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“Chaotic Dynamics of a Piecewise Linear Model of
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Chaotic Dynamics of a Piecewise Linear Model of Credit Cycles*

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Abstract

We develop a simple piecewise linear overlapping generations model exhibiting endogenous business cycles, which is based on Matsuyama's (2007) model of credit cycles. Some sort of "noise" representing information imperfection is shown to transform the Matsuyama model into a continuous, eventually expanding, piecewise linear map on the interval, which is tractable enough to investigate the dynamics in depth by using the techniques of the Frobenius-Perron operators to find observable chaos. While, according to the analysis of Asano et al. (2012), the Matsuyama model exhibits periodic cycles of arbitrarily large period, it is essentially not capable of chaotic dynamics. However, our model exhibits ergodic chaos with some robustness.

JEL Classification Numbers: E32; O14; O41; C62
Key words: Matsuyama model; credit cycle; piecewise linearity; chaotic dynamics; ergodic chaos

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1 Introduction

The overlapping generations (OLG, hereafter) model has been widely used as a general equilibrium model in many fields of macroeconomics. As one of the major research concerns in macroeconomic dynamics, the endogenous business cycles have also been investigated using the OLG framework, initiated by Benhabib and Day (1982) and Grandmont (1985). They find that even without external shocks, nonlinearities inherent in the underlying economic systems can cause complicated, in particular *chaotic*, dynamics.

From a technical viewpoint, however, it is not necessarily easy to detect and characterize chaotic behaviors in a given deterministic nonlinear dynamic economic model, especially when the long-run observability of irregular perpetual fluctuations is concerned, even if the model is described by a single one-dimensional difference equation. One apparent exception appears when the economic model is *piecewise linear.* A piecewise linear dynamic model has much to offer. Among other features, it facilitates *analytical tractability and dynamic richness at the same time.* Even if piecewise linear modeling seems to be an extreme simplification, it is very beneficial in the above sense as long as it is a tolerable approximation of the real world.

Furthermore, there are empirical affinities of piecewise linear models to the *threshold autoregressive* (TAR) models used in nonlinear time series analysis. See Tong and Lim (1980) and Tong (1983) for TAR models and Chan and Tong (1986) for the *smooth threshold autoregressive* models.[1] That is, a theoretical piecewise linear economic model free from external shocks can be thought of as a deterministic counterpart of the TAR model. In this sense, the development of deterministic piecewise linear models in the general equilibrium framework[2] will give some sound microeconomic foundations for the TAR models.

In the literature of optimal growth, which is thought of as another endogenous business cycle theory in the general equilibrium framework, there are some two-sector models with Leontief technology in which optimal transition functions exhibit piecewise linearity, giving rise to ergodic chaos (that is, complex dynamics with

[1]For applications to economics, for example, see Teräsvirta and Anderson (1992), Clements and Krolzig (1998), van Dijk et al. (2002), and Teräsvirta et al. (2005). See also Teräsvirta et al. (2010).

[2]Of course, there are a few discrete-time piecewise linear economic models exhibiting complicated dynamics in non-general equilibrium approaches. See, for instance, Hommes (1991) for several piecewise linear models with Hicksian (i.e., ceiling and floor) nonlinearity, Matsumoto (2001, 2005) for cobweb and inventory dynamics, Huang (2005) for cobweb dynamics, and Ishida and Yokoo (2004) and Yokoo and Ishida (2008) for a macroeconomic model with binary investment choice (see below for more detail).

observability in the long run). This piecewise linearity allows us to characterize complicated optimal growth paths in depth. See e.g. Nishimura et al. (1994) and Nishimura and Yano (1995) for more detail.

Recent studies have investigated complex dynamics of *piecewise smooth* growth cycle models intensively. Piecewise smooth modeling can be regarded as an intermediate modeling between the two extremes: piecewise linear modeling and smooth modeling. See Gardini et al. (2008)[3] and Matsuyama et al. (2016),[4] who employ the relatively new theory of border-collison bifurcation to show that their macroeconomic models can exhibit complicated transition to chaotic behavior. Admittedly, the complex dynamics of the piecewise linear map of the interval has been well understood since long before at least by mathematical experts, and therefore we have seemingly little to add to the literature.[5] On the other hand, piecewise linear modeling is recognized to give analytical results of complex dynamics in a sharp and clear way. Taking this fact into account and regarding that there seems to be no counterpart[6] to the chaotic piecewise linear optimal growth model in the OLG literature, at least to the best knowledge of the authors, it must be still worth filling the gap and developing a piecewise linear OLG model exhibiting chaotic behavior. Therefore, the purpose of this paper is to explore this task.

Among recent studies related to this paper, Matsuyama (2007) proposes an OLG model with endogenous technology switch caused by financial imperfection, and shows that the model can generate several growth patterns. In Matsuyama's (2007) model, agents face borrowing constraints due to financial imperfection, and each agent can choose to be either an entrepreneur or a lender. Furthermore, multiple investment technologies are assumed to be available. The market interest rate affects entrepreneurs' choice of technology and the market rate varies over time depending on the level of capital. This implies that the entrepreneurs' choice of technology changes endogenously, which gives rise to richer dynamics compared to other models in the literature on endogenous business cycles. Although the model proposed by

[3]Gardini et al. (2008) investigate the piecewise smooth growth cycle model of Matsuyama (1999). By incorporating misperception or observation errors into Matsuyama (2007), this paper investigates the piecewise linear model of Matsuyama (2007), which exhibits chaos based on a simpler model than that of Gardini et al. (2008).

[4]Matsuyama et al. (2016) analyze the piecewise smooth model of endogenous credit cycles developed by Matsuyama (2013). Contrary to Matsuyama et al. (2016) in which chaos is exhibited by a complicated model, by incorporating misperception or observation errors into Matsuyama (2007), this paper analyzes the piecewise linear model of Matsuyama (2007), exhibiting chaos within a simpler framework.

[5]For example, see Day and Pianigiani (1991) and the references therein.

[6]Umezuki and Yokoo (2017) provide another piecewise linear OLG model in a simpler setting than the present paper.

Matsuyama (2007) is relatively simple, it leads to various phenomena, such as credit traps, credit collapses, leapfrogging, credit cycles, and growth miracles. As such, Asano et al. (2012) analyze the dynamic property of the macroeconomic model proposed by Matsuyama (2007) in depth, and show that the model can be analyzed within the framework of the neuron model studied by Hata (1982, 1989).[7]

Furthermore, Asano et al. (2012) show that the model can exhibit either periodic fluctuations or fluctuations which are *chaotic* in some sense.[8] It is important to notice that chaos in Asano et al. (2012) occurs only on the set of parameter values of measure zero.[9] In the very sense, chaos is virtually *not observable* in the models of Asano et al. (2012) and, consequently, Matsuyama (2007). Once such a pathological parameter value for the occurrence of Hata's chaos is somehow chosen, however, any initial condition leads to a complicated long-run behavior. In the latter sense, Hata's chaos *is* observable, but such a case hardly occurs. Observability of chaos in terms of both sets of parameter values and initial conditions is important because it can be thought to capture the "recurrent but not periodic" nature of business cycles in the deterministic framework. Therefore, when we talk about the observability of chaos in the long run, we have to pay attention to both the state space and the parameter space. To avoid confusion, when we talk about observability in terms of the parameter values, we sometimes use closely related or synonymous terms such as *abundance* or *robustness*.[10]

Some existing studies need mentioning from the viewpoint of analytical techniques. Ishida and Yokoo (2004) develop a macroeconomic model in which firms face a binary choice problem in investment, and show that due to piecewise linearity, the model exhibits periodic cycles. Yokoo and Ishida (2008) modify the model by introducing imperfect observability,[11] and provide a mechanism by which obser-

[7]See also Hata (2014) for more recent development of this research.

[8]Asano et al. (2012) show the existence of chaos in the sense of Hata (1982, 1989). For the definition of chaos by Hata (1982, 1989), see Section 2 for further details.

[9]The Hausdorff dimension of the middle third Cantor set is positive, while that of the set of parameter values of Hata's chaos is zero, which says that the latter set is literally "thin." We thank Yosuke Umezuki for pointing out this fact. See also Hata (2014) and Umezuki and Yokoo (2017) for this topic.

[10]As economic applications of observable chaos in the general equilibrium framework, for example, Matsuyama (1991) analyzes endogenous fluctuations in an infinite-horizon model with money-in-the-utility-function. Nishimura and Yano (1995) investigate the possibility of ergodically chaotic optimal capital accumulation. While Matsuyama (1991) and Nishimura and Yano (1995) deal with one-dimensional models, Yokoo (2000) investigates a two-dimensional model in an OLG setting to show that the model can exhibit a strange attractor (i.e., observable chaos in the above double meaning) due to a homoclinic bifurcation.

[11]By imperfect observability, we mean that the state variables are observed with some noise, for example, $\hat{x}_t = x_t + \sigma\varepsilon_t$, where x_t is a state variable at time t, $\sigma > 0$ is a constant, and ε_t

vation errors lead to chaotic fluctuations. That is, Yokoo and Ishida (2008) show that observation errors or what they call *misperception* can be a source of observable chaos in economic systems.

Since we hardly know the true state of the world with precision, especially when aggregate amounts are concerned, it is natural to assume that such insufficient information affects decision making, in particular, about whether an agent chooses to become an entrepreneur or not. Therefore, it is of interest to incorporate such information imperfection or imperfect observability into Matsuyama's (2007) or equivalently Asano et al.'s (2012) framework under perfect observation to see how the dynamic patterns change.

The model proposed in this paper can be thought of as an extension of the model in Asano et al. (2012), which is a special case of Matsuyama's (2007), along the line of Yokoo and Ishida (2008). As a result, we transform Matsuyama's (2007) original model, together with misperception or observation errors, into a piecewise linear model, which is tractable enough to investigate the dynamics in depth by using the techniques of the Frobenius-Perron operators to find invariant measures (i.e., observable chaos) as in Yokoo and Ishida (2008).[12] Indeed, by specifying the set of parameters for some kind of Markov properties,[13] we can easily establish and characterize chaotic dynamics with observability in terms of initial conditions in more detail. Imposing Markov properties on models seems rather restrictive at first glance, however, this will be relaxed to the extent that such chaos is shown to be abundant in terms of the set of parameter values.

The organization of this paper is as follows. Based on Matsuyama (2007) and Asano et al. (2012), Section 2 provides a benchmark model, in which the productivity of agents is perfectly observable. Section 3 provides the main model of this paper, in which the productivity of agents is imperfectly observable. Section 4 considers further specifications of our model discussed in Section 3. Section 5 analyzes chaotic dynamics in detail. Section 6 gives concluding remarks. Some derivations are relegated to the Appendix.

is a random variable at time t. Therefore, perfect observability corresponds to the case in which relevant macroeconomic state variables are observed without any noise, that is, $\sigma = 0$.

[12]For example, see Boyarsky and Góra (1997, Chapter 4) or subsection 5.2 in this paper for further details. For applications of the Frobenius-Perron operator to piecewise linear economic models in different contexts, see e.g. Matsumoto (2001, 2005) and Huang (2005).

[13]To this end, we adopt some theory from dynamical systems theory related to ergodic theory. For example, see Boyarsky and Góra (1997).

2 The Model under Perfect Observation

In this section, based on Asano et al. (2012), directly following Matsuyama (2007), we consider the situation in which the returns generated by entrepreneurs' projects are perfectly observable. In the following sections, we extend the perfectly observable framework to an imperfectly observable one in which the returns generated by entrepreneurs' projects are observed with some noise.

A final good is produced by the following constant returns to scale technology:

$$Y_t = AF(K_t, L_t),$$

where K_t and L_t denote physical capital and labor at time t, respectively. Let $y_t = Y_t/L_t$, $k_t = K_t/L_t$, and $f(k_t) = F(k_t, 1)$. Then,

$$y_t = Af(k_t).$$

We also suppose that $f' > 0$ and $f'' < 0$. For simplicity, similar to Asano et al. (2012), we specify $f(k_t)$ as

$$f(k_t) = Ak_t^{\alpha}, \;\; 0 < \alpha < 1.$$

Since we assume that the labor market is competitive, the real wage, w_t, is as follows:

$$w_t = w(k_t) = (1 - \alpha)Ak_t^{\alpha}.$$

It is also assumed that physical capital fully depreciates in one period.

Similar to Matsuyama (2007) and Asano et al. (2012), we consider the Diamond overlapping generations model, in which agents live two periods. In each period, a new generation of potential entrepreneurs, which is a unit measure of homogeneous agents, is born with one unit of labor and lives two periods. In the young period, they supply labor and earn $w_t = w(k_t)$. They only consume in the old period. We assume that, in the young period, each agent born at time t can choose to become either a lender or an entrepreneur. On one hand, if she chooses to become a lender, then she lends all of her earnings and obtains $r_{t+1}w_t$ in the old period in the competitive credit market, where r_{t+1} denotes the real interest rate. On the other hand, if she chooses to become an entrepreneur, then she can choose from two types of projects, a type 1 project and a type 2 project. A type i ($i = 1, 2$) project transforms m_i units of the final good in period t into m_iR_i units of physical capital in period $t+1$. When $m_i > w_t$, she must borrow $m_i - w_t$ at rate r_{t+1}. When $m_i \leq w_t$, the project can be entirely self-financed and $w_t - m_i$ is lent. By the existence of credit constraints, each agent can pledge only up to a constant fraction of the project revenue for repayment, $\lambda_i m_i R_i f'(k_{t+1})$, where $0 \leq \lambda_i \leq 1$. The parameter, λ_i, captures the credit market friction.[14] Since the lender is assumed to know this, the lender would lend only up to $\lambda_i m_i R_i f'(k_{t+1})/r_{t+1}$. The agent must satisfy the following borrowing constraint:

$$\lambda_i m_i R_i f'(k_{t+1}) \geq r_{t+1}(m_i - w_t) \Leftrightarrow r_{t+1} \leq \frac{\lambda_i m_i R_i f'(k_{t+1})}{m_i - w_t} = \frac{R_i f'(k_{t+1})}{(1 - (w_t/m_i))/\lambda_i}$$

for $i = 1, 2$. A larger λ_i implies a weaker credit constraint. If $\lambda_i = 1$, then the agent can borrow up to the present discounted value of the project revenue, $m_i R_i f'(k_{t+1})/r_{t+1}$. If $\lambda_i = 0$, then the agent cannot borrow the money she needs and must self-finance the project entirely.

Since the agents can be lenders, the following profitability condition must be satisfied:

$$f'(k_{t+1}) m_i R_i - r_{t+1}(m_i - w_t) \geq r_{t+1} w_t \Leftrightarrow f'(k_{t+1}) R_i \geq r_{t+1}.$$

This condition states that the agents borrow and run a type i project if and only if earnings from investment are greater than those from lending.

The equilibrium interest rate satisfies the following:

$$r_{t+1} = \max\left\{\frac{R_1 f'(k_{t+1})}{\max\{1, (1 - (w_t/m_1))/\lambda_1\}}, \frac{R_2 f'(k_{t+1})}{\max\{1, (1 - (w_t/m_2))/\lambda_2\}}\right\}.$$

For credit cycles to appear in Matsuyama's (2007) model without noise, we assume the following inequalities as in Asano et al. (2012):

$$R_2 > R_1 > \lambda_2 R_2 > \lambda_1 R_1 \quad \text{and} \quad \frac{m_1}{m_2} < \frac{1 - (\lambda_2 R_2/R_1)}{1 - \lambda_1} < 1. \tag{1}$$

The first assumption implies that there exist trade-offs between productivity and pledgeability.[15] Under the first and second assumptions, the two graphs intersect twice as in Figure 4 in Matsuyama (2007, p.512).

Remember that project 1 is adopted if

$$\frac{R_2}{\max\{1, (1 - (w_t/m_2))/\lambda_2\}} \leq \frac{R_1}{\max\{1, (1 - (w_t/m_1))/\lambda_1\}}, \tag{2}$$

and that project 2 is adopted otherwise. Thus, by solving

$$\frac{R_2}{(1 - (w(k)/m_2))/\lambda_2} = \frac{R_1}{(1 - (w(k)/m_1))/\lambda_1}$$

[14] For this formulation of the credit market imperfection, see also Matsuyama (2004).
[15] See Matsuyama (2007) for details.

for $k = k_C$, where

$$w = w(k) = (1-\alpha)Ak^{\alpha},$$

we obtain

$$k_C = \left[\frac{m_1 m_2(\lambda_2 R_2 - \lambda_1 R_1)}{(1-\alpha)A(m_2\lambda_2 R_2 - m_1\lambda_1 R_1)}\right]^{1/\alpha}.$$

Similarly, by solving

$$\frac{R_2}{(1-(w(k_{CC})/m_2))/\lambda_2} = R_1,$$

we have

$$k_{CC} = \left[\frac{m_2(1-\lambda_2 R_2/R_1)}{(1-\alpha)A}\right]^{1/\alpha}.$$

For the later use, we define

$$k_D = \left[\frac{m_1(1-\lambda_1)}{(1-\alpha)A}\right]^{1/\alpha},$$

which is obtained by equalizing the components in the denominator of the right-hand side of (2), i.e.,

$$1 = (1-(w(k_D)/m_1))/\lambda_1.$$

Similarly, we also define

$$k_N = \left[\frac{m_2(1-\lambda_2)}{(1-\alpha)A}\right]^{1/\alpha}$$

by doing the same for the left-hand side of (2). Under assumptions (1), we have $k_C < k_D < k_{CC} < k_N$ (see Fig.1).

+++ insert Fig.1 around here +++

Under these assumptions, the model turns out to be given by

$$k_{t+1} = \begin{cases} R_2(1-\alpha)Ak_t^{\alpha} & \text{if } 0 < k_t < k_C, \\ R_1(1-\alpha)Ak_t^{\alpha} & \text{if } k_C \le k_t \le k_{CC}, \\ R_2(1-\alpha)Ak_t^{\alpha} & \text{if } k_{CC} < k_t \end{cases} \tag{3}$$

and can be verified to have credit cycles. As is also verified that every trajectory generated by (3) is eventually trapped in the interval $[R_1(1-\alpha)Ak_C^{\alpha}, R_2(1-\alpha)Ak_C^{\alpha}] \subset [0, k_{CC}]$, called a *trapping interval*, we find a variable change that transforms the trapping interval into the unit interval $[0,1]$. To provide a qualitative analysis of the model in the long run, Asano et al. (2012) transform (3), dropping off the third equation, into a tractable form by taking the logarithms of both the sides of (3), which gives

$$\log k_{t+1} = \begin{cases} \log(R_2(1-\alpha)A) + \alpha \log k_t, & \text{if } 0 < k_t < k_C, \\ \log(R_1(1-\alpha)A) + \alpha \log k_t, & \text{if } k_C \leq k_t \leq k_{CC}. \end{cases} \tag{4}$$

By defining a new variable x_t by

$$x_t = \frac{1}{\log(R_2/R_1)} \left[\log k_t - \log(R_1(1-\alpha)Ak_C^{\alpha})\right], \tag{5}$$

(4) can be transformed into the following piecewise linear difference equation:[16]

$$x_{t+1} = \begin{cases} 1 + \alpha(x_t - \gamma), & \text{if } x_t < \gamma, \\ \alpha(x_t - \gamma), & \text{if } x_t \geq x_t, \end{cases} \tag{6}$$

where

$$\gamma = \frac{\log\left(k_C^{1-\alpha}/R_1(1-\alpha)A\right)}{\log(R_2/R_1)}.$$

Asano et al. (2012) show that γ can take any value within the range $(0, 1)$. As pointed out by Asano et al. (2012), (6) is the same as the Caianiello equation analyzed by Hata (1982, 1989). Based on Hata's (1982, 1989) results, Asano et al. (2012) show that the macroeconomic model by Matsuyama (2007) mentioned above can exhibit periodic or "chaotic" fluctuations. As such, we restate the results by Asano et al. (2012) for comparison and the reader's convenience in the following proposition.

Proposition 1. *(Asano et al. (2012) with some modifications[17]) For any rational number $p/q \in (0, 1)$, where integers p and q are mutually prime, there exists a closed interval $\Delta(p/q)$ such that for any $\gamma \in \Delta(p/q)$, (6) exhibits a globally attracting period-q cycle. Moreover, let $E_0 = [0, 1] \setminus \bigcup_{0<p/q<1} \Delta(p/q)$, which is of measure zero. Then, for any $\gamma \in L_0$, (6) exhibits chaos in the sense of Hata (1982, 1989).*

Note that, unlike any other definitions of chaos in the present paper, chaos in the sense of Hata (1982, 1989) lacks the condition of the density of periodic points. Instead, Hata's chaos exhibits (i) expansivity (which implies sensitive dependence on initial conditions) and (ii) topological transitivity (see also Asano et al. (2012)). However, for the purpose of analyzing the observability of chaos in the long run, we consider the notion of *observable chaos* (or *ergodic chaos*), as defined in Section 5.

[16]Based on a similar form to (4), Ishida and Yokoo (2004) develop a business cycle model, and show that it can generate asymmetric periodic cycles for arbitrary periods.

[17]Also see Hata (1982, 1989).

3 The Model under Imperfect Observation

The main model in this paper builds on that with no uncertainty described in the previous section. To do this, we suppose that a project with a higher rate of return is riskier. To capture this idea in an easier way, we suppose, à la Yokoo and Ishida (2008), that entrepreneurs perceive the rate of return from project 2, which earns higher returns than project 1, with some "noise." For simplicity, we assume that entrepreneur i perceives R_2 to be $\hat{R}_{2,i}$, which we formulate as[18]

$$\hat{R}_{2,i} = (1 + \sigma\varepsilon_i)R_2, \quad \sigma \geq 0, \tag{7}$$

where $\varepsilon = \varepsilon_i$ is a stochastic variable, whose support is the interval $[-1, 1]$, independently drawn by entrepreneur i from an identical distribution. The distribution function will be specified later for our purpose. On one hand, the disturbance term, ε_i, represents the observational uncertainty by which entrepreneur i is affected. On the other hand, the constant, σ, is related to the variance of the disturbance term: the larger σ is, the riskier the project. When $\sigma = 0$, that is, when there is no uncertainty[19] involved, the model becomes as that analyzed in Asano et al. (2012). We assume that the variance of the disturbance term ε_i is normalized to one, which enables us to measure the degree of uncertainty by σ.

By the similar arguments of Matsuyama (2007) and Asano et al. (2012), project 1 is adopted by entrepreneur i if and only if

$$\frac{\hat{R}_{2,i}}{\max\{1, (1 - (w_t/m_2))/\lambda_2\}} \leq \frac{R_1}{\max\{1, (1 - (w_t/m_1))/\lambda_1\}}. \tag{8}$$

Then, by (7), Inequality (8) is rewritten as

$$\varepsilon_i \leq \frac{1}{\sigma}\left[\frac{R_1 \max\{1, (1 - w(k_t)/m_2)/\lambda_2\}}{R_2 \max\{1, (1 - w(k_t)/m_1)/\lambda_1\}} - 1\right] \equiv \rho(k_t).$$

Since $k_C < k_D < k_{CC} < k_N$, the function $\rho(k_t)$ restricted to the interval $[0, k_{CC}]$ has one and only one kink at $k_t = k_D$. Let ρ_L denote the restriction of ρ to the interval

[18]Here we introduce *multiplicative* noise. *Additive* noise formulated such as $\hat{R}_{2,i} = R_2 + \sigma\varepsilon_i$ can be a possible alternative, which, however, makes no essential difference for the outcomes.

[19]Interpretations of ε and σ are open to dispute. Given ε, σ may be regarded as the level of noise or the level of rationality of the agent. In any case, such formulations of "noise" often appear when agents face a discrete choice problem. For a comprehensive textbook on discrete choice theory, see Anderson et al. (1992). This theory is intensively used in stochastic evolution in games. See e.g. Sandholm (2010) for the use of noise in evolutionary game theory. For another application of discrete choice theory in relation of chaotic dynamics, see e.g. Brock and Hommes (1997) for adaptively rational equilibrium.

$[0, k_D]$ and ρ_R that to $[k_D, k_{CC}]$. Then, we have

$$\begin{aligned} \rho_L(k_t) &= \frac{1}{\sigma}\left[\frac{R_1(1-w(k_t)/m_2)/\lambda_2}{R_2(1-w(k_t)/m_1)/\lambda_1} - 1\right] \quad \text{and} \qquad (9)\\ \rho_R(k_t) &= \frac{1}{\sigma}\left[\frac{R_1}{R_2\lambda_2}(1-w(k_t)/m_2) - 1\right]. \qquad (10)\end{aligned}$$

Since $\operatorname{sign}\rho_L'(k_t) = \operatorname{sign}(m_2 - m_1) > 0$ and $\operatorname{sign}\rho_R'(k_t) = \operatorname{sign}(-w'(k_t)) < 0$, wherever the derivatives exist, the graph of ρ_L is upward-sloping and that of ρ_R is downward-sloping. Also note that by (1),

$$\begin{aligned} \rho_L(0) &= \frac{1}{\sigma}\left[\frac{\lambda_1 R_1}{\lambda_2 R_2} - 1\right] < 0 \quad \text{and}\\ \rho_L(k_D) &= \rho_R(k_D) = \frac{1}{\sigma}\left[1 - \frac{\lambda_2 R_2}{R_1} - (1-\lambda_1)\frac{m_1}{m_2}\right] > 0 \end{aligned}$$

for any $\sigma > 0$.

Let G be the cumulative distribution function for the stochastic variable ε, that is, $G(x) = \text{Prob}(\varepsilon \le x)$. Therefore, introducing uncertainty into original model (3) gives a generalized dynamic equation:

$$k_{t+1} = \left[R_1 G(\rho(k_t)) + R_2(1 - G(\rho(k_t)))\right] w(k_t). \qquad (11)$$

4 Piecewise-Linearization of the Model

As the form of (11) is still too general to characterize its dynamics in detail, we need to further specify its functional form. First, for $\sigma > 0$ small enough, we can define k_L and k_R by solving

$$\rho_L(k_L) = -1 \quad \text{and} \quad \rho_L(k_R) = 1,$$

where $\rho_L(k_t)$ is given by (9) for $k_t \in [0, k_D]$. Further computations show that

$$\begin{aligned} k_L &= k_L(\sigma) = \left[\frac{m_1 m_2((1-\sigma)\lambda_2 R_2 - \lambda_1 R_1)}{A(1-\alpha)((1-\sigma)m_2\lambda_2 R_2 - m_1\lambda_1 R_1)}\right]^{1/\alpha} \quad \text{and} \qquad (12)\\ k_R &= k_R(\sigma) = \left[\frac{m_1 m_2((1+\sigma)\lambda_2 R_2 - \lambda_1 R_1)}{A(1-\alpha)((1+\sigma)m_2\lambda_2 R_2 - m_1\lambda_1 R_1)}\right]^{1/\alpha} \qquad (13)\end{aligned}$$

for

$$0 < \sigma < \frac{\lambda_2 R_2 - \lambda_1 R_1}{\lambda_2 R_2}. \qquad (14)$$

It is easy to check that $k'_L(\sigma) < 0$ and $k'_R(\sigma) > 0$. Note that $k_L < k_C < k_R$ and that $\lim_{\sigma\to 0} k_L(\sigma) = \lim_{\sigma\to 0} k_R(\sigma) = k_C$. Since ρ_L is defined on $[0, k_D]$, it must also hold that

$$k_R \leq k_D$$

or

$$\sigma \leq \frac{m_1\lambda_1 R_1 + (m_2 - m_1)R_1 - m_2\lambda_2 R_2}{m_2\lambda_2 R_2}. \tag{15}$$

Next, we consider the case in which the function $\rho(k_t)$ is "well-behaved" in that once $\rho(k_t)$ exceeds 1 as k_t increases, it never falls below 1 until k_t reaches the right endpoint of the trapping interval. As we show below, we can take the interval

$$T = [\underline{k}, \overline{k}] \equiv [R_1(1-\alpha)Ak_L^\alpha, R_2(1-\alpha)Ak_R^\alpha] \tag{16}$$

as the trapping interval for (11) if G is appropriately specified. Let $\tilde{k}_R$ be defined as the solution of

$$\rho_R(\tilde{k}_R) = 1,$$

where ρ_R is given by (10). Therefore, for ρ to be well-behaved in the above sense, it suffices to require that

$$\overline{k} < \tilde{k}_R,$$

which can be rewritten as

$$A < \hat{A} \equiv \frac{m_2(1-(1+\sigma)\lambda_2 R_2/R_1)}{1-\alpha}\left[\frac{(1-\sigma)m_2\lambda_2 R_2 - m_1\lambda_1 R_1}{R_2 m_1 m_2((1-\sigma)\lambda_2 R_2 - \lambda_1 R_1)}\right]^\alpha. \tag{17}$$

Note that the fraction within the parentheses in the last expression is well defined if (14) is assumed and that $\hat{A}$ is positive if

$$\sigma < \frac{R_1 - \lambda_2 R_2}{\lambda_2 R_2}. \tag{18}$$

This situation is depicted in Fig.2.

+++ insert Fig.2 of the graph of ρ including $\bar{k}$ and $\tilde{k}_R$ around here +++

Assuming (17) and (18) for now, we introduce, analogously to (5), the following variable transformation, which maps the interval T given by (16) to the unit interval $[0, 1]$:

$$x_t = h(k_t) = \frac{1 + \alpha\,(\gamma_R - \gamma_L)}{\log(R_2/R_1)}\left[\log k_t - \log(R_1(1-\alpha)Ak_R^\alpha)\right], \tag{19}$$

where

$$\gamma_L = \gamma_L(\sigma) = \frac{\log \frac{k_L k_R^{-\alpha}}{R_1(1-\alpha)A}}{\log \frac{R_2}{R_1}\left(\frac{k_R}{k_L}\right)^{-\alpha}} \quad \text{and} \quad \gamma_R = \gamma_R(\sigma) = \frac{\log \frac{k_R^{1-\alpha}}{R_1(1-\alpha)A}}{\log \frac{R_2}{R_1}\left(\frac{k_R}{k_L}\right)^{-\alpha}}.$$

For the later use, note that

$$\gamma_L'(\sigma) < 0, \quad \gamma_R'(\sigma) > 0, \quad \text{and}$$

$$\lim_{\sigma\to 0} \gamma_L(\sigma) = \lim_{\sigma\to 0} \gamma_R(\sigma) = \frac{\log(k_C^{1-\alpha}/R_1(1-\alpha)A)}{\log(R_2/R_1)} \equiv \gamma.$$

Using the variable change given by (19), the model defined in (3) is transformed, in the long run, into

$$x_{t+1} = \begin{cases} 1 + \alpha(x_t - \gamma_L) & \text{if } x_t < \gamma_L, \\ \psi(x_t) & \text{if } \gamma_L \le x_t \le \gamma_R, \\ \alpha(x_t - \gamma_R) & \text{if } \gamma_R < x_t, \end{cases} \tag{20}$$

where the shape of $\psi(x_t)$ depends on the distribution function, G, of the stochastic variable, ε. If (17) would fail to be satisfied, then the value of G would be less than 1 on some interval of $[k_R, \overline{k}]$, so that the third equation of (20) would be distorted.

As our model, compared to Matsuyama's (2007) original model, is intended to be as tractable and have, at the same time, as rich dynamic properties as possible, we set the dynamic equation given by (20) to be continuous and *piecewise linear.*[20] This requires that ψ be of the following linear form:

$$\psi(x) = \frac{\gamma_R - x}{\gamma_R - \gamma_L},$$

which in turn implies that the cumulative distribution function, $G(y)$, for $y \in [-1, 1]$, must satisfy the following equation:

$$[R_1 G(\rho(k)) + R_2(1 - G(\rho(k)))]\, w(k) = h^{-1}(\psi(h(k))). \tag{21}$$

Letting $y = \rho(k)$ and solving (21) for $G(y)$, we obtain

$$G(y) = \begin{cases} 0 & \text{if } y < -1, \\ \frac{1}{R_2 - R_1}\left[R_2 - h^{-1}(\psi(h(\rho_L^{-1}(y))))/w(\rho_L^{-1}(y))\right] & \text{if } -1 \le y < 1, \\ 1 & \text{if } 1 \le y. \end{cases}$$

The graph of $G(y)$ is plotted in Fig.3.

[20] It is straightforward to make the model given by (20) piecewise smooth, but its analysis may be much harder.

+++ insert Fig.3 of a graph of G +++

In summary, we obtain the following proposition.

Proposition 2. *Suppose that the variance of the disturbance term, σ, satisfies (14) and (18), and that (17) is satisfied. Then, the cumulative distribution function, $G(y)$, for $y \in [-1, 1]$, is as follows.*

$$G(y) = \begin{cases} 0 & \textit{if } y < -1, \\ \frac{1}{R_2 - R_1}\left[R_2 - h^{-1}(\psi(h(\rho_L^{-1}(y))))/w(\rho_L^{-1}(y))\right] & \textit{if } -1 \leq y < 1, \\ 1 & \textit{if } 1 \leq y. \end{cases}$$

Notice that the cumulative distribution function, G, varies with parameters, especially when including σ, to keep the model piecewise linear. Using this distribution function, the long-run model we analyze in this paper turns out to be the map: $\varphi : I = [0, 1] \to I$ defined by

$$x_{t+1} = \varphi(x_t) = \begin{cases} 1 + \alpha(x_t - \gamma_L) \equiv \varphi_L(x_t) & \text{if } 0 \leq x_t < \gamma_L, \\ (\gamma_R - x_t)/(\gamma_R - \gamma_L) \equiv \varphi_M(x_t) & \text{if } \gamma_L \leq x_t < \gamma_R, \\ \alpha(x_t - \gamma_R) \equiv \varphi_R(x_t) & \text{if } \gamma_R \leq x_t \leq 1. \end{cases} \tag{22}$$

Since γ_L and γ_R in (22) depend on the variance parameter, σ, so does φ. To stress this dependence, we sometimes write φ_σ. As uncertainty vanishes, that is, as σ tends to 0, map φ_σ becomes in the limit as follows:

$$x_{t+1} = \varphi_0(x_t) = \begin{cases} 1 + \alpha(x_t - \gamma) & \text{if } 0 \leq x_t < \gamma, \\ \alpha(x_t - \gamma) & \text{if } \gamma \leq x_t \leq 1, \end{cases} \tag{23}$$

which is essentially the same model as studied by Asano et al. (2012).

5 Chaotic Dynamics

In the following, we show that the model given by (22) is capable of generating chaotic behaviors in some sense. First, note that the study of Asano et al. (2012) shows that for any integer $q > 1$, there is a $\gamma \in (0, 1)$ such that the limiting map, φ_0, given by (23) exhibits a periodic attractor of period q. On the other hand, by a variation of the Li-Yorke Theorem (Li and Yorke (1975)), it is known that if a continuous map on the interval has a periodic point whose period is not 2 to the power of n for any integer $n > 0$, then it is chaotic in the sense of Li-Yorke (Li and Yorke (1975)). To avoid confusion, we call this type of chaos *topological chaos* hereafter. Therefore, it is not surprising that the continuous map, φ_σ, given by (22), is topologically chaotic when φ_0, which has a discontinuity, has a non-2^n periodic point and when σ is positive but small enough.

However, it is well recognized that the existence of topological chaos does not assure the observability of complexity in the long run. Therefore, if we want to reproduce recurrent but not periodic fluctuations observed in business cycles using a model in the deterministic framework, we can establish the observability of chaos. As such, we show below that our model exhibits *observable chaos* or *ergodic chaos* under specific parametric conditions and that chaos of this type is not rare but rather abundant in some sense.

5.1 Markov Property

We first present some mathematical definitions related to the Markov property. For more details, see e.g. Boyarsky and Góra (1997, Chapters 6 and 9).

Let $I = [0, 1]$ and let $\tau : I \to I$ be a transformation of I onto itself with τ^n denoting the n-fold composition of τ with itself. Let $\mathcal{P}$ be a finite partition of I given by the points $0 = a_0 < a_1 < \cdots < a_n = 1$. For $i = 1, 2, \cdots, n$, let $I_i = (a_{i-1}, a_i)$ and denote the restriction of τ to I_i by τ_i. If τ_i is a homeomorphism from I_i onto some connected union of intervals of $\mathcal{P}$, then τ is said to be *Markov*. The partition $\mathcal{P} = \{I_i\}_{i=1}^n$ is referred to as a *Markov partition with respect to* τ. If each τ_i is linear on I_i, then we say that τ is a *piecewise linear Markov map*. We also say that a piecewise differentiable map (not necessarily Markov) τ is (piecewise) *expanding* if $\inf |\tau'(x)| > 1$ on each I_i wherever the derivative exists. If there is an integer $n \geq 1$ such that $\inf |(\tau^n)'(x)| > 1$ on each I_i wherever the derivative exists, then τ is said to be *eventually expanding*.

Since the piecewise linear map φ given by (22) has an N-shaped graph and, hence, two kinks excluding the endpoints of interval $[0, 1]$, a simple consideration reveals that the number of endpoints of the Markov partition needs to be strictly larger than 4. Therefore, we first show that $\varphi : I \to I$ is Markov with a partition given by a period-5 cycle $\{0, c, \gamma_L, \gamma_R, 1\}$ such that[21]

$$0 = \varphi^5(0) < \varphi^3(0) = c < \varphi(0) = \gamma_L < \varphi^4(0) = \gamma_R < \varphi^2(0) = 1. \tag{24}$$

For later use, let $I_1 = (0, c)$, $I_2 = (c, \gamma_L)$, $I_3 = (\gamma_L, \gamma_R)$, and $I_4 = (\gamma_R, 1)$ be the elements of the Markov partition of φ, which are numbered from left to right.

[21] There is another possible period-5 Markov cycle such that $0 < \gamma_L < \gamma_R < c < 1$. However, since exhausting possible cycles is out of our scope, we do not consider such a case.

To calculate the period-5 cycle given by (4), γ_L and γ_R must solve the following equations:

$$\varphi_L(0) = \gamma_L \quad \text{and} \quad \varphi_L(\varphi_R(1)) = \gamma_R,$$

yielding

$$\gamma_L^* = \gamma_{5,L}^* = \frac{1}{1+\alpha} \quad \text{and} \quad \gamma_R^* = \gamma_{5,R}^* = \frac{1+\alpha^2+\alpha^3}{1+\alpha+\alpha^2+\alpha^3}. \tag{25}$$

Note also that $c^* = \varphi_R(1) = \alpha^2/(1+\alpha+\alpha^2+\alpha^3)$ and $\gamma_{5,R}^* - \gamma_{5,L}^* = \alpha^3/(1+\alpha)(1+\alpha^2)$.

Direct but tedious computations show that $\gamma_L = \gamma_L^*$ and $\gamma_R = \gamma_R^*$ are attained by choosing σ and A suitably. In fact, using (12) and (13) to solve

$$\gamma_L(\sigma) = \gamma_L^* \quad \text{and} \quad \gamma_R(\sigma) = \gamma_R^*$$

for A and σ, we obtain special values, A^* and σ^*, for which (25) is realized. See the Appendix for the actual representations of A^* and σ^*.

For σ^* given by (30) to satisfy constraints (14), (15), and (18), it suffices to require that

$$\sqrt{\frac{(m_2\lambda_2R_2 - m_1\lambda_1R_1)(\lambda_2R_2 - \lambda_1R_1)}{m_2\lambda_2R_2}} < \min\left\{\frac{m_1}{m_2}\lambda_1R_1 + \left(1 - \frac{m_1}{m_2}\right)R_1 - \lambda_2R_2, \lambda_2R_2 - \lambda_1R_1, R_1 - \lambda_2R_2\right\}. \tag{26}$$

Note that the above inequality is independent of A and α. We can verify that the set of parameter values satisfying (1) and (26) contains a non-trivial open set in the parameter space. In fact, by considering, for instance,

$$m_1 = 0.2, \quad m_2 = 1.0, \quad \lambda_1 = 0.1, \quad \lambda_2 = 0.2, \quad R_1 = 4, \quad \text{and} \quad R_2 = 10,$$

we can check that the inequality given by (26) is satisfied.

Furthermore, it can be verified that (26) implies $A^* < \hat{A}$ for given σ^* (see the Appendix for a verification of the inequality). This fact implies that only if (26) together with (1) are satisfied, then the piecewise-linearization given by (20) is justified.

Now we show that model (22) exhibits chaotic dynamics with observability. To characterize the chaotic behavior here, we employ some theory from dynamical systems theory related to ergodic theory. For mathematical notions which are not or only roughly explained here, see Boyarsky and Góra (1997) for more details.

We now present some notions related to observable chaos used here. Let $I = [0,1]$ and let $\mathcal{B}$ be the Borel σ-algebra of $[0,1]$.[22] Given a measurable function $\tau : I = [0,1] \to I$, a measure μ is said to be *invariant* under τ (or τ preserves μ) if $\mu(\tau^{-1}(E)) = \mu(E)$ for all measurable sets $E \in \mathcal{B}$.[23] We say that a measure μ is *absolutely continuous* with respect to a measure ν if $\nu(E) = 0$ implies $\mu(E) = 0$. The existence of an absolutely continuous invariant measure is important from an economic point of view, since it assures, unlike topological chaos, the observability of recurrent but not periodic fluctuations in the long run and describes the asymptotic distribution of economic states over the course of a business cycle. By the existence of an absolutely continuous invariant measure, we define *observable chaos*. An absolutely continuous invariant measure corresponds to the notion of (non-periodic) *attractor* in topological dynamical systems theory. We say that a measurable function $\tau : I \to I$ preserving the measure μ is *ergodic* if $\tau^{-1}(E) = E$ implies $\mu(E) = 0$ or $\mu(I \setminus E) = 0$. This implies that an invariant set is a zero-measure set such as a periodic orbit or is of full measure, that is, the measure can no longer be decomposed.[24]

Proposition 3. *(observable chaos on a period-5 Markov partition) Let $\gamma_L^* = \gamma_{5,L}^*$ and $\gamma_R^* = \gamma_{5,R}^*$ as in (25). Let $\sigma = \sigma^* = \sigma(\gamma_L^*, \gamma_R^*)$ as in (30) and let $A = A^* = A(\sigma^*)$ as in (31). Then, $\varphi : I \to I$ defined by (22) exhibits observable chaos in the following (stronger) sense: it admits a unique (hence ergodic) invariant probability measure μ which is absolutely continuous with respect to the Lebesgue measure.*

Proof. By the Folklore Theorem (see e.g. Boyarsky and Góra (1997, Theorem 6.1.1.)), we need to check aperiodicity and eventual expandingness of φ. For aperi-

[22] Let X be a set and let 2^X denote the power set of X. A non-empty class of subsets of 2^X is a *σ-algebra* if (a) $X \in \mathcal{M}$, (b) $A \in \mathcal{M}$ implies $A^c \in \mathcal{M}$, and (c) $\langle A_i \rangle_{i=1}^{\infty} \subset \mathcal{M}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$ implies $\cup_{i=1}^{\infty} A_i \in \mathcal{M}$, where A^c and $\emptyset$ denote the complement of A and the empty set, respectively. If X is any metric space, or more generally any topological space, then the σ-algebra generated by the family of all open sets in X is called the *Borel σ-algebra* on X. A function $\mu : \mathcal{M} \to [0, \infty]$ is a *measure* if (a) $\mu(\emptyset) = 0$ and (b) $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for any sequence $\langle A_i \rangle_{i=1}^{\infty}$ of disjoint sets in $\mathcal{M}$. In addition, if a measure μ satisfies (c) $\mu(X) = 1$, then μ is called a *probability measure*.

[23] If X is a set and $\mathcal{M} \subset 2^X$ is a σ-algebra, then $(X, \mathcal{M})$ is called a *measurable space* and the sets in $\mathcal{M}$ are called *measurable sets*. If μ is a measure on $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a *measure space*. If μ is a probability measure on $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a *probability space* or a *normalized measure space*. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. A function $\tau : X \to Y$ is called *$(\mathcal{M}, \mathcal{N})$-measurable* or just *measurable* if $f^{-1}(E) = \{x \in X | f(x) \in E\} \in \mathcal{M}$ for all $E \in \mathcal{N}$. Let $(X, \mathcal{M}, \mu)$ be a normalized measure space and let $\tau : X \to X$ preserve μ. Then, $(X, \mathcal{M}, \mu, \tau)$ is called a *dynamical system*.

[24] Let $(X, \mathcal{M}, \mu, \tau)$ be a dynamical system. A set $E \in \mathcal{M}$ is called *τ-invariant* or just *invariant* if $\tau^{-1}(E) = E$.

odicity, we need to check that for each I_i there is n_i such that $\varphi^{n_i}(\bar{I}_i) = \bar{I}$, where $\bar{I}_i$ denotes the closure of I_i. It is easy to see by construction of φ that

$$\varphi^2(\bar{I}_1) = \varphi^4(\bar{I}_2) = \varphi(\bar{I}_3) = \varphi^3(\bar{I}_4) = \bar{I}.$$

For eventual expandingness, notice that every point $\varphi^n(x) \in I$, which is not on an endpoint of I_i, will visit I_3 at least once for every fourth iterate. Therefore, for $x \in I$ and for $j = L$ or R,

$$\left|(\varphi^4)'(x)\right| \geq |\varphi_j'|^3 |\varphi_M'| = \frac{\alpha^3}{\gamma_R^* - \gamma_L^*} = 1 + \alpha + \alpha^2 + \alpha^3 > 1,$$

whenever the derivatives exist. Therefore, φ^4 is piecewise expanding or φ is eventually expanding. $\square$

By the same argument, we can construct another Markov partition on a period-7 cycle such that

$$0 = \varphi^7(0) < \varphi^4(0) < \varphi(0) < \varphi^5(0) < \varphi^2(0) = \gamma_L < \varphi^6(0) = \gamma_R < \varphi^3(0) = 1.$$

Thus, solving

$$\varphi^2(0) = \varphi_L^2(0) = \gamma_L \quad \text{and} \quad \varphi^3(1) = \varphi_L^2(\varphi_R(1)) = \gamma_R$$

for γ_L and γ_R, we obtain

$$\begin{aligned} \gamma_{7,L}^* &= \frac{1+\alpha}{1+\alpha+\alpha^2} \quad \text{and} \\ \gamma_{7,R}^* &= \frac{1+\alpha+\alpha^3-\alpha(1+\alpha)\gamma_{7,L}^*}{1+\alpha^3} = \frac{1+\alpha+\alpha^3+\alpha^4+\alpha^5}{1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5}. \end{aligned} \tag{27}$$

Note also that $\gamma_{7,R}^* - \gamma_{7,L}^* = \alpha^5 / \sum_{i=0}^5 \alpha^i > 0$. As such, we have the following proposition.

Proposition 4. *(observable chaos on a period-7 Markov partition) The same assertion as in Proposition 3 holds if γ_L and γ_R are replaced by $\gamma_{7,L}^*$ and $\gamma_{7,R}^*$, respectively, as defined by (27).*

Fig.4 and Fig.5 depict the situations where the map φ is Markov with a period-5 cycle and a period-7 cycle, as described in Propositions 1 and 2.

+++ insert Fig.4 and Fig.5 of period-5 and period-7 Markov ++++

We can readily extend Propositions 3 and 4 to a more general case of a period-$(2n+3)$ Markov partition for $n \geq 1$. By solving

$$\varphi^n(0) = \varphi_L^n(0) = \gamma_L \quad \text{and} \quad \varphi^{n+1}(1) = \varphi_L^n(\varphi_R(1)) = \gamma_R,$$

we can derive

$$\begin{aligned} \gamma^*_{2n+3,L} &= \frac{1-\alpha^n}{1-\alpha^{n+1}} \quad \text{and} \\ \gamma^*_{2n+3,R} &= \left[\sum_{i=0}^{n+1} \alpha^i - \alpha^n - \alpha\gamma^*_{2n+3,L} \sum_{i=0}^{n-1} \alpha^i\right] \Big/ \left[1+\alpha^{n+1}\right] \\ &= \left[\sum_{i=0}^{2n+1} \alpha^i - \alpha^n\right] \Big/ \left[\sum_{i=0}^{2n+1} \alpha^i\right] = 1 - \frac{\alpha^n(1-\alpha)}{1-\alpha^{2(n+1)}}, \end{aligned} \tag{28}$$

with

$$\gamma^*_{2n+3,R} - \gamma^*_{2n+3,L} = \frac{(1-\alpha)\alpha^{2n+1}}{1-\alpha^{2(n+1)}}.$$

Analogously to Propositions 3 and 4, we can summarize our results in this subsection as follows.

Proposition 5. *(observable chaos on a period-$(2n+3)$ Markov partition) The same assertion as in Proposition 3 holds, if for $n \geq 1$, γ_L and γ_R are replaced by $\gamma^*_{2n+3,L}$ and $\gamma^*_{2n+3,R}$, respectively, as defined by (28).*

5.2 Matrix Representation of Chaotic Behavior

When the piecewise linear map given by (22) is Markov, the dynamics can be analyzed in more detail.

Let m be the normalized Lebesgue measure on $I = [0,1]$. Let L_1 be a space of all integrable functions defined on the interval $I = [0,1]$. Let $\tau : I \to I$ be a nonsingular map, where τ is said to be *nonsingular* if $m(\tau^{-1}(E)) = 0$ whenever $m(E) = 0$ for a measurable set E. The *Frobenius-Perron operator* $P_\tau : L^1 \to L^1$ is defined as a unique (up to almost everywhere equivalence) function such that for $f \in L_1$,

$$\int_E P_\tau f dm = \int_{\tau^{-1}(E)} f dm$$

for any measurable $E \subset I$. The existence and the uniqueness of $P_\tau f$ follow from the Radon-Nikodym Theorem.[25] The Frobenius-Perron operator $P_\tau f$ is shown to be

[25] Let τ be nonsingular and define $\mu(E) = \int_{\tau^{-1}(E)} f dm$, where $f \in L^1$ and E is an arbitrary

a linear operator. That is, for any α, $\beta \in \mathbb{R}$ and any f, $g \in L^1$, $P_\tau(\alpha f + \beta g) = P_\tau f + P_\tau g$ almost everywhere. Notice that f is invariant if and only if $P_\tau f = f$ almost everywhere.

Let $\mathcal{P}= \{I_i\}_{i=1}^n$ be a fixed partition of I and let S denote the class of all functions that are piecewise constant on partition $\mathcal{P}$. That is, $f \in S$ if and only if

$$f = \sum_{i=1}^{n} \pi_i \chi_{I_i} \equiv \pi^f = (\pi_1, \ldots, \pi_n)^T,$$

where χ is the indicator function, $\pi_1, \cdots, \pi_n$ are some constants, T denotes transpose, and f is identified with a column vector π^f.

By the theorems of Boyarsky and Scarowsky (1979) and Boyarsky and Góra (1997, Theorem 9.2.1.), if τ is piecewise linear Markov on partition $\mathcal{P}= \{I_i\}_{i=1}^n$, then there is an $n \times n$ matrix M_τ such that $P_\tau f = M_\tau^T \pi^f$ for every $f \in S$ and π^f is the column vector obtained from f. Here, the matrix $M_\tau = (m_{ij})_{1 \le i,j \le n}$ is given by

$$m_{ij} = \frac{b_{ij}}{|\tau_i'|}$$

with

$$b_{ij} = \begin{cases} 1 & \text{if} \quad I_j \subset \tau(I_i), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, by Boyarsky and Góra (1997, Theorem 9.4.1.), if a piecewise linear Markov map τ is eventually expanding, then τ is known to admit an invariant density function that is piecewise constant on the Markov partition $\mathcal{P}$. The τ-invariant density f can be obtained as a fixed point of $P_\tau f = f$. Using the matrix representation, the density $f = \pi^f$ is obtained by solving

$$M_\tau^T \pi^f = \pi^f,$$

which corresponds to the eigenvector associated with the eigenvalue of modulus 1 of matrix M_τ.

Now, let us apply the above results to our model. We first examine the simplest case of the period-5 Markov partition described in Proposition 3. We observe that on the partition $\mathcal{P}= \{I_i\}_{i=1}^4$, the following holds:

$$I_3 \subset \varphi(I_1),\ I_4 \subset \varphi(I_2),\ \cup_{i=1}^4 I_i \subset \varphi(I_3), \text{ and } I_1 \subset \varphi(I_4).$$

measurable set. The nonsingularity of τ means that $m(\tau^{-1}(E)) = 0$ whenever $m(E) = 0$ for a measurable set E, which implies $\mu(E) = 0$. Thus, m is absolute continuous with respect to μ. Therefore, by the Radon-Nikodym Theorem, there exists a unique function $\phi \in L^1$ such that, for any measurable set E, $\mu(E) = \int_E \phi dm$. By setting $P_\tau f = \phi$, the existence and the uniqueness of $P_\tau f$ follow. See Boyarsky and Góra (1997, pp.74-78).

For simplicity of numbering partitioning intervals, we set $\tilde{\mathcal{P}}= \{J_i\}_{i=1}^4$ by the following permutation: $\mathcal{J} = \Pi\mathcal{I}$ or

$$\begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}, \tag{29}$$

where $\mathcal{J} = (J_1, J_2, J_3, J_4)^T$, $\mathcal{I} = (I_1, I_2, I_3, I_4)^T$, and Π is represented as above.

Under the re-numbered partition $\tilde{\mathcal{P}}= \{J_i\}_{i=1}^4$ defined by (29), we see that

$$J_2 \subset \varphi(J_1),\ J_3 \subset \varphi(J_2),\ J_4 \subset \varphi(J_3), \text{ and } \cup_{i=1}^4 J_i \subset \varphi(J_4).$$

Noting that $|\varphi_i'| \equiv |\varphi'_{|J_i}| = \alpha \in (0,1)$ for $i \neq 4$ and that $|\varphi_4'| = (\gamma^*_{5,R} - \gamma^*_{5,L})^{-1} = (1+\alpha+\alpha^2+\alpha^3)/\alpha^3 \equiv \beta^{-1} > 1$, we obtain

$$M_\varphi = \begin{pmatrix} 0 & \alpha^{-1} & 0 & 0 \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & \alpha^{-1} \\ \beta & \beta & \beta & \beta \end{pmatrix}.$$

Solving $M_\varphi^T \pi = \pi$ for π, where $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)^T$ are the constants of the piecewise constant function on $\tilde{\mathcal{P}}= \{J_i\}_{i=1}^4$, we can define the unique (up to constant multiples) φ-invariant density:

$$\pi = (\alpha^3, \alpha^2+\alpha^3, \alpha+\alpha^2+\alpha^3, 1+\alpha+\alpha^2+\alpha^3)^T.$$

We summarize the result in the following proposition.

Proposition 6. *Let μ be the invariant measure given in Proposition 3. Then, its probability density ψ, i.e., the function such that*

$$\mu(E) = \int_E \psi dm,$$

for any measurable set $E \subset I$ is represented by

$$\psi(x) = \Delta^{-1} \sum_{i=1}^4 \pi_i \chi_{J_i}(x)$$

with $\pi_i = \sum_{j=4-i}^3 \alpha^j$ and $\Delta = \sum_{i=1}^4 \pi_i |J_i| = (\sum_{i=1}^4 \alpha^{4-i} \sum_{j=i}^4 \alpha^{j-1}) / \sum_{i=1}^4 \alpha^{i-1}$.

See the Appendix for a computation of Δ.

Note that we can easily obtain the invariant density on the original partition $\mathcal{P}$, denoted $\hat{\pi}$, by $\hat{\pi} = \Pi^{-1}\pi = \Pi^T\pi$. In this case, we have

$$\hat{\pi} = (\alpha + \alpha^2 + \alpha^3, \alpha^3, 1 + \alpha + \alpha^2 + \alpha^3, \alpha^2 + \alpha^3)^T.$$

It is straightforward to extend the above proposition to the case of higher-period.

Proposition 7. *Let μ be the invariant measure given in Proposition 5. Moreover, $\mathcal{P} = \{I_i\}_{i=1}^{2(n+1)}$ is the corresponding Markov partition. Then, its probability density ψ is represented by*

$$\psi(x) = \Delta^{-1} \sum_{i=1}^{2(n+1)} \pi_i \chi_{J_i}(x)$$

with $\pi_i = \sum_{j=2(n+1)-i}^{2n+1} \alpha^j$ and

$$\Delta = \sum_{i=1}^{2(n+1)} \pi_i |J_i| = \left(\sum_{i=1}^{2(n+1)} \alpha^{2(n+1)-i} \sum_{j=i}^{2(n+1)} \alpha^{j-1} \right) \Big/ \sum_{i=1}^{2(n+1)} \alpha^{i-1},$$

Here, $\mathcal{J} = (J_1, J_2, \ldots, J_{2(n+1)})^T$ is a permutation of $\mathcal{I} = (I_1, I_2, \ldots, I_{2(n+1)})^T$ via $\mathcal{J} = \Pi\mathcal{I}$, where Π is a permutation matrix given by

$$\Pi = (p_{i,j}) = \begin{cases} 1 & \textit{if} \quad 1 \le i \le n+1 \quad \textit{and} \quad j = 2i, \\ 1 & \textit{if} \quad n+1 < i \le 2(n+1) \quad \textit{and} \quad j = 2(i-n-1)-1, \\ 0 & \textit{otherwise.} \end{cases}$$

Fig.6 draws the histogram generated by computer-simulated trajectories of the period-5 piecewise linear Markov map described in Proposition 6. It can be seen that as the number of iterations increases, the histogram approaches the theoretically obtained invariant density. Fig.7 gives the counterpart for the period-7 Markov case.

+++ insert Fig.6 of the invariant density of period-5 Markov and its computer-simulated version +++

+++ insert Fig.7 of the invariant density of period-7 Markov and its computer-simulated version +++

It is worth noticing that the establishment of the Markov property for a given economic system not only serves as a tool of detecting observable chaos but also enables us to study the nature of business (or credit) cycles in more depth. For instance, the invariant density tells us how often the economy visits a certain range of the state over the course of a business cycle or what probability the economy moves with from a state to another. However, it can be argued that the parameters for which the Markov property is established would be so pathological that it would not be worth investigating. One refutation to this is presented in the next section, where observable chaos detected through the Markov property is not necessarily isolated in the parameter space, but rather abundant at least in our framework.

5.3 Abundance and Sudden Disappearance of Observable Chaos

Hitherto, we have investigated the dynamics of the model, in which the corresponding map exhibits the Markov property. However, it seems to restrict the set of parameter values too much. We thus want to check some kind of *robustness* or fragility of the dynamic features obtained above by characterizing the dynamics *near* the parameter values for which the map is Markov.

We first consider the period-5 Markov case and its perturbations with respect to σ. We show that the qualitative dynamic patterns change drastically when σ passes through the Markov value, σ^*.

Proposition 8. *(bifurcation from an attracting period-5 cycle to observable chaos) Let $A = A^*$, where $A^* = A(\sigma^*)$ and σ^* are given as in Proposition 3. Then, for $\sigma \in (0, \sigma^*)$, the map φ_σ given by (22) exhibits an attracting period-5 cycle coexisting with topological chaos. Moreover, there exists $\varepsilon > 0$ such that for each $\sigma \in [\sigma^*, \sigma^* + \varepsilon)$, the corresponding map φ_σ exhibits observable chaos.*

Proof. Recall that a decrease (increase) in the value of parameter σ shrinks (widens, respectively) the interval $[\gamma_L(\sigma), \gamma_R(\sigma)]$. Now, fix $A = A^*$ and $\sigma = \sigma^*$ so that the map is Markov with the period-5 Markov partition as in Proposition 3. By a graphical argument, we can easily see that a decrease in σ makes attracting the period-5 cycle, by which the period-5 Markov partition is otherwise defined. By Li-Yorke (1975), topological chaos immediately follows from the fact that φ_σ is continuous for $\sigma > 0$ and that the existence of a periodic point whose period is not 2^n.

On the other hand, a slight increase in σ will destroy the Markov property, but keep the map eventually expanding. In fact, for σ slightly smaller than σ^*, $\varphi^2(1) \in (\gamma_L(\sigma), \gamma_R(\sigma))$ and $\varphi^2(\gamma_R(\sigma)) = \varphi(0) \in (\gamma_L(\sigma), \varphi^2(1)) \subset (\gamma_L(\sigma), \gamma_R(\sigma))$.

Noticing also that $\varphi((\varphi^2(1), \gamma_L(\sigma))) \subset (\gamma_L(\sigma), 1)$, it follows by continuity that for any $x \in I$,

$$|(\varphi^4)'(x)| \geq |\varphi'_{L\,\mathrm{or}\,R}|^3 |\varphi'_M| = \alpha^3/(\gamma_R(\sigma) - \gamma_L(\sigma)) > 1,$$

wherever the derivative exists. This shows that, for σ arbitrarily close to σ^* with $\sigma \geq \sigma^*$, the piecewise linear map φ_σ is eventually expanding. By the theorems of Lasota and Yorke (1973), for such a map φ_σ, there exists an absolutely continuous invariant measure, which means that the map exhibits observable chaos. $\square$

The above argument can be extended to a little bit more general case to obtain the following result.

Proposition 9. *(bifurcation from an attracting period-*$(2n+3)$ *cycle to observable chaos) Let* $A = A^*$*, where* $A^* = A(\sigma^*)$ *and* σ^* *are given as in Proposition 5. Then, for* $\sigma \in (0, \sigma^*)$*, the map* φ_σ *given by (22) exhibits an attracting period-*$(2n+3)$ *cycle coexisting with topological chaos. Moreover, there exists* $\varepsilon > 0$ *such that, for each* $\sigma \in [\sigma^*, \sigma^* + \varepsilon)$*, the corresponding map* φ_σ *exhibits observable chaos.*

To intuitively understand the proposition above, it is useful to use a computer to draw bifurcation diagrams with respect to σ (see Fig.8 and Fig.9). In Fig.8, parameter A is fixed at A^* for the period-5 Markov property. First, we look at $\sigma = \sigma^*$, at which φ_σ exhibits the period-5 Markov property. By ergodicity, the trajectory with transient iterates being omitted covers the whole interval $[0, 1]$ (i.e., the vertical line at $\sigma = \sigma^*$). For each σ near σ^* with $\sigma^* < \sigma$, a similar situation seems to occur, which means that observable chaos persists for all nearby σ's larger than σ^*. In this sense, observable chaos is *abundant* in our model. On the other hand, a slight decrease in σ from σ^* annihilates observable chaos and gives rise to a periodic attractor of period 5 instead. Fig.9 is a bifurcation diagram associated with the period-7 Markov case, in which the transition from a period-7 attracting cycle to chaos is observed as σ increases.

+++ insert Fig.8 and Fig.9 of bifurcation w.r.t. σ with A^* fixed +++

To figure out how observable chaos suddenly disappears as soon as σ falls below σ^*, see Figures 11−13. In each of these figures, an enlargement of the graph of the fifth iterate of φ_σ, φ_σ^5, (see Fig.10) is depicted for a slightly different value of σ, with $A = A^*$ being fixed. Moreover, see Fig.12, which draws a graph of $\varphi_{\sigma^*}^5$. Since, at $\sigma = \sigma^*$, the period-5 Markov cycle appears on the set of kinks by construction and is a subset of fixed points of φ_σ^5, such a kink is on the 45 degree line as plotted in Fig.12, which is an enlargement of Fig.10. For $\sigma \in (\sigma^*, \sigma^* + \epsilon)$, where $\epsilon > 0$ is a sufficiently small number, the Markov property no longer holds and, as a result, such a kink deviates from the 45 degree line, but φ_σ is still eventually expanding and hence observably chaotic (see Fig.13). For $\sigma \in (\sigma^* - \epsilon, \sigma^*)$, however, the kink itself deviates from the 45 degree line in the opposite direction, so that the deviation gives birth to two new (transverse) intersections of the graph of φ_σ^5 with the 45 degree line, which are fixed points of φ_σ^5 (see Fig.11). At this time of the birth of intersections, a new less steep line segment is created as well, on which one of the two newly born intersections is located. This fixed point is the attracting periodic point described in Proposition 9, and it attracts nearby trajectories, which would behave chaotically and densely otherwise. Therefore, observable chaos disappears suddenly as σ drops below σ^*.

+++ insert Figures 10−13 of sudden disappearance of chaos +++

6 Concluding Remarks

Based on the model analyzed by Asano et al. (2012), which is essentially equivalent to the original credit cycle model proposed by Matsuyama (2007) as long as permanent fluctuations are concerned, we developed an OLG model of endogenous business cycles. By specifying the distribution of "noise" representing imperfect observability, we obtained a continuous piecewise linear model, for which we showed that, using the Markov property, observable chaos is detected and described by its invariant measures.

We now present some concluding remarks. First, compared to piecewise smooth models investigated in the literature, for example, by Gardini et al. (2008) and Matsuyama et al. (2016), piecewise linear models are much easier to analyze, and the complicated dynamics can be characterized in a sharper and clearer way. On the other hand, piecewise smooth modeling seems more straightforward than piecewise linear modeling because the latter tends to require more restrictive assumptions in general, which suggests that there is a trade-off between them.

Furthermore, we relaxed the parametric restriction for the Markov property by considering perturbations with respect to parameter σ, representing the level of noise or the intensity of choice. Our results showed that observable chaos found at the Markov parameter values persists against such perturbations at least in one direction of the σ value, which implies that the chaos detected in our model is observable not only for parameter values, but also for initial states. The existence of an absolutely continuous invariant measure assures that, for a large set of initial conditions, chaotic behavior appears as a long-run outcome. In our model, unlike in Asano et al. (2012), such a dynamic property is robust against perturbations of parameter values, in that for a large set of parameter values observable chaos appears.

However, it must also be recognized that (eventually) expanding piecewise linear models are likely to generate overabundance of observable chaos, compared to smooth or piecewise smooth models. In fact, observable chaos in our model appears for any parameter values of an interval. Thus, such overabundance itself is an artifact of piecewise linear modeling, which we however believe will not take away from the value of our model.

Finally, the parameter values for which the model is Markov were demonstrated to represent bifurcation values, through which qualitative behaviors change drastically. This suggests that the Markov property is useful not only in detecting chaotic behaviors in the given model, but also in identifying the set of parameter values for which “structural changes” (i.e., bifurcations) occur.

Appendix

Computation of σ^* and A^*

Given γ_L^* and γ_R^* ($\gamma_L^* < \gamma_R^*$), we can solve the following the simultaneous equations

$$\begin{aligned} \gamma_L^* &= \gamma_L(\sigma), \\ \gamma_R^* &= \gamma_R(\sigma) \end{aligned}$$

for σ and A to obtain

$$\sigma^* = \sigma(\gamma_L^*, \gamma_R^*) = \frac{-B + \sqrt{B^2 + D}}{2m_2\lambda_2^2 R_2^2} \leq \frac{\sqrt{D}}{2m_2\lambda_2^2 R_2^2}, \tag{30}$$

where

$$\begin{aligned} C &= (R_2/R_1)^{\frac{\alpha(\gamma_R^* - \gamma_L^*)}{1+\alpha(\gamma_R^* - \gamma_L^*)}} > 1, \\ B &= (m_2 - m_1)\lambda_1\lambda_2 R_1 R_2(1 + C)/(C - 1) > 0, \quad \text{and} \\ D &= 4m_2\lambda_2 R_2(m_2\lambda_2 R_2 - m_1\lambda_1 R_1)(\lambda_2 R_2 - \lambda_1 R_1) > 0. \end{aligned}$$

Given σ^* defined by (30), the solution for A will be represented by

$$A^* = A(\sigma^*) = U/V, \tag{31}$$

where

$$\begin{aligned} U &= \frac{m_1 m_2(((1+\sigma^*)\lambda_2 R_2 - \lambda_1 R_1))}{(1+\sigma^*)m_2\lambda_2 R_2 - m_1\lambda_1 R_1} > 0 \quad \text{and} \\ V &= (1-\alpha)(m_1 m_2 R_2)^\alpha \left[((1-\sigma^*)\lambda_2 R_2 - \lambda_1 R_1)/((1-\sigma^*)m_2\lambda_2 R_2 - m_1\lambda_1 R_1)\right]^\alpha. \end{aligned}$$

Notice that in order for A^* to be well-defined and positive if

$$\sigma^* < \frac{\lambda_2 R_2 - \lambda_1 R_1}{\lambda_2 R_2},$$

which is the constraint given by (14).

Verification of $A^* < \hat{A}$

We show that $A^* < \hat{A}$ when $\sigma = \sigma^*$ satisfying (26) is given. By (17) and (31), we have

$$\frac{A^*}{\hat{A}} = \frac{(1+\sigma^*)\lambda_2 R_2 - \lambda_1 R_1}{[(1+\sigma^*)\frac{m_2}{m_1}\lambda_2 R_2 - \lambda_1 R_1][1 - (1+\sigma^*)\lambda_2 R_2/R_1]}. \tag{32}$$

Suppose $A^* \geq \hat{A}$, then by (32), a simple computation shows

$$\sigma^* \geq \frac{m_1\lambda_1 R_1 + (m_2 - m_1)R_1 - m_2\lambda_2 R_2}{m_2\lambda_2 R_2},$$

which contradicts to (26). Thus, we obtain $A^* < \hat{A}$.

Computation of Δ

Direct computations reveal:

$$\begin{aligned}
|J_1| &= |I_2| = \frac{1}{(1+\alpha)(1+\alpha^2)}, \quad \pi_1 = \alpha^3, \\
|J_2| &= |I_4| = \frac{\alpha}{(1+\alpha)(1+\alpha^2)}, \quad \pi_2 = \alpha^2 + \alpha^3, \\
|J_3| &= |I_1| = \frac{\alpha^2}{(1+\alpha)(1+\alpha^2)}, \quad \pi_3 = \alpha + \alpha^2 + \alpha^3, \\
|J_4| &= |I_3| = \frac{\alpha^3}{(1+\alpha)(1+\alpha^2)}, \quad \pi_4 = 1 + \alpha + \alpha^2 + \alpha^3.
\end{aligned}$$

Thus, we obtain

$$\Delta = \sum_{i=1}^{4} \pi_i |J_i| = \frac{\sum_{i=1}^{4} \alpha^{4-i} \sum_{j=i}^{4} \alpha^{j-1}}{(1+\alpha)(1+\alpha^2)},$$

where the subscript i of π_i is associated with the renumbered partition $\tilde{\mathcal{P}} = \{J_i\}_{i=1}^4$.

References

[1] Anderson, S.P., A. de Palma, and J.-F. Thisse (1992): *Discrete Theory of Product Differentiation*, The MIT Press.

[2] Asano, T., T. Kunieda, and A. Shibata (2012): "Complex Behaviour in a Piecewise Linear Dynamic Macroeconomic Model with Endogenous Discontinuity," *Journal of Difference Equations and Applications* 18, 1889-1898.

[3] Benhabib, J. and R.H. Day (1982): "A Characterization of Erratic Dynamics in the Overlapping Generations Model," *Journal of Economic Dynamics and Control* 4, 37-55.

[4] Boyarsky, A. and P. Góra (1997): *Laws of Chaos: Invariant Measures and Dynamical Systems in One Dimension*, Springer.

[5] Boyarsky, A. and M. Scarowsky (1979): "On a Class of Transformations which Have Unique Absolutely Continuous Invariant Measures," *Transactions of the American Mathematical Society* 225, 243-262.

[6] Brock, W.A. and C.H. Hommes (1997): "A Rational Route to Randomness," *Econometrica* 65, 1059-1095.

[7] Chan, K.S. and H. Tong (1986): "On Estimating Thresholds in Autoregressive Models," *Journal of Time Series Analysis* 7, 179-190.

[8] Clements, M.P. and H.-M. Krolzig (1998): "A Comparison of the Forecast Performance of Markov-switching and Threshold Autoregressive Models of US GNP," *Econometrics Journal* 1, C47-C75.

[9] Day, R.H. and G. Pianigiani (1991): "Statistical Dynamics and Economics," *Journal of Economic Behavior and Organization* 16, 37-83.

[10] Gardini, L., I. Sushko, and A.K. Naimzada (2008): "Growing through Chaotic Intervals," *Journal of Economic Theory* 143, 541-557.

[11] Góra, P. (2009): "Invariant Densities for Piecewise Linear Maps of the Unit Interval," *Ergodic Theory and Dynamical Systems* 29, 1549-1583.

[12] Grandmont, J. (1985): "On Endogenous Competitive Business Cycles," *Econometrica* 53, 995-1045.

[13] Hata, M. (1982): "Dynamics of Caianiello's Equation," *Journal of Mathematics of Kyoto University* 22, 155-173.

[14] Hata, M. (1998): *Chaos in Neural Network Models* (in Japanese), Asakura, Tokyo.

[15] Hata, M. (2014): *Neurons: A Mathematical Ignition*, World Scientific.

[16] Hommes, C. (1991): *Chaotic Dynamics in Economic Models: Some Simple Case Studies*, Walters-Nordhoff, Groningen.

[17] Huang, W. (2005): "On the Statistical Dynamics of Economics," *Journal of Economic Behavior and Organization* 56, 543-565.

[18] Ishida, J. and M. Yokoo (2004): "Threshold Nonlinearities and Asymmetric Endogenous Business Cycles," *Journal of Economic Behavior and Organization* 54, 175-189.

[19] Lasota, A. and J.A. Yorke (1973): "On the Existence of Invariant Measures for Piecewise Monotonic Transformations," *Transactions of the American Mathematical Society* 186, 481-488.

[20] Li, T.-Y. and J.A. Yorke (1975): "Period Three Implies Chaos," *American Mathematical Monthly* 82, 985-992.

[21] Matsumoto, A. (2001): "Can Inventory Chaos be Welfare Improving?," *International Journal of Production Economics* 71, 31-43.

[22] Matsumoto, A. (2005): "Density Function of Piecewise Linear Transformation," *Journal of Economic Behavior and Organization* 56, 631-653.

[23] Matsuyama, K. (1991): "Endogenous Price Fluctuations in an Optimizing Model of a Monetary Economy," *Econometrica* 59, 1617-1631.

[24] Matsuyama, K. (1999): "Growing through Cycles," *Econometrica* 67, 335-347.

[25] Matsuyama, K. (2007): "Credit Traps and Credit Cycles," *American Economic Review* 97, 503-516.

[26] Matsuyama, K. (2013): "The Good, the Bad, and the Ugly: An Inquiry into the Causes and Nature of Credit Cycles," *Theoretical Economics* 8, 623-651.

[27] Matsuyama, K., Sushko, I., and L. Gardini (2016): "Revisiting the Model of Credit Cycles with Good and Bad Projects," *Journal of Economic Theory* 163, 525-556.

[28] Nishimura, K., G. Sorger, and M. Yano (1994): "Ergodic Chaos in Optimal Growth Models with Low Discount Rates," *Economic Theory* 4, 705-717.

[29] Nishimura, K. and M. Yano (1995): "Non-linear Dynamics and Chaos in Optimal Growth: An Example," *Econometica* 63, 981-1001.

[30] Sandholm, W.H. (2010): *Population Games and Evolutionary Dynamics*, The MIT Press.

[31] Teräsvirta, T. and H.M. Anderson (1992): "Characterizing Nonlinearities in Business Cycles Using Smooth Transition Autoregressive Models," *Journal of Applied Econometrics* 7, S119-S136.

[32] Teräsvirta, T., D. van Dijk, and M.C. Medeiros (2005): "Linear Model, Smooth Transition Autoregressions, and Neural Networks for Forecasting Macroeconomic Time Series: A Re-examonation," *International Journal of Forecasting* 21, 755-774.

[33] Teräsvirta, T., D. Tjøstheim, and C.W.J. Granger (2010): *Modelling Nonlinear Economic Time Series*, Oxford University Press.

[34] Tong, H. (1983): *Threshold Models in Nonlinear Time Series Analysis*, Lecture Notes in Statistics No. 21, Springer.

[35] Tong, H. and K.S. Lim (1980): "Threshold Autoregression, Limit Cycles and Cyclical Data," *Journal of Royal Statistical Society* Series B, 42, 245-292.

[36] Umezuki, Y. and M. Yokoo (2017): "A Simple Model of Growth Cycles with Technology Choice," mimeo, Okayama University.

[37] van Dijl, D., P.H. Franses, and R. Paap (2002): "A Nonlinear Long Memory Model, with an Application to US Unemployment," *Journal of Econometrics* 110, 135-165.

[38] Yokoo, M. (2000): "Chaotic Dynamics in a Two-dimensional Overlapping Generations Model," *Journal of Economic Dynamics and Control* 24, 909-934.

[39] Yokoo, M. and J. Ishida (2008): "Misperception-Driven Chaos: Theory and Policy Implications," *Journal of Economic Dynamics and Control* 32, 1732-1753.

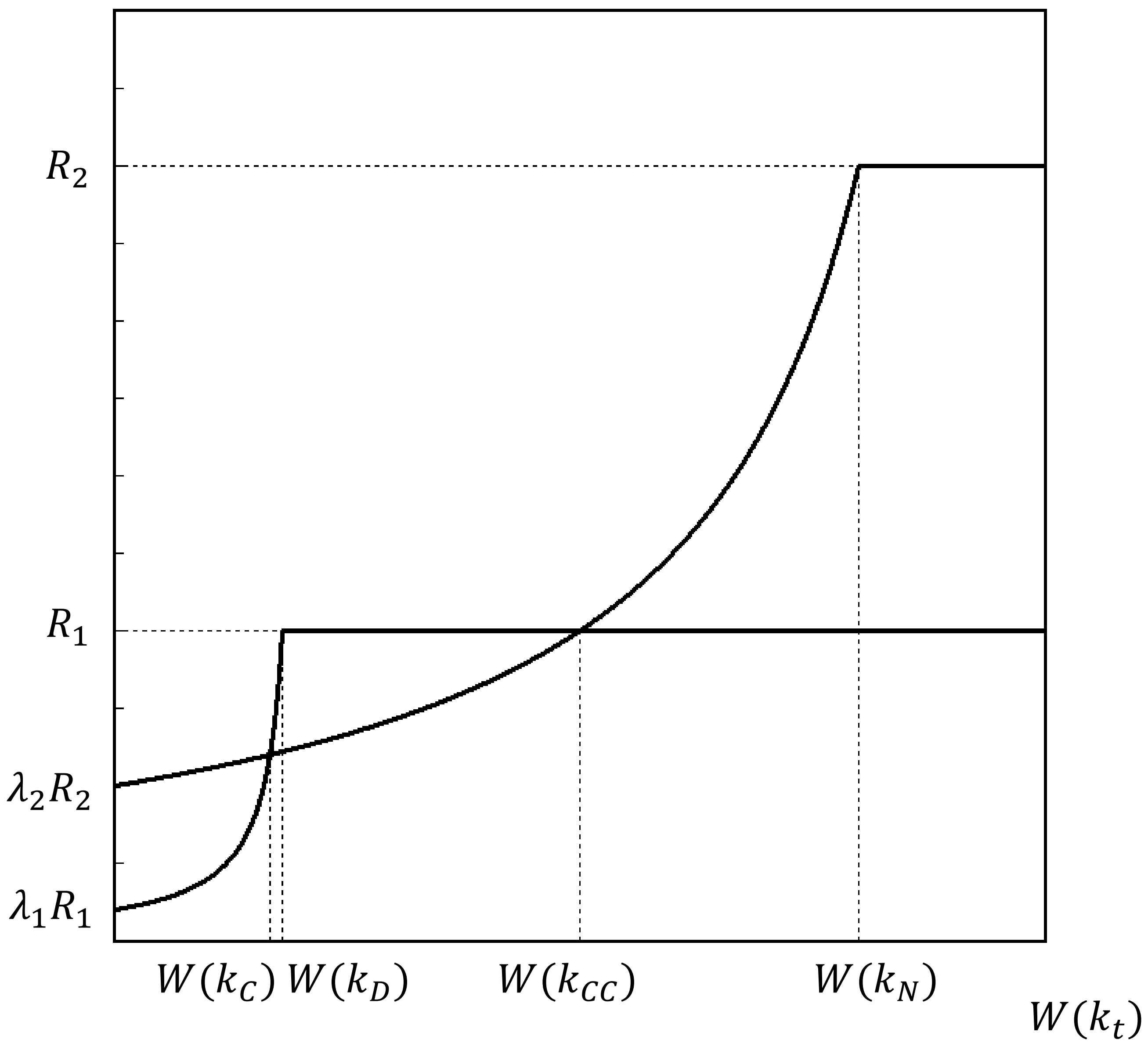

Figure 1: Choice of projects

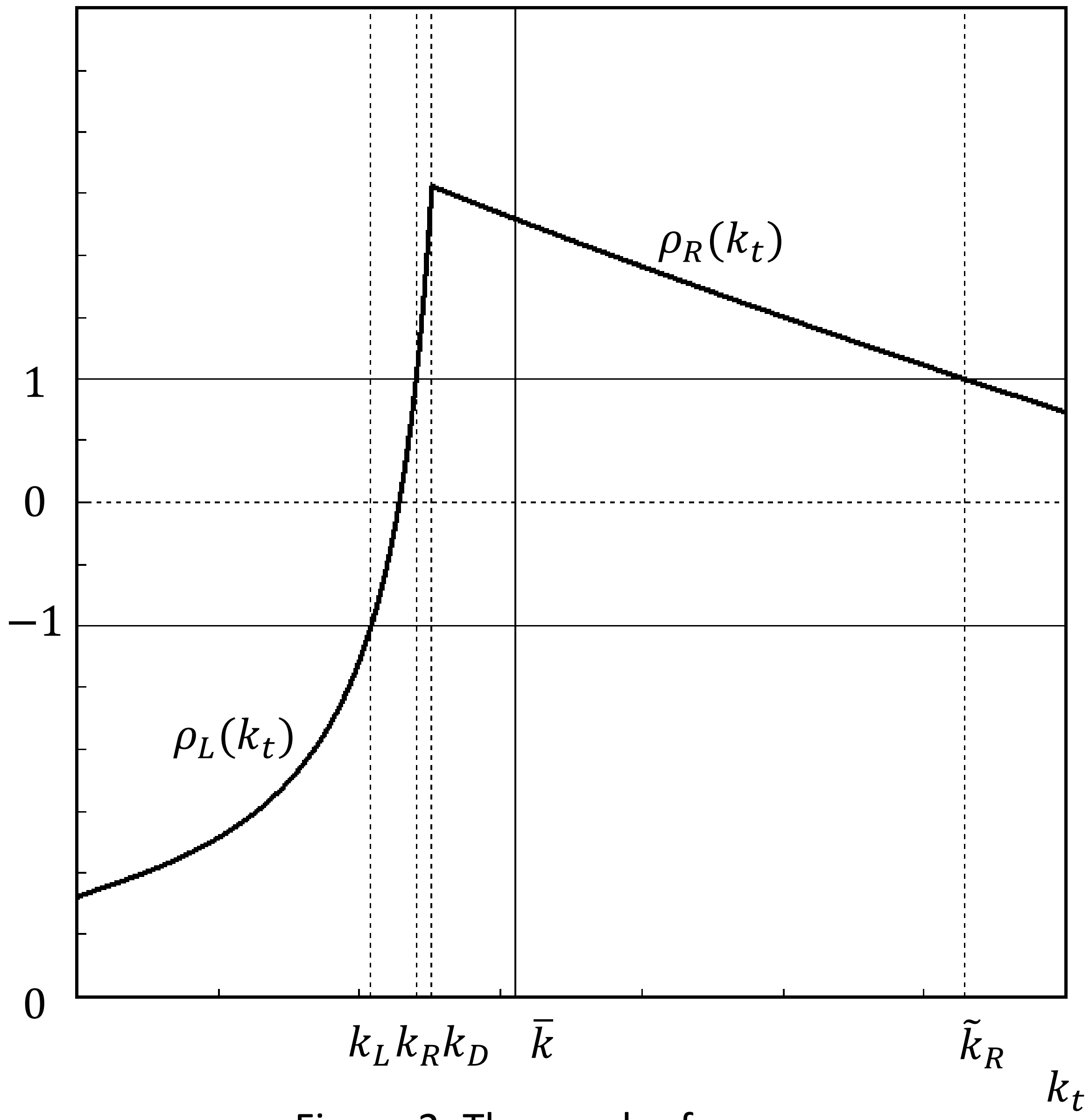

Figure 2: The graph of ρ

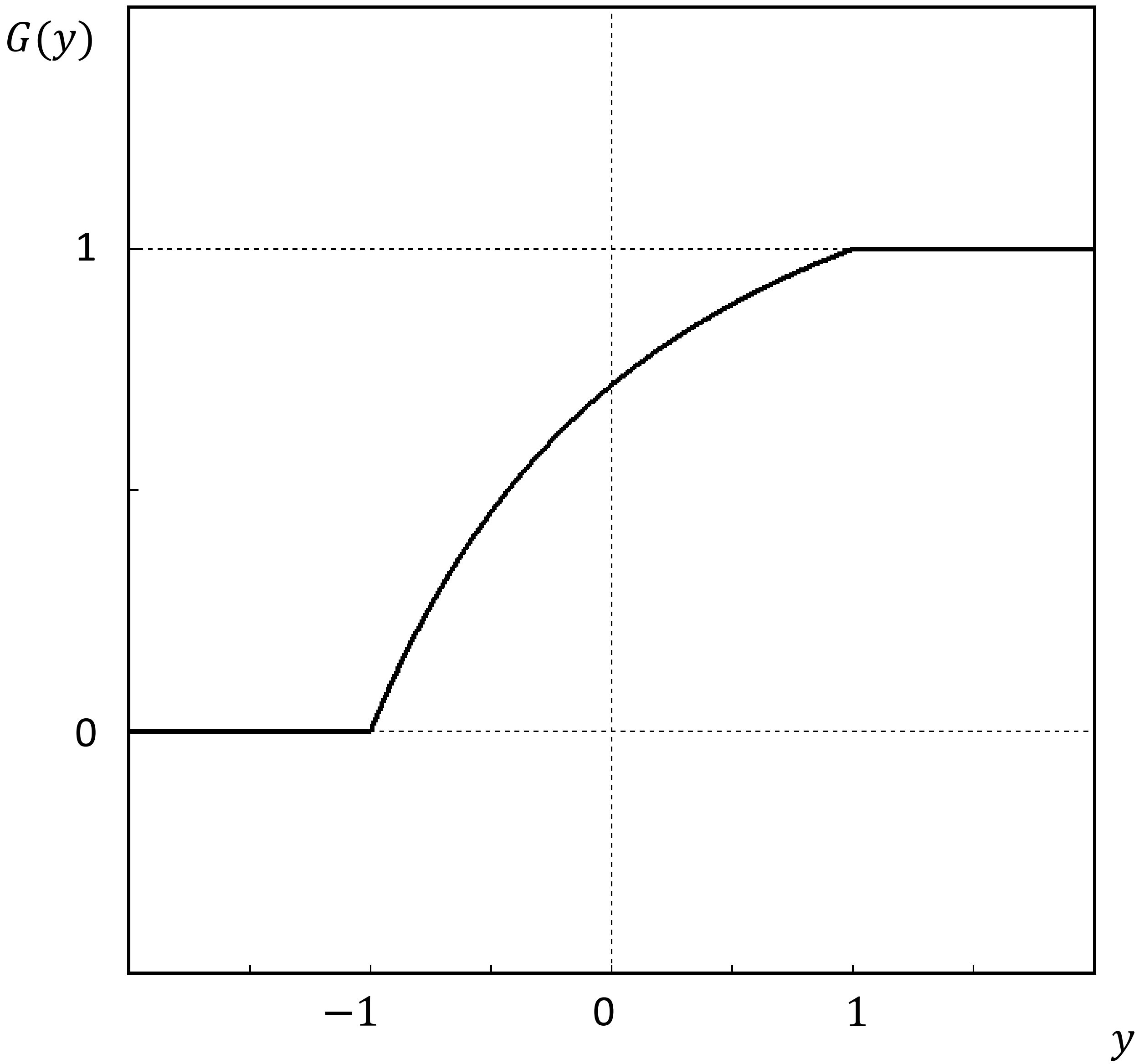

Figure 3: The cumulative distribution function G

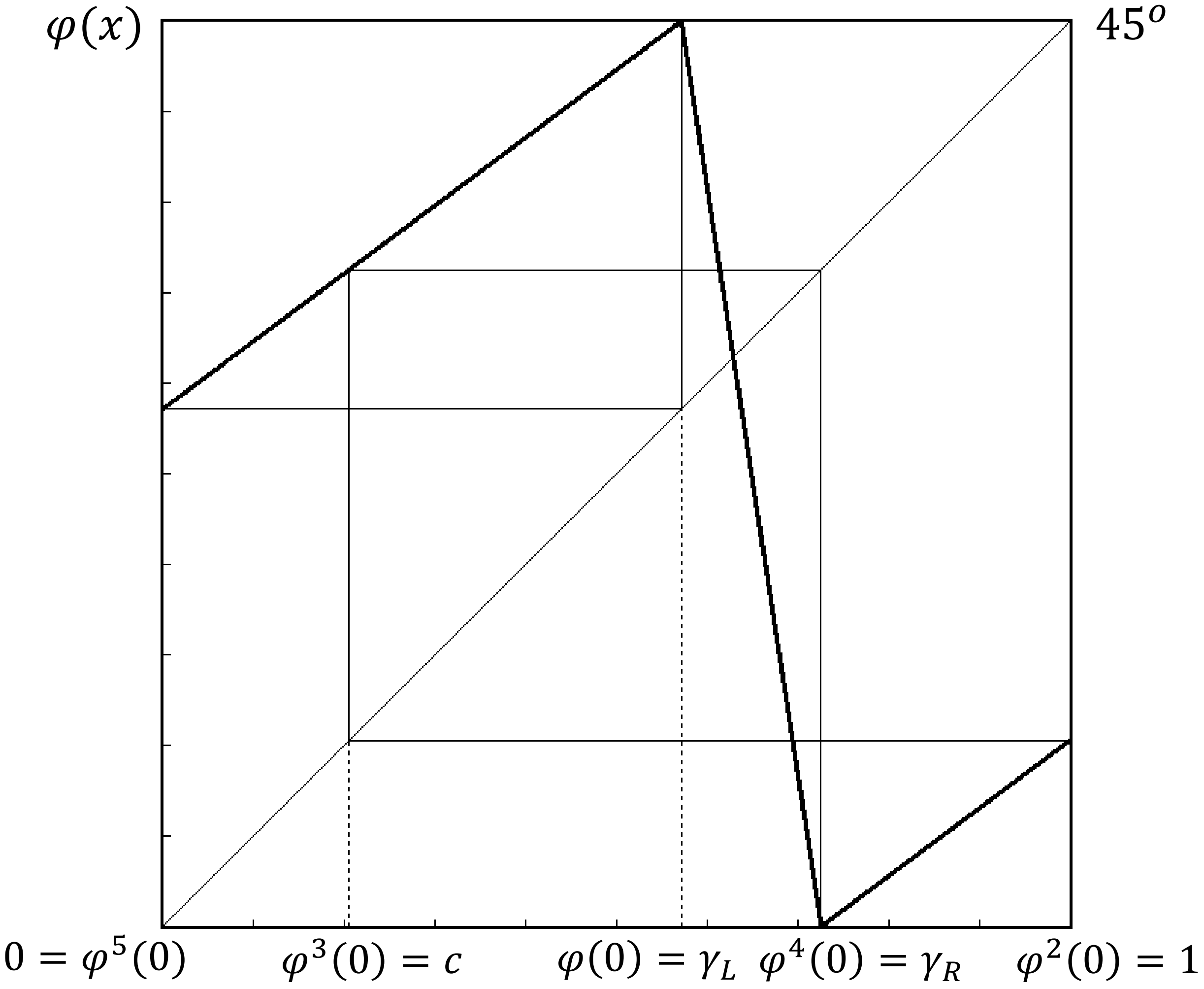

Figure 4: Period-5 Markov property

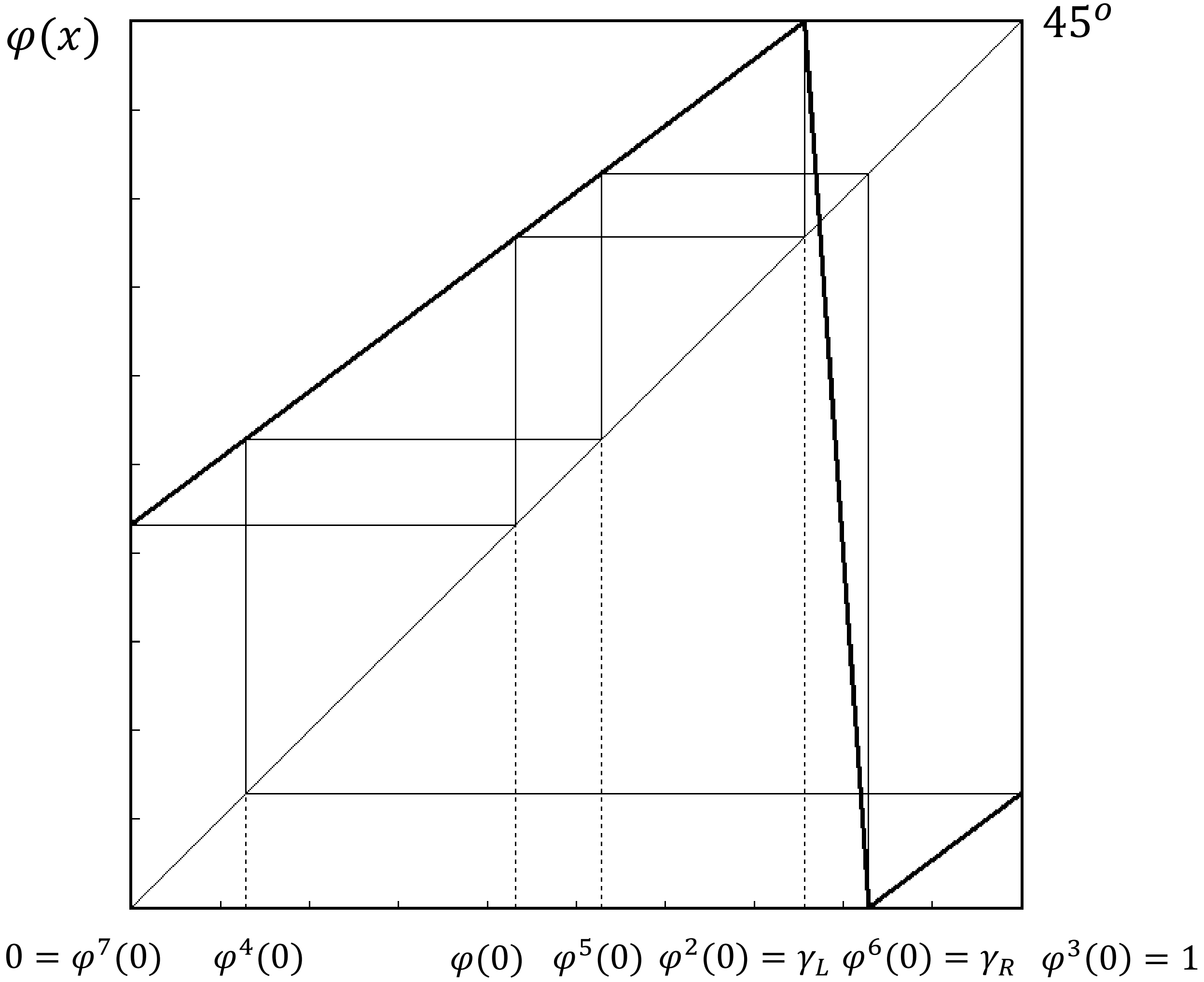

Figure 5: Period-7 Markov property

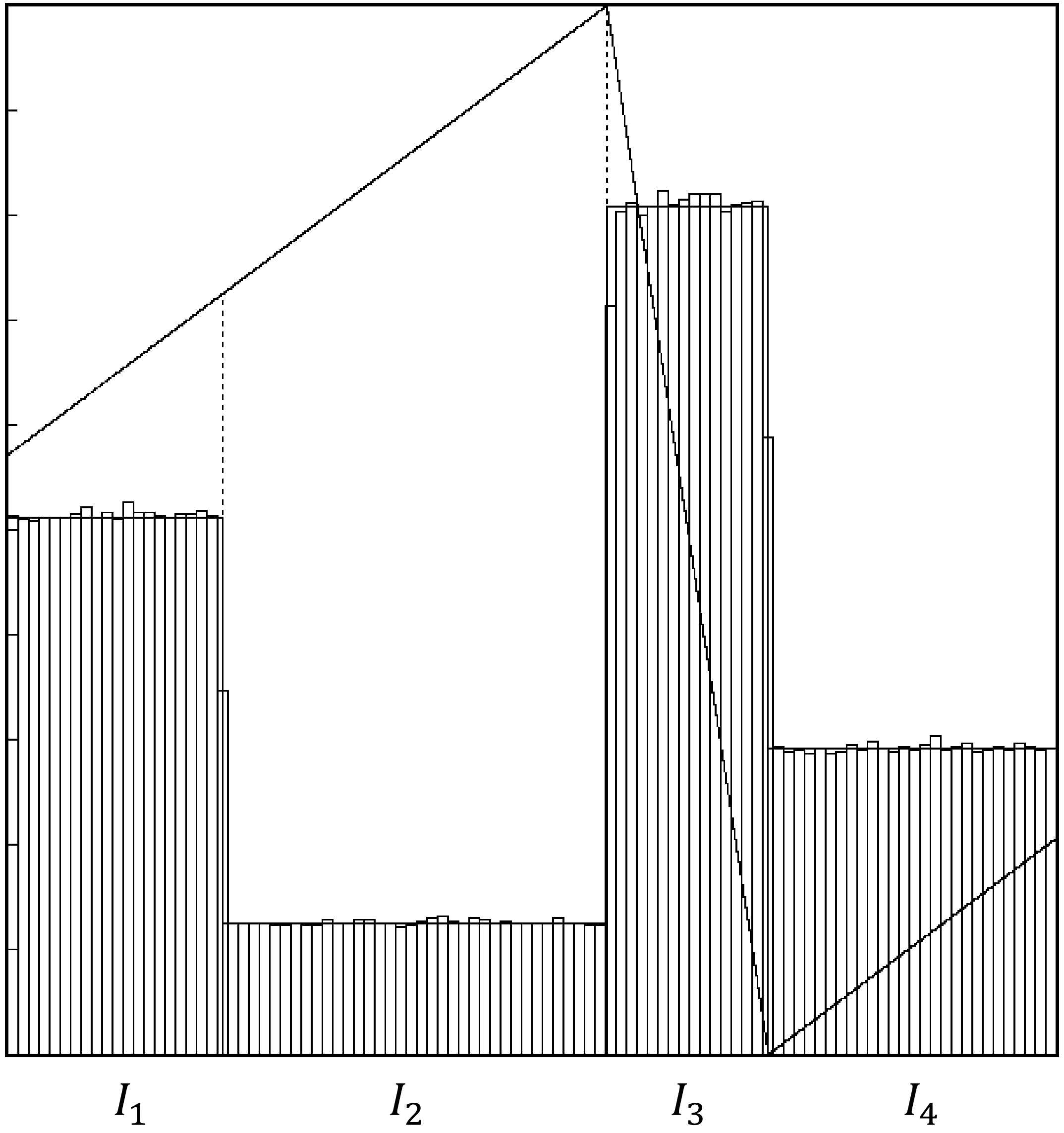

Figure 6: The invariant density for the period-5 Markov map and the simulated histogram of 10^6 iterations

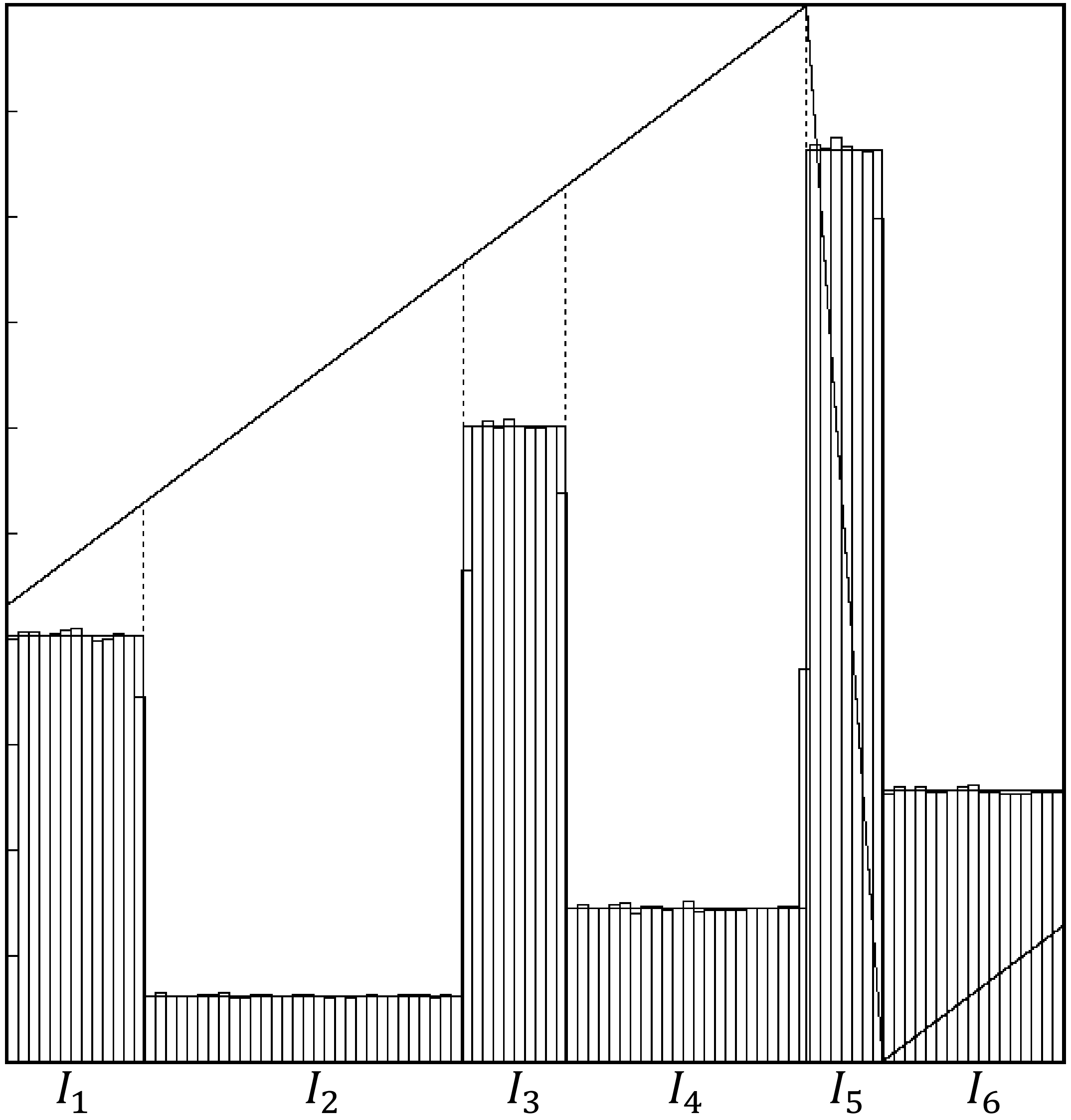

Figure 7: The invariant density for the period-7 Markov map and the simulated histogram of 10^6 iterations

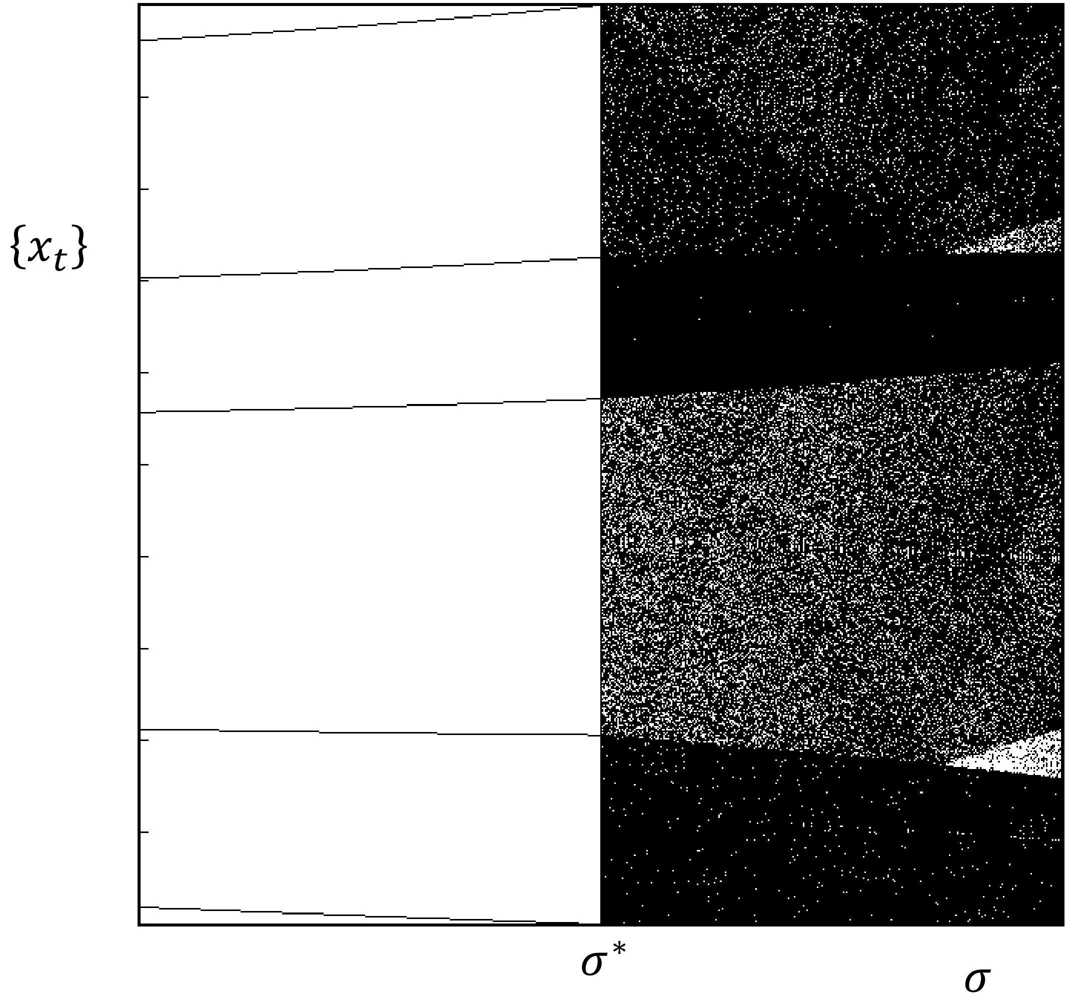

Figure 8: Bifurcation diagram with respect to σ around the period-5 Markov parameter

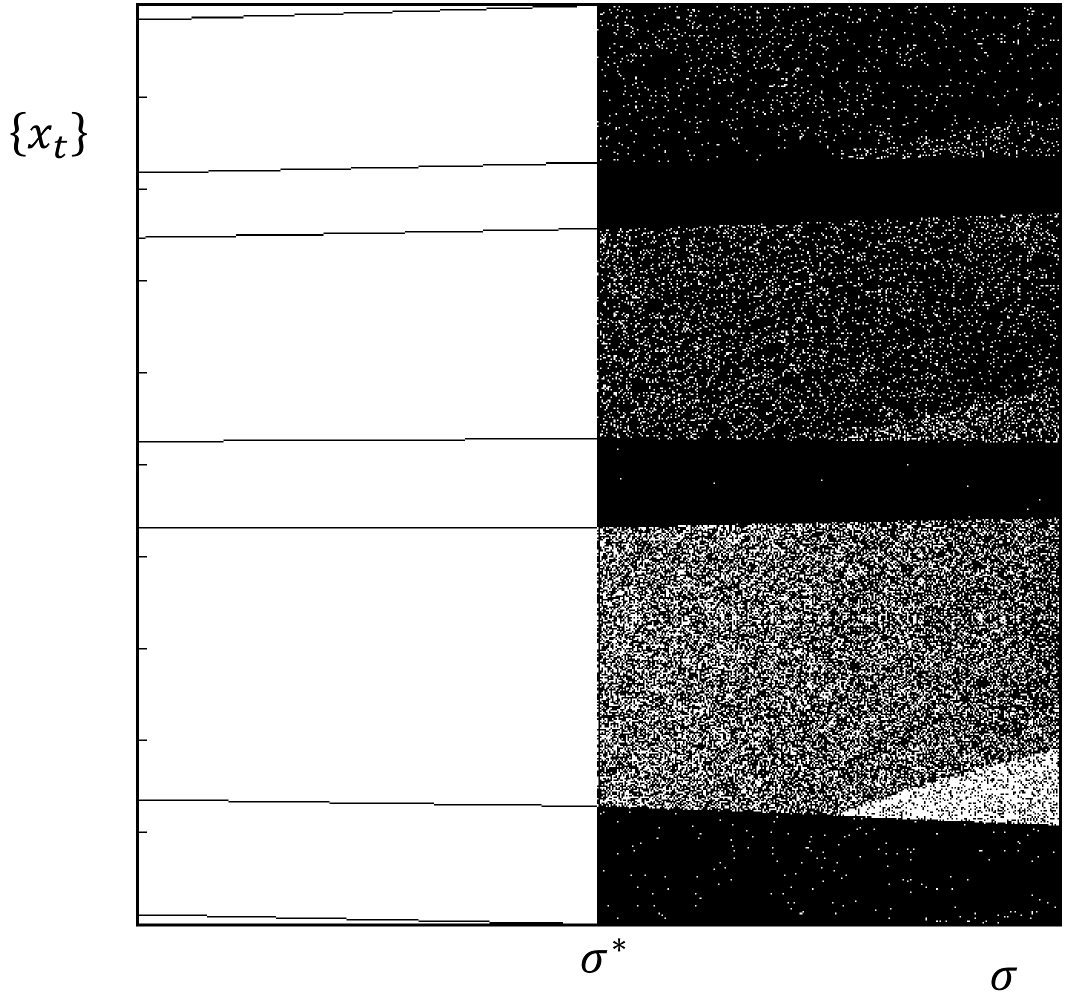

Figure 9: Bifurcation diagram with respect to σ around the period-7 Markov parameter

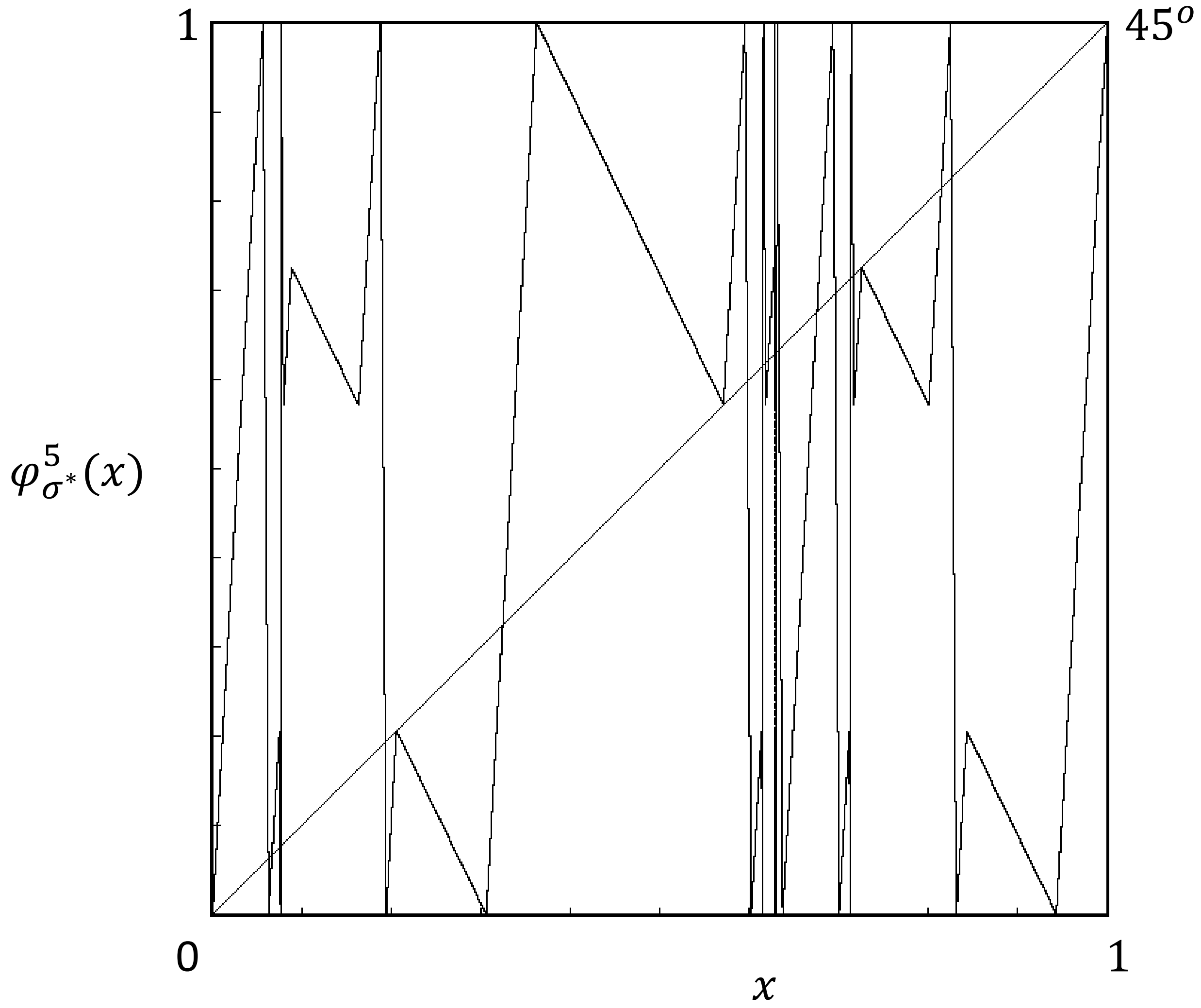

Figure 10: The graph of $\varphi^5_{\sigma^*}$

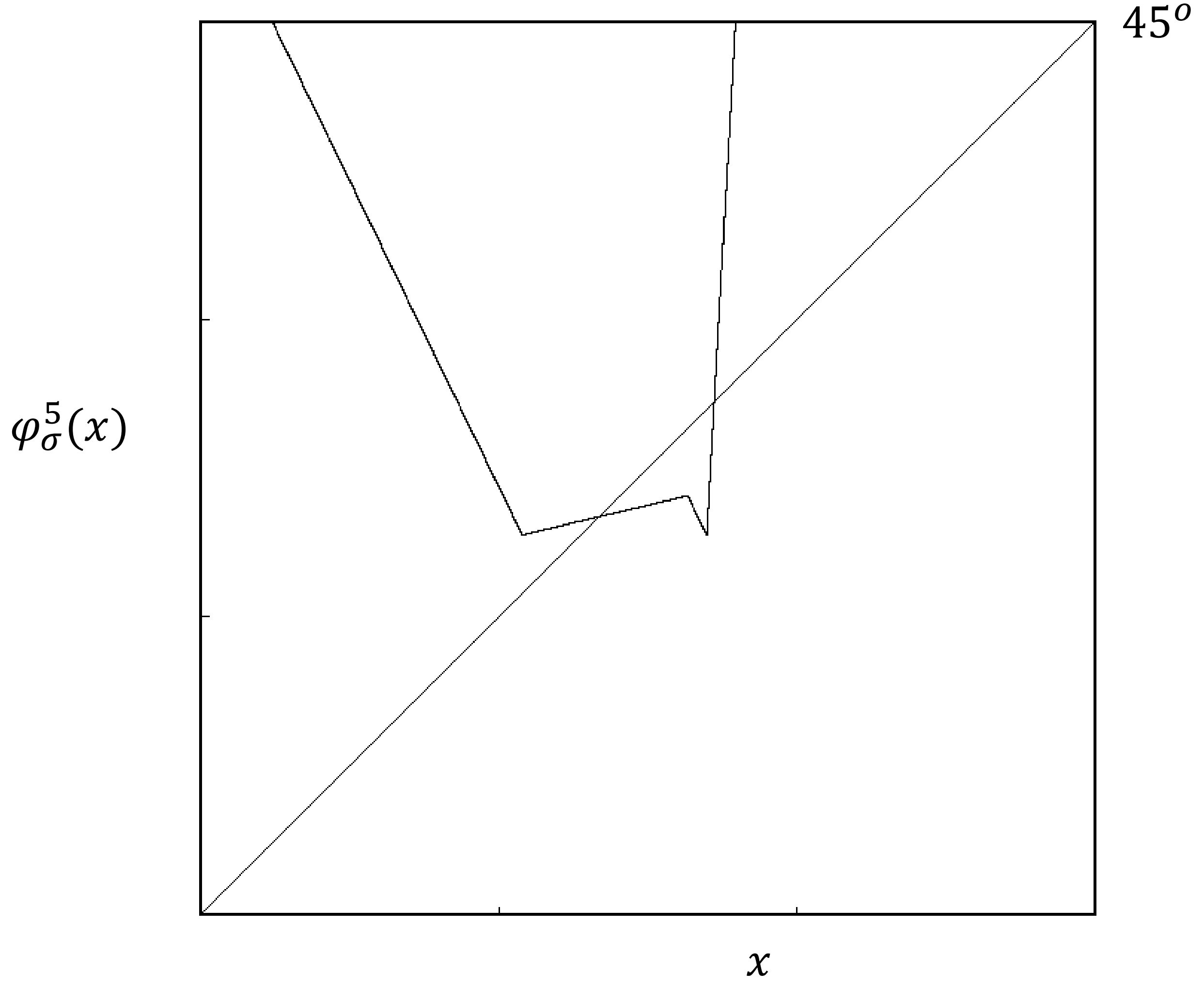

Figure 11: An enlargement: for $\sigma < \sigma^*$

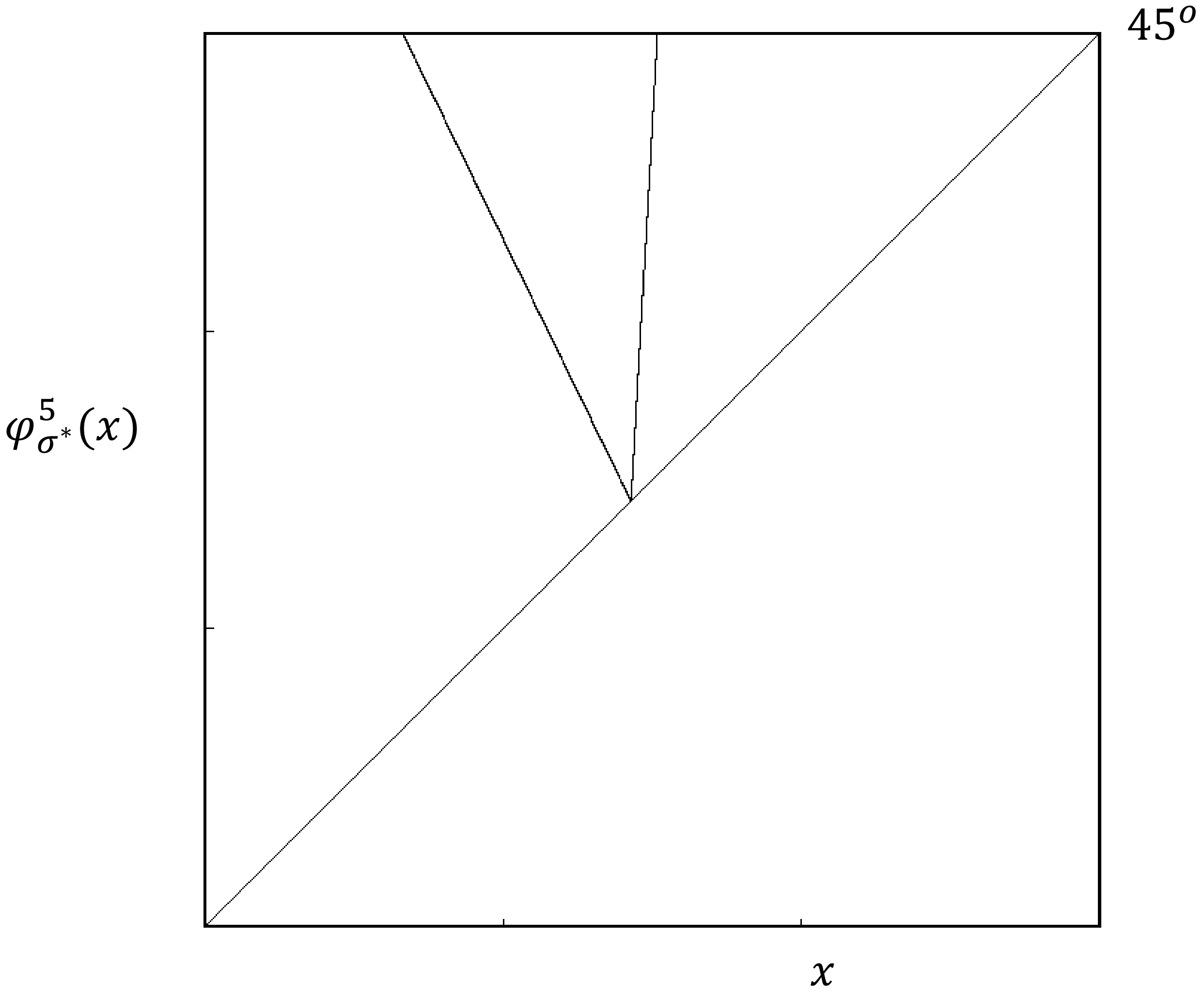

Figure 12: An enlargement: for $\sigma = \sigma^*$

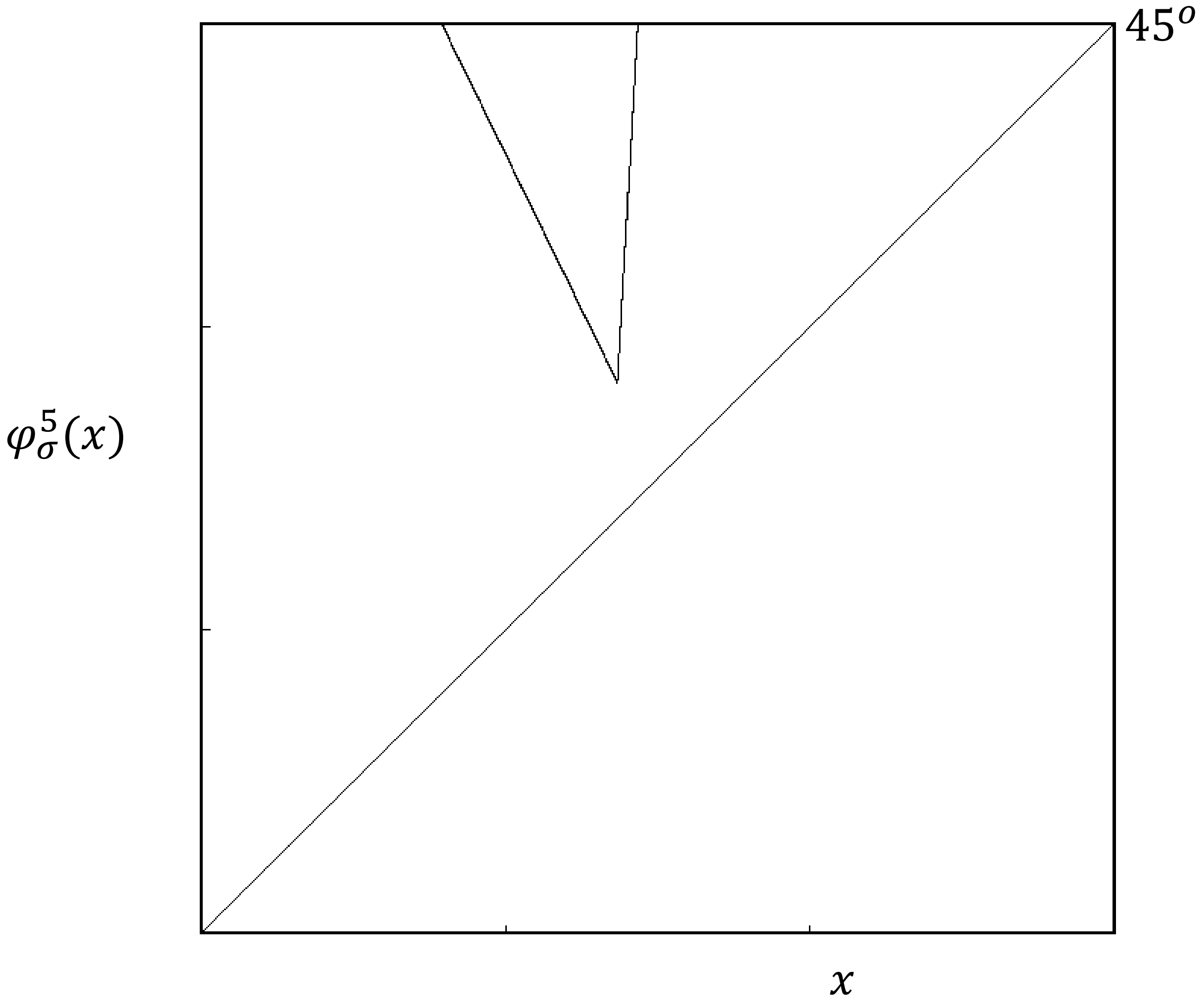

Figure 13: An enlargement: for $\sigma > \sigma^*$