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“Portfolio Allocation Problems between Risky
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Portfolio Allocation Problems between Risky and Ambiguous Assets ^{*}

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Abstract

This paper considers a portfolio allocation problem between a risky asset and an *ambiguous asset*, and investigates how greater ambiguity aversion influences the optimal proportion invested in the two assets. We derive several sufficient conditions under which greater ambiguity aversion decreases the optimal proportion invested in the ambiguous asset. Furthermore, we consider an international diversification problem as an application and show that ambiguity aversion partially resolves the home bias puzzle.

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1 Introduction

In the real world, it is difficult to precisely predict what will happen in the future. In particular, in financial markets, it is difficult for investors to accurately foresee returns on assets. Therefore, it is worth investigating how investors diversify their wealth across different assets under *uncertainty*. The notion of uncertainty has been investigated in the literature since Keynes (1921) and Knight (1921) from two perspectives: *risk* and *ambiguity*. While *risk* is a situation in which the beliefs of a decision maker (DM) are captured by a unique probability measure, *ambiguity* is a situation in which a DM's beliefs are not pinned down by a unique probability measure because of a lack of information. When investors choose between different assets, their knowledge of future returns is critical. When they know the return of the investment for sure, we can consider it a safe asset.¹ If different returns are possible, but investors know the distribution over these returns, the asset is risky.² When different returns are possible, but investors have only incomplete knowledge of probabilities, we would classify the asset as ambiguous. Existing studies have addressed the choice between two risky assets (Hadar and Seo (1988, 1990) and Chiu et al. (2012)) and between a safe and an ambiguous asset (Gollier (2011), Huang and Tzeng (2018)). The third combination, which is the choice between a risky and an ambiguous asset, has not been considered thus far. The current paper fills this gap in the literature. By incorporating the notions of both risk and ambiguity into portfolio selection problems and by introducing some notions of stochastic dominance to capture shifts in returns on assets,³ we investigate

¹Throughout this paper, to avoid confusion, we say that an asset whose return is known with certainty is *safe*, rather than riskless or risk-free.

²Throughout this paper, we say that an asset whose return is captured by a unique probability measure is *risky* and an asset whose return is not captured by a unique probability measure is *ambiguous*.

³For a survey of stochastic dominance, see Levy (1992). For applications of stochastic dominance to portfolio strategies, in particular, second-order stochastic dominance, see Roman et al. (2013). Recent studies of stochastic dominance in operations research and management science include Post and Kopa (2013), Eeckhoudt et al. (2016), and Fang and Post (2017).

how the presence of ambiguity affects optimal portfolio allocation.

As explained above, the notion of ambiguity is helpful to the understanding of investors' behaviors in financial markets. From the empirical viewpoint, the extant studies in the literature have shed light on the importance of ambiguity. Epstein and Schneider (2008) investigate the effects of bad news and good news on investors' behaviors, and show that under ambiguity, investors overvalue negative information and undervalue positive information. Kelsey et al. (2010) investigate the profitability of momentum strategies (buying past winners and selling past losers) in stock trading under ambiguity. Using the US stock market and accounting data, Kelsey et al. (2010) identify that negative momentum is greater than positive momentum in terms of magnitude and persistency of portfolio returns, and that such asymmetric patterns depend on ambiguity. In another recent study, Driouchi et al. (2018) investigate the behavior of US index put option holders during the pre-crisis and credit crunch period 2006-2008. Driouchi et al. (2018) find evidence of ambiguity in the US index options market during 2006-2008 and measure the effect of ambiguity on realized index volatility that is implied directly from observed option prices. Based on portfolio data from a large financial institution in France, Bianchi and Tallon (2018) show that ambiguity averse investors are relatively more exposed to the French than to the international stock market. This result implies that ambiguity aversion plays a significant role in explaining home bias in equity markets. Let us consider an investor who plans to purchase equities in her local and foreign markets. Here, we assume that she confronts more difficulty predicting returns on foreign equities than on local equities. In this situation, returns on foreign equities are more ambiguous for the investor than those on local equities, which is captured appropriately by risk and ambiguity. As another work related to our motivation in this paper, using a representative sample of about 300 Dutch investors in the De Nederlandse Bank Household Survey, Anantanasuwong et al. (2019) elicit ambiguity attitudes toward a familiar company stock, a local stock index, a foreign stock index, and a crypto currency. Anantanasuwong et al. (2019) identify that individuals' perceptions about ambiguity levels differ substantially depending on the type of asset, which implies that the same investor may perceive high

ambiguity about foreign stocks or indices and perceive low ambiguity about local ones.

Ellsberg (1961) shows experimentally that DMs typically dislike situations where they cannot assign a unique probability measure. This behavior, which is called *ambiguity aversion*, cannot be explained within the framework of expected utility theory. To overcome the shortcomings of expected utility theory pointed out by Ellsberg (1961), many preference representations, also known as *ambiguity models*, have been proposed. For example, Gilboa and Schmeidler (1989) propose max-min expected utility theory (MEU), and Schmeidler (1989) proposes Choquet expected utility theory (CEU). Studies of CEU in operations research include Chateauneuf (1994), Gilboa and Schmeidler (1994, 1995), and Ghirardato and Marinacci (2001).⁴ In this paper, we adopt the smooth ambiguity model by Klibanoff et al. (2005) as our ambiguity model.⁵ This is because the smooth ambiguity model can differentiate the DMs' attitude towards ambiguity from their perception of ambiguity, which implies that the smooth ambiguity model is more general than MEU and CEU. Furthermore, because the smooth ambiguity model has a “double” expected utility form, it is more tractable than most of ambiguity models.

Several studies in the literature on portfolio selection problems are worth mentioning. Hadar and Seo (1988, 1990) derive conditions on utility functions when returns on assets are shifted by first-order stochastic dominance (FSD). Conditions on utility functions can be removed by concepts stronger than FSD, such as monotone likelihood ratio dominance (MLRD) by Landsber and Meilijson (1990) and reversed hazard ratio dominance (RHRD) by Kijima and Ohnishi (1996). Kijima and Ohnishi (1996) provide a systematic method for stochastic dominance that is useful for portfolio allocation problems. While these papers analyze portfolio selection problems within the framework of expected utility theory, Gollier (2011), Osaki and Schlesinger (2014), and this paper investigate portfolio selection problems within the framework of the

⁴Borgonovo et al. (2018) study and provide a method to connect operational risk management with the theoretical background of decision theory.

⁵Particularly, we adopt Neilson's (2010) model that is a special case of Klibanoff et al. (2005). Neilson's (2010) model is popular in applications.

smooth ambiguity model. Gollier (2011) introduces ambiguity into returns on an asset and derives sufficient conditions under which any increase in ambiguity aversion decreases the purchase of the ambiguous asset. While Gollier (2011) analyzes portfolios consisting of one safe asset and one ambiguous asset, we analyze portfolios consisting of one risky asset and one ambiguous asset. As mentioned in the example above, in financial markets, it is appropriate to analyze portfolios consisting of one risky asset and one ambiguous asset. Osaki and Schlesinger (2014) do not introduce ambiguity into returns on an asset, but instead investigate background ambiguity. As in Osaki and Schlesinger (2014), we consider situations with different levels of ambiguity. The distinction between Osaki and Schlesinger (2014) and this paper is that the exposure to ambiguity is exogenous in Osaki and Schlesinger (2014), whereas it is endogenous in this paper. Finally, in related works from the viewpoint of the proof of Theorem 1, Peter (2019) uses a similar argument to show that ambiguity aversion raises precautionary saving, and Peter and Ying (2018) find that ambiguity aversion lowers insurance demand in the presence of ambiguity about contract nonperformance.⁶

2 Portfolio Allocation between a Risky and an Ambiguous Asset

In this section, we present a portfolio allocation problem based on the smooth ambiguity model by Klibanoff et al. (2005) in which an investor is faced with both risk and ambiguity.

To simultaneously analyze the effects of risk and ambiguity on portfolio choices, we consider an investor who allocates her wealth w between a risky asset and an ambiguous asset.⁷ The return on the risky asset is denoted by the random variable \tilde{x} whose probability distribution function F is defined over

⁶We acknowledge an anonymous reviewer who points out these works and provides an idea of an elementary proof of Theorem 1.

⁷In the literature, for example, Gollier (2011) considers one safe asset and one ambiguous asset.

the bounded support $[a, b]$ with $a < 0 < b$. The ambiguity of the return is represented by the second-order probability approach based on Segal (1987) and Klibanoff et al. (2005). There are n possibilities of the return on the ambiguous asset which are indexed by $\theta \in \Theta = \{1, \dots, n\}$. The possible returns on the ambiguous asset are denoted by \tilde{y}_θ , $\theta = 1, \dots, n$. The probability distribution function of \tilde{y}_θ is denoted by G_θ and is defined over the bounded support $[a, b]$. For simplicity, all possible probability distribution functions are assumed to be defined over the same support $[a, b]$. The investor attaches the second-order probability $\{q_1, \dots, q_n\}$ to the index set Θ . The return on the risky asset \tilde{x} and the possible returns on the ambiguous asset \tilde{y}_θ are assumed to be independent. The investor's preferences are assumed to be represented by the smooth ambiguity model by Klibanoff et al. (2005).

The investor chooses her portfolio allocation $(w - k, k)$ to maximize the welfare from the terminal wealth. Here, $w - k$ is the amount invested in the risky asset and k is the amount invested in the ambiguous asset. Her objective is to maximize the following:

$$V(k) = \sum_{\theta=1}^n q_\theta \phi(E[u((w - k)\tilde{x} + k\tilde{y}_\theta)]).$$

We assume that u is strictly increasing and strictly concave, that is, $u' > 0$ and $u'' < 0$, and ϕ is strictly increasing and concave, that is, $\phi' > 0$ and $\phi'' \leq 0$. The attitude towards ambiguity is captured by the curvature of ϕ . The concavity captures an investor's ambiguity aversion in the sense that she dislikes any mean-preserving spread of the expected utility of u . The linearity of ϕ captures her ambiguity neutrality in the sense that ambiguity degenerates to the single return on the asset, $\tilde{y}_O \stackrel{d}{=} \sum_{\theta=1}^n q_\theta \tilde{y}_\theta$, where $\stackrel{d}{=}$ indicates equality in distribution. The linearity of ϕ plays a significant role in applications (see Section 4).

The optimal portfolio allocation k^* is the solution of the following first-order condition (FOC):

$$V'(k^*) = \sum_{\theta=1}^n q_\theta \phi'(E[u((w - k^*)\tilde{x} + k^*\tilde{y}_\theta)])E[(\tilde{y}_\theta - \tilde{x})u'((w - k^*)\tilde{x} + k^*\tilde{y}_\theta)] = 0.$$

The second-order condition is satisfied by the concavities of u and ϕ . We suppose that $V'(0) > 0$ and $V'(w) < 0$, that is, the investor allocates a positive amount of her wealth to each asset.

We define k^θ as follows:

$$k^\theta = \operatorname{argmax}_k E[u((w - k)\tilde{x} + k\tilde{y}_\theta)], \quad \theta = 1, \dots, n, \quad (1)$$

where k^θ denotes the ex-post optimal portfolio allocation given θ .

3 Effects of Ambiguity Aversion on Optimal Portfolio Allocation

In this section, we provide the main result of this paper and an informal proof. A formal proof is relegated to Appendix.

3.1 Main Result

In the smooth ambiguity model, the attitude towards ambiguity is captured by the curvature of ϕ , and the concavity of ϕ captures an investor's ambiguity aversion. As shown by Klibanoff et al. (2005), greater ambiguity aversion is characterized by an increasing and concave transformation of ϕ . We examine how greater ambiguity aversion affects the optimal portfolio allocation. Let A and B be two investors, and let ϕ_A and ϕ_B be their ambiguity attitudes, respectively. Define the objective functions for $i = A, B$ as follows:

$$V_i(k) = \sum_{\theta=1}^n q_\theta \phi_i(E[u((w - k)\tilde{x} + k\tilde{y}_\theta)]).$$

Moreover, let k^A and k^B be their optimal portfolio allocations which must satisfy the first-order condition:

$$V'_i(k^i) = \sum_{\theta=1}^n q_\theta \phi'_i(E[u((w - k^i)\tilde{x} + k^i\tilde{y}_\theta)]) E[(\tilde{y}_\theta - \tilde{x})u'((w - k^i)\tilde{x} + k^i\tilde{y}_\theta)] = 0, \quad i = A, B.$$

Suppose that investor A is more ambiguity averse than investor B in the sense that there exists an increasing and concave function t such that $\phi_A = t \circ \phi_B$, and they are identical except for ambiguity aversion.

We introduce some definitions and notation before stating the main result. To obtain a clear result, we consider the situation in which the possible returns on the ambiguous asset are ranked by FSD. Let \tilde{y}_i and \tilde{y}_j be random variables for $i, j \in \Theta = \{1, \dots, n\}$. We say that \tilde{y}_j is greater than \tilde{y}_i in the sense of FSD, denoted by $\tilde{y}_i \lesssim_{\text{FSD}} \tilde{y}_j$, if $G_i(y) \geq G_j(y)$ holds for any $y \in [a, b]$ and every $i, j \in \Theta$ with $i < j$, where G_θ denotes the probability distribution function of \tilde{y}_θ for $\theta \in \Theta$. The Arrow-Pratt measure of relative risk aversion is defined by $R(z) = -zu''(z)/u'(z)$.⁸ This terminology is used within the framework of expected utility theory. We also use it in the smooth ambiguity model. The following result shows that, under some conditions, greater ambiguity aversion decreases the optimal amount of investment.

Theorem 1. *Greater ambiguity aversion decreases the optimal portfolio allocation, $k^A \leq k^B$, if the possible returns on an ambiguous asset $\{\tilde{y}_1, \dots, \tilde{y}_n\}$ are ranked by FSD and $R(z) \leq 1$.*

Proof. The proof is relegated to Appendix B. □

Gollier (2011, Proposition 2(1)) presents the same conditions in the portfolio problem that consists of a safe asset and an ambiguous asset. Because the safe asset in Gollier (2011) is replaced with a risky asset in Theorem 1, Theorem 1 can be viewed as a generalization of Gollier (2011). Recall that the optimal portfolio allocation for an ambiguity averse investor is denoted by k^* . When we set $\sum_{\theta=1}^n q_\theta \tilde{y}_\theta \stackrel{d}{=} \tilde{y}_O$, the optimal portfolio allocation for an expected utility maximizer is denoted by k^O . This value k^O corresponds to the optimal portfolio allocation for an ambiguity neutral investor. In other words, ambiguity neutral investors reduce compound lotteries so that the problem becomes a choice between two risky assets. The optimal portfolio allocation k^O must satisfy:

$$k^O = \operatorname{argmax}_k E[u((w - k)\tilde{x} + k\tilde{y}_O)] = \operatorname{argmax}_k q_\theta E[u((w - k)\tilde{x} + k\tilde{y}_\theta)].$$

By supposing investor A is ambiguity averse and investor B is ambiguity neutral, Corollary 1 follows from Theorem 1 as a special case.

⁸See Arrow (1965) and Pratt (1964).

Corollary 1. *The existence of ambiguity aversion decreases the optimal portfolio allocation, $k^* \leq k^O$, if the possible returns on an ambiguous asset $\{\tilde{y}_1, \dots, \tilde{y}_n\}$ are ranked by FSD and $R(z) \leq 1$.*

As a special case, Corollary 1 shows how the existence of ambiguity aversion affects the optimal portfolio allocation compared with ambiguity neutrality. Corollary 1 plays a significant role in Section 4. It should also be noted that by formally defining the notions of being more ambiguous, Jewitt and Mukerji (2017) investigate what makes one act *more ambiguous* than another one and provide more general definitions of greater ambiguity. See Jewitt and Mukerji (2017) for details.

To gain intuitive understanding, we consider the following example in which there are only two indices $\Theta = \{1, 2\}$. Define $U(k, \theta) = E[u((w - k)\tilde{x} + k\tilde{y}_\theta)]$ and $g(\theta, k) = E[(\tilde{y}_\theta - \tilde{x})u'((w - k)\tilde{x} + k\tilde{y}_\theta)]$. Recall that investor A is more ambiguity averse than investor B . By using the Radon-Nikodym derivatives $\{\hat{q}_1^A, \hat{q}_2^A\}$, the FOC for investor A can be written as follows:

$$V'_A(k^A) = \hat{q}_1^A(k^A)g(1, k^A) + \hat{q}_2^A(k^A)g(2, k^A) = 0, \quad (2)$$

where

$$\hat{q}_\theta^A(k) = \frac{q_\theta \phi'_A(U(k, \theta))}{q_1 \phi'_A(U(k, 1)) + q_2 \phi'_A(U(k, 2))}, \theta = 1, 2.$$

Similarly, by using the Radon-Nikodym derivatives $\{\hat{q}_1^B, \hat{q}_2^B\}$, the FOC for investor B can be written as follows:

$$V'_B(k^B) = \hat{q}_1^B(k^B)g(1, k^B) + \hat{q}_2^B(k^B)g(2, k^B) = 0,$$

where

$$\hat{q}_\theta^B(k) = \frac{q_\theta \phi'_B(U(k, \theta))}{q_1 \phi'_B(U(k, 1)) + q_2 \phi'_B(U(k, 2))}, \theta = 1, 2.$$

Without loss of generality, we assume that

$$U(k^A, 1) \leq U(k^A, 2). \quad (3)$$

We can then show that more ambiguity averse investors put greater weight on a lower expected utility than on a higher expected utility. That is, we can

show that:

$$U(k^A, 1) \leq U(k^A, 2) \Leftrightarrow \hat{q}_1^A(k^A) \geq \hat{q}_1^B(k^A), \quad (4)$$

(equivalently, $\hat{q}_2^A(k^A) \leq \hat{q}_2^B(k^A)$ because $\hat{q}_1^i(k^A) = 1 - \hat{q}_2^i(k^A)$, $i = A, B$),

because ϕ_A is an increasing and concave transformation of ϕ_B , that is, there exists an increasing and concave transformation t such that $\phi_A = t \circ \phi_B$.⁹ It follows from (2) that the sign of $g(\theta, k^A)$ is different for $\theta = 1$ and $\theta = 2$. That is, either

$$g(1, k^A) \leq 0 \leq g(2, k^A) \quad (5)$$

or

$$g(1, k^A) \geq 0 \geq g(2, k^A). \quad (6)$$

For the former case, because the objective function is concave, it follows by combining (4) and (5) that:

$$\begin{aligned} 0 &= V'_A(k^A) \\ &= \hat{q}_1^A(k^A)g(1, k^A) + \hat{q}_2^A(k^A)g(2, k^A) \\ &\leq \hat{q}_1^B(k^A)g(1, k^A) + \hat{q}_2^B(k^A)g(2, k^A) \\ &= V'_B(k^A) \\ &\Leftrightarrow k^A \leq k^B. \end{aligned}$$

This is an intuitive case in which greater ambiguity aversion decreases the amount invested in the ambiguous asset. However, such an intuitive result might not follow because (5) does not necessarily hold, even assuming FSD implies (3) as shown by, for example, Hadar and Seo (1990). Therefore, additional conditions are required to show that greater ambiguity aversion decreases the optimal portfolio allocation. Theorem 1 provides one such sufficient condition. In subsections 3.3 and 3.4, we present other sufficient conditions.

⁹See Appendix in detail.

For the latter case, because the objective function is concave, it follows by combining (4) and (6) that:

$$\begin{aligned}
0 &= V'_A(k^A) \\
&= \hat{q}_1^A(k^A)g(1, k^A) + \hat{q}_2^A(k^A)g(2, k^A) \\
&\geq \hat{q}_1^B(k^A)g(1, k^A) + \hat{q}_2^B(k^A)g(2, k^A) \\
&= V'_B(k^A) \\
&\Leftrightarrow k^A \geq k^B.
\end{aligned}$$

This is a counterintuitive case in which greater ambiguity aversion increases the amount invested in the ambiguous asset.

As a final remark, we discuss the assumption of the independence. Kijima and Ohnishi (1996) show that (5) holds for FSD, even though the return on the risky asset \tilde{x} , and each possible return on the ambiguous asset \tilde{y}_θ are dependent. However, the convolution property cannot be guaranteed to hold.¹⁰ If this property does not hold, (4) may be reversed, that is, it is possible that $\hat{q}_1^A(k^A) \leq \hat{q}_1^B(k^A)$. In this case, we obtain the counterintuitive result that states that greater ambiguity increases the amount invested in the ambiguous asset. However, if the convolution property holds, we obtain Theorem 1 in the case of the dependence.

3.2 Intuition

In this subsection, we provide the intuition for Theorem 1. From Theorem 1, greater ambiguity aversion does not necessarily decrease the demand for an ambiguous asset. A set of the sufficient conditions is that the returns on an ambiguous asset $\{\tilde{y}_1, \dots, \tilde{y}_n\}$ are ranked by FSD and the Arrow-Pratt measure of relative risk aversion is smaller than unity. It should be noted that the first-order condition of the optimal allocation problem can be understood as an Euler equation under ambiguity. The Euler equation is distorted by ambiguity aversion, and the distortion is pessimistic in the sense that ambiguity averse investors assign more weight to worse indices. Within the framework of the

¹⁰For the definition of the convolution property, see Appendix A.

expected utility theory, it is known that pessimistic deteriorations in beliefs do not always decrease the demand for risky assets.¹¹ Moreover, the deteriorations in beliefs have a substitution effect and a wealth effect. These conflicting effects determine whether the demand for the ambiguous asset decreases or not. Therefore, some sufficient conditions are required to show that more ambiguity aversion reduces the demand for an ambiguous asset.

3.3 Monotone Likelihood Ratio Dominance and Reversed Hazard Ratio Dominance

When the possible returns on an ambiguous asset are ranked by FSD, we can conclude that the presence of ambiguity decreases the optimal portfolio allocation for investors whose Arrow-Pratt measure of relative risk aversion is less than unity. The condition $R(z) \leq 1$ is assumed in determining the effect of FSD shifts on various decision problems. See, for example, Fishburn and Porter (1976). However, as pointed out by Meyer and Meyer (2005), it is unclear whether this condition is reasonable from an empirical viewpoint. We also question whether empirical observations under expected utility theory can be directly applied to the smooth ambiguity model, even though $R(z) \leq 1$ is viewed as a reasonable property. In this subsection, based on the motivations above, we introduce MLRD and RHRD as stronger notions of stochastic dominance than FSD. First, we provide the definition of MLRD.

Definition 1. Let \tilde{y}_i and \tilde{y}_j be random variables for $i, j \in \Theta$. Then, \tilde{y}_j is greater than \tilde{y}_i in the sense of monotone likelihood ratio dominance (MLRD), denoted by $\tilde{y}_i \lesssim_{\text{MLRD}} \tilde{y}_j$, if $g_j(t)/g_i(t) \geq g_j(s)/g_i(s)$ for any $s, t \in [a, b]$ with $s < t$, where G_θ and g_θ denote the probability distribution function of \tilde{y}_θ and the probability density function of \tilde{y}_θ for $\theta \in \Theta$, respectively.

Applying MLRD to rank the possible returns on assets, we obtain the following proposition corresponding to Theorem 1 without assuming $R(z) = -zu''(z)/u'(z) \leq 1$. Landsberger and Meilijson (1990) show that $k^i \leq k^j$ for

¹¹For example, see Rothschild and Stiglitz (1971), Fishburn and Porter (1976), and Hardar and Seo (1990).

$\tilde{y}_i \lesssim_{\text{MLRD}} \tilde{y}_j$ for $i, j \in \Theta$ with $i < j$ for any nondecreasing utility function u . Recall that k^i is defined by (1) for $i = 1, \dots, n$. Similar to Theorem 1 and Corollary 1, in the following analyses, we assume that \tilde{x} and \tilde{y}_i are independent, and \tilde{x} and \tilde{y}_j are independent for $i, j \in \Theta$ with $i < j$. Now, we obtain the following proposition.

Proposition 1. *Let u be any nondecreasing function. Then, greater ambiguity aversion decreases the optimal portfolio allocation, $k^A \leq k^B$ if the possible returns on the asset $\{\tilde{y}_1, \dots, \tilde{y}_n\}$ are ranked by MLRD.*

As is clear from Proposition 1, we do not need to impose the concavity of utility functions. In other words, this proposition can be applied to the value function of Kahneman and Tversky (1979) if the second-order condition is satisfied.

Next, we consider RHRD that is weaker than MLRD, which is shown, for example, in Eeckhoudt and Gollier (1995).¹²

Definition 2. Let \tilde{y}_i and \tilde{y}_j be random variables for $i, j \in \Theta$. Then, \tilde{y}_j is greater than \tilde{y}_i in the sense of RHRD, denoted by $\tilde{y}_i \lesssim_{\text{RHRD}} \tilde{y}_j$, if $G_j(t)/G_i(t) \geq g_j(t)/g_i(t)$ for any $t \in [a, b]$, where G_θ and g_θ denote the probability distribution function of \tilde{y}_θ and the probability density function of \tilde{y}_θ for $\theta \in \Theta$, respectively.¹³

Kijima and Ohnishi (1996) show that $k^i \leq k^j$ for $\tilde{y}_i \lesssim_{\text{RHRD}} \tilde{y}_j$ for $i, j \in \Theta$ with $i < j$ for any nondecreasing and concave utility function u . Thus, the following proposition is in order.

Proposition 2. *Let u be any nondecreasing and concave function. Then, greater ambiguity aversion decreases the optimal portfolio allocation, $k^A \leq k^B$ if the possible returns on the asset $\{\tilde{y}_1, \dots, \tilde{y}_n\}$ are ranked by RHRD.*

¹²In Eeckhoudt and Gollier (1995), RHRD is referred to as monotone probability ratio order.

¹³See Eeckhoudt and Gollier (1995, Lemma 2). Eeckhoudt and Gollier (1995, Lemma 1) also show that RHRD is stronger than FSD.

3.4 Higher-Order Increases in Risk

The notion of higher-order increases in risk is introduced by Ekern (1980), and has been analyzed, for example, by Eeckhoudt and Schlesinger (2006) and Jindapon and Neilson (2007). In this subsection, we show that the result in this paper also applies to higher-order increases in risk.

Definition 3. For $\theta \in \Theta = \{1, \dots, n\}$, let G_θ be probability distribution functions of random variables \tilde{y}_θ with supports contained in $[a, b]$. Define the functions by

$$\begin{aligned} G_\theta^1(x) &= G_\theta(x) \text{ and} \\ G_\theta^k(x) &= \int_a^x G_\theta^{k-1}(t) dt \end{aligned}$$

for $x \in [a, b]$, $\theta \in \Theta$, and $k = 2, \dots, N$, where the function G_θ^n denotes the n -th moment of G_θ . Let \tilde{y}_i and \tilde{y}_j be random variables for $i, j \in \Theta$. Then, \tilde{y}_j is greater than \tilde{y}_i in the sense of N -th degree risk, denoted by $\tilde{y}_i \lesssim_{N\text{-risk}} \tilde{y}_j$, if $G_j^N(y) \geq G_i^N(y)$ and $G_i^n(b) = G_j^n(b)$ for $n = 1, \dots, N - 1$.

Note that if \tilde{y}_j is greater than \tilde{y}_i in the sense of N -th degree risk, then the first $(N - 1)$ -moments of G_i and G_j are equal. It is worth mentioning that $N = 2$ corresponds to an increase in risk in the sense of Rothschild and Stiglitz (1970), and $N = 3$ corresponds to an increase in downside risk in the sense of Menezes et al. (1980).

Chiu et al. (2010) show that $k^i \leq k^j$ if $\tilde{y}_i \lesssim_{N\text{-risk}} \tilde{y}_j$, $(-1)^n u^n(x) \leq 0$ for $n = N, N + 1$, and $-xu^{N+1}(x)/u^N(x) \leq N$, where the utility function u is assumed to be strictly increasing and infinitely continuously differentiable, and u^n denotes the n -th derivative of u . Thus, the following proposition is in order.

Proposition 3. *Greater ambiguity aversion decreases the optimal portfolio allocation, $k^A \leq k^B$ if the possible returns on the asset $\{\tilde{y}_1, \dots, \tilde{y}_n\}$ are ranked by N -th degree risk, $(-1)^n u^n(x) \leq 0$ for $n = N, N + 1$, and $-xu^{N+1}(x)/u^N(x) \leq N$.*

4 Applications

In this section, we provide two further results as applications of Corollary 1 that compare an ambiguity averse investor with an ambiguity neutral investor because this result is suitable for applications, especially the home bias puzzle. First, we consider the home bias puzzle based on the smooth ambiguity model. Second, for the purpose of extending the *50% rule for portfolio allocation problems*, we consider conditions under which the 50% rule holds for the portfolio allocation problem with a risky asset and an ambiguous asset in the smooth ambiguity model. The 50% rule for portfolio allocation problems was investigated by Hadar and Seo (1988, 1990) and Clark and Jokung (1999).

4.1 The Home Bias Puzzle

In this subsection, we apply Corollary 1 to an international diversification problem, which provides a solution to *the home bias puzzle* from the viewpoint of ambiguity.

French and Poterba (1991) observe the tendency for investors to hold more equities in their home country than in foreign countries, which is contrary to theoretical results obtained from macroeconomic models. This is called *the home bias puzzle*. This puzzle cannot be explained by standard macroeconomic models within the framework of expected utility theory.¹⁴ As pointed out in Introduction, based on portfolio data from a large financial institution in France, Bianchi and Tallon (2018) show that ambiguity averse investors are relatively more exposed to the French than to the international stock market. This result implies that ambiguity aversion plays a significant role in explaining home bias in equity markets. Anantanasuwong et al. (2019) elicit ambiguity attitudes toward a familiar company stock, a local stock index, a foreign stock index, and a crypto currency, using a representative sample of about 300 Dutch investors in the De Nederlandse Bank Household Survey. Anantanasuwong et al. (2019) identify that individuals' perceptions about ambiguity levels dif-

¹⁴For a survey of the home bias puzzle, see Lewis (1999). For recent studies of the home bias puzzle, see Solnik and Zuo (2012, 2017).

fer substantially depending on the type of asset, which implies that the same investor may perceive high ambiguity about foreign stocks or indices and perceive low ambiguity about local ones. In a related work, Boyle et al. (2012) analytically show that in the presence of ambiguity about returns on other assets, investors hold a large amount of the familiar asset, and also show that investors who are familiar with particular assets and sufficiently ambiguous about all other assets hold only the familiar assets. Therefore, ambiguity may play an important role in explaining the home bias puzzle in equity markets.¹⁵ We investigate how the difference between individual investors with insufficient information and institutional investors with much information explains the home bias puzzle.

Let us consider an ambiguity averse individual investor who allocates her wealth w between a domestic asset and a foreign asset. The investor possesses enough information to quantify the return on the domestic asset using a single probability distribution, but she does not have enough information to quantify the return on the foreign asset similarly. In this situation, the domestic asset is risky, and the foreign asset is ambiguous. This setting is the same as the previous section. The return on the domestic asset is denoted by \tilde{x} , and the return on the foreign asset is represented by n possible returns on the asset $\{\tilde{y}_1, \dots, \tilde{y}_n\}$ and the associated second-order probability $\{q_1, \dots, q_n\}$. In this setting, the optimal portfolio allocation is determined by:

$$k^* = \operatorname{argmax}_k \sum_{\theta=1}^n q_{\theta} \phi(E[u((w-k)\tilde{x} + k\tilde{y}_{\theta})]).$$

We assume that institutional investors usually estimate returns on assets from historical data and assign unique probability distributions to these returns. The optimal portfolio allocation is given by:

$$k^O = \operatorname{argmax}_k E[u((w-k)\tilde{x} + k\tilde{y}^O)].$$

Applying Corollary 1 in the previous section to this setting, we find that $k^O \geq k^*$. That is, the individual investor purchases more of the domestic asset than

¹⁵Epstein and Miao (2003) explain the home bias puzzle under ambiguity within the framework of MEU.

the foreign asset compared with the optimal portfolio allocation derived by institutional investors. This is because an individual investor with insufficient information about the foreign asset considers the existence of ambiguity on the return on the foreign asset, and thus avoids investing in the foreign asset.

4.2 50% Rule

In this subsection, we investigate conditions under which the 50% rule holds for the portfolio allocation problem with a risky asset and an ambiguous asset. The so-called *demand problem* named by Kijima and Ohnishi (1996) has received attention in the literature. Setting $w = 1$, the portfolio allocation problem is formulated as follows:

$$V(k) = \sum_{\theta=1}^n q_{\theta} \phi(E[u((1-k)\tilde{x} + k\tilde{y}_{\theta})]). \quad (7)$$

Because $V'(0) > 0$ and $V'(1) < 0$, the optimal portfolio allocation k^* is an interior solution in $[0, 1]$.

Suppose that a risk averse investor can allocate her initial wealth to two risky assets, and the optimal allocation of one risky asset is denoted by $k \in [0, 1]$. Suppose, also, that her preferences are represented by the expected utility. Conditions for the optimal portfolio allocation to be $k \leq 0.5$ have been investigated in the literature. This is called the *50% rule for portfolio allocation problems*.¹⁶ By restricting the class of utility functions, Hadar and Seo (1988, Theorems 4 and 5) derive necessary and sufficient conditions for the optimal portfolio allocation to be $k \leq 0.5$. Clark and Jokung (1999) generalize Hadar and Seo (1988, Theorem 3) and derive sufficient conditions on the conditional distributions of the two risky assets under which the optimal portfolio allocation of one risky asset is less than 0.5. For the purpose of extending the 50% rule for portfolio allocation problems, based on the smooth ambiguity model, we investigate conditions under which the 50% rule holds

¹⁶The previous studies examine conditions under which the optimal portfolio allocation for one asset is greater than 50%, $k \geq 0.5$. Because it is essentially identical, their results are restated as $k \leq 0.5$, to agree with the settings in this paper.

for the portfolio allocation problem with a risky asset and an ambiguous asset. From Corollary 1, we have:

$$V'(0.5) \leq 0 \Leftrightarrow k^* \leq 0.5,$$

by putting $w = 1$ and $k^O = 0.5$ when the possible returns on the asset $\{\tilde{y}_1, \dots, \tilde{y}_n\}$ are ranked by FSD and $R(z) \leq 1$. Recall that $R(z) = -zu''(z)/u'(z)$ denotes the Arrow-Pratt measure of relative risk aversion. These are the conditions for the 50% rule for the portfolio allocation problem with a risky asset and an ambiguous asset. We summarize this argument in the following corollary.

Corollary 2. *Suppose that an investor's objective function is represented by Equation (7), and that $\tilde{x} \stackrel{d}{=} \sum_{\theta=1}^n q_{\theta} \tilde{y}_{\theta}$. The 50% rule holds, that is, $k^* \leq 0.5$ if the possible returns on the asset $\{\tilde{y}_1, \dots, \tilde{y}_n\}$ are ranked by FSD and $R(z) \leq 1$.*

This result can be applied to every compound return on the asset \tilde{y} for which $k^O = 0.5$. It can also be applied to other stochastic dominance relations mentioned in subsections 3.3 and 3.4 by imposing appropriate conditions on the utility function u .

5 Conclusion

This paper considers a portfolio allocation problem between a risky asset and an ambiguous asset. We determine conditions under which an investor decreases the optimal portfolio allocation for the ambiguous asset. The conditions are imposed on the investor's utility function u and the stochastic dominance relations of $\{\tilde{y}_1, \dots, \tilde{y}_n\}$. For FSD, the investor with an Arrow-Pratt measure of relative risk aversion less than unity decreases the portfolio allocation of the risky asset when ambiguity is incorporated into the model. We also investigate the effect of ambiguity on the optimal portfolio allocation based on MLRD, RHRD, and higher-order increases in risk introduced by Ekern (1980). Finally, our analyses can be applied to an international diversification problem providing a potential explanation of the home bias puzzle. Furthermore, we

extend the 50% rule for portfolio allocation problems of Hadar and Seo (1988, 1990) and Clark and Jokung (1999) based on the smooth ambiguity model.

This paper assumes that the return on the risky asset \tilde{x} and the possible returns on the ambiguous asset \tilde{y}_θ are independent. This assumption enables us to apply the convolution property to our analyses. However, it is appropriate to assume that these assets are dependent. We leave this extension for future research.

Appendix

Appendix A

We provide the definition of convolution in probability theory based on Billingsley (1995, p.266) and Lehmann (2005, p.103).

Definition 4. Let \tilde{x} and \tilde{y} be independent random variables with probabilities μ and ν , respectively, and let P and Q be the corresponding probability distribution functions. The *convolution* of P and Q is defined by

$$H(z) \equiv \int_a^b Q(z-x)dP(x). \quad (8)$$

It can be shown that H is a probability distribution function. It can also be shown that if two random variables \tilde{x} and \tilde{y} with probability distribution functions P and Q are independent, then $\tilde{x} + \tilde{y}$ has the probability distribution function H defined by (8). Next, we introduce the *convolution property*.

Definition 5. A stochastic order \lesssim_{st} satisfies the *convolution property* if $\tilde{x} + \tilde{y}_i \lesssim_{st} \tilde{x} + \tilde{y}_j$ for any random variable \tilde{x} such that \tilde{x} and \tilde{y}_i are independent and \tilde{x} and \tilde{y}_j are independent.

Appendix B. Proof of Theorem 1

Before providing the proof of Theorem 1, we present the following two lemmas. As in Kijima and Ohnishi (1996, Proposition 3.3), the convolution property holds for FSD, which shows the following lemma.

Lemma 1. Let \tilde{x} and \tilde{y}_i be independent, and \tilde{x} and \tilde{y}_j be independent for $i, j \in \Theta$ with $i < j$. Let $\{\tilde{y}_i, \dots, \tilde{y}_n\}$ be ranked by FSD. Let $k \in [0, w]$. Then,

$$E[u((w-k)\tilde{x} + k\tilde{y}_i)] \leq E[u((w-k)\tilde{x} + k\tilde{y}_j)].$$

Lemma 2. (Hadar and Seo (1990)) Suppose that a) $u' > 0$, $u'' \leq 0$, b) \tilde{x}^i and \tilde{y} are independent for $i = 1, 2$, and c) $E[u((w-k_i)\tilde{x}^i + k_i\tilde{y})]$ is maximized at k_i^* . Then, $k_1^* \leq k_2^*$ for any $\tilde{x}^2 \lesssim_{FSD} \tilde{x}^1$ if and only if $u'(z)z$ is non-decreasing if and only if $R(z) \leq 1$.

The following lemma follows from Lemma 2.

Lemma 3. *Let \tilde{x} and \tilde{y}_i be independent, and \tilde{x} and \tilde{y}_j be independent for $i, j \in \Theta$ with $i < j$. Let $\tilde{y}_i \lesssim_{\text{FSD}} \tilde{y}_j$ for $i, j \in \Theta$ with $i < j$. Then, $k^i \leq k^j$ if $R(z) \leq 1$.*

Now, we are in a position to show Theorem 1.

Proof of Theorem 1. Let $V_i(k) = \sum_{\theta=1}^n q_\theta \phi_i(E[u((w-k)\tilde{x} + k\tilde{y}_\theta)])$ be the objective functions for $i = A, B$. Let $\phi_A = t \circ \phi_B$ where t is an increasing and concave function. Define $U(k, \theta) = E[u((w-k)\tilde{x} + k\tilde{y}_\theta)]$ and $g(\theta, k) = E[(\tilde{y}_\theta - \tilde{x})u'((w-k)\tilde{x} + k\tilde{y}_\theta)]$. The optimal portfolio allocation for investor B must satisfy

$$V'_B(k^B) = \sum_{\theta=1}^n q_\theta \phi'_B(U(k^B, \theta))g(\theta, k^B) = 0.$$

By the concavity of the objective function, it suffices to show that the sign of $V'_A(k^B) = \sum_{\theta=1}^n q_\theta \phi'_A(U(k^B, \theta))g(\theta, k^B)$ is negative. Because $\phi_A = t \circ \phi_B$, $V'_A(k^B)$ can be rewritten as follows:

$$V'_A(k^B) = \sum_{\theta=1}^n q_\theta t'(\phi_B(U(k^B, \theta)))\phi'_B(U(k^B, \theta))g(\theta, k^B)$$

Now, $t'(\phi_B(U(k^B, \theta)))$ is decreasing in θ because, as θ increases, $(w-k)\tilde{x} + k\tilde{y}_\theta$ improves in the sense of FSD by Lemma 1, so that $U(k^B, \theta)$ increases in θ , and ϕ is increasing in θ because ϕ is increasing by assumption, but the concavity of t implies that $t'(\phi_B(U(k^B, \theta)))$ is decreasing in θ . From Lemma 3, k^θ is increasing in θ if $R(z) \leq 1$. Thus, we obtain that, for $k^i \leq k^B \leq k^i + 1$,

$$\begin{cases} g(\theta, k^B) \leq 0 & \text{for } \theta \in \{1, \dots, i\} \\ g(\theta, k^B) \geq 0 & \text{for } \theta \in \{i+1, \dots, n\}. \end{cases}$$

With this decomposition in mind, and noting that t' is decreasing in θ , we

obtain the following:

$$\begin{aligned}
& V'_A(k^B) \\
&= \sum_{\theta=1}^i q_\theta t'(\phi_B(U(k^B, \theta))) \phi'_B(U(k^B, \theta)) g(\theta, k^B) + \sum_{\theta=i+1}^n q_\theta t'(\phi_B(U(k^B, \theta))) \phi'_B(U(k^B, \theta)) g(\theta, k^B) \\
&\leq t'(\phi_B(U(k^B, i))) \sum_{\theta=1}^i q_\theta \phi'_B(U(k^B, \theta)) g(\theta, k^B) + t'(\phi_B(U(k^B, i))) \sum_{\theta=i+1}^n q_\theta \phi'_B(U(k^B, \theta)) g(\theta, k^B) \\
&= t'(\phi_B(U(k^B, i))) V'_B(k^B) = 0.
\end{aligned}$$

Because we show that $V'_A(k^B) = \sum_{\theta=1}^n q_\theta \phi'_A(U(k^B, \theta)) g(\theta, k^B)$ is negative, the proof is completed. \square

Appendix C. Derivation of (4)

Let $\phi_A = t \circ \phi_B$, where t is increasing and concave. Then, we can rewrite

$$\begin{aligned}
& q_1 \phi'_A(U(k^A, 1)) + q_2 \phi'_A(U(k^A, 2)) \\
&= q_1 t'(\phi_B(U(k^A, 1))) \phi'_B(U(k^A, 1)) + q_2 t'(\phi_B(U(k^A, 2))) \phi'_B(U(k^A, 2)).
\end{aligned}$$

Because $U(k^A, 1) \leq U(k^A, 2)$, ϕ_i is increasing for $i = A, B$, and t' is decreasing by t 's concavity, it holds that

$$t'(\phi_B(U(k^A, 1))) \geq t'(\phi_B(U(k^A, 2))).$$

Because ϕ_i is unique up to a positive affine transformation for $i = A, B$, we can obtain the following normalization,

$$q_1 \phi'_A(U(k^A, 1)) + q_2 \phi'_A(U(k^A, 2)) = q_1 \phi'_B(U(k^A, 1)) + q_2 \phi'_B(U(k^A, 2)), \quad (9)$$

which implies that the following inequalities must be satisfied:

$$t'(\phi_B(U(k^A, 1))) \geq 1 \geq t'(\phi_B(U(k^A, 2))).$$

From the first inequality,

$$\phi'_B(U(k^A, 1)) \leq t'(\phi_B(U(k^A, 1))) \phi'_B(U(k^A, 1)) = \phi'_A(U(k^A, 1))$$

holds. Now, we obtain that

$$\begin{aligned}
& \phi'_B(U(k^A, 1)) \leq \phi'_A(U(k^A, 1)) \\
\Leftrightarrow & \frac{\phi'_B(U(k^A, 1))}{q_1\phi'_B(U(k^A, 1)) + q_2\phi'_B(U(k^A, 2))} \\
\leq & \frac{\phi'_A(U(k^A, 1))}{q_1\phi'_A(U(k^A, 1)) + q_2\phi'_A(U(k^A, 2))} \\
\Leftrightarrow & \hat{q}_1^A(k^A) \geq \hat{q}_1^B(k^A),
\end{aligned}$$

where the first equivalence follows from (9). Therefore, we complete the proof.

Appendix D. Proofs of Propositions 1 and 2

We can show Propositions 1 and 2 based on the proof of Theorem 1. For that purpose, it suffices to show that the results corresponding to Lemmas 1 and 3 hold for MLRD and RHRD.

First, Lemma 1 holds for MLRD and RHRD because both MLRD and RHRD are stronger than FSD.

Second, as in the main text, the result corresponding to Lemma 3 can be shown by Landsberger and Meilijson (1990, Proposition 2) for MLRD, and the result corresponding to Lemma 3 can be shown by Kijima and Ohnishi (1996, Theorem 4.12 and its Corollary 4.7) for RHRD. Thus, the proofs of Propositions 1 and 2 are completed.

Appendix E. Proof of Proposition 3

Similar to Propositions 1 and 2, we can show Proposition 3 based on the proof of Theorem 1. For that purpose, it suffices to show that the results corresponding to Lemmas 1 and 3 hold for N -th degree risk.

First, we show the result corresponding to Lemma 3. Because \tilde{x} and \tilde{y}_θ are independent for any $\theta \in \Theta$, the convolution of F and G_θ is

$$H_\theta(z) = \int_a^b G_\theta(z-x)dF(x),$$

where F and G_θ denote the probability distribution functions of \tilde{x} and \tilde{y}_θ , respectively. It can be shown that the convolution H is also a probability

distribution function. Let us define $G_\theta^n(y-x) = \int_a^y G_\theta^{n-1}(t-x)dt$. By Fubini's theorem, we can rewrite the probability distribution function as

$$H_\theta^n(z) = \int_a^b G_\theta^n(z-x)dF(x).$$

Note that $\tilde{y}_i \lesssim_{n\text{-risk}} \tilde{y}_j$ is equivalent to $H_i^n(z) = \int_a^b G_i^n(z-x)dF(x) \geq \int_a^b G_j^n(z-x)dF(x) = H_j^n(z)$, that is, $\tilde{x} + \tilde{y}_i \lesssim_{n\text{-risk}} \tilde{x} + \tilde{y}_j$. From the convolution property, the claim is proved.

Second, the result corresponding to Lemma 3 can be shown by Chiu et al. (2010) for N -th degree risk. Thus, the proof of Proposition 3 is completed.

References

- [1] Anantanasuwong, K., R. Kouwenberg, O. S. Mitchell, and K. Peijnenburg (2019): “Ambiguity Attitudes about Investments: Evidence from the Field,” Wharton Pension Research Council Working Papers, University of Pennsylvania.
- [2] Arrow, K. J. (1965): *Aspects of the Theory of Risk-Bearing*, Helsinki: Yrjö Jahnsonin Saatio.
- [3] Bianchi, M. and J.-M. Tallon (2018): “Ambiguity Preferences and Portfolio Choices: Evidence from the Field,” forthcoming in *Management Science*.
- [4] Billingsley, P. (1995): *Probability and Measure*, 3rd edition, Wiley.
- [5] Borgonovo, E., V. Cappelli, F. Maccheroni, and M. Marinacci (2018): “Risk analysis and Decision Theory: A Bridge,” *European Journal of Operational Research* 264, 280-293.
- [6] Boyle, P., L. Garlappi, R. Uppal, and T. Wang (2012): “Keynes Meets Markowitz: The Trade-Off Between Familiarity and Diversification,” *Management Science* 58, 253-272.
- [7] Chateauneuf, A. (1994): “Modeling Attitudes towards Uncertainty and Risk through the Use of Choquet Integral,” *Annals of Operations Research* 52, 3-20.
- [8] Chiu, W. H., L. Eeckhoudt and B. Rey (2012): “On Relative and Partial Risk Attitudes: Theory and Implications,” *Economic Theory* 50, 151-167.
- [9] Clark, E., and O. Jokung (1999): “A Note on Asset Proportions, Stochastic Dominance, and the 50% Rule,” *Management Science* 45, 1724-1727.
- [10] Driouchi, T., L. Trigeorgis, and R. H. Y. So (2018): “Option Implied Ambiguity and Its Information Content: Evidence from the Subprime Crisis,” *Annals of Operations Research* 262, 463-491.

- [11] Eeckhoudt, L., A. M. Fiori, and E. R. Gianin (2016): “Loss-averse Preferences and Portfolio Choices: An Extension,” *European Journal of Operational Research* 249, 224-230.
- [12] Eeckhoudt, L. and C. Gollier (1995): “Demand for Risky Assets and the Monotone Probability Ratio Order,” *Journal of Risk and Uncertainty* 11, 113-122.
- [13] Eeckhoudt, L. and H. Schlesinger, (2006): “Putting Risk in its ‘Proper Place,’” *American Economic Review* 96, 280-289.
- [14] Ekern, S. (1980): “Increasing N th Degree Risk,” *Economics Letters* 6, 329-333.
- [15] Ellsberg, D. (1961): “Risk, Ambiguity, and the Savage Axioms,” *Quarterly Journal of Economics* 75, 643-669.
- [16] Epstein, L. G. and J. Miao (2003): “A Two-Person Dynamic Equilibrium under Ambiguity,” *Journal of Economic Dynamics and Control* 27, 1253-1288.
- [17] Epstein, L. G. and M. Schneider (2008): “Ambiguity, Information Quality, and Asset Pricing,” *Journal of Finance* 63, 197-228.
- [18] Fang, Y. and T. Post (2017): “Higher-degree Stochastic Dominance Optimality and Efficiency,” *European Journal of Operational Research* 261, 984-993.
- [19] Fishburn, P. C. and R. B. Porter (1976): “Optimal Portfolios with One Safe and One Risky Asset: Effects of Changes in Rate of Return and Risk,” *Management Science* 22, 1064-1073.
- [20] French, K. R. and J. M. Poterba (1991): “Investor Diversification and International Equity Markets,” *American Economic Review* 81, 221-226.
- [21] Ghirardato, P. and M. Marinacci (2001): “Risk, Ambiguity, and the Separation of Utility and Beliefs,” *Mathematics of Operations Research* 26, 864-890.

- [22] Gilboa, I. and D. Schmeidler (1989): "Maxmin Expected Utility with Non-Unique Priors," *Journal of Mathematical Economics* 18, 141-153.
- [23] Gilboa, I. and D. Schmeidler (1994): "Additive Representations of Non-additive Measures and the Choquet Integral," *Annals of Operations Research* 52, 43-65.
- [24] Gilboa, I. and D. Schmeidler (1995): "Canonical Representation of Set Functions," *Mathematics of Operations Research* 20, 197-212.
- [25] Gollier, C. (2011): "Portfolio Choice and Asset Prices: the Comparative Statics of Ambiguity Aversion," *Review of Economic Studies* 78, 1329-1344.
- [26] Hadar, J. and T. K. Seo (1988): "Asset Propotions in Optimal Portfolios," *Review of Economic Studies* 55, 459-468.
- [27] Hadar, J. and T. K. Seo (1990): "The Effects of Shifts in a Return Distribution on Optimal Portfolios," *International Economic Review* 31, 721-736.
- [28] Huang, Y-C, and L. Y. Tzeng (2018): "A Mean-preserving Increase in Ambiguity and Portfolio Choices," *Journal of Risk and Insurance* 85, 993-1012.
- [29] Jewitt, I. and S. Mukerji (2017): "Ordering Ambiguous Acts," *Journal of Economic Theory* 171: 213-267.
- [30] Jindapon, P. and W. S. Neilson (2007): "Higher-order Generalizations of Arrow-Pratt and Ross Risk Aversion: A Comparative Statics Approach," *Journal of Economic Theory* 136, 719-728.
- [31] Kahneman, D. and A. Tversky (1978): "Prospect Theory: An Analysis of Decision under Risk," *Econometrica* 47, 263-291.
- [32] Kelsey, D., R. Kozhan, and W. Pang (2010): "Asymmetric Momentum Effects under Uncertainty," *Review of Finance* 15, 603-631.

- [33] Keynes, J. M. (1921): *A Treatise on Probability*, London: MacMillan.
- [34] Kijima, M. and M. Ohnishi (1996): "Portfolio Selection Problems via the Bivariate Characterization of Stochastic Dominance Relations," *Mathematical Finance* 6, 237-277.
- [35] Klibanoff, P., M. Marinacci, and S. Mukerji (2005): "A Smooth Model of Decision Making under Ambiguity," *Econometrica* 73, 1849-1892.
- [36] Knight, F. H. (1921). *Risk, Uncertainty and Profit*, Hart, Schaffner and Marx.
- [37] Landsberger, M. and I. Meilijson (1990): "Demand for Risky Financial Assets: A Portfolio Analysis," *Journal of Economic Theory* 50, 204-213.
- [38] Lehmann, E. L. (2005): *Testing Statistical Hypotheses*, 3rd edition, Springer.
- [39] Levy, H. (1992): "Stochastic Dominance and Expected Utility: Survey and Analysis," *Management Science* 38, 555-593.
- [40] Lewis, K. K. (1999): "Trying to Explain Home Bias in Equities and Consumption," *Journal of Economic Literature* 37, 571-608.
- [41] Menezes, C., C. Geiss and J. Tressler (1980): "Increasing Downside Risk," *American Economic Review* 70, 921-932.
- [42] Meyer, D. J. and J. Meyer (2005): "Relative Risk Aversion: What do We Know?," *Journal of Risk and Uncertainty* 31, 243-262.
- [43] Neilson, W. (2010): "A Simplified Axiomatic Approach to Ambiguity Aversion," *Journal of Risk and Uncertainty* 41: 113-124.
- [44] Osaki, Y. and H. Schlesinger (2014): "Portfolio Choice and Ambiguous Background Risk," Working Paper, University of Alabama. Available at <http://hschlesinger.people.ua.edu/current-working-papers.html>
- [45] Peter, R. (2019): "Revisiting Precautionary Saving under Ambiguity," *Economics Letters* 174: 123-127.

- [46] Peter, R. and J. Ying (2018): “Do you Trust your Insurer? Ambiguity about Contract Nonperformance and Optimal Insurance Demand,” forthcoming in *Journal of Economic Behavior and Organization*.
- [47] Post, T. and M. Kopa (2013): “General Linear Formulations of Stochastic Dominance Criteria,” *European Journal of Operational Research* 230, 321-332.
- [48] Pratt, J. W. (1964): “Risk Aversion in the Small and the Large,” *Econometrica* 32, 122-136.
- [49] Roman, D., G. Mitra, and V. Zverovich (2013): “Enhanced Indexation Based on Second-order Stochastic Dominance,” *European Journal of Operational Research* 228, 273-281.
- [50] Rothschild, M. and J. E. Stiglitz (1970): “Increasing Risk: I. A Definition,” *Journal of Economic Theory* 2, 225-243.
- [51] Rothschild, M. and J. E. Stiglitz (1971): “Increasing Risk: II. Its Economic Consequences,” *Journal of Economic Theory* 3, 66-84.
- [52] Schmeidler, D. (1989): “Subjective Probability and Expected Utility without Additivity,” *Econometrica* 57, 571-587.
- [53] Segal, U. (1987): “The Ellsberg Paradox and Risk Aversion: An Anticipated Utility Approach,” *International Economic Review* 28, 175-202.
- [54] Solnik, B. and L. Zuo (2012): “A Global Equilibrium Asset Pricing Model with Home Preference,” *Management Science* 58, 273-292.
- [55] Solnik, B. and L. Zuo (2017): “Relative Optimism and the Home Bias Puzzle,” *Review of Finance* 21, 2045-2074.