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"ENDOGENOUS BUSINESS CYCLES WITH BUBBLES"

Shintaro Asaoka

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Shintaro Asaoka*

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Abstract

This study examines the existence of bubbles in an economy with a low growth rate. By using an overlapping-generations model with Matsuyama's (1999) production sector, it is shown a bubble exists in an economy with a low growth rate. If consumers can borrow assets when they are young, then there is a unique cycle with a bubble moving back and forth between two phases. In one phase, the output growth rate is low and innovation occurs. In the other phase, the output growth rate is high and there is no innovation. Therefore, a bubble also exists in an economy with a high growth rate. On the contrary, cycles cannot emerge if consumers save assets when they are young.

*Institute of Economic Research, Kyoto University, Sakyo-Ku, Kyoto 606-8501, Japan
E-mail : asaoka@kier.kyoto-u.ac.jp

1 Introduction

Bubbles have been a main topic of the economic growth literature since the seminal studies of Samuelson (1958) and Tirole (1985), which show that bubbles improve the problem of dynamic inefficiency in an overlapping-generations model. Thus, it is important to analyze the existence of bubbles in a growing economy. Recently, Martin and Ventura (2012) provide a model in which the economic growth rate increases with the expansion of a bubble. In fact, in real economies, many countries have experienced economic booms with bubbles. For example, the Japanese economy was growing when asset prices were inflated from 1985 to 1989. The economic boom in the United States from 2000 to 2006 came with rising prices of real estate[1]. Therefore, some may believe that bubbles exist during high economic performance. However, this belief is not correct and their studies do not imply this belief immediately.

In contrast to the examples mentioned above, there are cases in which bubbles exist in an economic downturn. In China, for example, housing prices kept rising sharply from 2015 while the GDP growth rate was declining and low compared with the level of 2006 (OECD, 2017). Therefore, it is possible that bubbles exist in an economy whose economic performance is low. Although it has been shown that an expansion of a bubble slows down economic growth in Tirole (1985), this result does not explain the existence of bubbles in an economy with low economic performance[2].

The main purpose of this study is to provide a theory for the existence of bubbles when the economic growth rate is low. To show this, we provide an overlapping-generations model with endogenous growth. Thus, we also reexamine the conditions for the existence of a steady-state equilibrium with bubbles in an endogenous growth model.

Our analysis is based on Tirole (1985) and Grossman and Yanagawa (1993). To achieve the research aims mentioned above, we add two extensions to their model. First, we consider two cases of bubbles : one is analyzed by Tirole (1985) and the other by Benhabib and Laroque (1988) and Kojima (2012a,b). In Tirole's (1985) model, consumers in their youth purchase an asset to save, which can form a bubble in the long run. On the other hand, Benhabib and Laroque (1988) and Kojima (2012a,b) set

[1] Martin and Ventura (2012) point out these facts.

[2] Grossman and Yanagawa (1993) and Futagami and Shibata (2000) also show this result.

up overlapping-generations models in which consumers borrow an asset (or money) when they are young[3]. In particular, Kojima (2012a,b) show that a steady-state equilibrium with a bubble can exist if the interest rate exceeds the growth rate in the steady-state equilibrium without bubbles. This result implies that the steady-state equilibrium with a bubble is dynamically inefficient[4]. Hereafter, Tirole's (1985) model is called the Samuelson–Tirole case and the model of Benhabib and Laroque (1988) and Kojima (2012a,b) is called the Benhabib–Laroque–Kojima case[5]. Second, we introduce the production sector provided by Matsuyama (1999), which generates business cycles endogenously. Matsuyama (1999) considers that two growth engines, capital accumulation and innovation, can capture different phases in a single economy. In order to show this, he introduces temporary monopoly power when innovators produce new goods in the lab equipment model of Rivera-Batiz and Romer (1991). Matsuyama's (1999) results show that an economy achieves growth through cycles, moving back and forth between two phases. In one phase, there is no innovation and the growth rates of investment and output are high. In the other phase, there is high innovation and the growth rates of investment and output are low. Therefore, his framework is suited to the analysis of endogenous business cycles.

This study obtains three major results. First, we derive necessary and sufficient conditions for the existence of the steady-state equilibrium with a bubble in the capital accumulation phase and the innovation phase. In particular, when consumers save (borrow) an asset when they are young, there is a steady-state equilibrium with a bubble if the growth rate is higher (lower) than the interest rate. This result is similar to that of Kojima (2012a). Second, the steady state is locally determinate in the Samuelson–Tirole case and is either locally indeterminate or locally determinate in the Benhabib–Laroque–Kojima case. Finally, and most importantly, there is a bubble in an economy with a low output growth rate. Specifically, there are bubbles in both economies of low output growth rate and high output growth rate

[3]Kojima's (2012a,b) framework is similar to Benhabib and Laroque's (1988) model.

[4]In Kojima's (2012a,b) model, bubbles remedy the problem of inadequate capital accumulation, which is dynamic efficiency. Thus, the expansion of a bubble raises capital accumulation and output growth rate.

[5]Following Gale (1973), an economy is called the Samuelson case if the growth rate is higher than the interest rate and the classical case if the interest rate is higher than the growth rate. Therefore, we can classify bubbles as two types: Tirole's results correspond to the Samuelson case and Kojima's results correspond to the classical case.

through endogenous business cycles. In addition, the cycle emerges only in the Benhabib–Laroque–Kojima case and then its steady state is locally determinate, that is, the economy never achieves growth through cycles in the Samuelson–Tirole case.

The rest of the paper is organized as follows. In the next Section 2, we present the endogenous growth model. In Section 3, we analyze the equilibrium dynamics. We show the conditions of existence of the steady-state equilibrium with bubbles and discuss the stability of this equilibrium. Furthermore, we show the conditions of existence of the business cycle with bubbles. Section 4 concludes. Finally, some proofs are shown in the Appendix.

2 The model

In this section, we develop an overlapping-generations model that is based on Grossman and Yanagawa (1993). It introduces two aspects to our analysis. First, we consider two cases of bubbles, one being that consumers save an asset when they are young and the other is that consumers borrow an asset to save when they are young. Second, we introduce Matsuyama's (1999) production sector of the economy.

2.1 Households

We consider the standard overlapping-generations model in discrete time ($t = 0, 1, 2, ...$), and L households live for two periods. Households, born at time t, supply one unit of labor when they are young and receive a wage income from a production sector, where w_t represents the real wage. In addition, households allocate income to consumption goods, c_t, savings, s_t, and an asset B_{t+1}. Let r_{t+1} be the return factor of the asset between time t and time $t+1$. The old consume goods d_{t+1} by using savings and selling the asset. The utility function of the individual born at time t is given by $u(c_t, d_{t+1}) = \alpha \log c_t + \beta \log d_{t+1}$, where $\alpha, \beta > 0$. Savings by an individual born at time t are determined by the following maximization problem:

$$\begin{array}{rc} & \max u\left(c_t, d_{t+1}\right), \\ \text{s.t.} & c_t + s_t + \frac{B_{t+1}}{L} \leq w_t, \\ & d_{t+1} \leq r_{t+1}\left(s_t + \frac{B_{t+1}}{L}\right). \end{array} \tag{1}$$

Following Tirole (1985), Benhabib and Laroque (1988), Grossman and Yanagawa (1993), and Kojima (2012a,b), we assume that there is an asset whose fundamental value is zero. Let M and p_t be the asset's supply and price at time t. Then, $B_t := p_t M$ is the aggregate value of the asset at time t. Using the no-arbitrage condition between bubbles and other assets, $p_{t+1}/p_t = r_t$, yields

$$B_{t+1} = r_t B_t \tag{2}$$

We assume that M is positive or negative, as in Benhabib and Laroque (1988) and Kojima (2012a,b). We say that $M > 0$ is the Samuelson–Tirole case. On the other hand, we call the negative bubble asset, namely, $M < 0$, the Benhabib–Laroque–Kojima case.

2.2 Production Sector

Following Matsuyama (1999), final goods Y_t are in a perfectly competitive market and are produced by labor and intermediate goods. The production function of Y_t is

$$Y_t = D_0 \left(\int_0^{A_t} X_t(i)^{1-\frac{1}{\sigma}} di \right) L^{\frac{1}{\sigma}}, \tag{3}$$

where D_0 is total factor productivity, $X_t(i)$ represents the input of the i-th intermediate good at time t, $[0, A_t]$ is the range of variety, and $\sigma \in (1, +\infty)$. The first-order conditions can be expressed as

$$w_t = \frac{1}{\sigma} \frac{Y_t}{L} \tag{4}$$

$$p_t\left(i\right) = \left(1 - \frac{1}{\sigma}\right) D_0 X_t\left(i\right)^{-\frac{1}{\sigma}} L^{\frac{1}{\sigma}} \tag{5}$$

We consider the existing intermediate goods sector i, $i \in [0, A_{t-1}]$. Suppose that this sector is competitive. The profit-maximization problem is $\pi_t^c(i) = \max p_t^c(i) X_t^c(i) - a r_t X_t^c(i)$, where a is the marginal cost. Hence, we obtain $p_t^c := p_t^c(i) = a r_t$ for all $i \in [0, A_{t-1}]$.

On the other hand, we assume that the new intermediate goods sector i,$i \in [A_{t-1}, A_t]$, is monopolistic. In this sector, the new intermediate goods are produced by the marginal cost, a, and a fixed cost, F, to innovate new goods. Thus, the profit-maximization problem is $\pi_t^m(i) = \max p_t^m(i) X_t^m(i) - r_t \left(aX_t^m(i) + F\right)$. Then, we obtain $p_t^m := p_t^m(i) = a\sigma r_t/(\sigma-1)$ for all $i \in [A_{t-1}, A_t]$. Since all the intermediate goods are symmetrical , (5) yields

$$\frac{X_t^c}{X_t^m} = \left(\frac{p_t^c}{p_t^m}\right)^{-\sigma} = \left(1 - \frac{1}{\sigma}\right)^{-\sigma}. \tag{6}$$

From $\pi_t^m < 0$, if and only if $aX_t^m < (\sigma-1)F$, the free-entry condition ensures

$$aX_t^m \leq (\sigma-1)F, A_t > A_{t-1}, \left(aX_t^m - (\sigma-1)F\right)\left(A_t - A_{t-1}\right) = 0. \tag{7}$$

Let K_t be all capital in this economy. Then, the resource constraint at time t is

$$K_t = A_{t-1}aX_t^c + \left(A_t - A_{t-1}\right)\left(aX_t^m + F\right). \tag{8}$$

(6), (7), and (8) yield

$$aX_t^c = \left(1 - \frac{1}{\sigma}\right)^{-\sigma} aX_t^m = \min\left\{\frac{K_t}{A_{t-1}}, \sigma\theta F\right\} \tag{9}$$

and

$$A_t = A_{t-1} + \max\left\{0, \frac{K_t}{\sigma F} - \theta A_{t-1}\right\}, \tag{10}$$

where $\theta := (1 - 1/\sigma)^{1-\sigma} \in (1, e), e = 2.71828....$

Using (7), (9), (10), and (3) can be rewritten as

$$Y_t = \begin{cases} D\left(\theta\sigma F A_{t-1}\right)^{\frac{1}{\sigma}} K_t^{1-\frac{1}{\sigma}} & K_t \leq \theta\sigma F A_{t-1} \\ DK_t & K_t \geq \theta\sigma F A_{t-1} \end{cases}, \tag{11}$$

where

$$D := \frac{D_0}{a}\left(\frac{aL}{\theta\sigma F}\right)^{\frac{1}{\sigma}}. \tag{12}$$

Let $q_t := K_t/\theta\sigma F A_{t-1}$, and we obtain

$$\frac{Y_t}{K_t} = D\phi\left(q_t\right), \tag{13}$$

$$\frac{Y_t}{\theta\sigma F A_{t-1}} = Dq_t\phi\left(q_t\right), \tag{14}$$

$$\frac{A_t}{A_{t-1}} = \psi(q_t), \tag{15}$$

where

$$\phi(q_t) = \begin{cases} q_t^{-\frac{1}{\sigma}} & q_t \leq 1 \\ 1 & q_t \geq 1 \end{cases} \tag{16}$$

and

$$\psi(q_t) = \begin{cases} 1 & q_t \leq 1 \\ 1 + (q_t - 1)\theta & q_t \geq 1 \end{cases}. \tag{17}$$

Following Matsuyama (1999), the economy is in the Solow regime if $q_t \leq 1$ and in the Romer regime if $q_t \geq 1$. There is no innovation and all intermediate goods are competitively supplied in the Solow regime. On the other hand, in the Romer regime , final goods and existing intermediate goods are competitive but innovated intermediate goods are a monopoly.

3 Equilibrium dynamics

In this section, we provide the results of this study. First, we derive the equilibrium dynamics of our model presented in Section 2. Then, a necessary and sufficient condition for the existence of steady-state equilibrium with a bubble is provided. Moreover, we study the dynamic property of our model around the steady-state equilibrium with a bubble. Second, we show the existence of period 2 cycles with bubbles. Thus, we obtain the main result that there is a bubble in an economy with a low growth rate.

3.1 Steady-state Equilibrium

By the optimization condition of (1), the resulting savings are written as

$$s_t + \frac{B_{t+1}}{L} = \frac{\beta}{\alpha + \beta} w_t. \tag{18}$$

Then, the budget constraint for the young and old holds in the equation.

All new savings by the young are invested in capital,

$$L s_t = K_{t+1}. \tag{19}$$

Thus, in equilibrium, $Y_t = r_t K_t + w_t L$ and

$$r_t = \left(1 - \frac{1}{\sigma}\right) \frac{Y_t}{K_t} \tag{20}$$

hold.

We denote $b_t := B_t / \theta \sigma F A_{t-1}$. By using (4), (13), (14), (15), (19), and (20), (18) and (2) can be written as follows:

$$q_{t+1} = D \frac{\phi(q_t)}{\psi\left(q_t\right)} \left(\frac{\beta}{\alpha + \beta} \frac{1}{\sigma} q_t - \left(1 - \frac{1}{\sigma}\right) b_t \right) \tag{21}$$

and

$$b_{t+1} = D \left(1 - \frac{1}{\sigma}\right) b_t \frac{\phi(q_t)}{\psi\left(q_t\right)}. \tag{22}$$

Therefore, (21) and (22) are a complete dynamic system with respect to q_t and b_t in this economy.

We can easily verify that (21) and (22) have a unique steady state.

Proposition 1 1. *Assume that* $(1 - 1/\sigma) D > 1$.

(a) *Consider the Samuelson–Tirole case.* $\sigma - 1 < \beta / (\alpha + \beta)$ *if and only if there is a unique steady state,* (q^*, b^*)*, in the Romer regime, given by the following (23).*

(b) *Consider the Benhabib–Laroque–Kojima case.* $\sigma - 1 > \beta / (\alpha + \beta)$ *if and only if there is a unique steady state,* (q^*, b^*)*, in the Romer regime, given by the following (23).*

$$(q^*, b^*) = \left(\frac{1}{\theta} \left(\left(1 - \frac{1}{\sigma}\right) D - 1 \right) + 1, q^* \left(\frac{\beta}{\alpha + \beta} \frac{1}{\sigma - 1} - 1 \right) \right). \tag{23}$$

Furthermore, this steady state is a balanced growth path:

$$\frac{Y_{t+1}}{Y_t} = \frac{A_t}{A_{t-1}} = \frac{K_{t+1}}{K_t} = \frac{B_{t+1}}{B_t} = \left(1 - \frac{1}{\sigma}\right) D. \tag{24}$$

2. *Assume that* $(1 - 1/\sigma) D < 1$.

(a) Consider the Samuelson–Tirole case. $\sigma - 1 < \beta / (\alpha + \beta)$ *if and only if there is a unique steady state,* (q^{**}, b^{**})*, in the Solow regime, given by the following (25).*

(b) Consider the Benhabib–Laroque–Kojima case. $\sigma - 1 > \beta / (\alpha + \beta)$ *if and only if there is a unique steady state,* (q^{**}, b^{**})*, in the Solow regime, given by the following (25).*

$$(q^{**}, b^{**}) = \left(\left(\left(1 - \frac{1}{\sigma} \right) D \right)^{\sigma}, q^{**} \left(\frac{\beta}{\alpha + \beta} \frac{1}{\sigma - 1} - 1 \right) \right). \tag{25}$$

Furthermore, the economy does not grow in this steady state, which is a neoclassical stationary path.

The steady-state equilibrium without bubbles in the Romer (Solow) regime is a balanced growth path (neoclassical stationary path). Let $\hat{g}^*$ and $\hat{g}^{**}$ be the capital growth rate in the steady state without the bubble in the Romer regime and the Solow regime, which yields

$$\hat{g}^* = D \frac{\beta}{\sigma (\alpha + \beta)}, \hat{g}^{**} = 1. \tag{26}$$

$\hat{r}^*$ and $\hat{r}^{**}$ denote the interest rate in the steady state without bubbles in the Romer regime and the Solow regime, respectively. Then,

$$\hat{r}^* = D \left(1 - \frac{1}{\sigma} \right), \hat{r}^{**} = \frac{(\alpha + \beta)(\sigma - 1)}{\beta}. \tag{27}$$

Therefore, we obtain the following relation:

$$\hat{g}^* \quad \lesseqqgtr \quad \hat{r}^* \Longleftrightarrow \frac{\beta}{\alpha + \beta} \lesseqqgtr \sigma - 1, \tag{28}$$

$$\hat{g}^{**} \quad \lesseqqgtr \quad \hat{r}^{**} \Longleftrightarrow \frac{\beta}{\alpha + \beta} \lesseqqgtr \sigma - 1. \tag{29}$$

Combining the above relations and Proposition 1, bubbles can exist if and only if the growth rate is higher (lower) than the interest rate in the Samuelson–Tirole (Benhabib–Laroque–Kojima) case. This result is similar to that of Kojima (2012a).

Next, we consider the local dynamics of (21) and (22). Since B_t is not predetermined while K_t and A_t are predetermined, the economy is locally determinate if the steady state is a saddle point and locally indeterminate if the steady state is a sink point. Thus, we show the following results.

Proposition 2 *1. Assume that $(1-1/\sigma)\,D > 1$. Table 1 summarizes the results of the stability for each types of steady state.*

Table 1

Romer regime	*Samuelson–Tirole*	*Benhabib–Laroque–Kojima*
$\left(1-\frac{1}{\sigma}\right) D > \theta - 1$	*Saddle point (Determinate)*	*Sink point (Indeterminate)*
$\left(1-\frac{1}{\sigma}\right) D < \theta - 1$	*non-existent*	*Saddle point (Determinate)*

2. Assume that $(1-1/\sigma)\,D < 1$. Table 2 summarizes the results of the stability for each type of steady state.

Table 2

Solow regime	*Samuelson–Tirole*	*Benhabib–Laroque–Kojima*
	Saddle point (Determinate)	*Sink point (Indeterminate)*

Proof. See Appendix. ■

3.2 Cycles with Bubbles

We show that there are period 2 cycles in the Benhabib–Laroque–Kojima case.

Proposition 3 *Suppose that $(1-1/\sigma)D > 1$. $(1-1/\sigma)D < \theta - 1$ and $\beta/(\alpha+\beta) < \sigma - 1$ if and only if there are unique period 2 Cycles, such as $q^L < 1 < q^H$ and $\left(q^L, b^L\right) \neq \left(q^H, b^H\right)$ in the Benhabib–Laroque–Kojima case, satisfying the following equations.*

$$\zeta\left(q^H\right) := q^L = \left(\frac{\left(1-\frac{1}{\sigma}\right)^2 D^2}{\psi\left(q^H\right)}\right)^{\sigma}, \tag{30}$$

$$\left(1-\frac{1}{\sigma}\right) Dq^H = \psi\left(q^H\right)\zeta\left(q^H\right), \tag{31}$$

$$b^L = \left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma-1} - 1\right)\zeta\left(q^H\right), \tag{32}$$

$$b^H = \left(\frac{\beta}{\alpha+\beta} \frac{1}{\sigma-1} - 1 \right) q^H. \tag{33}$$

Proof. See Appendix. ■

g_X denotes the growth rate of varable X. In the same way as Proposition 2 (Matsuyama, 1999), we obtain $g_A = 1 < D(1-1/\sigma) < D(1-1/\sigma)\phi\left(q^L\right) = g_K = g_Y = g_B$ in the Solow regime and $g_A = \psi\left(q^H\right) > D(1-1/\sigma) = g_K = g_Y = g_B$ in the Romer regime. Therefore, we get a result that output growth rate in the Solow regime is higher than in the Romer regime[6]. This result implies that bubbles exist in the economy of low output growth rate, Moreover, there is a bubble in the economy with a high output growth rate.

From Propositions 2 and 3, the local determinacy of the steady-state equilibrium is established and global indeterminacy holds in the economy of the Benhabib–Laroque–Kojima case. Moreover, Proposition 3 implies that period 2 cycles never emerge in the Samuelson–Tirole case.

4 Conclusion

Combining the framework of Tirole (1985) and Grossman and Yanagawa (1993) with Matsuyama's (1999) production sector, we have explored the existence of bubbles in countries with low economic performance. Specifically, we have shown the existence of a business cycle with a bubble, which moves back and forth between the phase of low output growth rate and the phase of high output growth rate. In addition, we have shown that this cycle emerges in the Benhabib–Laroque–Kojima case but not in the Tirole–Samuelson case . Moreover, necessary and sufficient conditions are provided for the existence of an equilibrium steady state with the bubble in both the Samuelson–Tirole case and the Benhabib–Laroque–Kojima case.

The present study has focused only on the existence of bubbles in relation to economic performance and has ignored the effects of bubbles on economic growth rate, as considered by Martin and Ventura (2012). Therefore, clarifying the effects of bubbles on economic growth in each type of economic performance is an interesting direction for future research.

[6] In the bubble-free economy, $q_{t+1} = D\frac{\phi(q_t)}{\psi(q_t)}\frac{\beta}{\alpha+\beta}\frac{1}{\sigma}q_t$ holds. Applying Proposition 2 (Matsuyama, 1999) to the bubble-free economy, we also get this result.

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Appendix

Before providing proofs of the proposition, we show a lemma of parameters.

Lemma 1 *We assume that* $(1-1/\sigma)\,D>1$*. Then,*

$$\sigma-1\leq\frac{\beta}{\alpha+\beta}\Longrightarrow\left(1-\frac{1}{\sigma}\right)D\geq\theta-1. \tag{34}$$

Proof. We assume that $(1-1/\sigma)D<\theta-1$. Then, $1<(1-1/\sigma)D<\theta-1$, which yields $2<\theta<e$. On the other hand, by using $0<\beta/(\alpha+\beta)<1$ and $\sigma-1\leq\beta/(\alpha+\beta)$, σ is in $(1,2)$. Since θ is an increasing function with respect to σ, $\theta\in(1,2)$ as $\sigma\in(1,2)$. This result contradicts $2<\theta<e$. ■

Proof of Proposition 2

As a first step, we compute the Jacobian matrix. We obtain

$$\frac{\partial q_{t+1}}{\partial q_t}=\begin{cases}\frac{D}{\sigma}\frac{1}{\psi(p_t)^2}\left(\theta(\sigma-1)\,b_t-(\theta-1)\,\frac{\beta}{\alpha+\beta}\right) & \text{Romer regime}\\ \left(1-\frac{1}{\sigma}\right)D\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma}\phi(q_t)-b_t\phi'(q_t)\right) & \text{Solow regime}\end{cases}, \tag{35}$$

$$\frac{\partial q_{t+1}}{\partial b_t}=\begin{cases}-\left(1-\frac{1}{\sigma}\right)D\frac{1}{\psi(q_t)} & \text{Romer regime}\\ -\left(1-\frac{1}{\sigma}\right)D\phi(q_t) & \text{Solow regime}\end{cases}, \tag{36}$$

$$\frac{\partial b_{t+1}}{\partial q_t}=\begin{cases}-\left(1-\frac{1}{\sigma}\right)D\theta b_t\frac{1}{\psi(q_t)^2} & \text{Romer regime}\\ \left(1-\frac{1}{\sigma}\right)Db_t\phi'(q_t) & \text{Solow regime}\end{cases}, \tag{37}$$

$$\frac{\partial b_{t+1}}{\partial b_t}=\begin{cases}\left(1-\frac{1}{\sigma}\right)D\frac{1}{\psi(q_t)} & \text{Romer regime}\\ \left(1-\frac{1}{\sigma}\right)D\phi(q_t) & \text{Solow regime}\end{cases}. \tag{38}$$

Therefore, in each steady state, we have

$$J_{11}:=\left.\frac{\partial q_{t+1}}{\partial q_t}\right|_{\text{steady state}}=\begin{cases}\frac{1}{\sigma-1}\frac{\beta}{\alpha+\beta}-1-\frac{1}{D}\frac{\sigma}{\sigma-1}(\theta-1) & \text{Romer regime}\\ \frac{1}{\sigma}\left(\frac{\beta}{\alpha+\beta}\frac{\sigma}{\sigma-1}-1\right) & \text{Solow regime}\end{cases}, \tag{39}$$

$$J_{12}:=\left.\frac{\partial q_{t+1}}{\partial b_t}\right|_{\text{steady state}}=\begin{cases}-1 & \text{Romer regime}\\ -1 & \text{Solow regime}\end{cases}, \tag{40}$$

$$J_{21}:=\left.\frac{\partial b_{t+1}}{\partial q_t}\right|_{\text{steady state}}=\begin{cases}-\frac{\sigma}{\sigma-1}\frac{1}{D}\left(\left(1-\frac{1}{\sigma}\right)D+\theta-1\right)\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma-1}-1\right) & \text{Romer regime}\\ -\frac{1}{\sigma}\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma-1}-1\right) & \text{Solow regime}\end{cases}, \tag{41}$$

$$J_{22}:=\left.\frac{\partial b_{t+1}}{\partial b_t}\right|_{\text{steady state}}=\begin{cases}1 & \text{Romer regime}\\ 1 & \text{Solow regime}\end{cases}. \tag{42}$$

We can write the Jacobian matrix evaluated at each steady state as

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}. \tag{43}$$

The eigenvalues of J are the solution of the following function

$$\begin{aligned} 0 &= \lambda^2 - (J_{11}+1)\lambda + J_{11} + J_{21} \qquad (44) \\ &= \begin{cases} \left(\lambda - \frac{1}{\sigma-1}\frac{\beta}{\alpha+\beta}\right)\left(\lambda + \frac{\sigma}{\sigma-1}\frac{\theta-1}{D}\right) & \text{Romer regime} \\ \left(\lambda - \frac{1}{\sigma-1}\frac{\beta}{\alpha+\beta}\right)\left(\lambda - \left(1-\frac{1}{\sigma}\right)\right) & \text{Solow regime} \end{cases}. \qquad (45) \end{aligned}$$

From Lemma 1 and $\sigma \in (1, +\infty)$, we obtain the result of this proposition.

Proof of Proposition 3

Substituting $((q_t, b_t), (q_{t+1}, b_{t+1})) = \left(\left(q^L, b^L\right), \left(q^H, b^H\right)\right)$ for (21) and (22), we obtain

$$q^H = D\left(q^L\right)^{-\frac{1}{\sigma}}\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma}q^L - \left(1-\frac{1}{\sigma}\right)b^L\right), \tag{46}$$

$$b^H = D\left(1-\frac{1}{\sigma}\right)b^L\left(q^L\right)^{-\frac{1}{\sigma}}. \tag{47}$$

On the other hand, substituting $((q_t, b_t), (q_{t+1}, b_{t+1})) = \left(\left(q^H, b^H\right), \left(q^L, b^L\right)\right)$ for (21) and (22), we obtain

$$q^L = D\frac{1}{1+(q^H-1)\theta}\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma}q^H - \left(1-\frac{1}{\sigma}\right)b^H\right), \tag{48}$$

$$b^L = D\left(1-\frac{1}{\sigma}\right)b^H\frac{1}{1+(q^H-1)\theta}. \tag{49}$$

Combining (46), (47), (48), and (49), we obtain (30), (31), (32), and (33).

Next, we show that q^L and q^H satisfy $0 < q^L < 1 < q^H$. By (30),

$$0 < q^L < 1 \Longleftrightarrow \underline{q} := \frac{1}{\theta}\left(x^2 - 1\right) + 1 < q^H, \tag{50}$$

where $x := (1-1/\sigma)D$, and $\underline{q} > 1$ by $(1-1/\sigma)D > 1$. Let (31) be defined as

$$f\left(q^H\right) := \psi\left(q^H\right)\zeta\left(q^H\right) - \left(1-\frac{1}{\sigma}\right)Dq^H. \tag{51}$$

Since

$$f'\left(q^H\right) < 0, \tag{52}$$

$$\lim_{q^H \to \infty} f\left(q^H\right) = -\infty \tag{53}$$

and

$$f\left(\underline{q}\right) > 0 \Longleftrightarrow \left(1 - \frac{1}{\sigma}\right) D < \theta - 1, \tag{54}$$

there is a solution, q^H, satisfying $1 < \underline{q} < q^H$. Moreover, we obtain

$$b^L < 0, b^H < 0 \Longleftrightarrow \frac{\beta}{\alpha + \beta} < \sigma - 1. \tag{55}$$

Therefore, $\left(\left(q^L, b^L\right), \left(q^H, b^H\right)\right)$ comprises the period 2 cycle with bubbles.

Conversely, we consider that (30), (31), (32), and (33) are given. Similarly, from (50), (52), (53), (54), and (55), we obtain $(1 - 1/\sigma)D < \theta - 1$ and $\beta/(\alpha + \beta) < \sigma - 1$.