KIER DISCUSSION PAPER SERIES

KYOTO INSTITUTE
OF
ECONOMIC RESEARCH

http://www.kier.kyoto-u.ac.jp/index.html

Discussion Paper No.599

Bargaining Set and Anonymous Core without the Monotonicity Assumption

by

Chiaki Hara

November 2004

KYOTO UNIVERSITY

KYOTO, JAPAN

Bargaining Set and Anonymous Core without the Monotonicity Assumption

Chiaki Hara* †
Institute of Economic Research, Kyoto University
and
Faculty of Economics, University of Cambridge

November 29, 2004

Abstract

We give an example of an atomless exchange economy in which consumers' preference relations are not monotone and in which the bargaining set of Mas-Colell (1989) consists of all allocations satisfying resource constraints, although the set of all Walrasian equilibrium allocations, the core, and the anonymous core of Hara (2002) are all empty. We also give an equivalence theorem for the anonymous core when the preference relations may not be monotone.

JEL Classification: C62, C71, D41, D51, D82, Q53.

Keywords: Atomless exchange economies, core, bargaining set, anonymous core, equivalence theorems.

1 Introduction

For an exchange economy with an atomless space of consumers, we consider two solution concepts based on the core, the most commonly used solution concept in cooperative game theory. The first one is the bargaining set of Mas-Colell (1989), which was proposed to investigate to what extent the possibility of subsequent counter-objections could undermine the feasibility (or *justifiability* in this case) of an objection. The second one is the anonymous core of Hara (2002), which was proposed to investigate to what extent the presence of private information could undermine the feasibility (or *anonymity* in this case) of an objection.[1] For both solution concepts, the set of unobjectionable outcomes may be larger than the core, and, as such, the principal question was whether these sets coincide with the set of all Walrasian equilibrium allocations.

In any atomless exchange economy satisfying the standard assumptions (such as the completeness, transitivity, continuity, and local non-satiability of preference relations and the strict positivity of initial endowment vectors), the set of Walrasian equilibrium allocations coincides with the core, the core is included in the anonymous core, and the anonymous core

*Institute of Economic Research, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto, 606-8501, Japan. Phone: +81 (0)75 753 7149. Fax: +81 (0)75 753 7148. Email: hara@kier.kyoto-u.ac.jp

†I am grateful to Manabu Toda, Akira Yamazaki, seminar participants at University of Exeter and Kyoto University, participants at the 13th European Workshop on General Equilibrium Theory in Venice, and an anonymous referee.

[1] An application of the anonymous core to the second-best insurance model was discussed in Section 5 of Hara (2002).

is included in the bargaining set. In particular, the equivalence between the core and the set of all Walrasian equilibrium allocations holds even when preference relations need not be monotone. Mas-Colell (1989) showed that if the preference relations are indeed strongly monotone, then the bargaining set coincides with the set of all equilibrium allocations. All the three solution concepts are, under the strong monotonicity assumption, equivalent in outcomes.

In this paper we deal with economies in which preference relations need not be monotone. It is important to incorporate non-monotone preference relations in a general equilibrium model to increase its applicability. A classical example of such an application is the problem of efficient allocation of garbage and toxic wastes via the price mechanism. Another example, which is equally classical but less well recognized in this context, is the Capital Asset Pricing Model with mean-variance utility functions, where the de-meaned market portfolio (the market portfolio with its mean subtracted) can most conveniently be regarded as a bad when embedded into a general equilibrium framework.

The contribution of this paper is twofold. First, we present an example in which the bargaining set is the set of all allocations satisfying the resource constraints, and, at the same time, the set of all Walrasian equilibrium allocations, the core, and the anonymous core are all empty. Second, we prove that, yet, the anonymous core coincides with the set of all Walrasian equilibrium allocations. In short, the equivalence between the bargaining set and the anonymous core fails drastically without the monotonicity assumption.

The example, taken from Hara (2004), who investigated the existence problem of Walarasian equilibria, highlights a weakness of the bargaining set: Its notion of a justifiable objection is so stringent that there may be no such objection, so that all allocations satisfying the resource constraints, even those which are not individually rational, may belong to the bargaining set. Anderson, Trockel, and Zhou (1997) also pointed out this type of weakness by giving an example of a sequence of replica economies in which there is a unique Walrasian equilibrium but the bargaining set eventually occupies the full measure of the set of all individually rational and Pareto efficient allocations having the equal treatment property of Debreu and Scarf (1963).

The intuition behind the example can roughly be explained as follows. As in the case of monotone preference relations, an objection is justifiable if and only if it is price-induced, in the sense that it is formed by consumers who can get (equally well off or) better off by agreeing to trade under some common price vector; and those who would get strictly worse off by doing so stay away from the objection. One of the commodities in the example economy is a *bad*, causing disutility to all consumers, for which the price would have to be strictly negative in any justifiable objection. Now, the crux of the example lies in the way in which the consumers' willingness to accept the bad is distributed: there always exists a group of consumers of a strictly positive measure who would get better off by accepting the bad, however small the absolute value of the (negative) price for the bad is and however favorable the status-quo allocation was. Moreover, then, the group's aggregate demand for the bad would become infinite. Since the infinite aggregate demand is never feasible in any objection, this means that there is no justifiable objection to any allocation. We can therefore conclude that every allocation belongs to the bargaining set.

An anonymous objection, on the other hand, can be formed by consumers who can get (equally well off or) better off by choosing from some menu, identical for all consumers in the economy, of net demands; and those who would get strictly worse off by doing so stay away from the objection. Since the menu can be arbitrary as long as it leads to a resource-feasible allocation, it can, in particular, set an upper bound on the quantity of the bad that each member is allowed to accept. Such an upper bound can guarantee that the aggregate demand is finite. Moreover, it turns out that for each non-Walrasian allocation, there exists an appropriate upper bound to form an anonymous core. We can therefore conclude that the equivalence theorem still holds for the anonymous core even without the monotonicity assumption.

2 Model

The set of consumers is given by an atomless complete probability measure space $(A, \mathscr{A}, \mu)$. Denote by L the number of commodities available in the economy. Denote by $\mathscr{P}$ be the set of all complete, transitive, continuous, and locally non-satiated preference relations defined on the consumption set $X = \boldsymbol{R}^L_+$. It is endowed with the relative topology of the closed convergence topology on the set of all closed subsets of $X \times X$, and also with the Borel σ-field induced from this topology. An *economy* is characterized by a mapping of A to $\mathscr{P}$, $a \mapsto \succsim_a$, which specifies each consumer's preference relation, and an integrable function $e : A \to \boldsymbol{R}^L_{++}$, which specifies each consumer's initial endowment vector. Note that we assume that every consumer is endowed with positive quantities of all commodities, to avoid complications arising from the failure of the so-called minimum income condition.

Definition 1 Let $B \in \mathscr{A}$ and $f : B \to X$, then f is an *allocation within* B if it is integrable and satisfies the resource feasibility constraint $\displaystyle\int_B f = \int_B e$. An allocation within A is also simply called an *allocation*.

Definition 2 A pair (f, p) of an allocation f and a price vector $p \in \boldsymbol{R}^L$ is a *Walrasian equilibrium* if for almost every $a \in A$ and every $x \in X$, $p \cdot f(a) \leq p \cdot e(a)$ and $p \cdot x > p \cdot e(a)$ whenever $x \succ_a f(a)$.

Definition 3 Let $f : A \to X$, which may or may not satisfy the resource constraint. A pair (C, g) of $C \in \mathscr{A}$ with $\mu(C) > 0$ and an allocation $g : C \to X$ within C is a *strong objection to* f if $\mu(\{a \in C \mid f(a) \succsim_a g(a)\}) = 0$. It is a *weak objection to* f if $\mu(\{a \in C \mid f(a) \succ_a g(a)\}) = 0$ and $\mu(\{a \in C \mid g(a) \succ_a f(a)\}) > 0$.

A weak objection differs from a strong objection in that only the former allows some of the members $a \in C$ to be indifferent between $f(a)$ and $g(a)$. A strong objection is thus a weak objection but the converse need not be true. Yet if preference relations are strongly monotone, then there is a strong objection whenever there is a weak one.

Definition 4 The *strong core* is the set of all allocations to which there is no weak objection. The *weak core* is the set of all allocations to which there is no strong objection.

If $f : A \to X$, $C \in \mathscr{A}$, and $g : C \to X$, then we define $\varphi_{f,C,g} : A \to X$ by

$$\varphi_{f,C,g}(a) = \begin{cases} g(a) & \text{if } a \in C, \\ f(a) & \text{if } a \in A \setminus C. \end{cases}$$

Note that $\varphi_{f,C,g}$ may not satisfy the resource feasibility constraint $\displaystyle\int_A \varphi_{f,C,g} = \int_A e$ even when f and g do so.

Definition 5 Let (C, g) be a weak objection to an allocation f. A pair (D, h) of $D \in \mathscr{A}$ with $\mu(D) > 0$ and an allocation $h : D \to X$ within D is a *strong counter-objection to* (C, g)

if it is a strong objection to $\varphi_{f,C,g}$. It is a *weak counter-objection to* (C, g) if it is a weak objection to $\varphi_{f,C,g}$.

The above definition of a strong counter-objection is nothing but Definition 2 of Mas-Colell (1989).

Definition 6 Let f be an allocation. An weak objection to f is *weakly justified* if there is no strong counter-objection to it. It is *strongly justified* if there is no weak counter-objection to it. The *strong bargaining set* is the set of all allocations to which there is no weakly justified weak objection. The *weak bargaining set* is the set of all allocations to which there is no strongly justified weak objection.

Every strongly justified weak objection is a weakly justified weak objection, but the converse need not be true. Hence the strong bargaining set is included in the the weak bargaining set, but the converse need not be true. Since Mas-Colell's notion of justifiability is the weak justifiability in the above definition, his bargaining set is the strong bargaining set in the above definition. Note that when defining both the strong and weak counter-objections, the (original) objection (C, g) to f is assumed to be a weak one. Mas-Colell (1989, Remarks 1 and 5) and Anderson, Trockel, and Zhou (1997, Section 2) pointed out that even when preference relations are strongly monotone, his equivalence theorem would not hold if the original objections were required to be strong ones. The difference between the strong and weak bargaining sets lie in the justifiability of original weak objections.

The definitions of an anonymous objection and the anonymous core are taken from Hara (2002).

Definition 7 A weak objection (C, g) to an allocation f is *anonymous* if there exists a $B \in \mathscr{A}$ such that $\mu(A \backslash B) = 0$, and, for every $a \in B$ and every $b \in B \cap C$, if $e(a)+(g(b)-e(b)) \in X$, then $\varphi_{f,C,g}(a) \succsim_a e(a) + (g(b) - e(b))$.

The anonymous objection embodies two anonymity requirements on the allocation g within the coalition B. For every $a \in B \cap C$, the condition $\varphi_{f,C,g}(a) \succsim_a e(a) + (g(b) - e(b))$ becomes $g(a) \succsim_a e(a)+(g(b)-e(b))$, which means that a cannot get better off by pretending to be any other member of C. For every $a \in B \backslash C$, the condition $\varphi_{f,C,g}(a) \succsim_a e(a)+(g(b)-e(b))$ becomes $f(a) \succsim_a e(a)+(g(b)-e(b))$, which means that a cannot get better off by pretending to be any member of C and is happy not to join C.

Definition 8 The *anonymous core* is the set of all allocations to which there is no anonymous weak objection.

3 Walrasian and Menu-Induced Objections

In this section we present two notions of objections which are useful for our subsequent analysis. The first, ingenious concept was given in Mas-Colell (1989).

Definition 9 (Mas-Colell) Let $f : A \to X$ be an allocation. A weak objection (C, g) to f is *Walrasian* if there exists a $p \in \boldsymbol{R}^L$ such that for almost every $a \in A$ and every $x \in X$, if $x \succ_a \varphi_{f,C,g}(a)$, then $p \cdot (x - e(a)) > 0$.

A Walrasian objection should be considered as a "price-induced" objection. Note first that by the assumption of locally non-satiated preference relations, for almost every $a \in A$ and every $x \in X$, if $x \succsim_a \varphi_{f,C,g}(a)$, then $p \cdot x \geq p \cdot e(a)$. In particular, since $p \cdot g(a) \geq p \cdot e(a)$

for every $a \in C$ and $\int_C g = \int_C e$, $p \cdot g(a) = p \cdot e(a)$ for almost every $a \in C$. Thus $g(a)$ is his demand under the price vector p. The definition of a Walrasian objection therefore implies that almost any member $a \in C$ cannot get worse off by rejecting $f(a)$ and participate in market transactions under p. It also implies that almost any non-member $a \in A \backslash C$ cannot get better off by rejecting the consumption vector $f(a)$ and participate in market transactions under p; and he would thus choose to retain the consumption vector $f(a)$. Note that if $\varphi_{f,C,g}$ satisfies the resource constraint, $\int_A \varphi_{f,C,g} = \int_A e$, then it is a Walrasian equilibrium allocation.

The following lemma is in the spirit of the second welfare theorem and similar to the statement that every core allocation is Walrasian. It was proved as Proposition 3 of Mas-Colell (1989) in the case of strongly monotone preferences. His proof method is applicable without the strong monotonicity. The dispensability of monotonicity was also mentioned in Remark 1 of Section 3 of Hildenbrand (1982). We shall therefore omit the proof of the following lemma.

Lemma 1 *Every weakly justified weak objection is Walrasian.*

The second concept of objections is a generalization of a Walrasian weak objection.

Definition 10 Let f be an allocation, (C, g) a weak objection to f, then (C, g) is *menu-induced* if there exists a subset Z of $\boldsymbol{R}^L$ such that $g(a) - e(a) \in Z$ for almost every $a \in C$ and, for almost every $a \in A$ and every $x \in X$, if $x \succ_a \varphi_{f,C,g}(a)$, then $x - e(a) \notin Z$.

If (C, g) is a Walrasian objection with a price vector p, then it is menu-induced by the budget set $\{z \in \boldsymbol{R}^L \mid p \cdot z \leq 0\}$ for net demands. The above definition differs from that of a Walrasian weak objection in that the former allows the menu set Z to be any subset of $\boldsymbol{R}^L$.

Lemma 2 *An weak objection is anonymous if and only if it is menu-induced.*

Proof of Lemma 2 It can be easily shown, and also follows from one direction of an equivalence theorem of Hammond (1979, Theorem 2) that if (C, g) is a menu-induced objection to f, then it is anonymous. For the converse, let the menu Z be $\{g(a) - e(a) \in \boldsymbol{R}^L \mid a \in B \cap C\}$, where B is as in Definition 7, then the anonymous weak objection f is induced by the menu Z. ///

To every allocation that is not individually rational, there exists an objection that is induced by the menu $Z = \{0\}$. Hence every allocation in the anonymous core is individually rational. As will become clear in Example 1, this can be seen as a first piece of evidence of the difference between the bargaining set and the anonymous core.

In the subsequent analysis of this paper, we consider a menu Z induced by both prices and upper bounds on the net demands. Finding these prices involves a argument similar to the proof of Proposition 2 of Mas-Colell (1989); and the use of a similar but different type of upper bounds can be found in the existence proofs for atomless economies, such as in Schmeidler (1969) and Hildenbrand (1970).

Lemmas 1 and 2 imply the following proposition.

Proposition 1 *The weak core is included in the anonymous core. The anonymous core is included in the bargaining set.*

4 Example on the Bargaining Set

The following example is taken from Hara (2004).

Example 1 Let A be the open interval $(0,1)$, $\mathscr{A}$ be the set of Lebesgue measurable subsets of A, and μ be the Lebesgue measure restricted on A. For each $a \in A$, let $\succsim_a$ be represented by a utility function $u_a(x) = x_1 - a\,(x_2)^2$ and $e(a) = (2,1)$.

In this example, for every $a \in A$, $e(a) \in \operatorname{int} X$ and u_a is smooth, strictly differentiably quasi-concave, and can be extended to the entire $\boldsymbol{R}^2$. It was shown in Hara (2004) that there is no Walrasian equilibrium in this economy. By the core equivalence theorem, therefore, the core of this economy is empty. The economy satisfies the assumptions of the equivalence theorem of the anonymous core (Theorem 1) in the next section, and hence the anonymous core is also empty.

Proposition 2 *In Example 1, the strong bargaining set consists of all allocations.*

Proof of Proposition 2 Let f be an allocation of Example 1. To prove the proposition, by Lemma 1, it suffices to show that there is no Walrasian weak objection to f. We shall do so by a contradiction argument, by supposing that there is a Walrasian weak objection (C,g) with a price vector p.

We can assume that $p_1 = 1$. If $p_2 \geq 0$, then $g_2(a) = 0$ for every $a \in C$, which is a contradiction to $\displaystyle\int_C g = \int_C e$. Since

$$\left| \frac{\dfrac{\partial u_a(e(a))}{\partial x_2}}{\dfrac{\partial u_a(e(a))}{\partial x_1}} \right| = 2a < 2$$

for every $a \in A$, if $p_2 \leq -2$, then $g_2(a) > e_2(a)$ for almost every $a \in C$, which is also a contradiction to $\displaystyle\int_C g = \int_C e$. Thus $-2 < p_2 < 0$. Thus $p \cdot e(a) > 0$, that is, the budget line going through $e(a) = (2,1)$ intersects not the vertical, but the horizontal, axis. By $p_2 < 0$, the utility-maximizing consumption bundle under the price vector p does not lie on the horizontal axis for any $a \in A$. Hence every consumer a's demand vector $h(a)$ under p is given by the first-order condition for the interior maximum:

$$h(a) = \left(2 - |p_2| + \frac{|p_2|^2}{2a}, \frac{|p_2|}{2a}\right).$$

Since $g(a) = h(a)$ for almost every $a \in C$ and g is integrable, the function $a \mapsto 1/a$ of C into R_{++} is integrable. On the other hand, the utility levels obtainable from $h(a)$ is

$$u_a(h(a)) = 2 - |p_2| + \frac{|p_2|^2}{4a}.$$

Then, for almost every $a \in A \setminus C$,

$$2 - a = u_a(e(a)) \leq u_a(h(a)) \leq u_a(f(a)) = f_1(a) - a(f_2(a))^2 \leq f_1(a),$$

and both the far left and right hand sides are integrable on $A \setminus C$. Thus the function $a \mapsto u_a(h(a))$ is integrable on $A \setminus C$. Thus the function $a \mapsto 1/a$ of $A \setminus C$ into R_{++} is integrable.

We have so far shown that the function $a \mapsto 1/a$ is integrable on both C and $A \setminus C$. It is therefore integrable on the entire A. But this is a contradiction. Hence there is no Walrasian weak objection to f. ///

5 Equivalence Theorem on the Anonymous Core

The crucial condition for our equivalence theorem is the following.

Definition 11 Let $v \in X$. The preference relation $\succsim_a$ is *strongly monotone in* v if, for every $x \in X$ and every $t > 0$, $x + tv \succ_a x$. It is *overridingly monotone in* v if, for every $x \in X$ and every $y \in X$, there exists a $t > 0$ such that $x + tv \succ_a y$.

The directional strong monotonicity is a substantial weakening of the usual strong monotonicity ($x \succ_a y$ whenever $x - y \in \boldsymbol{R}^L_+ \setminus \{0\}$). The overriding monotonicity is due to Broome (1972). If $\succsim_a$ is strictly convex, the strong monotonicity in v is implied by the overriding monotonicity in the same vector. The strong and overriding monotonicity in every $v \in \boldsymbol{R}^L_{++}$ is implied by the usual strong monotonicity. More generally, if a preference relation is proper in the direction of an open cone in X in the sense of Manelli (1991a, 1991b), then it is also strongly and overridingly monotone in every vector of the open cone. The preference relations in Example 1 are strongly and overridingly monotone in $v = (1, 0)$.

Theorem 1 *If there exists a $v \in X$ such that $\succsim_a$ is strongly monotone in v for every a in some $B \in \mathscr{A}$ with $\mu(A \setminus B) = 0$ and $\succsim_a$ is overridingly monotone in v for every a in some $B \in \mathscr{A}$ with $\mu(B) > 0$, then the anonymous core coincides with the set of all Walrasian equilibrium allocations.*

Proof of Theorem 1 By the core equivalence theorem, Proposition 1, and Lemma 2, it is sufficient to show that if an allocation f is not a Walrasian equilibrium allocation, then there is a menu-induced weak objection. Without loss of generality, we can assume that $\|v\| = 1$, where $\|\cdot\|$ denotes the Euclidean norm. Define $P = \{p \in \boldsymbol{R}^L \mid p \cdot v = 1\}$.

Step 1 For each $a \in A$, each positive integer n, and each $p \in P$, let $\xi^n(p, a)$ be the set of utility-maximizing net demands in the *truncated* budget set

$$\left\{ z \in \boldsymbol{R}^L \mid p \cdot z \leq 0 \text{ and } \|z - (v \cdot z)v\| \leq n \right\}. \tag{1}$$

We now show that $p \mapsto \xi^n(p, a)$ is a nonempty- and compact-valued and upper hemi-continuous correspondence. Indeed, since $p \cdot v = 1$, the orthogonal projection of $\{z \in \boldsymbol{R}^L \mid v \cdot z = 0\}$ onto $\{z \in \boldsymbol{R}^L \mid p \cdot z = 0\}$ is (linear and) surjective. Since

$$\left\{ z \in \boldsymbol{R}^L \mid p \cdot z = 0 \text{ and } \|z - (v \cdot z)v\| \leq n \right\} \tag{2}$$

is the image of $\{z \in \boldsymbol{R}^L \mid v \cdot z = 0 \text{ and } \|z\| \leq n\}$ under this projection, the set (2) is compact. Hence the utility-maximizing net demands in (2) constitute a nonempty- and compact-valued and upper hemi-continuous correspondence. Since $\succsim_a$ is strongly monotone in v for almost every $a \in A$, a net demand is utility-maximizing in (2) if and only if it is utility-maximizing in

$$\left\{ z \in \boldsymbol{R}^L \mid p \cdot z = 0 \text{ and } \|z - (v \cdot z)v\| \leq n \right\} - \{\lambda v \in X \mid \lambda \geq 0\}. \tag{3}$$

Since (1) includes (2) and coincides with (3), a net demand is utility-maximizing in (1) if and only if it is utility-maximizing in (2). Hence $p \mapsto \xi^n(p, a)$ is a nonempty- and compact-valued and upper hemi-continuous correspondence. Of course, $p \cdot z = 0$ for every $p \in P$, every $a \in A$, and every $z \in \zeta^n(p, a)$.

Step 2 Now let f be an allocation that is not a Walrasian equilibrium allocation. Then define

$$\zeta^n(p,a) = \begin{cases} \xi^n(p,a) & \text{if } e(a) + z \succ_a f(a) \text{ for every } z \in \xi^n(p,a), \\ \xi^n(p,a) \cup \{0\} & \text{if } e(a) + z \sim_a f(a) \text{ for every } z \in \xi^n(p,a), \\ \{0\} & \text{if } f(a) \succ_a e(a) + z \text{ for every } z \in \xi^n(p,a). \end{cases}$$

Then $p \mapsto \zeta^n(p, a)$ is a nonempty- and compact-valued and upper hemi-continuous correspondence. Moreover, for every $a \in A$ and $p \in P$, $\zeta^n(p, a)$ belongs to the set (2).

Let $B \in \mathscr{A}$ be such that $\mu(B) > 0$ and, for every $a \in B$, $\succsim_a$ is strongly and overridingly monotone in v. We shall now prove that for every $a \in B$ and every $t > 0$, there exists a positive integer m such that for every $p \in P$ and every $z \in \zeta^n(p, a)$, if $\|p\| \geq m$, then $v \cdot z \geq t$. In fact, by the overriding monotonicity and the continuity, there exist a $t > 0$ and an $s > 0$ such that $e(a) + tv + y \succ_a f(a)$ for every $y \in \boldsymbol{R}^L$ with $\|y\| \leq s$. For every $p \in P$ with $\|p\| \geq t/s$,

$$p \cdot \left(e(a) + tv - \frac{s}{\|p\|} p \right) = p \cdot e(a) + t - s\|p\| \leq p \cdot e(a).$$

Since $e(a) + tv - \dfrac{s}{\|p\|} p \succ_a f(a)$, this implies that $\zeta^n(p, a) = \xi^n(p, a)$. Since, for every $z \in \xi^n(p, a)$,

$$\|z\| \leq \|(v \cdot z)v\| + \|z - (v \cdot z)v\| \leq |v \cdot z| + n,$$

the standard argument is applicable along with the strong monotonicity in v to show that for every $t > 0$, there exists a positive integer m such that for every $p \in P$ and every $z \in \xi^n(p, a)$, if $\|p\| \geq m$, then $|v \cdot z| \geq t$. If $t > v \cdot e(a)$, then this implies that $v \cdot z \geq t$. By letting $m \geq t/s$ if necessary, we complete the proof.

Step 3 Now define

$$\zeta^n(p) = \int_A \zeta^n(p, a)\, \mu(da).$$

Since $\zeta^n(p, a)$ belongs to the set (2) for every $a \in A$ and $p \in P$, this integral is well defined and $p \cdot z = 0$ for every $p \in P$ and every $z \in \zeta^n(p)$. Moreover, $p \mapsto \zeta^n(p)$ is a nonempty- and compact-valued and upper hemi-continuous correspondence. By, for example, L.1.3 of Mas-Colell (1985), ζ^n is also convex-valued. We shall now prove that there exists a positive integer m such that for every $p \in P$ and every $z \in \zeta^n(p)$, if $\|p\| \geq m$, then $v \cdot z > 0$. To do so, for each m and t, define

$$\mathit{\Gamma}_{m,t} = \left(\{ z \in \boldsymbol{R}^L \mid v \cdot z \leq t \} \times \{ p \in P \mid \|p\| \geq m \} \times B \right) \cap \{ (z, p, a) \in \boldsymbol{R}^L \times P \times A \mid z \in \zeta^n(p, a) \}.$$

Then $\mathit{\Gamma}_{m,t} \in \mathscr{B}(\boldsymbol{R}^L) \otimes \mathscr{B}(P) \otimes \mathscr{A}$. Since the probability measure space $(A, \mathscr{A}, \mu)$ is complete, by D.II.(11) of Hildenbrand (1974), the projection of $\mathit{\Gamma}_{m,t}$ onto A, denoted by $B_{m,t}$, belongs to $\mathscr{A}$. Then $B_{m,t} \supseteq B_{m+1,t}$ and $\bigcap_{m=1}^{\infty} B_{m,t} = \varnothing$ by the result in the previous step. Hence, for every $t \geq 0$, there exists a positive integer m_t such that $\mu(B_{m_t,t}) < (1/2)\mu(B)$.

Now let $t > 0$, $\|p\| > m_t$, and $z^n : A \to \boldsymbol{R}^L$ be a measurable selection of $\zeta^n(p, \cdot)$. Then $v \cdot z^n(a) > t$ for every $a \in B \setminus B_{m_t,t}$. Hence

$$v \cdot \left(\int_A z^n \right) = \int_{B \setminus B_{m_t,t}} v \cdot z^n(a)\, \mu(da) + \int_{(A \setminus B) \cup B_{m_t,t}} v \cdot z^n(a)\, \mu(da) > \frac{\mu(B)}{2} t - v \cdot \left(\int_A e \right).$$

Hence, by taking t sufficiently large, we can make $v \cdot \left(\int_A z^n \right) > 0$. The proof is now completed.

By Lemma 3 in the appendix, therefore, there exists a $p^n \in P$ such that $0 \in \zeta^n(p^n)$. Let $z^n : A \to \boldsymbol{R}^L$ be a measurable selection of $\zeta^n(p, \cdot)$ such that $\int_A z^n = 0$.

Step 4 Without loss of generality, we can assume that the sequence of normalized price vectors $\|p^n\|^{-1} p^n$ converges to some $p \in \boldsymbol{R}^L$ with $\|p\| = 1$. Define

$$\begin{aligned} \Delta &= \{(z, a) \in \boldsymbol{R}^L \times A \mid p \cdot z < 0 \text{ and } e(a) + z \succ_a f(a)\}, \\ \Delta^n &= \{(z, a) \in \boldsymbol{R}^L \times A \mid p^n \cdot z \le 0,\ \|z - (v \cdot z)v\| \le n, \text{ and } e(a) + z \succ_a f(a)\}. \end{aligned}$$

Then $\bigcup_{n=1}^{\infty} \Delta^n \supseteq \Delta$. Denote by D the projection of Δ onto A and D^n the projection of Δ^n onto A, then $\bigcup_{n=1}^{\infty} D^n \supseteq D$. By D.II.(11) of Hildenbrand (1974), $D \in \mathscr{A}$ and $D^n \in \mathscr{A}$. Moreover, since f is not a Walrasian equilibrium allocation and $e(a) \in \operatorname{int} X$ for every $a \in A$, $\mu(D) > 0$. Hence there exists an n such that $\mu(D^n) > 0$. Then, for such an n, define $C^n = \{a \in A \mid z^n(a) \neq 0\}$ and $g^n : C^n \to X$ by $g^n(a) = e(a) + z^n(a)$, then (C^n, g^n) is a objection induced by the menu

$$Z = \left\{ z \in \boldsymbol{R}^L \mid p^n \cdot z \le 0 \text{ and } \|z - (v \cdot z)v\| \le n \right\}.$$

///

6 Conclusion

We have shown (Theorem 1) that in an atomless economy, the equivalence between the anonymous core and the set of Walrasian equilibrium allocations still holds even without the monotonicity assumption, as long as the directional strong monotonicity is imposed on almost every consumer and the overriding monotonicity is imposed on some set of consumers with positive measure (Definition 11). The equivalence fails drastically for the bargaining set of Mas-Colell (1989), however: We have given an example of an atomless economy with one good and one bad (Example 1), for which every resource-feasible allocation belongs to the bargaining set, while there is no Walrasian equilibrium.

There are a couple of other notions of objections worth investigating. The first one is the notion of an objection involving a *leader*, along the lines of Aumann and Maschler (1964), Davis and Maschler (1963), and Geanakoplos (1978), who is by definition never allured to join a counter-objection. Anderson (1998) proved convergence theorems for their bargaining sets and general sequences of finite economies. As this notion is not immediately applicable to an atomless economy, Mas-Colell (1989, Remark 2) instead considered a notion of a δ-objection, which involves a group of leaders with a positive but arbitrarily small measure and should thus be appropriate for the setting of this paper. Analysis on the validity of the equivalence principle with respect to δ-objections but without the monotonicity assumption would help clarify the role of the failure of the monotonicity assumption in our example (Example 1) for the bargaining set of Mas-Colell (1989).[2]

The second notion of objections worth investigating is the notion considered in (i) of Proposition 7.3.2 of Mas-Colell (1985), in which the allocation associated with an objection is required to be Walrasian within the coalition but some consumers outside the coalition may find it strictly beneficial to trade under the same prices as the coalition members. Such an objection need therefore not be a Walrasian objection in the sense of Definition 9 and the associated core-like solution set may hence be smaller than the bargaining set.

A key property to be noted in the proof of Theorem 1 is that the proportion of consumers within the objecting coalition C^n, constructed in Step 4, who do not receive the net demands under the price vector p^n can be made arbitrarily small by making n sufficiently large. It would therefore not be unreasonable to conjecture that our proof method may well turn out to be useful to establish the equivalence with respect to these notions of objections.

A Lemma on the Existence of an Equilibrium Price Vector

Lemma 3 *Let $v \in \boldsymbol{R}^L$ with $\|v\| = 1$, where $\|\cdot\|$ denotes the Euclidean norm, and define $P = \{p \in \boldsymbol{R}^L \mid p \cdot v = 1\}$. Let $\zeta : P \to \boldsymbol{R}^L$ be an nonempty-, convex-, and compact-valued and upper hemi-continuous correspondence satisfying the strict Walras law, that is, $p \cdot z = 0$ for every $p \in P$ and $z \in \zeta(p)$. Suppose that there exists a positive integer m such that for every $p \in P$ and every $z \in \zeta(p)$, if $\|p\| \geq m$, then $v \cdot z > 0$. Then there exists a $p \in P$ such that $0 \in \zeta(p)$.*

We prove this lemma in the essentially same way as Neuefeind (1980) and Hüsseinov (1999), but we do not impose any non-negativity constraints on prices.

Proof of Lemma 3 Let $P^* = \left\{ p \in P \mid \min_{z \in \zeta(p)} v \cdot z \leq 0 \right\}$ and denote by $\mathrm{co}\,(P^* \cup \{v\})$ the convex hull of $P^* \cup \{v\}$. Since $\mathrm{co}\,(P^* \cup \{v\})$ is a compact subset of P, there exists a convex and compact set Q whose relative interior (with respect to P) includes P^*. Then P^* and the relative boundary of Q (with respect to P) are disjoint.

By Proposition 3 of Hildenbrand (1974, I.B), there exists a convex and compact subset Z of $\boldsymbol{R}^L$ such that $\zeta(p) \subset Z$ for every $p \in Q$. Following the construction of Debreu (1956), we can show that there exists a $(p^*, z^*) \in Q \times Z$ such that $z^* \in \zeta(p^*)$ and

$$p^* \cdot z^* \geq p \cdot z^* \tag{4}$$

for every $p \in Q$. By the strict Walras law and $v \in Q$, this implies that $0 \geq v \cdot z^*$. Thus $p^* \in P^*$. Hence p^* belongs to the relative interior of Q. This and (4) together imply that $z^* = cv$ for some $c \in \boldsymbol{R}$. By the strict Walras law, if we take the inner products of both sides with p^*, then we obtain $c = 0$. Hence $z^* = 0$ and the proof is completed. ///

References

[1] Robert Anderson, 1998, Convergence of the Aumann-Davis-Maschler and Geanakoplos bargining sets, *Economic Theory* 11, 1–37.

[2] I owe this observation to an anonymous referee.

[2] Robert Anderson, Walter Trockel, and Lin Zhou, 1997, Nonconvergence of the Mas-Colell and Zhou bargaining sets, *Econometrica* 65, 1227–39.

[3] Robert J. Aumann, and Michael Maschler, 1964, The bargaining set for cooperative games, in: M. Dresher, L. S. Shapley, and A. W. Tucker, eds., *Advances in Game Theory* (Princeton University Press, Princeton) 443–476.

[4] John Broome, Approximate equilibrium in economies with indivisible commodities, *Journal of Economic Theory* 5, 224–249.

[5] Morton Davis, and Michael Maschler, 1963, Existence of stable payoff configurations for cooperative games, *Bulletin of the American Mathematical Society* 69, 106–110.

[6] Gerard Debreu and Herbert Scarf, 1963, A limit theorem on the core of an economy, *International Economic Review* 4, 236–246.

[7] John Geanakoplos, The bargaining set and nonstandard analysis, Chapter 3 of Ph.D. dissertation, Harvard University.

[8] Peter Hammond, 1979, Straightforward individual incentive compatibility in large economies, *Review of Economic Studies*, 46, 263–282.

[9] Chiaki Hara, 2002, The anonymous core of an exchange economy, *Journal of Mathematical Economics*, 38, 91–116.

[10] Chiaki Hara, 2004, Existence of equilibria in economies with bads, forthcoming in *Econometrica.*

[11] Werner Hildenbrand, 1970, Existence of equilibria for economies with production and a measure space of consumers, *Econometrica*, 38, 608–623.

[12] Werner Hildenbrand, 1974, *Core and Equilibria of a Large Economy* (Princeton University Press, Princeton).

[13] Werner Hildenbrand, 1982, Core of an Economy, in *Handbook of Mathematical Economics* vol. 2., edited by Kenneth J. Arrow and Michael D. Intriligator, North Holland.

[14] Farhad Hüsseinov, 1999, Boundary behavior of excess demand and existence of equilibrium, *Journal of Economic Theory*, 87, 434–449.

[15] Alejandro M. Manelli, 1991a, Monotonic preferences and core equivalence, *Econometrica* 59, 123–138.

[16] Alejandro M. Manelli, 1991b, Core convergence without monotone preferences and free disposal, *Journal of Economic Theory* 55, 400–415.

[17] Andreu Mas-Colell, 1985, *General Economic Equilibrium: A Differentiable Viewpoint* (Cambridge University Press, Cambridge).

[18] Andreu Mas-Colell, 1989, An equivalence theorem for a bargaining set, *Journal of Mathematical Economics* 18, 129–139.

[19] Wilhelm Neuefeind, 1980, Notes on existence of equilibrium proofs and the boundary behavior of supply, *Econometrica* 48, 1831–1837.

[20] David Schmeidler, 1969, Competitive equilibria in markets with a continuum of traders and incomplete preferences, *Econometrica* 37, 578–585.