carefully specify these two.

A Proof of Theorem 2

Throughout this appendix, we assume that the original utility function v_i and the cumulative distribution function G_i satisfies the conditions of Theorem 2, and u_i is the induced utility function derived from v_i and G_i. The case of $N_i = 0$ is not difficult to prove, so we concentrate on the case of $N_i \geq 1$. Then $\gamma_i < 1$ and v_i can be written in the form of

$$v_i(x_i) = \chi(1 - 1/\gamma_i)^{-1}(x_i - \underline{c}_i)^{1-1/\gamma_i} + \eta$$

for some constants $\chi > 0$ and η on some lower tail interval of $(\underline{c}_i, \infty)$. We can assume without loss of generality that $\eta = 0$. Then let $D^0 v_i = v_i$ and, for each positive integer $n < 1/\gamma_i$, we can inductively let $D^{-n} v_i : (\underline{c}_i, \infty) \to \boldsymbol{R}$ be a particular integral of $D^{-(n-1)} v_i : (\underline{c}_i, \infty) \to \boldsymbol{R}$, so that $(-1)^n D^{-n} v_i(x_i)$ can be written in the form of

$$(-1)^n D^{-n} v_i(x_i) = \frac{\chi_n}{(n+1) - 1/\gamma_i}(x_i - \underline{c}_i)^{(n+1)-1/\gamma_i}, \tag{4}$$

where χ_n is a positive constant dependent on n. All the integrals that appear in the sequel are up to the order of $N_i < 1/\gamma_i$ and understood to be these particular ones.

Theorem 2 will be proved after a series of lemmas.

Lemma 3 *For every $x_i > \underline{d}_i$,*

$$u_i(x_i) = \sum_{n=0}^{\infty} \kappa_n (-1)^n D^{-n} v_i(x_i + \underline{e}_i)$$
$$- \sum_{n=0}^{\infty} \left(\sum_{\ell=n}^{\infty} \kappa_\ell \frac{\overline{\delta}^{\ell-n}}{(\ell-n)!} \right) (-1)^n D^{-n} v_i(x_i + \underline{e}_i + \overline{\delta}) + \int_{\underline{e}_i+\overline{\delta}}^{\overline{e}_i} v_i(x_i + z_i)\, dG_i(z_i).$$

Proof of Lemma 3 For each non-negative integer m, each strictly positive integer n, and each $x_i > \underline{d}_i$, define

$$\mathscr{I}^{m,n}(x_i) = \int_{\underline{e}_i}^{\underline{e}_i+\overline{\delta}} (-1)^m D^{-m} v_i(x_i + z_i) \frac{(z_i - \underline{e}_i)^{n-1}}{(n-1)!}\, dz_i,$$

where $\overline{\delta} > 0$ is as in Theorem 2. We shall first prove that

$$\mathscr{I}^{m,n}(x_i) = (-1)^{m+n} D^{-(m+n)} v_i(x_i + \underline{e}_i) - \sum_{\ell=1}^{n} (-1)^{m+\ell} D^{-(m+\ell)} v_i(x_i + \underline{e}_i + \overline{\delta}) \frac{\overline{\delta}^{n-\ell}}{(n-\ell)!}. \quad (5)$$

Indeed, by definition,

$$\mathscr{I}^{m,1}(x_i) = \int_{\underline{e}_i}^{\underline{e}_i+\overline{\delta}} (-1)^m D^{-m} v_i(x_i + z_i)\, dz_i$$
$$= (-1)^{m+1} D^{-(m+1)} v_i(x_i + \underline{e}_i) - (-1)^{m+1} D^{-(m+1)} v_i\left(x_i + \underline{e}_i + \overline{\delta}\right). \quad (6)$$

By integration by parts, for $n \geq 2$,

$$\mathscr{I}^{m,n}(x_i) = \left[(-1)^m D^{-(m+1)} v_i(x_i + z_i) \frac{(z_i - \underline{e}_i)^{n-1}}{(n-1)!} \right]_{z_i=\underline{e}_i}^{z_i=\underline{e}_i+\overline{\delta}}$$
$$- \int_{\underline{e}_i}^{\underline{e}_i+\overline{\delta}} (-1)^m D^{-(m+1)} v_i(x_i + z_i) \frac{(z_i - \underline{e}_i)^{n-2}}{(n-2)!}\, dz_i$$
$$= \mathscr{I}^{m+1,n-1} - (-1)^{m+1} D^{-(m+1)} v_i\left(x_i + \underline{e}_i + \overline{\delta}\right) \frac{\overline{\delta}^{n-1}}{(n-1)!}. \quad (7)$$

Based on these results, we shall now prove (5) by an induction argument on n. The case of $n = 1$ follows immediately from (6). Now let $n \geq 2$ and suppose that (5) hold for $n - 1$ and every $m \geq 0$. Then, by (7),

$$\begin{aligned}
\mathscr{I}^{m,n}(x_i) &= \mathscr{I}^{m+1,n-1}(x_i) - (-1)^{m+1} D^{-(m+1)} v_i \left(x_i + \underline{e}_i + \overline{\delta}\right) \frac{\overline{\delta}^{n-1}}{(n-1)!} \\
&= (-1)^{m+n} D^{-(m+n)} v_i(x_i + \underline{e}_i) \\
&\quad - \sum_{\ell=1}^{n-1} (-1)^{m+1+\ell} D^{-(m+1+\ell)} v_i(x_i + \underline{e}_i + \overline{\delta}) \frac{\overline{\delta}^{n-1-\ell}}{(n-1-\ell)!} \\
&\quad - (-1)^{m+1} D^{-(m+1)} v_i \left(x_i + \underline{e}_i + \overline{\delta}\right) \frac{\overline{\delta}^{n-1}}{(n-1)!} \\
&= (-1)^{m+n} D^{-(m+n)} v_i(x_i + \underline{e}_i) - \sum_{\ell=1}^{n} (-1)^{m+\ell} D^{-(m+\ell)} v_i(x_i + \underline{e}_i + \overline{\delta}) \frac{\overline{\delta}^{n-\ell}}{(n-\ell)!}.
\end{aligned}$$

Equality (5) has thus been proved.

Now, by letting $m = 0$, we obtain

$$\begin{aligned}
&\int_{\underline{e}_i}^{\underline{e}_i + \overline{\delta}} v_i(x_i + z_i) \frac{(z_i - \underline{e}_i)^{n-1}}{(n-1)!} \, dz_i \\
=& (-1)^n D^{-n} v_i(x_i + \underline{e}_i) - \sum_{\ell=1}^{n} \frac{\overline{\delta}^{n-\ell}}{(n-\ell)!} (-1)^{\ell} D^{-\ell} v_i(x_i + \underline{e}_i + \overline{\delta}). \qquad (8)
\end{aligned}$$

Hence

$$\begin{aligned}
& u_i(x_i) \\
=& \int_{\underline{e}_i}^{\underline{e}_i + \overline{\delta}} v_i(x_i + z_i) \, dG_i(z_i) + \int_{\underline{e}_i + \overline{\delta}}^{\overline{e}_i} v_i(x_i + z_i) \, dG_i(z_i) \\
=& \kappa_0 v_i(x_i + \underline{e}_i) + \int_{\underline{e}_i}^{\underline{e}_i + \overline{\delta}} v_i(x_i + z_i) \left(\sum_{n=1}^{\infty} \kappa_n \frac{(z_i - \underline{e}_i)^{n-1}}{(n-1)!} \right) dz_i + \int_{\underline{e}_i + \overline{\delta}}^{\overline{e}_i} v_i(x_i + z_i) \, dG_i(z_i) \\
=& \kappa_0 v_i(x_i + \underline{e}_i) + \sum_{n=1}^{\infty} \kappa_n \left(\int_{\underline{e}_i}^{\underline{e}_i + \overline{\delta}} v_i(x_i + z_i) \frac{(z_i - \underline{e}_i)^{n-1}}{(n-1)!} \, dz_i \right) + \int_{\underline{e}_i + \overline{\delta}}^{\overline{e}_i} v_i(x_i + z_i) \, dG_i(z_i) \\
=& \kappa_0 v_i(x_i + \underline{e}_i) + \sum_{n=1}^{\infty} \kappa_n \left((-1)^n D^{-n} v_i(x_i + \underline{e}_i) - \sum_{\ell=1}^{n} (-1)^{\ell} D^{-\ell} v_i(x_i + \underline{e}_i + \overline{\delta}) \frac{\overline{\delta}^{n-\ell}}{(n-\ell)!} \right) \\
& + \int_{\underline{e}_i + \overline{\delta}}^{\overline{e}_i} v_i(x_i + z_i) \, dG_i(z_i) \\
=& \sum_{n=0}^{\infty} \kappa_n (-1)^n D^{-n} v_i(x_i + \underline{e}_i) \\
& - \sum_{n=1}^{\infty} \left(\sum_{\ell=n}^{\infty} \kappa_{\ell} \frac{\overline{\delta}^{\ell-n}}{(\ell-n)!} \right) (-1)^n D^{-n} v_i(x_i + \underline{e}_i + \overline{\delta}) + \int_{\underline{e}_i + \overline{\delta}}^{\overline{e}_i} v_i(x_i + z_i) \, dG_i(z_i).
\end{aligned}$$

Note that we swapped the order of the sum and the integral to obtain the third equality, and the order of the two sums over different indices to obtain the last equality. This is justified because $\kappa_n = 0$ for every sufficiently large n. ///

For future references, we need to take care of the case where the asymptotic cautiousness is larger than one.

Lemma 4 *Let $\underline{c} \in \boldsymbol{R}$, $\delta > 0$, and $v : (\underline{c}, \infty) \to \boldsymbol{R}$ be an infinitely many times differentiable function satisfying $v'(x) > 0$ and $v''(x) < 0$ for every $x \in (\underline{c}, \underline{c} + \delta]$. Let $s : (\underline{c}, \underline{c} + \delta] \to \boldsymbol{R}$ be the risk tolerance for v, that is, $s(x) = -v'(x)/v''(x)$. Suppose that $\lim_{x \to \underline{c}} s(x)$ exists and equals zero, and $\lim_{x \to \underline{c}} s'(x) > 1$. Then v is bounded from below on $(\underline{c}, \underline{c} + \delta]$.*

Proof of Lemma 4 For a sufficiently small $\varepsilon > 0$, taking $\delta > 0$ smaller if necessary, we can assume that $s(x) > \dfrac{x - \underline{c}}{1 - \varepsilon}$ for every $x \in (\underline{c}, \underline{c} + \delta]$. Then, for every $y \in (\underline{c}, \underline{c} + \delta]$,

$$\int_y^{\underline{c}+\delta} \frac{dw}{s(w)} < \int_y^{\underline{c}+\delta} \frac{1 - \varepsilon}{w - \underline{c}}\, dw = (1 - \varepsilon) \log \frac{\delta}{y - \underline{c}}.$$

Hence, for every $x \in (\underline{c}, \underline{c} + \delta)$,

$$\int_x^{\underline{c}+\delta} \exp\left(\int_y^{\underline{c}+\delta} \frac{dw}{s(w)} \right) dy < \int_x^{\underline{c}+\delta} \left(\frac{\delta}{y - \underline{c}} \right)^{1-\varepsilon} dy = \frac{\delta^{1-\varepsilon}}{\varepsilon} \left(\delta^{\varepsilon} - (x - \underline{c})^{\varepsilon} \right) < \frac{\delta}{\varepsilon}.$$

This means that v is abounded from below because

$$v(x) = \int_{\underline{c}+\delta}^{x} \exp\left(- \int_{\underline{c}+\delta}^{y} \frac{dw}{s(w)} \right) dy + v(\underline{c} + \delta)$$

for every $x > \underline{c}$. ///

Lemma 5 *Let $\underline{c} \in \boldsymbol{R}$, $\delta > 0$, and $v : (\underline{c}, \infty) \to \boldsymbol{R}$ be an infinitely many times differentiable function satisfying $v'(x) > 0$ and $v''(x) < 0$ for every $x \in (\underline{c}, \underline{c} + \delta]$. If v' is bounded from above on $(\underline{c}, \underline{c} + \delta]$, then v is bounded from below on $(\underline{c}, \underline{c} + \delta]$.*

Proof of Lemma 5 Since v' is bounded from above on $(\underline{c}, \underline{c} + \delta]$, the integral $\displaystyle\int_x^{\underline{c}+\delta} v'(y)\, dy$ is uniformly bounded from above over all $x \in (\underline{c}, \underline{c} + \delta]$. Since $v(x) = v(\underline{c} + \delta) - \displaystyle\int_x^{\underline{c}+\delta} v'(y)\, dy$, this implies that v is bounded from below on $(\underline{c}, \underline{c} + \delta]$. ///

Given our choice of particular integrals (4), the following one can be obtained by inductively applying Lemma 1. We omit a formal proof.

Lemma 6 *Let n be a nonnegative integer. If $n \lim_{x_i \to \underline{c}_i} s_i'(x_i) < 1$, then there exists an n-th order integral $D^{-n} v_i$ of v_i such that:*

1. *$D\left((-1)^n D^{-n} v_i \right)(x_i) \to \infty$ as $x \to \underline{c}_i$.*

2. *There exists a* $\delta^n > 0$ *such that* $D\left((-1)^n D^{-n} v_i\right)(x_i) > 0$ *and* $D^2\left((-1)^n D^{-n} v_i\right)(x_i) < 0$ *for every* $x_i \in (\underline{c}_i, \delta^n]$.

3. *Define* $t_i^n : (\underline{c}_i, \underline{c}_i + \delta^n] \to \boldsymbol{R}_{++}$ *by*

$$t_i^n(x_i) = -\frac{D^1((-1)^n D^{-n} v_i)(x_i)}{D^2((-1)^n D^{-n} v_i)(x_i)},$$

then $\lim_{x_i \to \underline{c}_i} t_i^n(x_i)$ *exists and equals zero.*

4. $\lim_{x_i \to \underline{c}_i} Dt_i^n(x_i)$ *exists and equals*

$$\frac{\lim_{x_i \to \underline{c}_i} s_i'(x_i)}{1 - n \lim_{x_i \to \underline{c}_i} s_i'(x_i)}.$$

Lemma 7 *Let* n *and* N *be two nonnegative integers such that* $n > N$*, and suppose that* $N \lim_{x_i \to \underline{c}_i} s_i'(x_i) < 1$*. Then both*

$$\lim_{x_i \to \underline{c}_i} \frac{D\left((-1)^n D^{-n} v_i\right)(x_i)}{D\left((-1)^N D^{-N} v_i\right)(x_i)} \text{ and } \lim_{x_i \to \underline{c}_i} \frac{D^2\left((-1)^n D^{-n} v_i\right)(x_i)}{D^2\left((-1)^N D^{-N} v_i\right)(x_i)}$$

exist and equal zero.

Proof of Lemma 7 We shall first prove that the limit

$$\lim_{x_i \to \underline{c}_i} \frac{D\left((-1)^n D^{-n} v_i\right)(x_i)}{D\left((-1)^N D^{-N} v_i\right)(x_i)} \tag{9}$$

exists and equals zero. Note first that $D\left((-1)^N D^{-N} v_i\right)(x_i) \to \infty$ as $x_i \to \underline{c}_i$ by the Inada condition for the case of $N = 0$ and by part 1 of Lemma 6 for the case of $N \geq 1$. If $\lim_{x_i \to \underline{c}_i} s_i'(x_i) < 1 < n \lim_{x_i \to \underline{c}_i} s_i'(x_i)$, then let $m < n$ be the positive integer such that $1/(m+1) < \lim_{x_i \to \underline{c}_i} s_i'(x_i) < 1/m$. If $\lim_{x_i \to \underline{c}_i} s_i'(x_i) > 1$, then let $m = 0$. Then, by assumption for the case of $m = 0$ and by part 4 of Lemma 6 for the case of $m \geq 1$, $\lim_{x_i \to \underline{c}_i} Dt_i^m(x_i) > 1$, where t_i^m is defined as in Lemma 6. Hence, by Lemma 4, $D((-1)^{m+1} D^{-(m+1)} v)$ is bounded from above. Thus, by applying Lemma 5 iteratively if necessary, we see that $D((-1)^n D^{-n} v)$ is bounded from above. Thus (9) follows.

Suppose now that $n \lim_{x_i \to \underline{c}_i} s_i'(x_i) < 1$. Then, by part 4 of Lemma 6,

$$\lim_{x_i \to \underline{c}_i} Dt_i^n(x_i) = \frac{\lim_{x_i \to \underline{c}_i} s_i'(x_i)}{1 - n \lim_{x_i \to \underline{c}_i} s_i'(x_i)} > \frac{\lim_{x_i \to \underline{c}_i} s_i'(x_i)}{1 - N \lim_{x_i \to \underline{c}_i} s_i'(x_i)} = \lim_{x_i \to \underline{c}_i} Dt_i^N(x_i).$$

Let γ^n and γ^N lie strictly between the above two values satisfying $\gamma^n > \gamma^N$. Let $\delta > 0$ be sufficiently small that

$$t_i^n(x_i) > \gamma^n(x_i - \underline{c}_i) \text{ and } t_i^N(x_i) < \gamma^N(x_i - \underline{c}_i)$$

for every $x_i \in (\underline{c}_i, \underline{c}_i + \delta]$. Note also that

$$D\left((-1)^N D^{-N} v_i\right)(x_i) = D\left((-1)^N D^{-N} v_i\right)(\underline{c}_i + \delta) \exp\left(\int_{x_i}^{\underline{c}_i+\delta} \frac{dw}{t_i^N(w)}\right),$$

and analogously for $D\left((-1)^n D^{-n} v_i\right)(x_i)$. Hence

$$\begin{aligned}
&\frac{D\left((-1)^n D^{-n} v_i\right)(x_i)}{D\left((-1)^N D^{-N} v_i\right)(x_i)} \\
=&\frac{D\left((-1)^n D^{-n} v_i\right)(\underline{c}_i + \delta)}{D\left((-1)^N D^{-N} v_i\right)(\underline{c}_i + \delta)} \exp\left(\int_{x_i}^{\underline{c}_i+\delta} \left(\frac{1}{t_i^n(w)} - \frac{1}{t_i^N(w)}\right) dw\right) \\
<&\frac{D\left((-1)^n D^{-n} v_i\right)(\underline{c}_i + \delta)}{D\left((-1)^N D^{-N} v_i\right)(\underline{c}_i + \delta)} \exp\left(\int_{x_i}^{\underline{c}_i+\delta} \left(\frac{1}{\gamma^n (w - \underline{c}_i)} - \frac{1}{\gamma^N (w - \underline{c}_i)}\right) dw\right) \\
=&\frac{D\left((-1)^n D^{-n} v_i\right)(\underline{c}_i + \delta)}{D\left((-1)^N D^{-N} v_i\right)(\underline{c}_i + \delta)} \exp\left(\left(\frac{1}{\gamma^n} - \frac{1}{\gamma^N}\right) \int_{x_i}^{\underline{c}_i+\delta} \frac{dw}{w - \underline{c}_i}\right) \\
=&\frac{D\left((-1)^n D^{-n} v_i\right)(\underline{c}_i + \delta)}{D\left((-1)^N D^{-N} v_i\right)(\underline{c}_i + \delta)} \left(\frac{x_i - \underline{c}_i}{\delta}\right)^{1/\gamma^N - 1/\gamma^n}.
\end{aligned}$$

Since $1/\gamma^N - 1/\gamma^n > 0$, the far right hand side converges to zero as $x_i \to \underline{c}_i$. The proof is thus completed for (9).

As for the ratio of the second derivatives, note that

$$\frac{D^2\left((-1)^n D^{-n} v_i\right)(x_i)}{D^2\left((-1)^N D^{-N} v_i\right)(x_i)} = \frac{D\left((-1)^{n-1} D^{-(n-1)} v_i\right)(x_i)}{D\left((-1)^{N-1} D^{-(N-1)} v_i\right)(x_i)}.$$

We can thus apply the previous result to $n-1$ and $N-1$ if $N \geq 1$. If $N = 0$, then

$$\frac{D^2\left((-1)^n D^{-n} v_i\right)(x_i)}{D^2\left((-1)^N D^{-N} v_i\right)(x_i)} = \frac{(-1) D v_i(x_i)}{D^2 v_i(x_i)} \frac{(-1)^n D^{-(n-2)} v_i(x_i)}{(-1) D v_i(x_i)} = s_i(x_i) \frac{D\left((-1)^{n-1} D^{-(n-1)} v_i\right)(x_i)}{D v_i(x_i)}.$$

Since $s_i(x_i)$ converges to zero and the fraction on the far right hand side converges to one (if $n = 1$) or zero (if $n \geq 1$ by the result on the ratio of the first derivatives) as $x_i \to \underline{c}_i$, this completes the proof. ///

Proof of Theorem 2 Differentiate both sides of Lemma 3 with respect to x_i, then we obtain

$$\begin{aligned}
u_i'(x_i) =& \sum_{n=0}^{\infty} \kappa_n D((-1)^n D^{-n} v_i)(x_i + \underline{e}_i) \\
&- \sum_{n=1}^{\infty} \left(\sum_{\ell=n}^{\infty} \kappa_\ell \frac{\overline{\delta}^{\ell-n}}{(\ell-n)!}\right) D((-1)^n D^{-n} v_i)(x_i + \underline{e}_i + \overline{\delta}) + \int_{\underline{e}_i+\overline{\delta}}^{\overline{e}_i} v_i'(x_i + z_i)\, dG_i(z_i).
\end{aligned}$$

Hence

$$\frac{u_i'(x_i)}{D((-1)^{N_i}D^{-N_i}v_i)(x_i+\underline{e}_i)}$$
$$= \sum_{n=N_i}^{\infty} \kappa_n \frac{D((-1)^n D^{-n}v_i)(x_i+\underline{e}_i)}{D((-1)^{N_i}D^{-N_i}v_i)(x_i+\underline{e}_i)}$$
$$-\sum_{n=1}^{\infty}\left(\sum_{\ell=n}^{\infty}\kappa_\ell\frac{\overline{\delta}^{\ell-n}}{(\ell-n)!}\right)\frac{D((-1)^n D^{-n}v_i)(x_i+\underline{e}_i+\overline{\delta})}{D((-1)^{N_i}D^{-N_i}v_i)(x_i+\underline{e}_i)}+\frac{\displaystyle\int_{\underline{e}_i+\overline{\delta}}^{\overline{e}_i} v_i'(x_i+z_i)\,dG_i(z_i)}{D((-1)^{N_i}D^{-N_i}v_i)(x_i+\underline{e}_i)}.$$

As $x_i \to \underline{d}_i$, $x_i+\underline{e}_i \to \underline{c}_i$. Hence, by part 1 of Lemma 6, $D((-1)^{N_i}D^{-N_i}v_i)(x_i+\underline{e}_i)\to\infty$. By Lemma 7, every fraction in the first term on the right hand side, except for $n = N_i$, converges to zero, and hence the infinite sum converges to κ_{N_i}. Also, as $x_i \to \underline{d}_i$, $x_i+\underline{e}_i+\overline{\delta} \to \underline{c}_i+\overline{\delta}$ and hence, for every n, $D((-1)^n D^{-n}v_i)(x_i+\underline{e}_i+\overline{\delta})$ remain bounded from above by $D((-1)^n D^{-n}v_i)(\underline{c}_i+\overline{\delta})$. Each fraction in the second term thus converges to zero, and so does the second term itself. The last term converges to zero because the numerator remains bounded from above by $v_i'(\underline{e}_i+\overline{\delta})$. Thus

$$\frac{u_i'(x_i)}{D((-1)^{N_i}D^{-N_i}v_i)(x_i+\underline{e}_i)} \to \kappa_{N_i}$$

as $x_i \to \underline{d}_i$. Thus, in particular, $u_i'(x_i)\to\infty$ as $x_i \to \underline{d}_i$.

We can analogously prove that

$$\frac{u_i''(x_i)}{D^2((-1)^{N}D^{-N}v_i)(x_i+\underline{e}_i)} \to \kappa_{N_i}$$

as $x_i \to \underline{d}_i$. Hence, by part 4 of Lemma 6, $\lim_{x_i\to\underline{d}_i} -\frac{u_i'(x_i)}{u_i''(x_i)}$ exists and equals

$$\lim_{x_i\to\underline{d}_i} -\frac{D((-1)^{N_i}D^{-N_i}v_i)(x_i+\underline{e}_i)}{D^2((-1)^{N_i}D^{-N_i}v_i)(x_i+\underline{e}_i)} = \frac{\lim_{x_i\to\underline{c}_i} s_i'(x_i)}{1-N_i \lim_{x_i\to\underline{c}_i} s_i'(x_i)}.$$

///

Remark 1 Theorem 2 assumes that $\kappa_n = 0$ for every sufficient large n in (3). This means that the cumulative distribution function G_i is polynomial on some lower tail interval of $[\underline{e}_i, \overline{e}_i]$. It would be nice if we could dispense with this assumption, so that G_i may simply be a real analytic function. We do not do this generalization, as we do not know any reasonable condition to guarantee that the order of the integral of the now infinite sum can still (3) be swapped in the proof of Lemma 3 when $\kappa_n \neq 0$ for infinitely many n.

Remark 2 Theorem 2 assumes that v_i is a HARA-utility function on some lower tail interval of $(\underline{c}_i, \infty)$. It would be nice to dispense with this assumption to accommodate general utility functions in our analysis. As we alluded to when referring to Proposition 3 of Huang (2002a), the only difficulty in this generalization is to guarantee the validity of part 4 of Lemma 6, and, in particular, the existence of $\lim_{x_i \to \underline{c}_i} Dt_i^n(x_i)$. One way to do so is to assume that v_i can be written, on some lower tail interval, as

$$v_i(x_i) = \sum_{m=0}^{\infty} \chi_n \frac{(x_i - \underline{c}_i)^{1-1/\gamma_m}}{1 - 1/\gamma_m} \tag{10}$$

for some sequence $(\gamma_0, \gamma_1, \gamma_2, \dots)$ of strictly positive numbers and some other sequence $(\chi_0, \chi_1, \chi_2, \dots)$ of nonnegative numbers such that $\chi_m = 0$ for every sufficiently large m and $N_i \gamma_m < 1$ for some m with $\chi_m > 0$. This means in short that v_i is a finite sum of HARA-utility functions, and the joint assumption with G_i must be met with the smallest γ_m involved in (10). We did not explicitly give this generalization to simplify our exposition.

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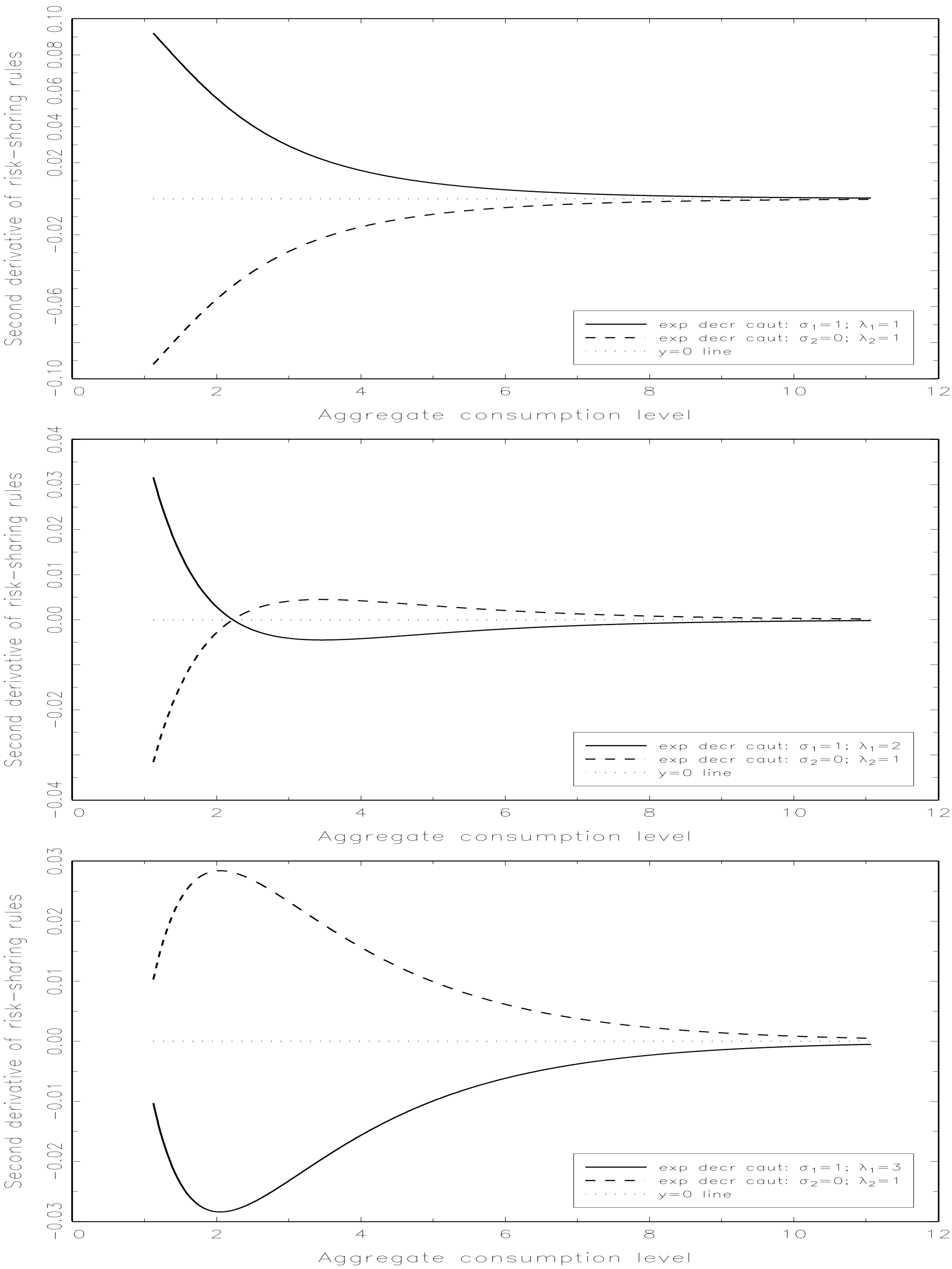

Figure 3: Second derivatives of risk-sharing in a two-consumer economy. Both consumers have the same original utility function v_i which is such that its cautiousness is exponentially decreasing $s_i'(x_i) = \exp(-x_i)$. Only one consumer (solid line) has a background risk. This background risk puts probability 1/2 on 1 and −1 each. The three graphs differ only in the weight given the consumer with background risk ($\lambda_1/\lambda_2 = 1, 2,$ and 3, respectively).

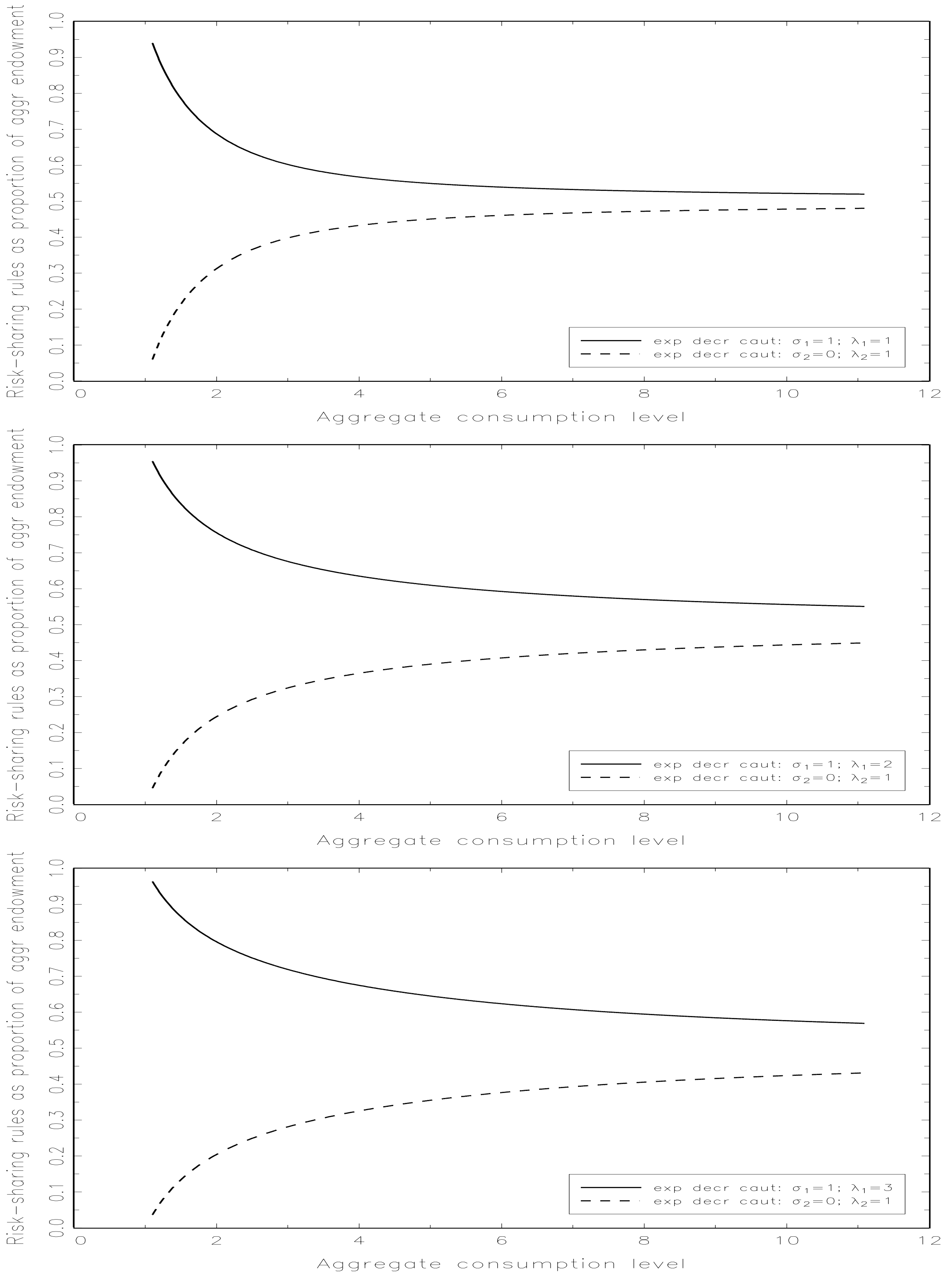

Figure 4: Risk-sharing rules as a proportion of aggregate endowment in a two-consumer economy. Both consumers have the same original utility function v_i which is such that its cautiousness is exponentially decreasing $s_i'(x_i) = \exp(-x_i)$. Only one consumer (solid curve) has a background risk. This background risk puts probability 1/2 on 1 and -1 each. The three graphs differ only in the weight given the consumer with background risk ($\lambda_1/\lambda_2 = 1, 2$, and 3, respectively).

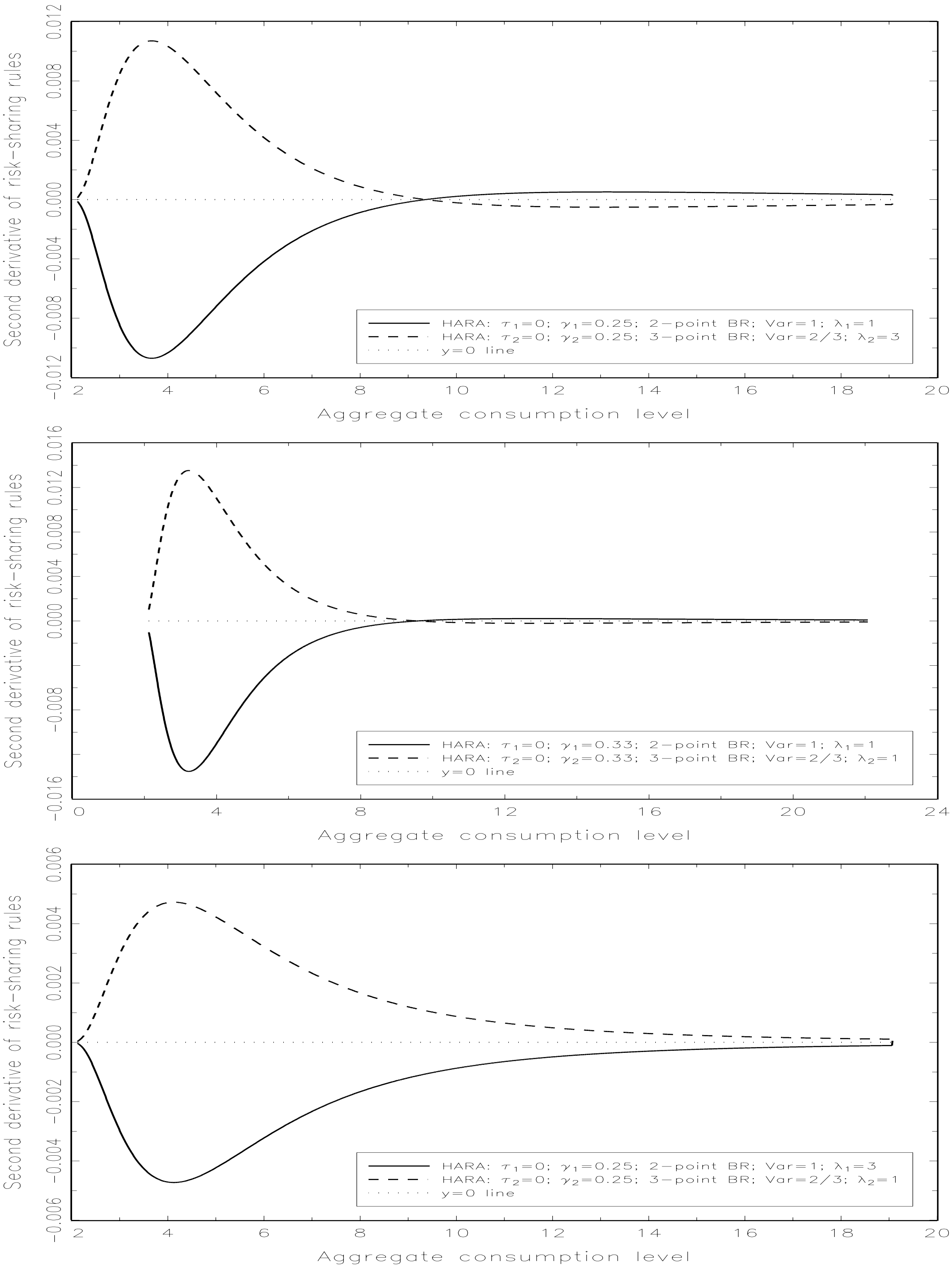

Figure 5: Second derivatives of risk-sharing in a two-consumer economy. Both consumers have the same original utility function v_i exhibiting constant relative risk aversion of 4). One consumer (solid curve) has a discrete two-point background risk (probability 1/2 on -1 and 1 each, which leads to a variance of 1). The other consumer (dashed curve) has a discrete three-point background risk (probability 1/3 on -1, 0, and 1 each, which leads to a variance of 2/3). The two graphs differ only in the weight given to the first consumer ($\lambda_1/\lambda_2 \in \{1/3, 1, 3\}$).

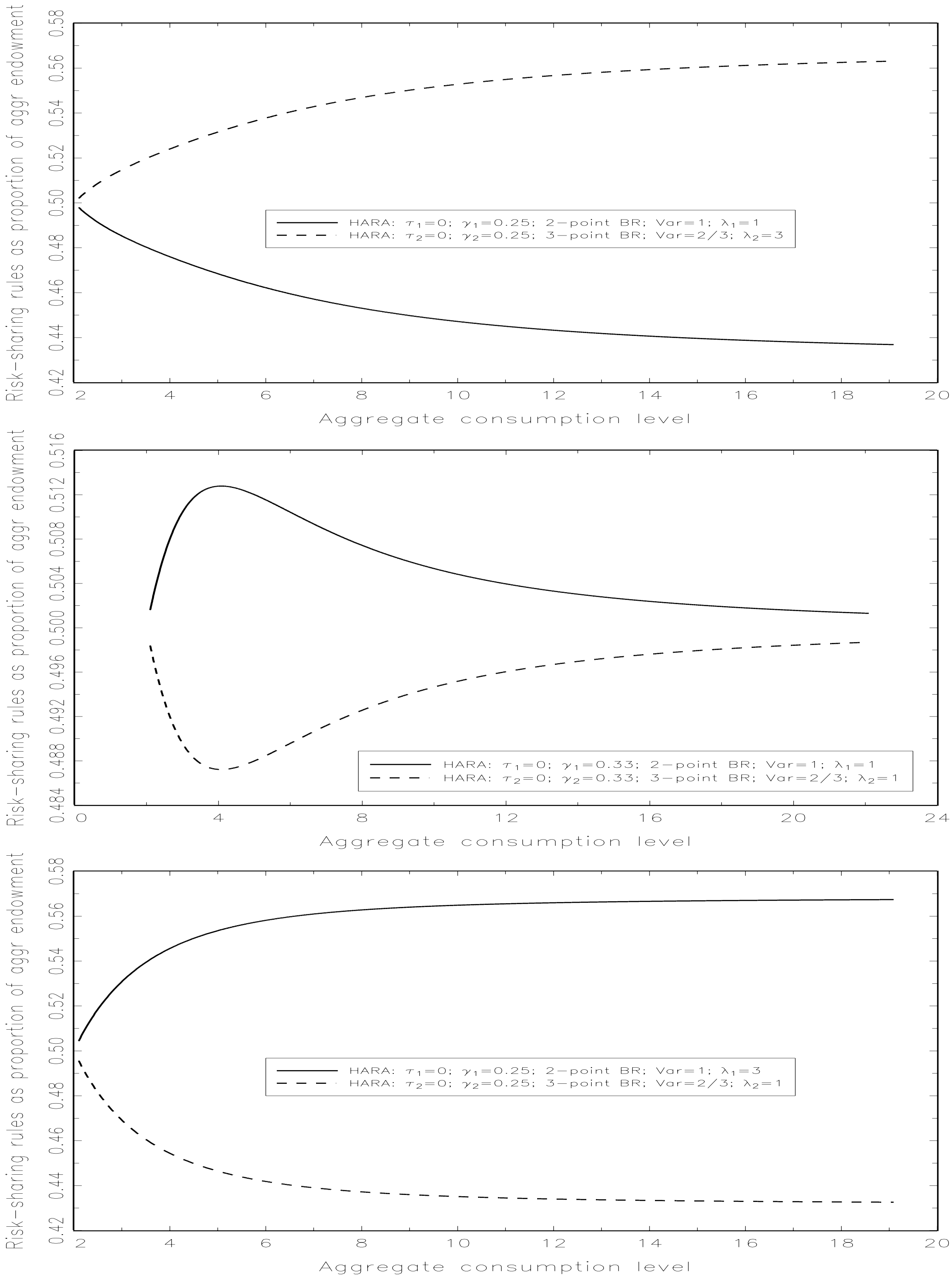

Figure 6: Risk-sharing rules as a proportion of aggregate endowment in a two-consumer economy. Both consumers have the same original utility function v_i exhibiting constant relative risk aversion 4. One consumer (solid curve) has a discrete two-point background risk (probability 1/2 on -1 and 1 each, which leads to a variance of 1). The other consumer (dashed curve) has a discrete three-point background risk (probability 1/3 on -1, 0, and 1 each, which leads to a variance of 2/3). The two graphs differ only in the weight given to the first consumer ($\lambda_1/\lambda_2 \in \{1/3, 1, 3\}$).