KIER DISCUSSION PAPER SERIES

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Discussion Paper No. 608

Probabilistically Sophisticated Multiple Priors.

by

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October 2005

KYOTO UNIVERSITY

KYOTO, JAPAN

17 October 2005

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Abstract

We characterize the intersection of the probabilistically sophisticated and multiple prior models. We show this class is strictly larger than the subjective expected utility model and that its elements can be generated from a generalized class of the ε-contaminated priors, which we dub the ε-contaminated/ γ-truncated prior.

JEL Classification: D81

Keywords: subjective probability, maximin expected utility, epsilon-contamined priors.

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*We thank Massimo Marinacci and Zvi Safra for their constructive comments and suggestions. We accept responsibility for any errors and shortcomings of the paper. Kajii acknowledges the financial support by Grant-in-Aid for the 21st Century COE Program and Grant-in-Aid for Scientific Research.

1 Introduction

To what extent can the maximin expected utility (MEU) model of Gilboa and Schmeidler (1989) and the probabilistically sophisticated model of Machina and Schmeidler (1992) coexist? The latter is a generalization of the models of de Finetti and Savage in which choice may be viewed as being based on beliefs that can be represented by a convex-ranged probability measure. This provides a foundation for the non-expected utility models under risk dealing with Allais-type paradoxes in the Savage framework of purely subjective uncertainty. The explicit motivation offered by Gilboa and Schmeidler (1989) for their maximin expected utility (also known as the 'multiple prior') model is to accommodate choice patterns such as those in the classic Ellsberg Paradoxes, where it can be shown that choice cannot be rationalized by beliefs that can be represented by 'additive' probabilities; that is, the Ellsberg paradox is not consistent with probabilistic sophistication. Thus, the question raised at the beginning is natural and important.

It was Marinacci (2002) who raised this important question, and established, under some conditions, that the only intersection of these two models is Savage's subjective expected utility (SEU) model. He maintains that the conditions are mild and thus he draws the conclusion that "once we wish to deal with Ellsberg-type phenomena with an MEU preference relation, we can no longer accommodate Allais-type phenomena via probabilistic sophistication, *even "locally" on the collection of unambiguous events.*" (Marinacci, 2002, p755, emphasis added).

In this paper, we fully characterize the class of MEU preferences which are probabilistically sophisticated on a given sufficiently rich collection of unambiguous events, but, unlike Marinacci, without any extra conditions. The class of such preferences obviously must include SEU preferences, but we show that it is strictly larger. We then derive Marinacci's result restricted to the MEU model as a corollary, by showing that among these preferences we identify, only SEU preferences satisfies Marinacci's condition. In this sense, the first contribution of this paper is a generalization of Marinacci's result applied to the MEU model, which provides a deeper understanding of the relation between MEU and probabilistic sophistication.

Obviously, our result above would be of little economic and decision theoretic interest if those probabilistically sophisticated MEU preferences other than SEU turned out to be pathological. We therefore investigate those probabilistically sophisticated MEU preferences, which do not satisfy Marinacci's condition.

The second contribution of this paper is that we characterize this class and show that they can be generated from a generalized class of the so called ε-contamination model; in fact, the basis for the class of probability sophisticated MEU preferences is essentially expressed by a two parameter family of sets of probability measures which is just one parameter richer than the ε-contamination model, which we dub the ε-contaminated (upper) γ-truncated prior. The ε-contamination model has been applied in economic applications.[1] Also, as we shall demonstrate, the ε-contaminated γ-truncated prior model is rich enough to accommodate Allais' type behavior.

We contend that the ε-contaminated γ-truncated prior model proposed in this paper is simple and easy to handle, and so it is also useful in applications. Although the Marinacci condition may appear mild in the particular setup he chose, it does rule out this important class of preferences, and hence it may not be so innocuous for the original question of identifying probabilistically sophisticated MEU.

2 Framework

The set-up consists of a set S of states of the world, a collection Σ of subsets of S and a set of consequences X.

For any $E \subset S$, let E^c denote its complement. We shall refer to any subset E in Σ as an *(unambiguous) event*. Marinacci (2002) focuses on Dynkin systems as the appropriate structure for the collection of unambiguous events and so we shall take Σ to be a Dynkin

[1] See for instance Carlier et al (2003), Lo (2000), Nishimura and Ozaki (2004).

system: that is, (i) $S \in \Sigma$, (ii) it is closed under complementation, that is, if $E \in \Sigma$ then so $E^c \in \Sigma$ (hence $\varnothing \in \Sigma$) and (iii) for any countable sequence of pairwise disjoint events, $E_n \in \Sigma$, $n = 1, 2, \ldots, E_i \cap E_j = \varnothing$, for all $i \neq j$, their countable union $\cup_{n=1}^{\infty} E_n$ is also in Σ.

A function $Q : \Sigma \to [0,1]$ is a *convex-ranged probability measure* if (i) $Q(S) = 1$, (ii) for any countable sequence of pairwise disjoint events, $\{E_n\}_{n=1}^{\infty}$, $Q(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} Q(E_n)$, and (iii) for all $Q(E) > 0$ and α in $(0,1)$, there exists an event $E' \subseteq E$ such that $Q(E') = \alpha Q(E)$. Let $\mathcal{P}$ be the set of all such measures.

Denote by Σ^* the smallest σ-field (that is, it also closed under intersections) containing Σ. By Dynkin's theorem, there is a unique extension of a convex-ranged probability measure Q on Σ^*. Abusing notation we shall simply call each element Q of P a probability measure, and treat it as if a measure on Σ^* in the sequel.

An act $f : S \to X$ is a $\Sigma-$ measurable function such that $f(S)$ is a finite set. The set of such acts is denoted by $\mathcal{F}$. The individual's preferences over acts is denoted by a binary relation $\succsim \subset \mathcal{F} \times \mathcal{F}$. With slight abuse of notation each $x \in X$ will also denote the constant act that yields x no matter which s in S obtains. Thus, $x \succsim f$ means the act f is not preferred to receiving the outcome x for sure.

For a given preference relation $\succsim$, an event is deemed null if there is indifference between any two acts that only differ on that event. That is, E is null if for any pair of acts $f, g \in \mathcal{F}$, $f(s) = g(s)$, for all $s \notin E$ implies $f \sim g$. An event E is deemed universal if its complement E^c is null.

Let $\mathcal{U}$ denote the set of utility indices, that is, the set of mappings of the form $u : X \to R$. For given $u \in \mathcal{U}$ and $f \in \mathcal{F}$, let $u \circ f$ denote the random variable, where $u \circ f(s) = u(f(s))$ for all s in S. Since the range of $f \in \mathcal{F}$ is finite, $u \circ f$ is $\Sigma-$ measurable and the integral $\int_S u \circ f \, dP$ is well defined and finite for any $P \in \mathcal{P}$ and any utility index $u \in \mathcal{U}$.

We consider the weak* topology on $\mathcal{P}$ with the set of these $\Sigma-$ measurable random variables of utilities as its dual: a sequence of probabilities $\{Q^n : n = 1, ...\}$ converges to $Q \in \mathcal{P}$ iff $\int_S u \circ f \, dQ^n \to \int_S u \circ f \, dQ$ for any $f \in \mathcal{F}$ and any $u \in \mathcal{U}$, which is equivalent to

$Q^n(E) \to Q(E)$ for all $E \in \Sigma$.[2] In particular, note that if every Q^n is absolutely continuous with respect to a probability P, so is the limit Q.

3 Probabilistic Sophistication

The intuitive idea behind an individual being probabilistic sophisticated with respect to a given probability measure P defined over a collection of *unambiguous* events is that the effect of assigning an outcome x to an event E that is deemed unambiguous depends solely on the resulting contribution of probability $P(E)$ the event makes to the overall probability $P(f^{-1}(x))$ of obtaining x, rather than on any specific "state-dependence" between x and E. That is, such events serve only as a randomization device, and they do not contain any further information about preferences. Thus in evaluating an unambiguous act, that is, an act that is measurable with respect to the collection of unambiguous events, a probabilistically sophisticated individual appears as if he first works out the probability distribution over outcomes (that is, a lottery) induced by the act, and then he evaluates the lottery without further regard to how this lottery is generated. Another way of saying this is that probabilistic sophistication requires any pair of unambiguous acts that are both mapped by P to the same lottery over outcomes should come from the same indifference class.

Definition 1 *An individual is said to be* probabilistically sophisticated *on $\mathcal{F}$ if there exists a unique convex-ranged probability measure $P \in \mathcal{P}$, such that for any pair of acts $f, g \in \mathcal{F}$,*

$$P\left(f^{-1}(x)\right) = P\left(g^{-1}(x)\right) \text{ for all } x \in X \Rightarrow f \sim g.$$

It is known that if an individual is probabilistically sophisticated, Ellsberg-type paradoxes do not arise. But since probabilistic sophistication imposes no restriction on the preferences over the induced lotteries, Allais-type paradoxes are consistent with probabilistic sophistication.[3]

[2] In other words, this is the relative topology on the set of convex ranged measures induced from the weak* topology of the set of all finitely additive measures.

[3] See Machina and Schmeidler (1992) and Grant (1995) for discussion.

The subjective expected utility (SEU) model is the special case of a probabilistically sophisticated individual whose preferences over acts can be completely determined by a probability measure P and a utility index u.[4] An SEU maximizer chooses among acts as if she first uses her utility index to map consequences to 'utilities' and then uses her subjective probability measure to determine the (cumulative) probability distribution over utilities (a unidimensional 'outcome' set). She then compares alternative acts solely on the basis of the means of the induced distributions over utilities.

In order to define this class formally, it is convenient to associate with each act the cumulative distribution over utilities induced by the utility index and the probability measure. That is, fix a utility index u and a probability measure $P \in \mathcal{P}$, and for each act $f \in \mathcal{F}$ denote by $F^P_{u \circ f}(\cdot)$ the cumulative distribution function over utilities induced by u and P, where for each z in $\mathbb{R}$,

$$F^P_{u \circ f}(z) := P(\{s \in S : u(f(s)) \leq z\}).$$

Note that by a change of variables, $\int_{\mathbb{R}} z dF^P_{u \circ f}(z) = \int_S u \circ f \, dP$.

Definition 2 *An individual is said to be a (Savage) subjective expected utility (SEU) maximizer if there exists a unique (up to positive affine transformations) utility index $u \in \mathcal{U}$, and a unique probability measure $P \in \mathcal{P}$, such that for all pair of acts $f, g \in \mathcal{F}$, $f \succsim g$ if and only if*

$$\int z dF^P_{u \circ f}(z) \geq \int z dF^P_{u \circ g}(z). \tag{1}$$

4 Maximin Expected Utility

As we noted in the introduction, the MEU model of Gilboa and Schmeidler has been used to generate preferences that can accommodate Ellsberg paradox patterns of behavior. We

[4] Strictly speaking, in the SEU setup Σ is also a $\sigma-$algebra. But we shall refer to the model here as SEU as well.

follow Marinacci and adopt the next definition as the analog of Gilboa and Schmeidler's MEU model in the Savage state-act framework of purely subjective uncertainty.[5]

Definition 3 *An individual is said to be a minimum expected utility (MEU) maximizer if there exists a unique (up to positive affine transformations) utility index $u \in \mathcal{U}$, and a unique non-empty weak*-compact and convex set $C \subset \mathcal{P}$, such that for all pair of acts $f, g \in \mathcal{F}$, $f \succsim g$ if and only if*

$$\min_{P \in C} \int z dF^{P}_{u \circ f}(z) \geq \min_{P \in C} \int z dF^{P}_{u \circ g}(z). \tag{2}$$

While probabilistic sophistication requires the existence of a unique probability measure to evaluate acts, the maximin expected utility model postulates a set of probability measures and the individual maximizes the minimum expected utility where the minimum is taken over the set of probabilities. Thus it may appear, and in fact we believe that it is a common "intuition" shared in the literature, that only a 'polar' case of an MEU maximizer exhibits probabilistic sophistication, namely an SEU maximizer, and that any non-trivial MEU preference relation, as it requires a multiplicity of underlying probability measures, cannot be probabilistically sophisticated. Indeed, Marinacci showed the following result:[6]

Corollary to Marinacci's Proposition 1 Suppose an individual is an MEU maximizer with an associated set $C \subset \mathcal{P}$ of 'multiple priors.' If there exists an event $E \in \Sigma$, such that

$$0 < \min_{P \in C} P(E) = \max_{P \in C} P(E) < 1,$$

[5] Cassadesus-Masanell et al (2000) provide an axiomatization of the MEU model in a setting of purely subjective uncertainty in which Σ is an algebra and X is connected and separable. The richness of the outcome space allows them to work with either an infinite or finite state space. We are unaware of any axiomatization of the MEU model in a setting analogous to that of Savage and that would as a consequence require the set of probability measures in the MEU representation all to be convex-ranged.

[6] Marrinacci actually establishes this result for any member of the $\alpha-$MEU family, for which $\alpha \neq 1/2$. An $\alpha-$MEU preference admits a representation of the form

$$V(f) = \alpha \min_{P \in C} \int z dF^{P}_{u \circ f}(z) + (1-\alpha) \max_{P \in C} \int z dF^{P}_{u \circ f}(z).$$

Clearly, the MEU family corresponds to $\alpha = 1$.

then the following two statements are equivalent:

1. The individual is probabilistically sophisticated.
2. The individual is a SEU maximizer.

In other words, this result confirms that probabilistic sophistication and MEU implies SEU, thus all the Allais-type paradoxes are not consistent with MEU, under the 'regularity' assumption that there is at least one event that is neither null nor universal and on which all the probabilities measures in C assign the same probability.

Obviously, the strength of the regularity assumption needs to be examined to appreciate the result above. Marinacci contends that it only entails there exists at least one proper 'unambiguous' event, and so it is mild and innocuous. This argument is especially convincing if it has been established that the decision maker's perception about ambiguity is represented by the particular set of probabilities. In other words, the argument is valid if one has already established that considerations about beliefs can be completely separated from considerations over utilities. Ghirardato and Marinacci (2002) propose a notion of absolute ambiguity aversion and an associated notion of ambiguity neutrality that builds upon a particular notion of comparative ambiguity aversion that entails such a separation. Epstein (1999), however, proposes a different notion of comparative ambiguity aversion that leads to a different notion of ambiguity neutrality that need not result in a complete separation.[7]

What the appropriate notion of comparative ambiguity aversion is, remains a contentious issue in the literature. But it is one that we do not need to directly confront in this paper. Rather we simply note that we shall show in the sequel that the class of probabilistically sophisticated MEU preferences is much larger than the class of SEU preferences, and it contains an important class of MEU preferences. We discuss the key idea in the next section.

[7] See also the discussion in Epstein & Zhang (2001).

5 Subjective Rank Dependent Expected Utility

A popular generalization of expected utility under risk is the rank dependent expected utility model (RDEU) of Quiggin (1982) and Yaari (1981). A Subjective Rank Dependent Expected Utility (SRDEU) maximizer like his SEU counterpart, acts as if he associates with each act the cumulative probability distribution over utilities induced by his utility index and his subjective probability measure. But before taking any expectations, he first *transforms* the cumulative probability distribution using his *probability transformation function.* By definition a probability transformation is a function $\phi : [0,1] \to [0,1]$, which is a non-decreasing function with $\phi(0) = 0$ and $\phi(1) = 1$. He then compares acts solely on the basis of the mean of each transformed induced distribution over utilities, which is $\phi \circ F^P_{u \circ f}(z)$. Note that since each act f is assumed to be finite range, $\phi \circ F^P_{u \circ f}(z)$ is a well defined cumulative distribution function.

Denote by $\mathcal{T}$ the set of probability transformation functions, and by $\mathcal{T}_{CON}$ the set of probability transformation functions that are concave.

Definition 4 *An individual is said to be a subjective rank-dependent expected utility (SRDEU) maximizer if there exists a unique (up to positive affine transformations) utility index $u : X \to \mathbb{R}$, a unique probability measure $P \in \mathcal{P}$, and a unique probability transformation function $\phi \in \mathcal{T}$, such that for all pair of acts $f, g \in \mathcal{F}$, $f \succsim g$ if and only if*

$$\int z d\, \phi \circ F^P_{u \circ f}(z) \geq \int z d\, \phi \circ F^P_{u \circ g}(z) \,. \tag{3}$$

Clearly, for any two acts f and g, if $P\left(f^{-1}(x)\right) = P\left(g^{-1}(x)\right)$ for all $x \in X$, then the distribution functions $\phi \circ F^P_{u \circ f}(z)$ and $\phi \circ F^P_{u \circ g}(z)$ are identical. So, a SRDEU maximizer is probabilistically sophisticated on $\mathcal{F}$.

An SRDEU model is known to be a special case of the Choquet Expected Utility model in which preferences are represented via the Choquet integral of utility with respect to a non-additive measure or *capacity* over a σ-field.[8] For the SRDEU model the associated

[8] See Wakker (1990).

capacity is given by $\nu_\phi(E) := 1 - \phi(1 - P(E))$, and this is convex if and only if ϕ is concave.[9] That is, one can write $\int zd\,\phi \circ F^P_{u\circ f}(z) = \int u \circ f d\nu_\phi$, where the latter integral is the Choquet integral. Recall our convention of treating P as a measure on Σ^* and hence ν_ϕ is also a capacity over Σ^*.

On the other hand, it is known[10] that when the capacity is convex the Choquet integral of a real-valued function admits an MEU representation: denote by Core(ν) the core of capacity ν, which is by definition the set of all finitely additive probability measures p with $p(E) \geq \nu(E)$ for all $E \in \Sigma$. It can be shown that Core(ν) is convex and compact. Then a capacity ν is convex iff (1) Core(ν) $\neq \varnothing$ and (2) for every bounded measurable function f,

$$\int f d\nu = \min_{p \in Core(\nu)} \int f dp. \tag{4}$$

Combining these results, we conclude:

Proposition 1 *For any $\phi \in \mathcal{T}_{CON}$, the associated SRDEU maximizer with utility function u and probability measure P is probabilistically sophisticated and also it is a MEU maximizer with the set of probabilities $C = Core(\nu_\phi)$.*

Notice that the result above suggests that there are probabilistically sophisticated MEU maximizers who are not SEU maximizers. But it does not characterize the whole class of probabilistically sophisticated MEU maximizers; that is, there may be other types of probabilistically sophisticated MEU maximizers. Also, it does not say if a SRDEU maximizer

[9] To see this, recall a capacity is convex, if for all $A, B \in \Sigma$,

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B).$$

If we set $p := P(A \cap B^c)$, $q := P(B \cap A^c)$, and $r := P(A^c \cap B^c)$, then for the SRDEU example, this condition, becomes, for all $p, q, r \geq 0$, s.t. $p + q + r \leq 1$,

$$\phi(r) + \phi(r + p + q) \leq \phi(r + q) + \phi(r + p),$$

or equivalently,

$$\phi(r + p + q) - \phi(r + p) \leq \phi(r + q) - \phi(r).$$

[10] See for instance Theorem 2.2 of Gilboa & Schmeidler (1995) and its references.

is of economic and decision theoretic interest. In the next section, we first examine the latter issue.

6 A Canonical Example: The Epsilon-Contaminated Gamma-Truncated-Prior

The purpose of this section is to present a simple two parameter class of SRDEU preferences, which is straightforward to handle but rich enough to accommodate various economic questions.

Consider the following two-parameter family of probability transformation functions:

$$\phi_{(\varepsilon,\gamma)}(p) = \begin{cases} 0 & \text{if } p = 0 \\ \varepsilon + (1-\varepsilon)\, p/(1-\gamma) & \text{if } p \in (0, 1-\gamma] \\ 1 & \text{if } p > 1-\gamma \end{cases} \text{, where } (\varepsilon,\gamma) \in [0,1)^2 .$$

Since $\phi_{(0,0)}$ is the identity function, this corresponds to an SEU maximizer. Note that $\phi_{(\varepsilon,\gamma)}$ is increasing and concave, thus it belongs to $\mathcal{T}_{CON}$. Thus by Proposition 1, an SRDEU maximizer with utility function u, probability measure P and probability transformation function $\phi_{(\varepsilon,\gamma)}$ admits an MEU representation for each $(\varepsilon,\gamma) \in [0,1)^2$.

We shall show below that for any (ε,γ), with $\varepsilon > 0$, the preferences exhibit the 'certainty' effect which is often given as an intuitive explanation for Allais paradox patterns of choices. That is, this class of SRDEU preferences can generate choice patterns consistent with those exhibited in standard Allais paradoxes.

To see this, it is convenient to define

$$\begin{aligned} \inf F &= \sup\{z : F(z) = 0\} \\ \text{and } {}_{\gamma}F(z) &= \begin{cases} 0 & \text{if } F(z) = 0 \\ F(z)/(1-\gamma) & \text{if } F(z) \in (0, 1-\gamma] \\ 1 & \text{if } F(z) > 1-\gamma \end{cases} \end{aligned}$$

In words, $\inf F$ is the greatest lower bound of the support of a random variable with a cumulative distribution function F, and ${}_{\gamma}F(\cdot)$ is the cumulative distribution function obtained by *truncating* the *upper* $\gamma-$ tail of that random variable.

Notice that for any cumulative distribution function F on $\mathbb{R}$, the transformed cumulative distribution function $\phi_{(\varepsilon,\gamma)} \circ F$ is given by

$$\phi_{(\varepsilon,\gamma)} \circ F(z) = \varepsilon\delta_{\inf F}(z) + (1-\varepsilon)\left[{}_{\gamma}F(z)\right],$$

where $\delta_{\hat{z}}$ denotes the cumulative distribution function of a degenerate random variable which assigns its unit probability to the real number $\hat{z}$. Applying this to the SRDEU formula (3), we have that the preferences can be represented by the following functional[11]

$$V(f) = \varepsilon \inf F^{P}_{u\circ f} + (1-\varepsilon)\int zd\left[{}_{\gamma}F^{P}_{u\circ f}(z)\right]. \tag{5}$$

In the table below we list how this functional evaluates four acts, f, g, f', g', that are each measurable with respect to the partition $\{A, B, C\}$ of S and for which $P(A) = 0.89$, $P(B) = 0.1$ and $P(C) = 0.01$. Without loss of generality we assume $u(5) = 1$, $u(0) = 0$ and $u(1) = v \in (0,1)$, and for the purposes of rationalizing Allais-paradox choice behavior we take γ to be less than 0.1. In the last column of the table, the utility of each act is computed from (3), or equivalently from (5).

act	Event			SRDEU $V(.)$
	A	B	C	
f	1	1	1	v
g	1	5	0	$(1-\varepsilon)\left[0.89v + (0.1-\gamma)\right]/(1-\gamma)$
f'	0	1	1	$(1-\varepsilon)(0.11-\gamma)v/(1-\gamma)$
g'	0	5	0	$(1-\varepsilon)(0.1-\gamma)/(1-\gamma)$

[11] The reason that neither $\varepsilon = 1$ nor $\gamma = 1$ are valid parameter values is that the probability measure P in the SRDEU representation is no longer unique, since any probability measure that is mutually and absolutely continous with P could be used in place of P.

For the SRDEU preference relation $\succsim$ that is rationalized by $V(.)$ to generate the classic Allais paradox, we require $f \succsim g$ and $g \succ f$. That is,

$$\begin{aligned} V(f) - V(g) &= \varepsilon v + (1-\varepsilon)\frac{[(0.11-\gamma)v - (0.1-\gamma)]}{(1-\gamma)} > 0 \\ \text{and } V(g') - V(f') &= (1-\varepsilon)\frac{[(0.1-\gamma) - (0.11-\gamma)v]}{(1-\gamma)} > 0. \end{aligned}$$

It is straightforward to verify that there exists a non-empty set of permissible values for the three parameters v, ε and γ for which these two inequalities hold. Indeed, if $f \sim g$, then

$$\varepsilon v = (1-\varepsilon)\frac{[(0.1-\gamma) - (0.11-\gamma)v]}{(1-\gamma)},$$

in which case $\varepsilon > 0$ implies $g' \succ f'$. So if there is close to indifference between the first pair a small perturbation of the parameters from expected utility exhibits the classic Allais paradox.

For the purpose of applications, it will be useful and also instructive to consider the set of extreme points of $\min_{p \in Core(\nu)} \int f dp$ in (4), that is, the set of measures in $Core(\nu)$ which actually achieve the minimum for some act f. Let $\mathcal{P}^P \subset \mathcal{P}$ denote the set of probability measures that are absolutely continuous with respect to P. That is, Q will be in $\mathcal{P}^P$, if for every $E \in \Sigma$, $P(E) = 0$ implies $Q(E) = 0$. And for each γ in $[0,1)$, let ${}_\gamma\mathcal{P}^P$ be the set of probability measures that can be obtained by updating the prior P conditional on an event for which P assigns at least $1-\gamma$. More formally, write $P(\cdot|E)$ for the conditional probability measure given event E with $P(E) > 0$: i.e., $P(A|E) = P(A \cap E)/P(E)$ for every $A \in \Sigma$. Then,

$${}_\gamma\mathcal{P}^P = \{Q \in \mathcal{P}^P : Q = P(\cdot|E) \text{ for some } E \in \Sigma \text{ with } P(E) \geq 1-\gamma\}.$$

Now consider the set of priors defined as

$$C^P_{(\varepsilon,\gamma)} := \varepsilon\left(\mathcal{P}^P\right) + (1-\varepsilon)\, co\left({}_\gamma\mathcal{P}^P\right),$$

where $co\left({}_\gamma\mathcal{P}^P\right)$ is the convex hull of ${}_\gamma\mathcal{P}^P$.

Notice that for the particular case of $\gamma = 0$, the set of priors $\varepsilon\left(\mathcal{P}^P\right) + (1-\varepsilon)\{P\}$ is sometimes referred to in the literature as an *epsilon-contaminated prior*. When $\varepsilon = 0$, the parameter γ defines a cut-off value of the upper tail distribution for the induced random utility $u \circ f$. So intuitively, we may view the general case with non-zero ε and γ, as defining sets of probability measures 'centered' around a focal prior belief P. Imprecision about this focal prior P manifests itself by the inclusion of the epsilon convex mixtures of certain measures that are absolutely continuous with this prior.

We shall refer to this two parameter family of MEU preferences the *The Epsilon-Contaminated Gamma-Truncated-Prior* model, as is defined below.

Definition 5 *An individual is said to be an $\varepsilon-$contaminated $\gamma-$truncated prior MEU maximizer if there exists a unique (up to positive affine transformations) utility index $u \in \mathcal{U}$, a probability $P \in \mathcal{P}$, and $(\varepsilon, \gamma) \in [0,1)^2$, such that for all pair of acts $f, g \in \mathcal{F}$, $f \succsim g$ if and only if*

$$\min_{Q \in C^P_{(\varepsilon,\gamma)}} \int z dF^Q_{u \circ f}(z) \geq \min_{Q \in C^P_{(\varepsilon,\gamma)}} \int z dF^Q_{u \circ g}(z). \tag{6}$$

We want to show that the preference relation defined above with utility index $u(\cdot)$ and parameters $(\varepsilon, \gamma) \in [0,1)^2$ coincides with an SRDEU preference relation with utility index $u(\cdot)$, underlying probability measure P and probability transformation function $\phi_{(\varepsilon,\gamma)}(\cdot)$. Hence in particular it shows an $\varepsilon-$contaminated $\gamma-$truncated prior MEU maximizer has an MEU representation as we have already argued. For this purpose, one can examine the core of $\nu_{\phi_{(\varepsilon,\gamma)}}$, which is equivalent to establish the following result.

Proposition 2 *For any act f, $\int zd\left[\phi_{(\varepsilon,\gamma)} \circ F^P_{u\circ f}(z)\right] = \min_{Q \in C^P_{(\varepsilon,\gamma)}} \int z dF^Q_{u\circ f}(z)$.*

Proof. We first establish that

$$\int zd\left[{}_\gamma F^P_{u\circ f}(z)\right] = \min_{Q \in {}_\gamma\mathcal{P}^P} \int z dF^Q_{u\circ f}(z) = \min_{Q \in co({}_\gamma\mathcal{P}^P)} \int uz dF^Q_{u\circ f}(z)$$

The second equality follows from the fact that ${}_\gamma\mathcal{P}^P$ is closed, and from the general property of a support function in convex analysis: the support function of a set is identical to the support function of the closed convex hull of the set. For the first equality, notice that

$$\int zd\left[{}_{\gamma}F_{u\circ f}^{P}(z)\right]=\min_{Q\in\,{}_{\gamma}\mathcal{P}^{P}}\int zdF_{u\circ f}^{Q}(z)$$

holds. Indeed, it is clear from the structure of ${}_{\gamma}\mathcal{P}^{P}$ that any minimizer Q^{*} of the right hand side must assign probability at most γ on the event where $u\circ f$ attains its maximum, but then by definition $\int zd\left[{}_{\gamma}F_{u\circ f}^{P}(z)\right]=\int zdF_{u\circ f}^{Q^{*}}(z)$.

Now the proposition can be established from the following calculation:

$$\begin{aligned}
\int zd\left[\phi_{(\varepsilon,\gamma)}\circ F_{u\circ f}^{P}(z)\right] &= \int zd\left\{\varepsilon\delta_{\inf F}(z)+(1-\varepsilon)\,{}_{\gamma}F(z)\right\}\\
&= \varepsilon\inf F_{u\circ f}^{P}+(1-\varepsilon)\int zd\left[{}_{\gamma}F_{u\circ f}^{P}(z)\right]\\
&= \varepsilon\left[\min_{Q\in\mathcal{P}^{P}}\int zdF_{u\circ f}^{Q}(z)\right]+(1-\varepsilon)\left[\min_{Q\in\,{}_{\gamma}\mathcal{P}^{P}}\int zdF_{u\circ f}^{Q}(z)\right]\\
&= \varepsilon\left[\min_{Q\in\mathcal{P}^{P}}\int zdF_{u\circ f}^{Q}(z)\right]+(1-\varepsilon)\left[\min_{Q\in co({}_{\gamma}\mathcal{P}^{P})}\int zdF_{u\circ f}^{Q}(z)\right]\\
&= \min_{Q\in C_{(\varepsilon,\gamma)}^{P}}\int zdF_{u\circ f}^{Q}(z).
\end{aligned}$$

■

To sum up, we have established that an MEU maximizer with the set of priors $C_{(\varepsilon,\gamma)}^{P}$ is an SRDEU maximizer, and that in particular such a preference relation exhibits probabilistic sophistication by Proposition 1.

Notice that for any $(\varepsilon,\gamma)\neq(0,0)$, Marinacci's regularity condition does not hold for the set $C_{(\varepsilon,\gamma)}^{P}$ and so his theorem has no bite in this class. Nevertheless, as we have shown, an ε-contaminated γ-truncated-prior MEU maximizer is probabilistically sophisticated, and can readily accommodate Allais paradox behavior. Since this class of preferences contains an important class of MEU preferences such as the epsilon contamination model, we are led to conclude that Marinacci's regularity condition, which rules out this class, is not so innocuous for the question he raised.

7 The Characterization

The purpose of this section is to give a complete characterization of the class of probabilistically sophisticated MEU model. As Proposition 1 suggests, it is larger than the class of the ε-contaminated γ-truncated-prior model. But we shall argue that in fact these canonical examples of 'contaminated priors' provide a basis to identify the set of probabilistically sophisticated MEU.

First, we show that if an MEU model is represented as an SRDEU as is the ε-contaminated γ truncated-prior model, then it must be expressed as an "envelope" of the ε-contaminated γ truncated-prior functionals.

Lemma 3 *Fix a prior belief P, a utility index u, and a concave probability transformation function ϕ. Let $C \subset \mathcal{P}$, be the set of priors for which*

$$\int zd\left[\phi \circ F_{u \circ f}^{P}(z)\right] = \min_{Q \in C} \int zdF_{u \circ f}^{Q}(z) \text{ for all } f \in \mathcal{F}.$$

Then there exists $D \subset (0,1]^2$, such that

$$C = \bigcap_{(\alpha,\gamma) \in D} C_{(\varepsilon,\gamma)}^{P}.$$

Proof. Consider the epigraph of ϕ on $(0,1]$, i.e., $\{(p,q) \in (0,1] \times \mathbb{R} : \phi(p) \geq q\}$. Since ϕ is concave, the epigraph is a closed convex set and thus it is the intersection of half spaces restricted on $(0,1] \times \mathbb{R}$ containing it. Notice that a half space restricted on $(0,1] \times \mathbb{R}$ is exactly the epigraph of $\phi_{(\varepsilon,\gamma)}$ for some (ε,γ). That is, if we define $D = \{(\varepsilon,\gamma) \in (0,1]^2 : \phi_{(\varepsilon,\gamma)}(p) \geq \phi(p) \text{ for all } p \in (0,1]\}$, we have $\phi = \inf_{(\varepsilon,\gamma) \in D} \phi_{(\varepsilon,\gamma)}$. Then, by construction, the corresponding set of measures C is exactly the intersection of $C_{(\varepsilon,\gamma)}^{P}$, i.e., $C = \bigcap_{(\alpha,\gamma) \in D} C_{(\varepsilon,\gamma)}^{P}$.
■

Essentially, we can build up any probabilistically sophisticated set of priors by taking intersections of $C_{(\varepsilon,\gamma)}^{P}$ sets and then take the convex hull of the unions of these intersections. In this sense the contaminated prior class discussed in the previous section provides a basis for the entire class of probabilistically sophisticated sets of multiple priors.

In general, the class of probabilistically sophisticated MEU maximizers is larger than the class of SRDEU maximizers. The following provides a characterization.

Theorem 4 *Fix a preference relation $\succsim$ that admits a non-trivial MEU representation*

$$V(f) = \min_{Q \in C} \int z dF^{Q}_{u \circ f}(z)$$

where $C \subset \mathcal{P}$ is a closed and convex set, and the range of $u(.)$ has non-empty interior. Then the following two statements are equivalent.

1. *The relation $\succsim$ is probabilistically sophisticated on $\mathcal{F}$.*

2. *There exists a unique convex-ranged probability measure $P : \Sigma \to [0,1]$, and a set of probability transformation functions $\Phi \subset \mathcal{T}_{CON}$, such that for all $f \in \mathcal{F}$*

$$V(f) = \min_{\phi \in \Phi} \int zd\left[\phi \circ F^{P}_{u \circ f}(z)\right]. \tag{7}$$

Proof. (1) $\Rightarrow$ (2). Let P be the probability with respect to which $\succsim$ is probabilistically sophisticated. We regard (S, P) a probability space. Denote by U the range of utility index u, i.e., $U = u(X)$. Hereafter we shall simply say a random variable for a random variable from S to U. For each act f, note that $u \circ f$ is a random variable. Conversely, for any given random variable $\hat{f}$, there is an act f with $u \circ f = \hat{f}$.

By the assumption of MEU preferences, the utility of an act f is determined by the corresponding random variable $u \circ f$ only. So we can unambiguously define a utility function V^* for random variables by the rule $V^*\left(\hat{f}\right) := V(f)$ with $\hat{f} = u \circ f$.[12]

We claim that V^* exhibits distribution invariance: that is, if two random variables $\hat{f}$ and $\hat{f}'$ induce the same lottery with respect to P then $V^*\left(\hat{f}\right) = V^*\left(\hat{f}'\right)$. To see this, for

[12] Notice that this follows from MEU assumption and it is *not* a direct consequence of probabilistic sophistication since $u \circ f = u \circ f'$ does not imply that the induced distributions of *outcomes* are identical. Assumption of MEU implies that outcomes are already transformed into utility indices, which in effect assumes the so called reduction principle. This does not come for free, since in general even a state dependent expected utility model can be probabilistically sophisticated if the reduction assumption is dropped. See Grant and Karni (2004).

each number $\hat{u}$ in the finite common range of $\hat{f}$ and $\hat{f}'$, fix an outcome $x_{\hat{u}} \in u^{-1}(\hat{u})$, and set $f(s) = x_{\hat{u}}$ for all $s \in \hat{f}^{-1}(\hat{u})$ and $f'(s) = x_{\hat{u}}$ for all $s \in \hat{f}'^{-1}(\hat{u})$. By construction, f and f' induce the same distribution over X with respect to P. Thus by probabilistic sophistication, $V(f) = V(f')$ and hence $V^*\left(\hat{f}\right) = V^*\left(\hat{f}'\right)$ by construction.

Following the proof of Lemma 4 of Safra & Segal (1998, pp 35-37),[13] there exists a set of concave transformation functions Φ such that $V^*\left(\hat{f}\right) = \min_{\phi\in\Phi} \int zd\left[\phi \circ F^P_{\hat{f}}(z)\right]$. Thus we have by construction $V(f) = \min_{\phi\in\Phi} \int zd\left[\phi \circ F^P_{u\circ f}(z)\right]$ as desired. ■

(2) ⇒ (1) For any pair of acts $f, g \in \mathcal{F}$, if $P(f^{-1}(x)) = P(g^{-1}(x))$ for all $x \in X$, then $F^P_{u\circ f}(z) = F^P_{u\circ g}(z)$ for all $z \in \mathbb{R}$. Thus we have

$$V(f) = \min_{\phi\in\Phi} \int zd\left[\phi \circ F^P_{u\circ f}(z)\right] = \min_{\phi\in\Phi} \int zd\left[\phi \circ F^P_{u\circ g}(z)\right] = V(g),$$

and hence $f \sim g$, as required. ■

A few remarks are due. First, as we asserted in Introduction, Marrinacci's Result applied to the MEU model is a corollary to this result. To see this, notice that given expression (4) and Proposition 1, we see that condition (2) above implies that every $Q \in C$ is a member of the core of capacity $\nu_\phi(\cdot) := 1 - \phi(1 - P(\cdot))$, and thus $\nu_\phi(E) = \min_{Q\in C} Q(E)$ for any $E \in \Sigma$.[14] Choose any $\phi \in \Phi$. Now assume in addition that there exists an event $E \in \Sigma$, such that $0 < \min_{Q\in C} Q(E) = \max_{Q\in C} Q(E) < 1$. Since each $Q \in C$ is a probability measure so that $Q(E) + Q(S\backslash E) = 1$, this additional condition implies that $\nu_\phi(E) + \nu_\phi(S\backslash E) = 1$. Thus by the definition of ν_ϕ, $\phi(1 - P(E)) + \phi(P(E)) = 1$. This is possible only if ϕ is linear, since ϕ is concave and $\phi(0) = 0$ and $\phi(1) = 1$ In conclusion, any $\phi \in \Phi$ must be the linear function $\phi(z) = z$, and hence (7) induces SEU (1).

Secondly, note that the formula (7) is not necessarily SRDEU; (7) is a minimum of several

[13] Dana (2005, Corollary 3.3) proves a similar result. The proof of Lemma 4 of Safra & Segal first establishes that V^* in our model is well defined and satisfies distribution invariance, then show that V^* has representation (7). Strictly speaking, the proof needs to be modified since they consider the set of all non-negative random variables whereas the random variables in our model are finite ranged and have a restricted range U. But a careful reading will show that this can be readily acheived as long as U has a non-empty interior. A detailed proof can be found in Appendix.

[14] See Schmeidler (1972).

Choquet integrals, which is not necessarily comonotonic additive and hence not representable by a Choquet integral.[15] Thus we conclude that the class of probabilistically sophisticated MEU is even strictly larger than that of SRDEU.[16]

Finally, in view of Lemma 3, any probabilistically sophisticated MEU preferences can be built up from the ε-contaminated γ truncated-prior functionals. This suggests that this simple class of preferences is fundamental to study the implications of probabilistically sophisticated MEU, and hence the widely used ε-contamination model focuses on only the half of the whole picture.

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[15] See Schmeidler (1986).

[16] Maccheroni et al (2005) provide a characterization of "probabilistically sophisticated" variational preferences that are defined over the set of Anscombe-Aumann two-stage subjective "horse-race"/objective "roulette"-lotteries. This family of preferences includes the original Gilboa-Schmeidler MEU preferences as a special case. Since they are working in the Anscombe-Aumann setting, however, these preferences are not probabilistically sophisticated in the Machina-Schmeidler sense but rather exhibit 'second-order' probabilistic sophistication in the sense of Ergin & Gul (2002). In particular, the preference relation restricted to 'constant acts' (that is, restricted to the set of objective roulette lotteries) conforms to standard expected utility theory.

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Appendix

A Proof of Theorem: (1) $\Rightarrow$ (2)

The proof draws on Segal and Safra's (1998) analysis of preferences over lotteries that exhibit constant risk aversion. In particular it adapts the arguments used in the proof of Lemma 4 pp35-37.

Let P be the probability with respect to which $\succsim$ is probabilistically sophisticated. Let $C_{\min}(f) := \operatorname{argmin}_{Q\in C} \int z dF^{Q}_{u\circ f}(z)$, which is the set of minimizing probabilities.

Lemma 5 *Every $Q \in C$ is absolutely continuous with respect to P.*

Proof. Suppose not. That is, suppose $E \in \Sigma$, $P(E) = 0$ and $\max_{Q\in C} Q(E) = q > 0$. Let $[\hat{y}_E \hat{x}]$ denote the act for which $[\hat{y}_E \hat{x}](s) = \hat{y}$ if $s \in E$ and $[\hat{y}_E \hat{x}](s) = \hat{x}$ if $s \notin S$. By probabilistic sophistication $[\hat{y}_E \hat{x}] \sim \hat{x}$, since $P\left([\hat{y}_E \hat{x}]^{-1}(\hat{x})\right) = P\left(\hat{x}^{-1}(\hat{x})\right) = 1$ and $P\left([\hat{y}_E \hat{x}]^{-1}(x)\right) = P\left(\hat{x}^{-1}(x)\right) = 0$ for all $x \neq \hat{x}$. But from the MEU representation we have

$$\min_{Q\in C} \int z dF^{Q}_{u\circ\hat{x}}(z) = \min_{Q\in C} \int z d\delta_1(z) = 1 > q \times 0 + (1-q) \times 1 = \min_{Q\in C} \int z dF^{Q}_{u\circ[\hat{y}_E \hat{x}]}(z)$$

I.e. $\hat{x} \succ [\hat{y}_E \hat{x}]$, a contradiction. ■

First note that Lemma 5 implies that, for all $f \in \mathcal{F}$, $Q \in C$, and $z, z' \in \mathbb{R}$,

$$F^{P}_{u\circ f}(z) = F^{P}_{u\circ f}(z') \Rightarrow F^{Q}_{u\circ f}(z) = F^{Q}_{u\circ f}(z').$$

Fix an act $f \in \mathcal{F}$ and probability measure $Q \in C$, and define a function $\phi^{Q}_{f} : [0,1] \to [0,1]$ by the rule

$$\phi^{Q}_{f}(p) = \begin{cases} F^{Q}_{u\circ f}\left(\left[F^{P}_{u\circ f}\right]^{-1}(p)\right), & p \in Image\left(F^{P}_{u\circ f}\right) \\ 0, & p = 0 \\ \ell(p), & \text{otherwise,} \end{cases}$$

where ℓ is the piece-wise linear function, defined on the complement of the image of $F^{P}_{u\circ f}$, that makes ϕ^{Q}_{f} continuous on $[0,1]$. That is, if we write $u \circ f(S) = \{z_1, ..., z_n\}$ where

$z_1 < z_2 < \cdots < z_n$, and $E_i = (u \circ f)^{-1}(z_i)$ for each $i = 1, ..., n$, the graph of ϕ_f^Q is obtained by connecting the following $n+1$ points:

$$(0,0);(P(E_1),Q(E_1));\ldots\ ;\left(\sum\nolimits_{j=1}^{i} P(E_j),\sum\nolimits_{j=1}^{i} Q(E_j)\right);\ldots\ ;\ (1,1).$$

So, in addition to being continuous, by construction ϕ_f^Q is non-decreasing and onto and $F_{u\circ f}^{Q}(z) = \phi_f^Q \circ F_{u\circ f}^{P}(z)$ for every non-null point in $u\circ f(S)$. Thus $\int zdF_{u\circ f}^{Q}(z) = \int zd\left[\phi_f^Q \circ F_{u\circ f}^{P}(z)\right]$ holds. Fix $f \in F$ and $Q_f \in C_{\min}(f)$ arbitrarily. Since the property above holds for any $Q \in C$, we have in particular that:

$$V(f) = \min_{Q\in C}\left\{\int zdF_{u\circ f}^{Q}(z)\right\} = \int zdF_{u\circ f}^{Q_f}(z) = \int zd\left[\phi_f^{Q_f} \circ F_{u\circ f}^{P}(z)\right].$$

Note that these equalities do not depend on the choice of Q_f, and so we shall write Q_f for an arbitrarily chosen element of $C_{\min}(f)$ from now on.

Define $W : \mathcal{F} \to R$ by

$$W(f) = \min_{h\in\mathcal{F},\ Q\in C_{\min}(h)}\left\{\int zd\left[\phi_h^{Q} \circ F_{u\circ f}^{P}(z)\right]\right\}.$$

We shall show that $W(f) = V(f)$ for any $f \in \mathcal{F}$. By definition $W(f) \leq V(f)$ for any $f \in \mathcal{F}$. So suppose the strict inequality holds; that is, there exists f such that $W(f) = \bar{v} < v = V(f)$. Then by the construction of W, there exists $\bar{f} \in \mathcal{F}$ and $Q_{\bar{f}} \in C_{\min}\left(\bar{f}\right)$ such that

$$W(f) = \int zd\left[\phi_{\bar{f}}^{Q_{\bar{f}}} \circ F_{u\circ f}^{P}(z)\right] < V(f).$$

We first show that if the range of $u(\cdot)$ has a non-empty interior, then probabilistic sophistication implies that C exhibits the following symmetry property with respect to P.

Definition 6 (Symmetry) *A convex compact set $C \subseteq \mathcal{P}^P$ is* symmetric *with respect to P, if the following condition holds: for any $n > 1$, and any pair of n-event partitions of S, $\{E_1, \ldots, E_n\}$ and $\{\hat{E}_1, \ldots, \hat{E}_n\}$ such that $P(E_i) = P\left(\hat{E}_i\right) = 1/n$ for every $i = 1, \ldots, n$, for each $Q \in C$ and each permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, there exists $\hat{Q} \in C$ such that*

$$\hat{Q}\left(\hat{E}_i\right) = Q\left(E_{\sigma(i)}\right),\ i = 1, \ldots, n.$$

Lemma 6 *Suppose that the richness condition is satisfied. If the relation $\succsim$ generated by $V(f) = \min_{Q \in C} \int z dF^Q_{u \circ f}(z)$ is probabilistically sophisticated with respect to $P \in \mathcal{P}$, then C is symmetric with respect to P.*

Proof. Suppose that the symmetry condition is a violated: that is, there is a pair of n-event partitions of S, $\{E_1, \ldots, E_n\}$ and $\{\hat{E}_1, \ldots, \hat{E}_n\}$ such that $P(E_i) = P\left(\hat{E}_i\right) = 1/n$ for every $i = 1, \ldots, n$, and there is a $\bar{Q} \in C$ and a permutation $\bar{\sigma} : \{1, \ldots, n\} \to \{1, \ldots, n\}$, such that there exists no $\hat{Q} \in C$ which satisfies:

$$\hat{Q}\left(\hat{E}_i\right) = \bar{Q}\left(E_{\sigma(i)}\right), \ i = 1, \ldots, n.$$

We shall construct two acts $\bar{f}$ and $\hat{f}$ such that $F^P_{u \circ \bar{f}} = F^P_{u \circ \hat{f}}$ but $V\left(\hat{f}\right) > V\left(\bar{f}\right)$.

Denote by $\Delta_C := \{(Q(E_1), \ldots, Q(E_n)) : Q \in C\}$ and by $\hat{\Delta}_C := \left\{\left(Q\left(\hat{E}_1\right), \ldots, Q\left(\hat{E}_n\right)\right) : Q \in C\right\}$. By the convexity and compactness of C, Δ_C and $\hat{\Delta}_C$ are convex and compact subsets of $\Delta^{n-1} := \{(q_1, ..., q_n) : q_i \geq 0, \sum_{i=1}^n q_i = 1\}$. Notice that for an act f which is measurable with respect to $\{E_1, \ldots, E_n\}$, we have $V(f) = \min_{q \in \Delta_C} \sum_{i=1}^n q_i (u \circ f(E_i))$. Similarly, for any act $\hat{f}$ which is measurable with respect to $\{\hat{E}_1, \ldots, \hat{E}_n\}$, we have $V\left(\hat{f}\right) = \min_{\hat{q} \in \hat{\Delta}_C} \sum_{i=1}^n \hat{q}_i \left(u \circ \hat{f}\left(\hat{E}_i\right)\right)$.

For each vector $z \in \mathbb{R}^n$, write $z^{\bar{\sigma}}$ for the element which is obtained from z by permutation $\bar{\sigma}$, i.e., $z_i^{\bar{\sigma}} = z_{\bar{\sigma}(i)}$ for $i = 1, ..., n$. Let $\bar{q} = (\bar{q}_1, ..., \bar{q}_n) := \left(\bar{Q}(E_1), \ldots, \bar{Q}(E_n)\right) \in \Delta_C$. Then our hypothesis above implies that $\bar{q}^{\bar{\sigma}} \notin \hat{\Delta}_C$.

Since Δ_C and $\hat{\Delta}_C$ are compact and convex subsets of the hyperplane $\{z \in \mathbb{R}^n : \sum_{i=1}^n z_i = 1\}$, applying the separation theorem we can find $\bar{z} \in \mathbb{R}^n$ such that $\bar{q}^{\bar{\sigma}} \cdot \bar{z}^{\bar{\sigma}} < \min_{\hat{q} \in \hat{\Delta}_C} \hat{q} \cdot \bar{z}^{\bar{\sigma}}$. On the other hand, $\bar{q}^{\bar{\sigma}} \cdot \bar{z}^{\bar{\sigma}} = \bar{q} \cdot \bar{z}$ by construction, and since $\bar{q} \in \Delta_C$, we conclude that $\min_{q \in \Delta_C} q \cdot \bar{z} < \min_{\hat{q} \in \hat{\Delta}_C} \hat{q} \cdot \bar{z}^{\bar{\sigma}}$.

Since the interior of $\{u(x) : x \in X\}$ is non-empty by assumption, pick α in the interior of this set, and choose $\beta > 0$ small enough so that $\alpha + \beta \bar{z}_i \in \{u(x) : x \in X\}$ for $i = 1, .., n$. Pick for each $i = 1, ..., n$, $\bar{x}_i \in X$ such that $u(\bar{x}_i) = \alpha + \beta \bar{z}_i$. Then define acts $\bar{f}$ and $\hat{f}$ which are measurable with respect to $\{E_1, \ldots, E_n\}$ and $\{\hat{E}_1, \ldots, \hat{E}_n\}$, respectively, by the rule $\bar{f}(E_i) = \bar{x}_i$ and $\hat{f}\left(\hat{E}_i\right) = \bar{x}_i^{\bar{\sigma}}$ for $i = 1, ..., n$ By construction, $u \circ \bar{f}(E_i) = \alpha + \beta \bar{z}_i$ and $u \circ \hat{f}\left(\hat{E}_i\right) = \alpha + \beta \bar{z}_i^{\bar{\sigma}}$ for $i = 1, ..., n$, and $F^P_{u\circ\bar{f}} = F^P_{u\circ\hat{f}}$.

Finally, $V\left(\bar{f}\right) = \min_{Q\in C} \int z dF^Q_{u\circ\bar{f}}(z) = \min_{q\in\Delta_C} \sum_{i=1}^n q_i \left(u \circ \bar{f}(E_i)\right) =$
$\min_{q\in\Delta_C} \sum_{i=1}^n q_i (\alpha + \beta \bar{z}_i) < \min_{\hat{q}\in\hat{\Delta}_C} \sum_{i=1}^n \hat{q}_i (\alpha + \beta \bar{z}_i^{\bar{\sigma}}) = \min_{Q\in C} \int z dF^Q_{u\circ\hat{f}}(z) = V\left(\hat{f}\right)$. ■

Let $\overrightarrow{\mathcal{F}}^n$ be the set of all acts $g \in F$ such that $u \circ g$ has at most n different 'utility levels' $z_1 \leq \cdots \leq z_n$ and satisfy $P\left((u \circ g)^{-1}(z_i)\right) = 1/n$. An immediate implication of Lemma 6 is that if f and $\hat{f}$ are both in $\overrightarrow{\mathcal{F}}^n$ then for each $Q \in C$, there exists $\hat{Q} \in C$, such that $\phi^Q_f = \phi^{\hat{Q}}_{\hat{f}}$.

Now consider the situation where there exists n and $\hat{f}$ such that f, $\hat{f} \in \overrightarrow{\mathcal{F}}^n$, and $F^P_{u\circ\hat{f}} = F^P_{u\circ\bar{f}}$. By symmetry of C, $\phi^{\bar{Q}}_{\bar{f}} = \phi^{\hat{Q}}_{\hat{f}}$ which implies $W(f) = V(f)$, a contradiction.

If there is no such n then, by continuity, there exists n large enough for which there exist f^n, $\hat{f}^n \in \overrightarrow{\mathcal{F}}^n$ that satisfy

$$W(f^n) = \int z d\left[\phi^{Q_{f^n}}_{\hat{f}^n} \circ F^P_{u\circ f^n}(z)\right] < \bar{v} + 1/2(v - \bar{v})$$

and $V(f^n) > \bar{v} + 1/2(v - \bar{v})$, a contradiction.

Hence, for $\hat{\Phi} = \left\{\phi^{Q(h)}_{u\circ h} : h \in \mathcal{F}\right\}$

$$V(f) = \min_{\phi\in\hat{\Phi}} \int z d\left[\phi \circ F^P_{u\circ f}(z)\right].$$

It remains for us to show that each $\phi^{Q_f}_f$ is concave.

Lemma 7 *The function $\phi^{Q_f}_f$ is concave for every $f \in \mathcal{F}$.*

Proof. Suppose for some $f \in F$, $\phi^{Q_f}_f$ is not concave. Write $u \circ f(S) = \{z_1, ..., z_n\}$ where $z_1 < z_2 < \cdots < z_n$, and $E_i = (u \circ f)^{-1}(z_i)$ for each $i = 1, ..., n$. Then $\sum_{j=1}^i P(E_j) = F^P_{u\circ f}(z_i)$ and so $\left[F^P_{u\circ f}\right]^{-1}\left(\sum_{j=1}^i P(E_j)\right) = z_i$.Since $\phi^{Q_f}_f$ does not depend on P-null events, we can assume without loss of generality that $P(E_i) > 0$ for each $i = 1, ..., n$. By construction, $\phi^Q_{u\circ f}\left(\left(\sum_{j=1}^i P(E_j)\right)\right) = F^Q_{u\circ f}\left(\left[F^P_{u\circ f}\right]^{-1}\left(\sum_{j=1}^i P(E_j)\right)\right) = F^Q_{u\circ f}(z_i) = \sum_{j=1}^i Q(E_j)$ for each $i = 1, ..., n$.

From the structure of the graph of $\phi^{Q(f)}_{u\circ f}$, it is clear that $\phi^{Q(f)}_{u\circ f}$ is non-concave if and only if there exists $i \in \{1, \ldots, n-1\}$, such that

$$\frac{Q(E_i)}{P(E_i)} < \frac{Q(E_{i+1})}{P(E_{i+1})}.$$

Lemma 8 *Let Q be absolutely continuous with respect to the $P \in \mathcal{P}$, and let events A and B in Σ be disjoint and satisfy*

$$0 < \frac{Q(A)}{P(A)} < \frac{Q(B)}{P(B)} < \infty.$$

Then there exists events $\hat{A} \subseteq A$ and $\hat{B} \subseteq B$ with $P\left(\hat{A}\right) = P\left(\hat{B}\right)$ and $Q\left(\hat{A}\right) < Q\left(\hat{B}\right)$.

Proof. By the convex rangedness assumption, we can find events $A' \subseteq A$ and $B' \subseteq B$ such that both $P(A')$ and $P(B')$ are rational numbers, and $0 < Q(A')/P(A') < Q(B')/P(B') < \infty$. So it suffices to establish the statement for the case $P(A)$ and $P(B)$ are rationals. Find an $\alpha > 0$ such that $P(A) = m\alpha$ and $P(B) = n\alpha$ for some integers m and n. By convex valudedness assumption, we can find a partitions $\{A_i : i = 1, ..., m\}$ of A and $\{B_j : j = 1, ..., n\}$ of B such that $P(A_i) = P(B_j) = \alpha$ for any i and j.

Since $Q(A) = \sum_{i=1}^{n} Q(A_i)$, there must be at least one i^* such that $Q(A_{i^*})/P(A_{i^*}) = Q(A_{i^*})/\alpha \leq Q(A)/P(A)$. Similarly, there must be at least one j^* with $Q(B_{j^*})/\alpha \geq Q(B)/P(B)\}$. So set $\hat{A} = A_{i^*}$ and $\hat{B} = B_{j^*}$. By construction, $\hat{A} \subseteq A$ and $\hat{B} \subseteq B$ with $P\left(\hat{A}\right) = P\left(\hat{B}\right)$ $(= \alpha)$. Moreover, $Q\left(\hat{A}\right) \leq \alpha(Q(A)/P(A)) < \alpha(Q(B)/P(B)) \leq Q\left(\hat{B}\right)$, as desired. ■

From Lemma 8 it follows there exist events $\hat{A} \subseteq E_i$ and $\hat{B} \subseteq E_{i+1}$ with $P\left(\hat{A}\right) = P\left(\hat{B}\right)$ and $Q\left(\hat{A}\right) < Q\left(\hat{B}\right)$. So consider the act g defined to be

$$u \circ g(s) = \begin{cases} z_i & \text{if } s \in \hat{B} \\ z_{i+1} & \text{if } s \in \hat{A} \\ u \circ f(s) & \text{otherwise} \end{cases}$$

By construction, we have by direct calculation:

$$V(g) = \min_{Q \in C} \left\{ \int z dF_{u \circ g}^{Q}(z) \right\} \leq \int z dF_{u \circ g}^{Q(f)}(z) < \int z dF_{u \circ f}^{Q(f)}(z) = V(f).$$

But on the other hand, since $P\left(\hat{A}\right) = P\left(\hat{B}\right)$ and so by construction $u \circ g$ induces the same distribution of utility indices over $\{z_1, ..., z_n\}$. So by probabilistic sophistication it follows $V(f) = V(g)$, a contradiction. ■

From Lemma 7 we have that $\hat{\Phi} \subset \mathcal{T}_{CON}$. Let $Cl\left(\hat{\Phi}\right)$ denote the closure of $\hat{\Phi}$ By continuity we have

$$\min_{\phi \in \hat{\Phi}} \int zd\left[\phi \circ F^{P}_{u \circ f}(z)\right] = \min_{\phi \in Cl\left(\hat{\Phi}\right)} \int zd\left[\phi \circ F^{P}_{u \circ f}(z)\right]$$

Define $>_{CON}$ to be the partial ordering of 'more concave' defined over $\mathcal{T}_{CON}$. That is, for any pair $\phi, \phi' \in \mathcal{T}_{CON}$, $\phi >_{CON} \phi'$ if $\phi = \psi \circ \phi'$ for some $\psi \in \mathcal{T}_{CON}$ and $\phi \neq \phi'$. Notice that if $\phi >_{CON} \phi'$ then

$$\begin{aligned} \int zd\left[\phi \circ F^{P}_{u \circ f}(z)\right] &\leq \int zd\left[\phi' \circ F^{P}_{u \circ f}(z)\right], \text{ for all } f \in \mathcal{F} \\ \text{and } \int zd\left[\phi \circ F^{P}_{u \circ g}(z)\right] &< \int zd\left[\phi' \circ F^{P}_{u \circ g}(z)\right], \text{ for some } g \in \mathcal{F}. \end{aligned}$$

We thus construct our set of probability transformation functions by selecting the set of 'maximally' concave functions from $Cl\left(\hat{\Phi}\right)$. That is,

$$\Phi = \left\{\phi \in Cl\left(\hat{\Phi}\right) : \nexists \psi \in Cl\left(\hat{\Phi}\right), \psi >_{CON} \phi\right\}.$$

This yields,

$$V(f) = \min_{\phi \in \Phi} \left\{\int zd\left[\phi \circ F^{P}_{u \circ f}(z)\right]\right\},$$

as required. ■