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Abstract

This paper proposes a class of weak additivity concepts for an operator on the set of real valued functions on a finite state space Ω, which include additivity and comonotonic additivity as extreme cases. Let $\mathcal{E} \subseteq 2^{\Omega}$ be a collection of subsets of Ω. Two functions x and y on Ω are $\mathcal{E}$-cominimum if, for each $E \in \mathcal{E}$, the set of minimizers of x restricted on E and that of y have a common element. An operator I on the set of functions on Ω is $\mathcal{E}$-cominimum additive if $I(x+y) = I(x)+I(y)$ whenever x and y are $\mathcal{E}$-cominimum. The main result characterizes homogeneous $\mathcal{E}$-cominimum additive operators in terms of the Choquet integrals and the corresponding non-additive signed measures. As applications, this paper gives an alternative proof for the characterization of the E-capacity expected utility model of Eichberger and Kelsey (1999) and that of the multi-period decision model of Gilboa (1989).

JEL classification: C71, D81, D90.

Keywords: Choquet integral; comonotonicity; non-additive probabilities; capacities; cooperative games.

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1 Introduction

Consider an operator I on the set of real valued functions on a finite set Ω. It is well known that an operator I is homogeneous (i.e. $I(\lambda x) = \lambda I(x)$ for a function x on Ω and $\lambda > 0$) and additive (i.e. $I(x + y) = I(x) + I(y)$ for functions x and y on Ω) if and only if it is represented as the integral with respect to a signed measure v on Ω; that is, $I(x) = \int x dv$ for a function x on Ω.

In his seminal paper, Schmeidler (1986) considered a homogeneous operator that is additive on comonotonic functions. Two functions x and y on Ω are said to be comonotonic if $(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) \geq 0$ for all ω, $\omega' \in \Omega$. He showed that an operator I is homogeneous and additive on comonotonic functions (i.e. $I(x+y) = I(x)+I(y)$ whenever x and y are comonotonic) if and only if it is represented as the Choquet integral with respect to a non-additive signed measure v on Ω; that is, $I(x) = \int x dv$ for a function x on Ω with the understanding that the integral is the Choquet integral. In the decision theory under uncertainty, the utility function representable as a Choquet integral now constitutes one of the important benchmarks.

In this paper, we propose a class of weak additivity concepts for an operator on the set of real valued functions, which include both additivity and comonotonic additivity as extreme cases. To be precise, let $\mathcal{E} \subseteq 2^{\Omega}$ be a collection of subsets of Ω. Two functions x and y on Ω are said to be $\mathcal{E}$-cominimum if, for every $E \in \mathcal{E}$, the set of minimizers of x restricted on E and that of y have a common element. An operator I is said to be $\mathcal{E}$-cominimum additive if $I(x + y) = I(x) + I(y)$ whenever x and y are $\mathcal{E}$-cominimum.

For example, if $\mathcal{E}$ is empty or contains only singletons, then any two functions are trivially $\mathcal{E}$-cominimum. In this case, $\mathcal{E}$-cominimum additivity coincides with additivity. If $\mathcal{E}$ consists of all subsets of Ω, then any two comonotonic functions are $\mathcal{E}$-cominimum and conversely any two $\mathcal{E}$-cominimum functions are comonotonic. In this case, $\mathcal{E}$-cominimum additivity coincides with comonotonic additivity. Thus, in general, $\mathcal{E}$-cominimum additivity is stronger than comonotonic additivity but weaker than additivity.

The main result of the paper (Theorem 3) is a representation theorem for homogeneous operators satisfying $\mathcal{E}$-cominimum additivity, which we shall sketch in the following. Notice that since $\mathcal{E}$-cominimum additivity implies comonotonic additivity, a homogeneous $\mathcal{E}$-cominimum additive operator is represented by the Choquet integral with respect to a non-additive signed measure v by Schmeidler's theorem, a fortiori. Since v can be uniquely written as $v = \sum_{T \subseteq \Omega} \beta_T u_T$, where u_T is the so called unanimity game on $T \subseteq \Omega$, the characterization of the operator can be done in terms of coefficients $\{\beta_T\}_{T \subseteq \Omega}$. We say that $T \subseteq \Omega$ is $\mathcal{E}$-complete if, for any two points $\omega, \omega' \in T$, there exists $E \in \mathcal{E}$ satisfying $\{\omega, \omega'\} \subseteq E \subseteq T$; that is, any two elements are "connected" within T by an element of $\mathcal{E}$. The main result shows that a homogeneous operator is $\mathcal{E}$-cominimum additive if and only if $\beta_T = 0$ for every T which is not $\mathcal{E}$-complete. It also shows that this condition is equivalent to the condition that v is modular on a suitably defined collection of pairs of events: $v(T_1 \cup T_2) + v(T_1 \cap T_2) = v(T_1) + v(T_2)$ whenever the pair (T_1, T_2) belongs to the collection.

We shall supply two applications to decision models under uncertainty. The first is the E-capacity expected utility model of Eichberger and Kelsey (1999). The E-capacities include the so called ε-contamination as a special case. The second is the multi-period decision model of Gilboa (1989). For both decision models, we provide alternative proofs for the axiomatic characterization using our results directly.

The organization of this paper is as follows. Section 2 quotes some known results about the Choquet integrals and Schmeidler's theorem. Section 3 introduces $\mathcal{E}$-cominimum functions and studies properties of $\mathcal{E}$-complete events. Section 4 provides the main results and Section 5 discusses applications.

2 The Choquet integrals and Schmeidler's theorem

Let $\Omega = \{1, \ldots, n\}$ be a finite set of states of the world. A subset $E \subseteq \Omega$ is called an event. Denote by $\mathcal{F}$ the collection of all non-empty subsets of Ω, and by $\mathcal{F}_k$ the collection of subsets with k elements.

A set function $v : 2^\Omega \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *game* or a *non-additive signed measure*. Since each game is identified with a point in $\mathbb{R}^\mathcal{F}$, we denote by $\mathbb{R}^\mathcal{F}$ the set of all games. For a game $v \in \mathbb{R}^\mathcal{F}$, we use the following definitions:

- v is *monotone* if $E \subseteq F$ implies $v(E) \leq v(F)$ for all $E, F \in \mathcal{F}$.
- v is *additive* if $v(E \cup F) = v(E) + v(F)$ for all $E, F \in \mathcal{F}$ with $E \cap F = \emptyset$, which is equivalent to $v(E) + v(F) = v(E \cup F) + v(E \cap F)$ for all $E, F \in \mathcal{F}$.
- v is *convex* (or *supermodular*) if $v(E) + v(F) \leq v(E \cup F) + v(E \cap F)$ for all $E, F \in \mathcal{F}$.
- v is *normalized* if $v(\Omega) = 1$.
- v is *non-negative* if $v(E) \geq 0$ for all $E \in \mathcal{F}$.
- v is a *non-additive measure* if it is non-negative and monotone. A normalized non-additive measure is called a *capacity*.
- v is a *measure* if it is non-negative and additive. A normalized measure is called a *probability measure*.
- The *conjugate* of v, denoted by v', is defined as $v'(E) = v(\Omega) - v(\Omega \backslash E)$ for all $E \in \mathcal{F}$. Note that $(v')' = v$ and $(v + w)' = v' + w'$ for $v, w \in \mathbb{R}^\mathcal{F}$.

For $T \in \mathcal{F}$, let $u_T \in \mathbb{R}^\mathcal{F}$ be the *unanimity game* on T defined by the rule: $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. Let u'_T be the conjugate of u_T. Then $u'_T(S) = 1$ if $T \cap S \neq \emptyset$ and $u'_T(S) = 0$ otherwise. The following result is well known as the Möbius inversion in discrete and combinatorial mathematics (cf. Shapley, 1953).

Lemma 1 *The collection $\{u_T\}_{T\in\mathcal{F}}$ is a linear base for $\mathbb{R}^{\mathcal{F}}$. The unique collection of coefficients $\{\beta_T\}_{T\in\mathcal{F}}$ satisfying $v = \sum_{T\in\mathcal{F}} \beta_T u_T$, or equivalently $v(E) = \sum_{T\subseteq E} \beta_T$ for all $E \in \mathcal{F}$, is given by $\beta_T = \sum_{E\subseteq T}(-1)^{|T|-|E|}v(E)$.*

The collection of coefficients $\{\beta_T\}_{T\in\mathcal{F}}$ is referred to as the Möbius transform of v. If $v = \sum_{T\in\mathcal{F}} \beta_T u_T$, then the conjugate v' is given by $v' = \sum_{T\in\mathcal{F}} \beta_T u'_T$.

Denote by $\mathbb{R}^\Omega = \{x \,|\, x : \Omega \to \mathbb{R}\}$ the set of all real valued functions on Ω. Let $1_E \in \mathbb{R}^\Omega$ be the indicator function of an event $E \in \mathcal{F}$. We write $\min_E x = \min_{\omega\in E} x(\omega)$, $\arg\min_E x = \arg\min_{\omega\in E} x(\omega)$, $\max_E x = \max_{\omega\in E} x(\omega)$, $\arg\max_E x = \arg\min_{\omega\in E} x(\omega)$ for $E \in \mathcal{F}$ and $x \in \mathbb{R}^\Omega$.

Definition 1 For $x \in \mathbb{R}^\Omega$ and $v \in \mathbb{R}^{\mathcal{F}}$, the *Choquet integral* of x with respect to v is defined as

$$\int x dv = \int_{\underline{x}}^{\bar{x}} v(x \geq \alpha) d\alpha + \underline{x} v(\Omega), \tag{1}$$

where $\bar{x} = \max_\Omega x(\omega)$, $\underline{x} = \min_\Omega x(\omega)$, and $v(x \geq \alpha) = v(\{\omega \in \Omega \,|\, x(\omega) \geq \alpha\})$.

For example, the Choquet integral of an indicator function is $\int 1_E dv = \int_0^1 v(1_E \geq \alpha) d\alpha = v(E)$; the Choquet integral with respect to unanimity games and their conjugates are

$$\int x du_T = \int_{\underline{x}}^{\bar{x}} u_T(x \geq \alpha) d\alpha + \underline{x} u_T(\Omega) = \left[\min_T x - \min_\Omega x\right] + \min_\Omega x = \min_T x,$$
$$\int x du'_T = \int_{\underline{x}}^{\bar{x}} u'_T(x \geq \alpha) d\alpha + \underline{x} u'_T(\Omega) = \left[\max_T x - \min_\Omega x\right] + \min_\Omega x = \max_T x$$

because $u_T(x \geq \alpha) = 1$ if $\min_T x \geq \alpha$ and $u_T(x \geq \alpha) = 0$ otherwise, and $u'_T(x \geq \alpha) = 1$ if $\max_T x \geq \alpha$ and $u'_T(x \geq \alpha) = 0$ otherwise.

It is straightforward to see that the Choquet integral is linear in games:

$$\int x d(sv + tw) = s \int x dv + t \int x dw \text{ for all } x \in \mathbb{R}^\Omega,\ v, w \in \mathbb{R}^{\mathcal{F}}, \text{ and } s, t \in \mathbb{R}.$$

An important implication of the linearity is the following additive representation of the Choquet integral (cf. Gilboa and Schmeidler, 1994).

Lemma 2 *For $x \in \mathbb{R}^\Omega$ and $v = \sum_{T\in\mathcal{F}} \beta_T u_T \in \mathbb{R}^{\mathcal{F}}$,*

$$\int x dv = \sum_{T\in\mathcal{F}} \beta_T \int x du_T = \sum_{T\in\mathcal{F}} \beta_T \min_T x, \tag{2}$$
$$\int x dv' = \sum_{T\in\mathcal{F}} \beta_T \int x du'_T = \sum_{T\in\mathcal{F}} \beta_T \max_T x. \tag{3}$$

Lemma 2 says that the Choquet integral of x with respect to v can be represented as a weighted sum of all minima of x with respect to some possibly negative weights.

Two functions $x, y \in \mathbb{R}^\Omega$ are said to be *comonotonic* if $(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) \geq 0$ for all $\omega, \omega' \in \Omega$. Observe that two functions $x, y \in \mathbb{R}^\Omega$ are comonotonic if and only if $\arg\min_E x \cap \arg\min_E y \neq \emptyset$ for all $E \in \mathcal{F}$. Symmetrically, two functions $x, y \in \mathbb{R}^\Omega$ are comonotonic if and only if $\arg\max_E x \cap \arg\max_E y \neq \emptyset$ for all $E \in \mathcal{F}$.

If x and y are comonotonic then $\min_T(x + y) = \min_T x + \min_T y$ for all $T \in \mathcal{F}$. Thus, the Choquet integral is additive on comonotonic functions by Lemma 2:

$$\int (x+y)dv = \sum_{T \in \mathcal{F}} \beta_T \min_T (x+y) = \sum_{T \in \mathcal{F}} \beta_T \min_T x + \sum_{T \in \mathcal{F}} \beta_T \min_T y = \int xdv + \int ydv.$$

We say that an operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ satisfies *comonotonic additivity* provided it is additive on comonotonic functions, i.e., $I(x + y) = I(x) + I(y)$ whenever x and y are comonotonic. Thus, the Choquet integral satisfies comonotonic additivity. We say that an operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ is *homogeneous* (more precisely, positively homogeneous of degree one) provided $I(\lambda x) = \lambda I(x)$ for all $\lambda > 0$. It is easy to see that the Choquet integral is homogenous. Schmeidler (1986) showed that a homogeneous operator which satisfies comonotonic additivity must be the Choquet integral. The following is a slightly different version of Schmeidler's theorem.[1]

Theorem 1 *An operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ is homogenous and satisfies comonotonic additivity if and only if $I(x) = \int xdv$ for all $x \in \mathbb{R}^\Omega$ where $v \in \mathbb{R}^\mathcal{F}$ is defined by the rule $v(E) = I(1_E)$.*

Proof. This can be shown by just a minor modification of Schmeidler's proof. ■

3 Cominimum functions

We will study homogenous operators satisfying a property stronger than comonotonic additivity and weaker than additivity. For this purpose, we generalize the notion of comonotonic functions.

Remember that two functions $x, y \in \mathbb{R}^\Omega$ are comonotonic if and only if $\arg\min_E x \cap \arg\min_E y \neq \emptyset$ for all $E \in \mathcal{F}$. By replacing $\mathcal{F}$ with a collection of events $\mathcal{E} \subseteq \mathcal{F}$, we have a weaker notion of comonotonic functions.[2]

Definition 2 Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. Two functions $x, y \in \mathbb{R}^\Omega$ are said to be $\mathcal{E}$-*cominimum*, provided $\arg\min_E x \cap \arg\min_E y \neq \emptyset$ for all $E \in \mathcal{E}$. Two functions $x, y \in \mathbb{R}^\Omega$ are said to be $\mathcal{E}$-*comaximum*, provided $\arg\max_E x \cap \arg\max_E y \neq \emptyset$ for all $E \in \mathcal{E}$.

[1] Schmeidler (1986) assumed monotonicity instead of homogeneity. But this can be readily shown adopting his proof. In fact, since homogeneity is a consequence of monotonicity in his proof, our statement is less elegant. But with monotonicity, the resulting game is necessarily a capacity, which is inconvenient for us since we want to work with general games.

[2] Kojima (2004) was the first to consider a weaker notion of comonotonic functions in this direction. He introduced the notion of cominimum functions, which are $\{\Omega\}$-cominimum functions in this paper.

Note that x and y are $\mathcal{E}$-cominimum if and only if $-x$ and $-y$ are $\mathcal{E}$-comaximum. So in fact any result about $\mathcal{E}$-cominimum functions can be translated for $\mathcal{E}$-comaximum functions in a straightforward manner.

The following properties are immediate consequences of the definition:

- If two functions are $\mathcal{E}$-cominimum (resp. comaximum) then they are $\mathcal{E}'$-cominimum (resp. comaximum) for any $\mathcal{E}' \subseteq \mathcal{E}$.
- If two functions are both $\mathcal{E}$-cominimum (resp. comaximum) and $\mathcal{E}'$-cominimum (resp. comaximum) then they are $\mathcal{E} \cup \mathcal{E}'$-cominimum (resp. comaximum).
- Any two functions are $\mathcal{F}_1$-cominimum (comaximum) where $\mathcal{F}_1 = \{\{\omega\} \mid \omega \in \Omega\}$.
- Two functions are $\mathcal{E}$-cominimum (resp. comaximum) if and only if they are $\mathcal{E} \cup \mathcal{F}_1$-cominimum (resp. comaximum).
- The following statements are equivalent.
  - Two functions are comonotonic.
  - Two functions are $\mathcal{F}_2$-cominimum (comaximum) where $\mathcal{F}_2 = \{\{\omega, \omega'\} \mid \omega, \omega' \in \Omega\}$.
  - Two functions are $\mathcal{F}$-cominimum (comaximum).
  - Two functions are $\mathcal{E}$-cominimum (comaximum) for all $\mathcal{E} \subseteq \mathcal{F}$.

The last item above implies that even if $\mathcal{E} \neq \mathcal{E}'$, the collection of $\mathcal{E}$-cominimum pairs of functions may coincide with that of $\mathcal{E}'$-cominimum pairs. Among collections of events which induce the same pairs of cominimum functions, there is a special collection, the complete collection, which will play an important role in the main result of this paper.

Definition 3 Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. An event $T \in \mathcal{F}$ is $\mathcal{E}$-complete provided, for any two distinct points ω_1 and ω_2 in T, there is $E \in \mathcal{E}$ such that $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. The collection of all $\mathcal{E}$-complete events is called the $\mathcal{E}$-complete collection and denoted by $\Upsilon(\mathcal{E})$. A collection $\mathcal{E}$ is said to be complete if $\mathcal{E} = \Upsilon(\mathcal{E})$.

We adopt the term "complete" from an analogy to a complete graph.[3] For $T \in \mathcal{F}$, consider an undirected graph with a vertex set T where $\{\omega, \omega'\} \subseteq T$ is an edge if there is $E \in \mathcal{E}$ satisfying $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. This is a complete graph if and only if T is $\mathcal{E}$-complete.

As an operator, Υ is monotone in the sense that $\Upsilon(\mathcal{E}) \subseteq \Upsilon(\mathcal{E}')$ whenever $\mathcal{E} \subseteq \mathcal{E}'$. Note that any $E \in \mathcal{E}$ is $\mathcal{E}$-complete, i.e., $\mathcal{E} \subseteq \Upsilon(\mathcal{E})$, and any singleton is $\mathcal{E}$-complete trivially, i.e., $\mathcal{F}_1 \subseteq \Upsilon(\mathcal{E})$. The following results show that $\Upsilon(\mathcal{E})$ itself is complete and it serves as a canonical collection among collections which induce the same pairs of cominimum functions.

[3] We can also regard $(\Omega, \mathcal{E})$ as a hypergraph. The theory of hypergraph has the concept of completeness, which is different from that in this paper.

Lemma 3 *For any $\mathcal{E} \subseteq \mathcal{F}$, $\Upsilon(\mathcal{E})$ is complete, i.e., $\Upsilon(\mathcal{E}) = \Upsilon(\Upsilon(\mathcal{E}))$.*

Proof. Since $\Upsilon(\mathcal{E}) \subseteq \Upsilon(\Upsilon(\mathcal{E}))$ by the monotonicity of Υ, it is enough to show that $\Upsilon(\mathcal{E}) \supseteq \Upsilon(\Upsilon(\mathcal{E}))$. Let $T \in \mathcal{F}$ be $\Upsilon(\mathcal{E})$-complete, i.e., $T \in \Upsilon(\Upsilon(\mathcal{E}))$. Then, for any $\omega_1, \omega_2 \in T$, there is $E \in \Upsilon(\mathcal{E})$ such that $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. Since $E \in \Upsilon(\mathcal{E})$ is $\mathcal{E}$-complete, there is $E' \in \mathcal{E}$ such that $\{\omega_1, \omega_2\} \subseteq E' \subseteq E \subseteq T$. This implies that T is $\mathcal{E}$-complete and thus $T \in \Upsilon(\mathcal{E})$, which completes the proof. ■

Lemma 4 *Two functions are $\mathcal{E}$-cominimum if and only if they are $\Upsilon(\mathcal{E})$-cominimum.*

Proof. Since $\mathcal{E} \subseteq \Upsilon(\mathcal{E})$, $\Upsilon(\mathcal{E})$-cominimum functions are $\mathcal{E}$-cominimum. Conversely, let two functions x_1 and x_2 be $\mathcal{E}$-cominimum. Seeking a contradiction, suppose that these are not $\Upsilon(\mathcal{E})$-cominimum: that is, there is an $\mathcal{E}$-complete event $T \in \mathcal{F}$ such that $\arg\min_T x_1 \cap \arg\min_T x_2 = \emptyset$. Pick $\omega_1 \in \arg\min_T x_1$ and $\omega_2 \in \arg\min_T x_2$. Since T is $\mathcal{E}$-complete, there is $E \in \mathcal{E}$ with $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. Since x_1 and x_2 are $\mathcal{E}$-cominimum, there is $\omega^* \in \arg\min_E x_1 \cap \arg\min_E x_2$. But then $x_i(\omega^*) \leq x_i(\omega_i)$ for $i = 1, 2$, and thus $\omega^* \in \arg\min_T x_1 \cap \arg\min_T x_2$, which is a contradiction. ■

If two functions are indicator functions, the $\mathcal{E}$-cominimum relation naturally induces a relation on pairs of events. We shall pursue this idea in the following.

Definition 4 Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. A pair of events $(T_1, T_2) \subseteq \mathcal{F} \times \mathcal{F}$ with $T_1 \not\subseteq T_2$ and $T_2 \not\subseteq T_1$ are said to be a decomposition pair for $T \in \mathcal{F}$ in $\mathcal{E}$, provided $T_1 \cup T_2 = T$ and, for any $E \in \mathcal{E}$, $E \subseteq T$ implies $E \subseteq T_1$ or $E \subseteq T_2$ (or both). Denote by $W(\mathcal{E})$ the collection of all the decomposition pairs for some events in $\mathcal{E}$:

$$\begin{aligned} W(\mathcal{E}) = \{(T_1, T_2) \in \mathcal{F} \times \mathcal{F} \,|\, & T_1 \not\subseteq T_2 \text{ and } T_2 \not\subseteq T_1, \\ & E \subseteq T_1 \cup T_2 \text{ implies } E \subseteq T_1 \text{ or } E \subseteq T_2 \text{ for all } E \in \mathcal{E}\}. \end{aligned}$$

An event $T \in \mathcal{F}$ is $\mathcal{E}$-decomposable if there exists a decomposition pair for T in $\mathcal{E}$, i.e., $T = T_1 \cup T_2$ for some $(T_1, T_2) \in W(\mathcal{E})$.

The idea of decomposition is exactly the $\mathcal{E}$-cominimum relation restricted to indicator functions, as is shown next.

Lemma 5 *Let $T_1, T_2 \in \mathcal{F}$ be such that $T_1 \not\subseteq T_2$ and $T_2 \not\subseteq T_1$. Indicator functions 1_{T_1} and 1_{T_2} are $\mathcal{E}$-cominimum if and only if $(T_1, T_2) \in W(\mathcal{E})$.*

Proof. Suppose that $(T_1, T_2) \in W(\mathcal{E})$. Pick any $E \in \mathcal{E}$. If $E \subseteq T_1 \cup T_2$, then $E \subseteq T_1$ or $E \subseteq T_2$ and thus $\arg\min_E 1_{T_1} = E$ or $\arg\min_E 1_{T_2} = E$ must hold. In both cases, $\arg\min_E 1_{T_1} \cap \arg\min_E 1_{T_2} \neq \emptyset$ holds. If $E \subseteq T_1 \cup T_2$ does not hold, $\arg\min_E 1_{T_1} \cap \arg\min_E 1_{T_2} = E \backslash T_1 \cap E \backslash T_2 = E \backslash (T_1 \cup T_2) \neq \emptyset$. Therefore, 1_{T_1} and 1_{T_2} are $\mathcal{E}$-cominimum.

Conversely, assume that 1_{T_1} and 1_{T_2} are $\mathcal{E}$-cominimum. Suppose there is $E \in \mathcal{E}$ with $E \subseteq T_1 \cup T_2$ but $E \not\subseteq T_1$ and $E \not\subseteq T_2$. Then $\arg\min_E 1_{T_1} = E \backslash T_1 \subseteq (T_1 \cup T_2) \backslash T_1$ and $\arg\min_E 1_{T_2} = E \backslash T_2 \subseteq (T_1 \cup T_2) \backslash T_2$, thus $\arg\min_E 1_{T_1} \cap \arg\min_E 1_{T_2} = \emptyset$, contrary to the assumption. Thus, such an E cannot exist and so $(T_1, T_2) \in W(\mathcal{E})$. ■

As is then easily expected, $\mathcal{E}$-decomposability of an event is closely related to $\mathcal{E}$-completeness. Note that any singleton is not $\mathcal{E}$-decomposable trivially, and that any $E \in \mathcal{E}$ is not $\mathcal{E}$-decomposable. The latter implies that any $\mathcal{E}$-complete event, which is necessarily an element of $\Upsilon(\mathcal{E})$ by definition, is not $\Upsilon(\mathcal{E})$-decomposable. In fact, $\mathcal{E}$-decomposability and $\Upsilon(\mathcal{E})$-decomposability are equivalent as the following lemma show.

Lemma 6 $W(\mathcal{E}) = W(\Upsilon(\mathcal{E}))$.

Proof. Since $\mathcal{E} \subseteq \Upsilon(\mathcal{E})$, $W(\mathcal{E}) \supseteq W(\Upsilon(\mathcal{E}))$. We show $W(\mathcal{E}) \subseteq W(\Upsilon(\mathcal{E}))$. Suppose that $(T_1, T_2) \in W(\mathcal{E})$ and $(T_1, T_2) \notin W(\Upsilon(\mathcal{E}))$. The former implies that $T_1 \not\subseteq T_2$ and $T_2 \not\subseteq T_1$, and the latter implies that there exists $E \in \Upsilon(\mathcal{E})$ such that $E \subseteq T_1 \cup T_2$ but neither $E \subseteq T_1$ nor $E \subseteq T_2$. Thus, there exist $\omega_1, \omega_2 \in E$ such that $\omega_1 \in T_1 \backslash T_2$ and $\omega_2 \in T_2 \backslash T_1$. Since E is $\mathcal{E}$-complete, there exists $E' \in \mathcal{E}$ such that $\omega_1, \omega_2 \in E'$, which contradicts to the assumption that (T_1, T_2) is a decomposition pair for T in $\mathcal{E}$. ■

The next result shows that the decomposability is in fact the "complement" of the completeness.

Lemma 7 *An event $T \in \mathcal{F}$ is $\mathcal{E}$-complete if and only if T is not $\mathcal{E}$-decomposable. Consequently,*

$$\begin{aligned}\Upsilon(\mathcal{E}) &= \{T \in \mathcal{F} \,|\, T \neq T_1 \cup T_2 \text{ for any } (T_1, T_2) \in W(\mathcal{E})\}, \\ &= \mathcal{F} \backslash \{T_1 \cup T_2 \,|\, (T_1, T_2) \in W(\mathcal{E})\}.\end{aligned}$$

Proof. The "if" part is clear from the definition. We shall establish the "only if" part. Assume that T is not $\mathcal{E}$-complete. Then there exists two distinct points $\omega_1, \omega_2 \in T$ such that there exists no $E \in \mathcal{E}$ satisfying $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. Set $T_1 = T \backslash \{\omega_1\}$ and $T_2 = T \backslash \{\omega_2\}$. By construction, $T_1 \not\subseteq T_2$, $T_2 \not\subseteq T_1$, and $T_1 \cup T_2 = T$. Also, for any $E \in \mathcal{E}$, if $E \subseteq T_1 \cup T_2$ then $\{\omega_1, \omega_2\} \not\subseteq E$ and so $E \subseteq T_1$ or $E \subseteq T_2$ must hold by construction. Therefore, $(T_1, T_2) \in W(\mathcal{E})$ and thus T is $\mathcal{E}$-decomposable. ■

To conclude this section, we shall give a sufficient condition for completeness.

Lemma 8 *Suppose that $\mathcal{E}$ contains all the singleton events and satisfies the following property: if $E, E_1, \ldots, E_n \in \mathcal{E}$ satisfy $E \subseteq \bigcup_{i=1}^n E_i$ then $E \cup E_i \in \mathcal{E}$ for at least one $i \in \{1, \ldots, n\}$. Then, $\mathcal{E}$ is complete.*

Proof. Let $T \notin \mathcal{E}$. We want to show that T is not $\mathcal{E}$-complete. By Lemma 7, it suffices to show that T is $\mathcal{E}$-decomposable. Fix $\bar{\omega} \in T$, and let $T_1 \subseteq T$ be a maximal set containing $\bar{\omega}$ and included in $\mathcal{E}$. Since $T \notin \mathcal{E}$, T_1 must be a proper subset of T.

If $T_1 = \{\bar{\omega}\}$, then there is no event $E \in \mathcal{E}$ such that $\{\bar{\omega}\} \subsetneq E \subseteq T$. Then it is readily verified that T_1 and $T \backslash T_1$ constitute an $\mathcal{E}$-decomposition of T.

If $T_1 \neq \{\bar{\omega}\}$, then let $\mathcal{E}' = \{E \in \mathcal{E} \mid E \subseteq T \text{ and } E \nsubseteq T_1\}$. It must be true that $T_1 \nsubseteq \bigcup_{E \in \mathcal{E}'} E$. To see this, suppose that $T_1 \subseteq \bigcup_{E \in \mathcal{E}'} E$. Then, there exists $E \in \mathcal{E}'$ such that $T_1 \cup E \in \mathcal{E}$ by the assumption on $\mathcal{E}$. Since $E \subseteq T$ and $E \nsubseteq T_1$, we have $T \supseteq T_1 \cup E \supsetneq T_1$, which contradicts to the maximality of T_1.

Let $T_2 = (T \backslash T_1) \cup (\bigcup_{E \in \mathcal{E}'} E)$. We claim T_1 and T_2 is an $\mathcal{E}$-decomposition of T. By construction, $T_1 \cup T_2 = T$. As we noted above, $T_1 \subsetneq T$. Since $T_1 \nsubseteq \bigcup_{E \in \mathcal{E}'} E$, $T_2 \subsetneq T$, and hence $T_1 \nsubseteq T_2$ and $T_2 \nsubseteq T_1$. Finally, pick any $E \in \mathcal{E}$ with $E \subseteq T$ and suppose $E \nsubseteq T_1$. Then $E \in \mathcal{E}'$, and so $E \subseteq T_2$. Thus $(T_1, T_2) \in W(\mathcal{E})$, which completes the proof. ■

In practice, a stronger condition is also useful.

Lemma 9 *Suppose that $\mathcal{E}$ contains all the singleton events and satisfies the following property: for any $E_1, E_2 \in \mathcal{E}$, if $E_1 \cap E_2 \neq \emptyset$ then $E_1 \cup E_2 \in \mathcal{E}$. Then, $\mathcal{E}$ is complete.*

Proof. The condition above implies the property of Lemma 8; if $E \subset \cup_{i=1}^{n} E_i$, then for at least one i, $E \cap E_i \neq \emptyset$, and so $E \cup E_i \in \mathcal{E}$. ■

If $E \cap E' = \emptyset$ or $E \subseteq E'$ or $E' \subseteq E$ for all $E, E' \in \mathcal{E}$, then $\mathcal{E} \cup \mathcal{F}_1$ is complete. Especially, if $\mathcal{E}$ is a partition of Ω, then $\mathcal{E} \cup \mathcal{F}_1$ is complete.

Lemma 8, however, does not provide a necessary condition for completeness. For instance, let $\Omega = \{1, 2, 3, 4\}$, $\mathcal{E} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 4\}\}$. Then $\mathcal{E}$ is complete. Indeed, $\{2, 3\}$, $\{3, 4\}$ and $\{1, 4\}$ are not $\mathcal{E}$-complete since no element of $\mathcal{E}$ contains them. Thus any set which contains one of them is not $\mathcal{E}$-complete, thus any three points set and Ω are not $\mathcal{E}$-complete. But then $\mathcal{E}$ does not satisfy the condition of Lemma 8.

4 Cominimum additive operators

The notion of $\mathcal{E}$-cominimum (comaximum) functions induces the following additivity property of an operator $I : \mathbb{R}^\Omega \to \mathbb{R}$.

Definition 5 An operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ is $\mathcal{E}$-cominimum (resp. comaximum) additive provided $I(x + y) = I(x) + I(y)$ whenever x and y are $\mathcal{E}$-cominimum (resp. comaximum).

Since $\mathcal{E}$-cominimum (comaximum) additivity implies comonotonic additivity, we have the following corollary of Theorem 1.

Corollary 2 *An operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ is homogeneous and $\mathcal{E}$-cominimum (comaximum) additive for some $\mathcal{E} \subseteq \mathcal{F}$ if and only if $I(x) = \int x dv$ for any $x \in \mathbb{R}^\Omega$ where $v \in \mathbb{R}^\mathcal{F}$ is defined by the rule $v(E) = I(1_E)$.*

Therefore, a homogeneous, $\mathcal{E}$-cominimum (comaximum) additive operator is associated with a game v. As is easily expected, $\mathcal{E}$-cominimum (comaximum) additivity of an operator requires some further structure on the corresponding game v. To find the required structure, we shall focus on a game v, and say that v is $\mathcal{E}$-cominimum (comaximum) additive to mean that the corresponding operator is $\mathcal{E}$-cominimum (comaximum) additive.

Definition 6 A game v is said to be $\mathcal{E}$-cominimum additive (resp. $\mathcal{E}$-comaximum additive) provided $\int (x+y) dv = \int x dv + \int y dv$ whenever x and y are $\mathcal{E}$-cominimum (resp. $\mathcal{E}$-comaximum).

The following result gives a simple sufficient condition for $\mathcal{E}$-cominimum additivity.

Lemma 10 *Let $v = \sum_{T \in \mathcal{F}} \beta_T u_T \in \mathbb{R}^\mathcal{F}$ be a game. If $\beta_T = 0$ for all $T \notin \mathcal{E}$, then v is $\mathcal{E}$-cominimum additive.*

Proof. Let two functions x and y be $\mathcal{E}$-cominimum. Note that, for all $E \in \mathcal{E}$, $\arg\min_E x \cap \arg\min_E y \neq \emptyset$ and thus $\min_E (x+y) = \min_E x + \min_E y$. So using (2), we have

$$\int (x+y) dv = \sum_{T \in \mathcal{F}} \beta_T \min_T (x+y) = \sum_{T \in \mathcal{E}} \beta_T \min_T (x+y) = \sum_{T \in \mathcal{E}} \beta_T (\min_T x + \min_T y)$$
$$= \sum_{T \in \mathcal{E}} \beta_T \min_T x + \sum_{T \in \mathcal{E}} \beta_T \min_T y = \sum_{T \in \mathcal{F}} \beta_T \min_T x + \sum_{T \in \mathcal{F}} \beta_T \min_T y = \int x dv + \int y dv,$$

which completes the proof. ■

A natural question then is whether the converse is true, i.e., $\mathcal{E}$-cominimum additivity implies $\beta_T = 0$ for any $T \notin \mathcal{E}$. But in general, this is not true. Remember that two functions are comonotonic if and only if they are $\mathcal{F}_2$-cominimum where $\mathcal{F}_2$ is the set of all two-point events. Since the Choquet integral is additive on comonotonic functions, $\mathcal{F}_2$-cominimum additivity does not imply $\beta_T = 0$ for any $T \notin \mathcal{F}_2$.

If v is $\mathcal{E}$-cominimum additive then, by Lemma 5 and the definition of the Choquet integral,

$$v(T_1 \cup T_2) + v(T_1 \cap T_2) = \int (1_{T_1} + 1_{T_2}) dv = \int 1_{T_1} dv + \int 1_{T_2} dv = v(T_1) + v(T_2) \qquad (4)$$

for all $(T_1, T_2) \in W(\mathcal{E})$. We call this property the modularity for $\mathcal{E}$-decomposition pairs.

Definition 7 A game v is said to be modular for $\mathcal{E}$-decomposition pairs provided

$$v(T_1 \cup T_2) + v(T_1 \cap T_2) = v(T_1) + v(T_2) \text{ for all } (T_1, T_2) \in W(\mathcal{E}).$$

It turns out that $\mathcal{E}$-cominimum additivity and the modularity for $\mathcal{E}$-decomposition pairs are equivalent, which leads us to the complete characterization of $\mathcal{E}$-cominimum additivity.

Theorem 3 *Let $v = \sum_{T \in \mathcal{F}} \beta_T u_T \in \mathbb{R}^{\mathcal{F}}$ be a game. The following three statements are equivalent:* (i) *v is $\mathcal{E}$-cominimum additive;* (ii) *v is modular for $\mathcal{E}$-decomposition pairs;* (iii) *$\beta_T = 0$ for any $T \notin \Upsilon(\mathcal{E})$. Therefore, if $\mathcal{E}$ is complete, v is $\mathcal{E}$-cominimum additive if and only if $\beta_T = 0$ for any $T \notin \mathcal{E}$.*

Proof. (iii) $\Rightarrow$ (i). By Lemma 10, v is $\Upsilon(\mathcal{E})$-cominimum additive. By Lemma 4, two functions are $\Upsilon(\mathcal{E})$-cominimum if and only if they are $\mathcal{E}$-cominimum. Thus, v must be $\mathcal{E}$-cominimum additive.

(i) $\Rightarrow$ (ii). This is true by Lemma 5 and the definition of the Choquet integral, as in (4).

(ii) $\Rightarrow$ (iii). If $|T| = 1$, then T must be $\mathcal{E}$-complete, thus the statement is true vacuously. Let $k \geq 2$, and suppose as an induction hypothesis that for any T with $|T| \leq k-1$, if T is not $\mathcal{E}$-complete, $\beta_T = 0$. Let $|T| = k$ and assume that T is not $\mathcal{E}$-complete. Then T is $\mathcal{E}$-decomposable by Lemma 7, and so there exists $(T_1, T_2) \in W(\mathcal{E})$ such that $T = T_1 \cup T_2$.

Since $W(\mathcal{E}) = W(\Upsilon(\mathcal{E}))$ by Lemma 6, any $S \in \Upsilon(\mathcal{E})$ with $S \subsetneq T$ must be either $S \subseteq T_1$ or $S \subseteq T_2$ (or both, i.e., $S \subseteq T_1 \cap T_2$). Therefore, if $S \subseteq T$ satisfies $S \nsubseteq T_1$ and $S \nsubseteq T_2$, then $S \notin \Upsilon(\mathcal{E})$ and so $\beta_S = 0$ by the induction hypothesis, unless $S = T$. Now from the modularity for $\mathcal{E}$-decomposition pairs, we have

$$\begin{aligned} 0 &= v(T_1 \cup T_2) + v(T_1 \cap T_2) - v(T_1) - v(T_2) \\ &= \sum_{S \subseteq T} \beta_S + \sum_{S \subseteq T_1 \cap T_2} \beta_S - \sum_{S \subseteq T_1} \beta_S - \sum_{S \subseteq T_2} \beta_S \\ &= \sum_{S \subseteq T,\ S \nsubseteq T_1,\ S \nsubseteq T_2} \beta_S = \beta_T, \end{aligned}$$

which completes the proof. ■

The cominimum additivity is the conjugate of the comaximum additivity, and vice versa in the following sense.

Lemma 11 *A game v is $\mathcal{E}$-cominimum additive if and only if v' is $\mathcal{E}$-comaximum additive.*

Proof. Since $\min_{\omega \in T} -x(\omega) = -\max_{\omega \in T} x(\omega)$, we have $\int -x dv = -\int x dv'$ by (2) and (3). Thus, $\int (x+y) dv = \int x dv + \int y dv$ holds if and only if $\int ((-x) + (-y)) dv' = \int (-x) dv' + \int (-y) dv'$. So the result holds because x and y are $\mathcal{E}$-cominimum if and only if $-x$ and $-y$ are $\mathcal{E}$-comaximum. ■

Using the conjugation, an analogous characterization can be done for $\mathcal{E}$-comaximum additivity.

Corollary 4 *Let $v = \sum_{T \in \mathcal{F}} \gamma_T u'_T \in \mathbb{R}^{\mathcal{F}}$ be a game. The following three statements are equivalent:* (i) *v is $\mathcal{E}$-comaximum additive;* (ii) *$v(T_1 \cup T_2) + v(T_1 \cap T_2) = v(T_1) + v(T_2)$ for all $(T_1, T_2) \in \mathcal{F} \times \mathcal{F}$ with $(\Omega \backslash T_1, \Omega \backslash T_2) \in W(\mathcal{E})$.* (iii) *$\gamma_T = 0$ for any $T \notin \Upsilon(\mathcal{E})$. Therefore, if $\mathcal{E}$ is complete, v is $\mathcal{E}$-comaximum additive if and only if $\gamma_T = 0$ for any $T \notin \mathcal{E}$.*

Proof. Note that $v' = \sum_{T \in \mathcal{F}} \gamma_T u_T$. By Lemma 11, v is $\mathcal{E}$-comaximum additive if and only if v' is $\mathcal{E}$-cominimum additive. So the result follows from Theorem 3. ∎

A slight modification of Theorem 3 shows that the completeness is tight for our characterization.

Corollary 5 *The following statements are equivalent:* (i) *$\mathcal{E}$ is complete, i.e., $\Upsilon(\mathcal{E}) = \mathcal{E}$;* (ii) *For any game $v = \sum_{T \in \mathcal{F}} \beta_T u_T \in \mathbb{R}^{\mathcal{F}}$, v is $\mathcal{E}$-cominimum additive if and only if $\beta_T = 0$ for any $T \notin \mathcal{E}$.*

Proof. (i) $\Rightarrow$ (ii). This is a restatement of Theorem 3.

(ii) $\Rightarrow$ (i). Suppose that $\mathcal{E}$ is not complete. Then there is $T^* \notin \mathcal{E}$ which is $\mathcal{E}$-complete, i.e., $T^* \in \Upsilon(\mathcal{E})$. Consider a game $v = \sum_{T \in \mathcal{F}} \beta_T u_T = u_{T^*}$. Since $\beta_T = 0$ for every $T \notin \Upsilon(\mathcal{E})$, v is $\mathcal{E}$-cominimum additive by Theorem 3. On the other hand, if (ii) is true, v is not $\mathcal{E}$-cominimum additive because $\beta_{T^*} \neq 0$ and $T^* \notin \mathcal{E}$, which is a contradiction. ∎

5 Applications

5.1 The E-capacity and ε-contamination

Denote by $\Delta(\Omega)$ the set of all probability measures and by Π_E the set of probability measures assigning probability one to an event $E \in \mathcal{F}$, i.e., $\Pi_E = \{p \in \Delta(\Omega) \,|\, p(E) = 1\}$.

Definition 8 For $\pi \in \Delta(\Omega)$, $0 \leq \varepsilon \leq 1$, and $E \in \mathcal{F}$, the set of probability measures $\{(1-\varepsilon)\pi + \varepsilon p \,|\, p \in \Pi_E\}$ is referred to as the ε-contamination of π on E.

The notion of ε-contamination is old; it is discussed in the literature of robust estimation since Huber (1964). In economic applications, the ε-contamination is used with the maximin decision rule (Gilboa and Schmeidler, 1989) which evaluates a function x by the minimum of expected values with respect to the ε-contamination. The following result characterizes this decision rule,[4] which follows from a more general result we shall present later.

[4] Proposition 1 is a generalization of Kojima (2004) which shows that when $\mathcal{E} = \{\Omega\}$, $\mathcal{E}$-cominimum additivity is equivalent to ε-contamination.

Proposition 1 *Let $v \in \mathbb{R}^{\mathcal{F}}$ be a convex capacity and $E \in \mathcal{F}$ be an event. Then the following three statements are equivalent:* (i) *$\int(x+y)dv = \int xdv + \int ydv$ whenever $\arg\min_E x \cap \arg\min_E y \neq \emptyset$;* (ii) *there exist $\pi \in \Delta(\Omega)$ and $\varepsilon \in [0,1]$ such that $v = (1-\varepsilon)\pi + \varepsilon u_E$;* (iii) *there exist $\pi \in \Delta(\Omega)$ and $\varepsilon \in [0,1]$ such that $\int xdv = \min\{\int xdq \mid q = (1-\varepsilon)\pi + \varepsilon p,\ p \in \Pi_E\}$, i.e., the Choquet integral of x is the minimum of expected values with respect to the ε-contamination of π on E.*

The maximin decision rule with the ε-contamination of π on E is represented by the Choquet integral with respect to $v = (1-\varepsilon)\pi + \varepsilon u_E$.[5] Thus, we also call this capacity the ε-contamination of π on E.

Eichberger and Kelsey (1999) investigated the class of capacities which explains the Ellsberg paradox. They called these capacities the E-capacity, and the ε-contamination is a special case.

Definition 9 Let $E_1, \ldots, E_K$ be non-empty, disjoint subsets of Ω with $|E_k| \geq 2$ for each k. A capacity v is said to be an E-capacity with respect to $\mathcal{E} = \{E_1, \ldots, E_K\}$ if there exists a probability π and a number $\varepsilon \in [0,1]$, and probability assignment ρ on $\mathcal{E}$ (i.e. $\rho(E_k) \geq 0$ for each k and $\sum_{k=1}^K \rho(E_k) = 1$) such that $v = (1-\varepsilon)\pi + \varepsilon \sum_{k=1}^K \rho(E_k) u_{E_k}$.

Eichberger and Kelsey (1999) gave an axiomatic characterization of E-capacity, and so that of ε-contamination, a fortiori. The next result, which generalizes Proposition 1, is essentially Proposition 3.1 of Eichberger and Kelsey (1999), but we give an alternative proof based on our main result.[6]

Proposition 2 *Let $v \in \mathbb{R}^{\mathcal{F}}$ be a convex capacity. Let $E_1, \ldots, E_K$ be non-empty, disjoint subsets of Ω with $|E_k| \geq 2$ for each k. Let $\mathcal{E} = \{E_1, \ldots, E_K\}$. Then the following three statements are equivalent:* (i) *v is $\mathcal{E}$-cominimum additive;* (ii) *v is an E-capacity with respect to $\mathcal{E}$;* (iii) *there exists a probability π and numbers $\varepsilon_1, \ldots, \varepsilon_K \in [0,1]$ with $\sum_{k=1}^K \varepsilon_k \leq 1$ such that for any x, $\int xdv = \min\{\int xdq \mid q = (1-\sum_{k=1}^K \varepsilon_k)\pi + \sum_{k=1}^K \varepsilon_k p_k,\ p_k \in \Pi_{E_k}\}$.*

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{E}^* = \mathcal{E} \cup \mathcal{F}_1$. From Lemma 9, $\mathcal{E}^*$ is complete. So by Theorem 3, (i) implies that $\beta_T = 0$ for every $T \notin \mathcal{E}^*$ where $v = \sum_{T \in \mathcal{F}} \beta_T u_T$. Therefore, v must be of the form $v = \sum_{\omega \in \Omega} \beta_{\{\omega\}} u_{\{\omega\}} + \sum_k \beta_{E_k} u_{E_k}$, and this expression is unique. Since $v(\Omega) = 1$, we have $\sum_{\omega \in \Omega} \beta_{\{\omega\}} + \sum_{k=1}^K \beta_{E_k} = 1$. Since v is non-negative, for all $\omega \in \Omega$, $\beta_{\{\omega\}} = v(\{\omega\}) \geq 0$. We claim $\beta_{E_k} \geq 0$ for each k. To see this, write E_k as the union of non-empty disjoint sets, F_1 and F_2, which is possible because $|E_k| \geq 2$. Then by the convexity of v, and from the assumption that E_k's are disjoint, $\sum_{\omega \in E_k} \beta_{\{\omega\}} + \beta_{E_k} = v(E_k) \geq v(F_1) + v(F_2) = \sum_{\omega \in F_1} \beta_{\{\omega\}} + \sum_{\omega \in F_2} \beta_{\{\omega\}}$. Hence $\beta_{E_k} \geq 0$. Set $\varepsilon = \sum_{k=1}^K \beta_{E_k} = 1 - \sum_{\omega \in \Omega} \beta_{\{\omega\}}$. We show that $v = \sum_{\omega \in \Omega} \beta_{\{\omega\}} u_{\{\omega\}} + \sum_k \beta_{E_k} u_{E_k}$

[5] In fact, the core of $v = (1-\varepsilon)\pi + \varepsilon u_E$ coincides with the ε-contamination of π on E, which is a consequence of additivity of the core (cf. Danilov and Koshevoy, 2000).

[6] Ozaki and Nishimura (2003) gave an alternative axiomatization of the ε-contamination. Their axioms are not directly comparable with Eichberger and Kelsey (1999) or Kojima (2004).

is indeed the required expression. If $0 < \varepsilon < 1$, set $\rho(E_k) = \beta_{E_k}/\varepsilon$ for each k, and set $\pi = \frac{1}{1-\varepsilon}\sum_{\omega\in\Omega}\beta_{\{\omega\}}u_{\{\omega\}}$. If $\varepsilon = 0$, set $\pi = \sum_{\omega\in\Omega}\beta_{\{\omega\}}u_{\{\omega\}}$, and if $\varepsilon = 1$, set $\rho(E_k) = \beta_{E_k}$ for each k.

(ii) $\Rightarrow$ (iii): Assume $v = (1-\varepsilon)\pi + \varepsilon\sum_{k=1}^{K}\rho(E_k)u_{E_k}$. Using (2), for any x, $\int xdv = \int xd((1-\varepsilon)\pi + \varepsilon\sum_{k=1}^{K}\rho(E_k)u_{E_k}) = (1-\varepsilon)\int xd\pi + \varepsilon\sum_{k=1}^{K}\rho(E_k)\min_{E_k} x = (1-\varepsilon)\int xd\pi + \varepsilon\sum_{k=1}^{K}\rho(E_k)\min_{p_k\in\Pi_{E_k}}\int xdp_k$. Since E_k's are disjoint, this is equal to $\min\{\int xdq \mid q = (1-\varepsilon)\pi + \varepsilon\sum_{k=1}^{K}\rho(E_k)p_k,\ p_k \in \Pi_{E_k}\}$, so set $\varepsilon_k = \varepsilon\rho(E_k)$, and we have (iii) since $\sum_{k=1}^{K}\varepsilon_k = \varepsilon\sum_{k=1}^{K}\rho(E_k) = \varepsilon$.

(iii) $\Rightarrow$ (i): Let two functions x and y be $\mathcal{E}$-cominimum. Then $\min_{E_k}(x+y) = \min_{E_k} x + \min_{E_k} y$ for every k. Set $\varepsilon = 1-\sum_{k=1}^{K}\varepsilon_k$. We have $\int(x+y)dv = \min\{\int(x+y)dq \mid q = (1-\varepsilon)\pi + \sum_{k=1}^{K}\varepsilon_k p_k,\ p_k \in \Pi_{E_k}\} = (1-\varepsilon)\int(x+y)d\pi + \sum_{k=1}^{K}\varepsilon_k\min_{E_k}(x+y) = (1-\varepsilon)(\int xd\pi + \int yd\pi) + \sum_{k=1}^{K}\varepsilon_k(\min_{E_k} x + \min_{E_k} y) = (1-\varepsilon)\int xd\pi + \sum_{k=1}^{K}\varepsilon_k\min_{E_k} x + (1-\varepsilon)\int yd\pi + \sum_{k=1}^{K}\varepsilon_k\min_{E_k} y = \int xdv + \int ydv$, establishing $\mathcal{E}$-cominimum additivity of v. ■

Let us point out that although we started with a convex capacity for the sake of brevity, the results above can be translated to a "preference based" axiomatization of the E-capacity and the ε-contamination in a straightforward manner. Indeed, replace Schmeidler (1989)'s comonotonic independence axiom with the $\mathcal{E}$-cominimum additivity with $\mathcal{E} = \{E_1, \ldots, E_K\}$. Since $\mathcal{E}$-cominimum additivity implies comonotonic additivity, by Schmeidler's theorem, we have a utility function in the Choquet expected utility form with a convex capacity v. Then apply the result above to show that v is the E-capacity with respect to $\mathcal{E}$.

5.2 Multi-period decisions

We shall consider an axiomatic multi-period decision model developed by Gilboa (1991), which axiomatizes the following special form of utility:

$$\sum_{i=1}^{n} p_i x(i) + \sum_{i=2}^{n}\delta_i|x(i) - x(i-1)|, \tag{5}$$

where $p_1, \ldots, p_n$ and $\delta_2, \ldots, \delta_n$ are constants.[7] Interpret $\Omega = \{1, \ldots, n\}$ as a collection of time periods, and $x(1), \ldots, x(n)$ as a stream of income. The utility in (5) describes the value of the stream of income as an weighted average $\sum_{i=1}^{n} p_i x(i)$ plus an adjustment factor $\sum_{i=2}^{n}\delta_i|x(i) - x(i-1)|$ which measures the variations of the stream.

Let $\mathcal{E} = \{\{i, i+1\} \mid 1 \leq i < n\}$. Thus, $\mathcal{E}$ is the collection of adjacent time periods. Note that $\mathcal{E} \cup \mathcal{F}_1$ is complete since if $E \notin \mathcal{E} \cup \mathcal{F}_1$ then E must contain two points which are not adjacent.

Proposition 3 *Let $v = \sum_{T\in\mathcal{F}}\beta_T u_T \in \mathbb{R}^{\mathcal{F}}$ be a game, and define $\mathcal{E}$ as above. Then the following two statements are equivalent:* (i) *v is $\mathcal{E}$-cominimum additive;* (ii) *the Choquet integral with respect to v has the form* (5).

[7] We thank I. Gilboa for suggesting this application. This is a simplified version of the model studied in Gilboa (1991), which we adopted for ease of exposition.

Proof. Note that $|a-b| = a+b-2\min\{a,b\}$ for any $a,b \in \mathbb{R}$. So, (5) can be written as

$$\begin{aligned}\sum_{i=1}^{n} p_i x(i) + \sum_{i=2}^{n} \delta_i |x(i) - x(i-1)| &= \sum_{i=1}^{n} p_i x(i) + \sum_{i=2}^{n} \delta_i (x(i) + x(i-1) - 2\min\{x(i), x(i-1)\}) \\ &= \sum_{i=1}^{n} \beta_i x(i) + \sum_{i=2}^{n} \beta_{\{i-1,i\}} \min\{x(i), x(i-1)\}, \qquad (6)\end{aligned}$$

where $\beta_i = p_i + \delta_i$ for $i \in \{1, n\}$, $\beta_i = p_i + \delta_i + \delta_{i+1}$ for $i \in \{2, \ldots, n-1\}$, and $\beta_{\{i-1,i\}} = -2\delta_i$ for $i \in \{2, \ldots, n\}$.

Since $\mathcal{E} \cup \mathcal{F}^1$ is complete, by Theorem 3, (i) is equivalent to the condition that $\beta_T = 0$ unless T is a singleton or $T \in \mathcal{E}$. This is true if and only if the Choquet integral with respect to v is of the form (6). ■

References

Danilov, V. I., Koshevoy, G. A., 2000, Cores of cooperative games, superdifferentials of functions, and the Minkowski difference of sets, Journal of Mathematical Analysis and Applications 247, 1–14.

Eichberger, J., Kelsey, D., 1999, E-capacities and the Ellsberg paradox, Theory and Decision 46, 107–140.

Gilboa, I., 1989, Expectation and variation in multi-period decisions, Econometrica 57, 1153–1169.

Gilboa, I., Schmeidler, D., 1989, Maxmin expected utility with non-unique prior, Journal Mathematical Economics 18, 141–153.

Gilboa, I., Schmeidler, D., 1994, Additive representation of non-additive measures and the Choquet integral, Annals of Operations Research 52, 43–65.

Huber, P. J., 1964, Robust estimation of a location parameter, Annals of Mathematical Statistics 35, 73–101.

Kojima, H., 2004, ε-Contamination and comonotonic independence axiom, mimeo, Teikyo University.

Ozaki, H., Nishimura, K. G., 2003, An axiomatic approach to ε-contamination, forthcoming in Economic Theory.

Schmeidler, D., 1986, Integral representation without additivity, Proceedings of the American Mathematical Society 97, 255–261.

Schmeidler, D., 1989, Subjective probability and expected utility without additivity, Econometrica 57, 571–587.

Shapley, L. S., 1953, A value for n-person games, in: H. Kuhn and A. Tucker, eds., Contributions to the Theory of Games II (Princeton Univ. Press, NJ) p.p.307–317.