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"A Dynamic Mechanism Design with Overbooking, Different Deadlines, and Multi-unit Demands"

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A Dynamic Mechanism Design with Overbooking, Different Deadlines, and Multi-unit Demands*

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Abstract

This paper considers a dynamic mechanism design in which multiple objects with different consumption deadlines are allocated over time. Agents arrive over time and may have multi-unit demand. We characterize necessary and sufficient condition for periodic ex-post incentive compatibility and provide the optimal mechanism that maximizes the seller's expected revenue under regularity conditions. When complete contingent-contracts are available, the optimal mechanism can be interpreted as an "overbooking" mechanism. The seller utilizes overbooking for screening and price-discriminating advance agents. When agents demand multiple objects as complements, the seller may face a tradeoff between the last-minute price of the current object and the future profit.

Keywords: dynamic mechanism design, optimal auction, overbooking, price discrimination, revenue management
JEL classification codes: D82, D44

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1 Introduction

In recent years, an extensive literature on mechanism design examines dynamic allocation problems. A stream of the literature concerns mechanism design with dynamic populations, in which agents with fixed private information stochastically arrive over time. A typical example is various kinds of ticket sales for airplanes, trains, hotels, concerts, etc. The seller allocates a fixed capacity of goods to buyers by a certain deadline, when an airplane takes off or a concert is held. Buyers arrive at the market (or a mechanism) at different points in time. Both efficient and revenue-maximizing mechanisms are studied assuming myopic impatient agents (Gershkov and Moldovanu, 2009; 2010), patient agents (Board and Skrzypacz, 2016), and a mixture thereof (Pai and Vohra, 2013; Mierendorff, 2016).[1]

Another example is repeated perishable-goods sales. The seller allocates a number of perishable goods or service slots in each period to buyers arriving stochastically. Efficient mechanism design is examined by Bergemann and Valimaki (2010) and Said (2012), and revenue maximization is studied by Hinnosaar (2015).

Many applications including ticket sales for transportation and events are complex mixtures of these two types of problems. For example, airline companies simultaneously handle tickets for airplanes on different dates, and hotels make room reservations on arbitrary dates. Buyers have demands and arrive at different points in time and purchase tickets on different dates. Moreover, buyers may demand multiple objects whose sales deadlines (or consumption dates) are different. A round-trip traveler wants to purchase seats for both outbound and return flights on different dates. A hotel guest may need to stay for two or more nights. These complex preferences are naturally unknown to the seller, but they are private information of buyers.

Motivated by these applications, this paper examines the revenue-maximizing mechanism for a dynamic allocation problem with multiple heterogeneous objects and agents with multi-unit demands. Multiple heterogeneous objects are sequentially allocated over time. Agents stochastically arrive and have multidimensional type regarding their desired objects and valuations. We allow the seller to design complete

[1] See Bergemann and Said (2011) for an early review of mechanism design with dynamic populations. Revenue maximization under dynamic population is often referred to *revenue management* in management science literature. See Talluri and van Ryzin (2004) for example.

contracts contingent on future events. To make the problem tractable, we formulate a two-period allocation problem of two heterogeneous objects. Agents are classified into three demand types; unit demand for either object and multi-unit demand. We assume a multi-unit demand agent evaluates objects as complements.

We provide a necessary and sufficient condition for a direct revelation mechanism being incentive compatible. Myerson's (1981) canonical result is extended, and a mechanism is implementable if and only if the allocation policy is monotone in valuation and several other conditions hold. Revenue (and payoff) equivalence theorem holds. The seller's expected revenue is transformed into a virtual surplus form. The seller is allowed to sign a contingent contract with agents. Following the standard approach, we derive the "relaxed solution," which ignores implementability conditions. We then argue regularity conditions for the relaxed solution being optimal.

We show that the optimal mechanism looks like a sales mechanism using "overbooking." When contingent contracts are available, the seller determines the allocation of each object at its sales deadline. An advance agent demanding a future object is allocated if a "last-minute" agent having a high valuation does not arrive in the future. However, if such a last-minute agent arrives, the object is allocated to the new agent. That is, although the advance agent has booked the object beforehand, the seller still sells the object to the last-minute agent. When the last-minute agent purchases, the object is overbooked and the advance agent is bumped.

Because the advance agent faces a risk of being overbooked, his payment is discounted. To finely screen advance agents, the seller offers "multiple fare classes." A cheap, discount ticket is assigned a high risk of being overbooked, whereas an expensive ticket guarantees to allocate the object. The last-minute agent faces an expensive last-minute price, which depends on the "booking class" of the advance agent, because only a very profitable agent is welcome to the seller at that time.

When an agent exhibits multi-unit demand and complementarity, the sales mechanism with overbooking may generate a tradeoff to the seller. A contract for a multi-unit-demand agent is also discounted by the same overbooking risk with respect to the future object. When the seller expects a high option value raised from the last-minute agent in the future, the risk of overbooking is high. However, the high risk of overbooking decreases the value of multi-unit-demand agent for the current object because the current object is valuable only if the future object is allocated. Hence, the price for the bundle of objects is considerably discounted, so that the (last-minute) price of the current single object needs to be sufficiently low. To make the last-minute price of the current object high enough, the seller needs to increase the probability of allocation to the multi-unit-demand agent and relinquish the option value to some extent.

1.1 Related Literature

This paper contributes to the literature on dynamic mechanism design in terms of introducing multi-unit demands. In the literature, most studies focus on single-unit demand agents or single-unit demand in each period. Dizdar et al. (2011) is a notable exception that examines the revenue-maximizing mechanism for a fixed capacity sales problem with multi-unit demand agents. They consider a ticket sales problem where in each period an agent arrives and demands multiple units. In their model, goods are homogeneous and allocation rules are limited such that the seller allocates the requested quantity or not. In contrast, in our model goods are heterogeneous and allocated at different points in time, so that all the desired objects may not be allocated to an agent. This makes the mechanism design problem more complicated.

We provide a novel tractable model with multidimensional type. It is well known that multidimensional mechanism design is complex and Myerson's (1981) approach is not successful in general. Myerson's approach is applicable to our model because the valuation is assumed to be single dimensional even for agents with multi-unit demand. Multidimensional type similar to ours is considered by Lehmann et al. (2002) in the literature of multi-object auctions, and by Dizdar et al. (2011), Pai and Vohra (2013), and Mierendorff (2016) in the dynamic settings.

We argue that the optimal mechanism can be interpreted as an overbooking sales mechanism. Overbooking from the dynamic mechanism design point of view is studied by Ely et al. (2016). They consider that agents arrive stochastically and that advance agents gradually learn their valuation. They show that the seller may sell more tickets than its capacity to advance agents in the optimal mechanism, and the goods are rationed or reallocated subsequently by an auction.

Ely et al.'s (2016) model occupies in another stream of the dynamic mechanism design literature; mechanism design with dynamic information. In this stream, agents' types evolve over time, and each agent reports his type at each period. The dynamic information framework is developed by Courty and Li (2000) and Eso and

Szentes (2007) and generalized by Pavan et al. (2014). Contingent-contracts as in this paper are considered in their framework. In Ely et al. (2016), the object allocation is probabilistic because the valuation is still stochastic for advance agents and new agents may arrive in the future. Our model is simpler than Ely et al. in the sense that the allocation of the future object is probabilistic only because of the possibility of new agents in the future. The novelty of our model is the existence of an object allocated at an early date and possible multi-unit demands.

The remainder of the paper is organized as follows. In Section 2, we illustrate the main result using a simple example of air-ticket sales. In Section 3, we formulate the general model and mechanisms and define the equilibrium concept. We employ so-called periodic ex-post incentive compatibility as the equilibrium concept (Bergemann and Valimaki, 2010). In Section 4, we provide the necessary and sufficient condition for a mechanism being incentive compatible. In Section 5, we derive the optimal mechanism. We first consider the case of a single agent in each period. We provide regularity conditions for the relaxed solution being optimal. Then, we consider the case where many agents may arrive in each period. We provide conditions such that the regularity conditions for the single-agent case are sufficient to make the problem regular in the general case.

2 An Illustration: Sequential Air-Ticket Sales

For an illustration of our main model and results, consider that a monopoly airline company sells seats for two flights X and Y to travelers. Flight X is an "outbound flight" which departs at date 1, and flight Y is a "return flight" leaving at date 2. Two travelers, A and B, arrive to purchase tickets sequentially at date 1 and 2, respectively. Although traveler B wants to take flight Y, traveler A may want to take flight X and/or Y, which is unknown to the airline. Traveler A corresponds to one of the three travel types, $\{out, in, round\}$. The travel type *out* indicates an outbound traveler, who wants to take flight X only. The travel type *in* indicates an inbound traveler, who wants to take flight Y only. The travel type *round* indicates a round-trip traveler, wo wants to take both flights. Regardless of travel types, each traveler (including B) has a per-flight value v_i, which is private information of the travelers and uniformly distributed on $[0, 1]$.

Let us consider the airline's revenue maximization given that A's travel type is known to the airline. Only travelers' valuations are private information. First, suppose that A is an outbound traveler demanding flight X. By uniform distribution of valuations and textbook calculation[2], flight X (respectively, Y) is assigned to traveler A (resp., B) if and only if his value v_A (resp., v_B) is greater than 1/2. This optimal allocation is implemented by simply posting a price to each flight $p_X = p_Y = 1/2$.

Second, suppose that A is an inbound traveler demanding flight Y. Because both travelers demand the same flight and values are uniformly distributed, the airline maximizes her expected revenue by a second-price auction with a reserve price 1/2. Remembering that travelers arrive sequentially, however, the optimal allocation is also implemented by the following *sequential sales mechanism*: The airline offers traveler A a menu of contracts $\{z^{in}(v)\} = \{(\alpha^{in}(v), p^{in}(v))\}$, which is indexed by v. In a contract $z^{in}(v) = (\alpha^{in}(v), p^{in}(v))$, α^{in} determines the probability that flight Y is assigned to A, and p^{in} is the price of such a "lottery."[3] According to the contract that traveler A purchases, the airline posts a price to traveler B, $p^B(v)$, where v is the index of a contract that traveler A purchases. The flight is assigned to B if he purchases, so the probability of A's contract needs to satisfy

$$\alpha^{in}(v) + \Pr\{v_B > p^B(v)\} \leq 1.$$

Using a sequential sales mechanism, the optimal allocation is implemented when we set $\alpha^{in}(v) = v$, where $v \in [1/2, 1]$. Traveler A purchases contract v_A if $p^{in}(v)$ is equal to the expected payment in the optimal auction:

$$\begin{aligned} p^{in}(v) &= \Pr\{v_B < v\} E[\max\{1/2, v_B\} | v_B < v] \\ &= \frac{v^2}{2} + \frac{1}{8}, \end{aligned}$$

where $v \in [1/2, 1]$. The price to traveler B is specified as

$$p^B(v; in) = \begin{cases} 1/2 & \text{if A exits,} \\ v & \text{if A purchases } v. \end{cases}$$

[2]See Krishna (2010) for example. Flight X (respectively, Y) is assigned to traveler A (resp., B) if and only if he has a positive virtual valuation: $2v_i - 1 \geq 0$.

[3]We assume that the contract is non-refundable when the traveler is denied boarding in the end. We assume that the airline fully commits to contracts, so that it is without loss of generality to limit attention on non-refundable contracts.

Under the above specification, the associated allocation rule coincides with that in the second-price auction with the optimal reserve price. Therefore, the airline earns her maximum expected revenue by the Revenue Equivalence Theorem.

A similar sales mechanism can also be applied to a round-trip traveler. Suppose that A is a round-trip traveler demands both flights and flights are perfect complements: The value from either single flight only is zero. The airline offers traveler A a menu of contracts $\{z^{ro}(v)\} = \{(a_X^{ro}(v), \alpha_Y^{ro}(v), p^{ro}(v))\}$, which is indexed by v. In a contract $z^{ro}(v) = (a_X^{ro}(v), \alpha_Y^{ro}(v), p^{ro}(v))$, $a_X^{ro} \in \{0, 1\}$ determines allocation of flight X, α_Y^{ro} determines the probability that flight Y is assigned, and p^{ro} is the price of the contract. Given A is assigned the outbound flight X, the return flight is assigned to A if his *virtual valuation* is larger than that of traveler B. Because the total value $2v_A$ of a round-trip traveler is uniformly distributed on $[0, 2]$, A is assigned flight Y if and only if

$$4v_A - 2 > 2v_B - 1 \Leftrightarrow v_B < 2v_A - 1/2.$$

Traveler A with per-flight value $v_A \geq 1/2$ chooses contract v_A when we set $\alpha_Y^{ro}(v) = \min\{2v - 1/2, 1\}$ and

$$p^{ro}(v) = \begin{cases} 2v^2 & \text{if } v \in [1/2, 3/4) \\ 9/8 & \text{if } v \in [3/4, 1], \end{cases}$$

and the optimal allocation at date 2 is implemented. By perfect complementarity, flight X is assigned, $a_X^{ro}(v) = 1$, for any contract $v \in [1/2, 1]$. Finally, the price to traveler B is

$$p^B(v; ro) = \begin{cases} 1/2 & \text{if A exits} \\ 2v - 1/2 & \text{if A purchases } v \in [1/2, 3/4), \end{cases}$$

and the airline does not sell to B if A purchases $v \geq 3/4$. In the following sections, we will confirm that the combination of the above dynamic sales mechanisms for each travel type in fact maximizes the airline's expected revenue even if A's travel type is his private information.

One might interpret the above sequential sales mechanism as a dynamic sales mechanism using "overbooking." Interestingly, the above sequential sales mechanism has several similarities with real air-ticket sales practice. Consider the case where traveler A is an inbound or round-trip traveler. When traveler A has a value greater than 1/2, he purchases a ticket from the airline. However, although traveler A holds a ticket and the flight is full, the airline still sells a ticket to traveler B. The airline "oversells" tickets in this sense. When traveler B also purchases a ticket, the flight is overbooked and the allocation is rationed. Traveler A is denied boarding.

Second, the early buyer, traveler A, faces a variety of contracts, which can be regarded as multiple fare classes. For each ticket class v, a probability of being seated is determined. A discount ticket holder (having a low v) is more likely to be overbooked and bumped, whereas an expensive ticket holder (having a high v) is rarely bumped. This is consistent with the real reallocation mechanism for overbooking. Finally, the prices exhibit an increasing price trend, although the advance price is not unique. Suppose that traveler A is an inbound traveler. The ticket price at the early date is distributed between $p^{in}(1/2) = 1/4$ and $p^{in}(1) = 5/8$. The "last-minute" price just before departure is likely to be high and it is between 1/2 and 1.

This example shows that the seller screens and price-discriminates early buyers by using the risk of overbooking. Screening by the risk of overbooking enables the seller to allocate the good efficiently under the dynamic population and increase her expected revenue.

However, the combination of the optimal mechanisms under the known travel types may not be incentive compatible under another type distribution. To see this, consider that a round-trip traveler's per-flight value is uniformly distributed on $[0, 3/4]$ instead of $[0, 1]$. Then, a new incentive problem arises. We recalculate the optimal menu of contracts for a round-trip traveler, which is specified as for index $v \in [3/8, 3/4]$,

$$\alpha_Y^{ro}(v) = \min\{2v - 1/4, 1\},$$

$$p^{ro}(v) = \begin{cases} 2v^2 + \frac{3}{32} & \text{if } v \in [3/8, 5/8) \\ 7/8 & \text{if } v \in [5/8, 3/4], \end{cases}$$

and flight X is assigned, $a_X^{ro}(v) = 1$, for all contract $v \in [3/8, 3/4]$. Flight X is assigned if the round-trip traveler has a positive virtual value; $4v_A - 3/2 > 0 \Leftrightarrow v_A > 3/8$. Because a round-trip traveler with $v_A = 3/8$ completes his travel with probability 1/2, the ticket price for such a type is discounted and $p^{ro}(3/8) = 3/8$. The round-trip ticket price is lower than that of the "one-way" ticket for an outbound traveler: $p^{ro}(3/8) < 1/2 = p_X$. Hence, under the specified sales mechanism, an outbound traveler has an incentive to deviate and purchase the cheapest round-trip ticket.

The intuition of the problem is indeed general. Because buyers arrive sequentially, the seller wants to defer the allocation of goods consumed in the future until its deadline. The seller wants to collect buyers' information until the allocation deadline, and then allocates the good to the most profitable buyer. Then, the expected payment of the advance buyer should be low because of the uncertainty of the allocation. This is the case for buyers exhibiting multi-unit demand for different consumption deadlines. Accordingly, a buyer with multi-unit demand may be able to consume an object of early date with a low price. However, such an object should be high-priced due to certainty of the allocation, which conflicts with the incentive of buyers having a single-unit demand for a current object. The mechanism designer needs to design the optimal mechanism taking account of price monotonicity between a last-minute price of a current good and a discounted price of a long-term multi-unit contract.

3 Model

The monopolist seller allocates two objects 1 and 2 over two periods of time. There is no time discounting. One unit of each object is supplied.[4] Object $t \in \{1, 2\}$ is allocated and consumed at period t. In each period, a finite number of agents enter the mechanism. The set of entrants at period t is denoted by N^t, and the number of entrants $|N^t| \geq 0$ is an IID random variable at each period. The set of agents having entered by t is denoted by $\mathcal{N}^t \equiv \bigcup_{s \leq t} N^s$. An allocation of object t to agent i is denoted by $a_i^t \in \{0, 1\}$. An allocation of object t is denoted by $a^t = (a_i^t)_{i \in \mathcal{N}^t}$. An allocation a^t is feasible at t if $\sum_i a_i^t \leq 1$ and $a_i^t = 0$ for any i who is not in the mechanism at t.

Agents are risk neutral and have quasi-linear utility. Agents arriving at period 1 are classified into three *demand types*, which are denoted by $k_i \in \{1, 2, M\}$. An agent with demand type 1 demands object 1 only. He is short-lived and exits at the end of period 1. A type-1 agent i's payoff takes the form of $a_i^1 v_i - p_i$, where v_i is his value and p_i is his payment. An agent with demand type 2 demands object 2 only. A type-2 agent i is long-lived and his payoff takes the form of $a_i^2 v_i - p_i$. An agent with demand type M demands both objects. A type-M agent has a value v_i when he consumes both objects, whereas he has a value βv_i when he consumes object 1 only. Object 2 is valuable only when he consumes object 1. Thus, a type-M agent

[4]The number of units is straightforwardly extended to more than one.

i's payoff takes the form

$$\begin{cases} v_i - p_i & \text{if } a_i^1 = a_i^2 = 1, \\ \beta v_i - p_i & \text{if } a_i^1 = 1 \text{ and } a_i^2 = 0, \\ -p_i & \text{if } a_i^1 = 0, \end{cases} \tag{1}$$

where $\beta \in [0, 1)$ is a parameter specifying the extent to which the agent with multi-unit demand is willing to win the second object. We assume that object 1 is essential for type-M agents to have a value and that they exit the mechanism when they do not obtain object 1. For example, suppose that objects are seats on outbound and return flights and that a business traveler wants to make a round-trip. For the business traveler, the departing flight is essential and necessary for doing his job and making profits; whereas he can afford to find an alternative return transportation, should he not take the originally specified option, paying a cost of $(1 - \beta)v_i$.

To make the problem tractable, we assume β is identical for all type-M agents and known to the mechanism designer. Note that our model includes the case of perfect complements for type M, $\beta = 0$.

Both a demand type $k_i \in \{1, 2, M\}$ and a valuation $v_i \in \mathbb{R}_+$ are private information of an agent. A pair of an agent's demand type and valuation is called the agent's *type* and denoted by $\theta_i^1 = (v_i, k_i) \in \Theta^1 \equiv [0, \bar{v}] \times \{1, 2, M\}$, where $\bar{v} \leq \infty$.

All agents arriving at period 2 are demand type 2. When agent $j \in N^2$ pays an amount of p_j, then his payoff is $a_j^2 v_j - p_j$. The type of agent j at period 2 is denoted by $\theta_j^2 \in \Theta^2 \equiv [0, \bar{v}] \times \{2\}$ or simply by $v_j \in [0, \bar{v}]$, which is also private information of j.

The number of agents and their types are realized over time. The probability that $|N^t| = n$ is denoted by $\eta_t(n) \geq 0$. The types of agents are independently distributed. The type of period-1 agent i is drawn from a CDF F. The conditional distribution function is denoted by $F_k(v) = F(v|k)$ for $k \in \{1, 2, M\}$. The conditional distribution functions have density $f_k(v) > 0$. The type of period-2 agent is drawn from a CDF $F_2(v)$, which is the same as the conditional distribution of valuation of a type-2 agent at period 1. The hazard rate function of a conditional distribution F_k is denoted by

$$\lambda_k(v) = \frac{f_k(v)}{1 - F_k(v)}.$$

The standard regularity condition is assumed throughout the paper.

Assumption 1 For each $k \in \{1, 2, M\}$, the *virtual valuation* $\psi_k(v) \equiv v - 1/\lambda_k(v)$ is strictly increasing in v.

3.1 Deterministic Mechanisms

We focus on direct revelation mechanisms: agents report their types on their arrival. We further focus on deterministic mechanisms: the seller never randomizes allocations or payments. However, the allocation of object 2 can depend on type profile at period 2. In particular, type-2 and -M agents' allocation of object 2 is not determined at the time of their contracting (i.e., the time of arrival). However, we assume that payments are completed at the time of contracting. This is just for simplicity because there is a degree of freedom on the timing and distribution of payments. We can modify the payment rule so that it is sequential and depends on θ_j^2 at $t = 2$.

We consider that agents' arrivals are strategic: a period-1 agent may strategically delay his entry to the mechanism. Denote by $\emptyset$ the strategic delay by a period-1 agent. A report $\emptyset$ indicates that the mechanism designer does not identify the agent at period 1. If the agent delays his entry and makes a report at period 2, he is regarded as a period-2 agent. Let $\Theta_+^1 \equiv \Theta^1 \cup \{\emptyset\}$ be the extended message space of a period-1 agent including delaying. By the definition of utility, however, strategic delay matters only for those with demand type 2.

A type profile reported at period t is denoted by $\theta^t = (\theta_i^t)_{i \in N^t}$. A *direct mechanism* is defined as $\Gamma \equiv (a^1, p^1, a^2, p^2)$, where

- $a^1 : (\Theta_+^1)^{N^1} \to \{0, 1\}^{N^1}$ is an allocation function of object 1,
- $p^1 : (\Theta_+^1)^{N^1} \to \mathbb{R}^{N^1}$ determines payments of period-1 agents,
- $a^2 : (\Theta^2)^{N^2} \times (\Theta_+^1)^{N^1} \to \{0, 1\}^{\mathcal{N}^2}$ is an allocation function of object 2, and
- $p^2 : (\Theta^2)^{N^2} \times (\Theta_+^1)^{N^1} \to \mathbb{R}^{N^2}$ determines payments of period-2 agents.

When a report of a period-1 agent i is $\emptyset$, neither a_i^1 or p_i^1 is defined (alternatively, they are described as zero). A mechanism is *feasible* if $\sum_{i \in N^1} a_i^1(\theta^1) \leq 1$ and $\sum_{i \in \mathcal{N}^\in} a_i^2(\theta^1, \theta^2) \leq 1$ for all $\theta^1 \in \prod_{i \in N^1} \Theta^1$ and $\theta^2 \in \prod_{j \in N^2} \Theta^2$. It is natural to focus on allocation rules satisfying the following properties. It is clearly without loss of optimality because types are independently drawn.

Assumption 2 An allocation rule $a = (a^1, a^2)$ satisfies the following properties:

1. For all $i \in N^1$ with $\theta_i^1 = (v_i, 1)$, $a_i^2(\theta^2, \theta^1) = 0$,

2. for all $i \in N^1$ with $\theta_i^1 = (v_i, 2)$, $a_i^1(\theta^1) = 0$, and

3. for all $i \in N^1$ with $\theta_i^1 = (v_i, M)$, $a_i^2(\theta^1, \theta^2) = 1$ only if $a_i^1(\theta^1) = 1$.

The first and second terms require that an undesired object is never allocated to agents with single unit demand. The third term requires that object 2 is assigned to an agent with multi-unit demand only if he is assigned at $t = 1$.

Under the definition of a mechanism and assumptions, each demand-type-1 agent signs a contract $(a_i^1(\theta^1), p_i^1(\theta^1))$, which is a pair consisting of an object 1 allocation and a payment. For each demand-type-2 agent at period 1, a contract is $(a_i^2(\cdot, \theta^1), p_i^1(\theta^1))$, where $a_i^2(\cdot, \theta^1)$ specifies an allocation rule for object 2. For each demand-type-M agent, a contract is $(a_i^1(\theta^1), a_i^2(\cdot, \theta^1), p_i^1(\theta^1))$, which specifies an allocation of object 1, an allocation rule for object 2, and an associated payment. Finally, for each period-2 agent, a contract is $(a_i^2(\theta^2, \theta^1), p_i^2(\theta^2, \theta^1))$, which is similar to demand type 1.

We evaluate and define each agent's payoff at the end of his arrival period. Let $\hat{\theta}^t = (\hat{\theta}_i^t)_{i \in N^t}$ be a profile of reports at period t. Given a mechanism Γ, an associated payoff function is denoted by π_i^t for $i \in N^t$. Let us define

$$A_i(\hat{\theta}^1, k_i) \equiv \begin{cases} a_i^1(\hat{\theta}^1) & \text{if } k_i = 1 \\ \alpha_i^2(\hat{\theta}^1) & \text{if } k_i = 2 \\ a_i^1(\hat{\theta}^1)\big(\alpha_i^2(\hat{\theta}^1)(1-\beta) + \beta\big) & \text{if } k_i = M, \end{cases} \tag{2}$$

where k_i is agent i's true demand type and

$$\alpha_i^2(\hat{\theta}^1) \equiv E[a_i^2(\theta^2, \hat{\theta}^1)]$$

is the probability that object 2 is allocated. When a period-1 agent i with a true type $\theta_i^1 = (v_i, k_i)$ makes a report $\hat{\theta}_i^1 \neq \emptyset$, his payoff given a current type profile $\hat{\theta}^1$ is

$$\pi_i^1(\hat{\theta}^1, \theta_i^1) = A_i(\hat{\theta}^1, k_i)v_i - p_i^1(\hat{\theta}^1). \tag{3}$$

For a period-2 agent $j \in N^2$, his ex-post payoff given current and past type profiles is

$$\pi_j^2(\hat{\theta}^2, \hat{\theta}^1, \theta_j^2) = a_j^2(\hat{\theta}^2, \hat{\theta}^1)v_j - p_j^2(\hat{\theta}^2, \hat{\theta}^1).$$

When a period-1 agent i with $k_i = 2$ delays his arrival and makes a report $\hat{\theta}_i^2$ at period 2, his ex-post payoff is written as

$$\tilde{\pi}_i^2((\hat{\theta}_i^2, \hat{\theta}^2), (\emptyset, \hat{\theta}_{-i}^1), \theta_i^1) = a_i^2((\hat{\theta}_i^2, \hat{\theta}^2), (\emptyset, \hat{\theta}_{-i}^1))v_i - p_i^2((\hat{\theta}_i^2, \hat{\theta}^2), (\emptyset, \hat{\theta}_{-i}^1)).$$

The agent's payoff of delaying is defined by

$$\pi_i^1((\emptyset, \hat{\theta}_{-i}^1), \theta_i^1) = \begin{cases} \max_{\hat{\theta}_i^2} E[\tilde{\pi}_i^2((\hat{\theta}_i^2, \theta^2), (\emptyset, \hat{\theta}_{-i}^1), \theta_i^1)] & \text{if } k_i = 2 \\ \max_{\hat{\theta}_i^2} -E[p_i^2((\hat{\theta}_i^2, \theta^2), (\emptyset, \hat{\theta}_{-i}^1))] & \text{otherwise} \end{cases}. \tag{4}$$

3.2 Incentive Compatibility

We impose so-called periodic ex-post incentive compatibility for the equilibrium concept (Bergemann and Valimaki, 2010). That is, agents have no incentive to deviate from truth-telling after observing the current type profile, given that future agents report truthfully. PEPIC is equivalent to the standard ex-post (or dominant strategy) incentive compatibility for short-lived agents. However, it is not for long-lived agents because types of future agents are still uncertain at the end of the arrival period. Let $\Pi_i^1(\theta^1) = \pi_i^1(\theta^1, \theta_i^1)$ be the payoff under truth-telling for a period-1 agent $i \in N^1$, and let $\Pi_j^2(\theta^2, \theta^1) = \pi_j^2(\theta^2, \theta^1, \theta_j^2)$ be that for a period-2 agent $j \in N^2$.

Definition 1 A mechanism Γ is *periodically ex-post incentive compatible (PEPIC)* if for all $i \in N^1$, all $\theta^1 \in \prod_{N^1} \Theta^1$, and all $\hat{\theta}_i^1 \in \Theta_+^1$,

$$\Pi_i^1(\theta^1) \geq \pi_i^1((\hat{\theta}_i^1, \theta_{-i}^1), \theta_i^1),$$

and for all $j \in N^2$, all $\theta^1 \in \prod_{N^1} \Theta^1$, all $\theta^2 \in \prod_{N^2} \Theta^2$, and all $\hat{\theta}_i^2 \in \Theta^2$,

$$\Pi_j^2(\theta^2, \theta^1) \geq \pi_j^2((\hat{\theta}_j^2, \theta_{-j}^2), \theta^1, \theta_j^2).$$

Definition 2 A mechanism Γ is *periodically ex-post individually rational (IR)* if for all $i \in N^1$, all $j \in N^2$, all $\theta^1 \in \prod_{N^1} \Theta^1$, and all $\theta^2 \in \prod_{N^2} \Theta^2$, $\Pi_i^1(\theta^1) \geq 0$ and $\Pi_j^2(\theta^2, \theta^1) \geq 0$.

3.3 The Seller's Problem

Our objective is to find a PEPIC and IR mechanism that dynamically maximizes the seller's expected revenue. The seller's expected revenue is

$$R = E\Big[\sum_{i \in N^1} p_i^1(\theta^1) + \sum_{j \in N^2} p_j^2(\theta^2, \theta^1)\Big], \tag{5}$$

where expectation is taken over populations (N^1 and N^2) and types θ^t. The revenue maximization problem is written as[5]

$$\max E\Big[\sum_{i\in N^1} p_i^1(\theta^1) + \sum_{j\in N^2} p_j^2(\theta^2,\theta^1)\Big] \tag{6}$$
$$\text{subject to } PEPIC, IR, \text{ and } Feasibility.$$

4 Characterization of Incentive Compatibility

We derive equivalent conditions for incentive compatibility to apply the standard Myerson (1981) technique. To avoid messy exhibition, we exclude θ^1_{-i} and θ^2_{-i} from equations except for the formal description of the theorem in this section. For example, an allocation of object 1, $a_i^1(\theta_i^1,\theta^1_{-i})$ is simply denoted by $a_i^1(\theta_i^1)$ or $a_i^1(v_i,k_i)$. Similarly, $a_j^2(\theta^2,\theta^1)$ is replaced with $a_j^2(v_j,\theta^1)$: we consider the problem as if there is only one agent in the period.

Given that agents do not misreport their demand types, incentive compatibility is characterized in a standard manner. Myerson (1981) shows that a direct revelation mechanism is incentive compatible (in valuation) if and only if the allocation rule is monotone and payoff equivalence holds:

Proposition 1 (Myerson, 1981) *Suppose that agents report their true demand types. A direct revelation mechanism is PEPIC (in valuation) if and only if the following conditions hold:*

1. *(Value-Monotonicity 1) for all $i \in N^1$, $A_i(\theta_i^1, k_i)$ is weakly increasing in v_i for each $k_i \in \{1, 2, M\}$,*

2. *(Value-Monotonicity 2) for all $j \in N^2$, $a_j^2(v_j,\theta^1)$ is weakly increasing in v_j for each $\theta^1 \in \prod_{N^1}\Theta^1$,*

3. *(Payoff Equivalence) each agent's truthful payoff satisfies*

$$\Pi_i^1(\theta_i^1) = \Pi_i^1(0,k_i) + \int_0^{v_i} A_i((s,k_i),k_i)\mathrm{d}s \tag{7}$$

for $i \in N^1$, and

$$\Pi_j^2(\theta_j^2,\theta^1) = \Pi_j^2(0,\theta^1) + \int_0^{v_i} a_j^2(s,\theta^1)\mathrm{d}s \tag{8}$$

for $j \in N^2$.

[5]It is straightforward to find the socially optimal mechanism in this setting.

Value-Monotonicity is restated as for $i \in N^1$,

$$a_i^1(v_i, 1) = 1 \;\Rightarrow\; (\forall v_i' > v_i)\; a_i^1(v_i', 1) = 1, \tag{Mon-1}$$

$$\alpha_i^2(v_i, 2) \text{ is non-decreasing in } v_i, \tag{Mon-2}$$

$$a_i^1(v_i, M)\big(\alpha_i^2(v_i, M)(1-\beta) + \beta\big) \text{ is non-decreasing in } v_i. \tag{9}$$

When $\beta > 0$, (9) clearly requires that both $a_i^1(\cdot, M)$ and $\alpha_i^2(\cdot, M)$ are increasing. Hence, (9) is separated into two allocative monotonicity conditions:[6]

$$a_i^1(v_i, M) = 1 \;\Rightarrow\; (\forall v_i' > v_i)\; a_i^1(v_i', M) = 1, \tag{Mon-Ma}$$

$$\alpha_i^2(v_i, M) \text{ is non-decreasing in } v_i. \tag{Mon-Mb}$$

Note that when $\beta = 0$, we have $a_i^1(v_i, M)\alpha_i^2(v_i, M) = \alpha_i^2(v_i, M)$ for all v_i because $\alpha_i^2 > 0$ only if $a_i^1 = 1$ by Assumption 2.3. Hence, Value-Monotonicity for demand type M is characterized by only (Mon-Mb) when $\beta = 0$.

For $j \in N^2$ and all $\theta^1 \in \prod_{N^1} \Theta^1$, Value-Monotonicity indicates

$$a_j^2(v_j, \theta^1) = 1 \;\Rightarrow\; (\forall v_j' > v_j)\; a_j^2(v_j', \theta^1) = 1. \tag{Mon-22}$$

Note that PEPIC is equivalent to dominant strategy incentive compatibility for demand type 1 and period-2 agents. Because an allocation rule for these agents is deterministic and binary, we can define the *cutoff values* for demand types as

$$c_i^1(1) \equiv \inf\{v_i | a_i^1(v_i, 1) = 1\}, \tag{10}$$

$$c_j^2(\theta^1) \equiv \inf\{v_j | a_j^2(v_j, \theta^1) = 1\}. \tag{11}$$

The dominant-strategy incentive compatible payment rule is specified by a form of $p_i^t = a_i^t c_i^t + Z_i$, where Z_i is any constant variable.[7]

The incentive compatibility for demand types $k_i = 2, M$ at period 1 is Bayesian in the sense that allocation depends on future information. However, for demand type M, we also define the cutoff value of object 1 as

$$c_i^1(M) = \inf\{v_i | a_i^1(v_i, M) = 1\}. \tag{12}$$

In order to prevent period-1 agents from misreporting his demand type, we need to impose additional conditions on allocation rules. By Assumption 2, we can ignore

[6]Condition (9) does not imply the monotonicity of $\alpha_i^2(v_i, M)$ for v_i such that $a_i^1(v_i, M) = 0$. However, by Assumption 2.3, $\alpha_i^2 = 0$ for such v_i and the monotonicity over all v_i holds.

[7]If the infimum does not exist, let $c_i^t = \infty$.

deviations by an agent with $k_i = 1$ to $\hat{k}_i = 2$, by an agent with $k_i = 2$ to $\hat{k}_i = 1$, and by an agent with $k_i = M$ to $\hat{k}_i = 2$. Thus, we need to take account of deviations by a period-1 agent (1) from $k_i = 1$ to $\hat{k}_i = M$, (2) from $k_i = M$ to $\hat{k}_i = 1$, and (3) from $k_i = 2$ to $\hat{k}_i = M$. By Proposition 1, an allocation rule must be monotone in valuations and payment rule is uniquely specified by payoff equivalence formulas up to constants. Using properties in Proposition 1, additional conditions for PEPIC are specified.

To state the conditions, let us define $\bar{\alpha}_i^2(v, M) \equiv \lim_{s \to v+} \alpha_i^2(s, M)$. PEPIC requires the following conditions:

$$c_i^1(1) \leq \big((1-\beta)\bar{\alpha}_i^2(c_i^1(M), M) + \beta\big)c_i^1(M), \tag{Cond-1M}$$

$$\beta c_i^1(M) \leq c_i^1(1), \tag{Cond-M1}$$

and

$$\int_0^v \alpha_i^2(s, 2)\mathrm{d}s \geq \int_0^v \alpha_i^2\Big(\frac{s}{1-\beta}, M\Big)\mathrm{d}s - \beta c_i^1(M) \tag{Cond-2M}$$

for all $v \geq (1-\beta)c_i^1(M)$. See proof of Theorem 1 in Appendix for the derivation of them. (Cond-1M) prevents an agent with demand type 1 from reporting demand type M. (Cond-M1) prevents an agent with demand type M from reporting demand type 1. (Cond-2M) prevents an agent with demand type 2 from reporting demand type M.

Finally, we need to prevent a demand-type-2 agent at period 1 from delaying his entry. Consider a strategic delay of a period-1 agent with demand type 2. His ex-post payoff in the end is $\max\{v_i - c_i^2(\theta_{-i}^1), 0\}$. By payoff equivalence, the expected payoff under delaying is denoted by

$$\int_0^v \tilde{\alpha}_i^2(s, \theta_{-i}^1)\mathrm{d}s, \tag{13}$$

where $\tilde{\alpha}_i^2(v, \theta_{-i}^1) \equiv E[a_i^2((v, 2), \theta_{-i}^1)]$ and the expectation is taken over θ^2. The incentive compatibility requires that there exists some $d_i \geq 0$ and

$$\int_0^v \alpha_i^2(s, 2)\mathrm{d}s + d_i \geq \int_0^v \tilde{\alpha}_i^2(s, \theta_{-i}^1)\mathrm{d}s \tag{ND}$$

for all v. The term d_i represents the difference in constant terms in truthful payoffs.

It turns out that these conditions are necessary and sufficient for PEPIC. We do not exclude deviation by a demand-type-M agent to reporting demand type 2. This makes us drop IR condition from our characterization. The following theorem is our first main result that is the characterization of PEPIC in our model.

Theorem 1 *A mechanism Γ is PEPIC if and only if*

1. *there exist two functions $\underline{\Pi}_i^1(\theta_{-i}^1)$ and $\underline{\Pi}_j^2(\theta_{-j}^2, \theta^1)$, satisfying $\underline{\Pi}_i^1(\theta_{-i}^1) \geq E[\underline{\Pi}_i^2(\theta^2, \theta_{-i}^1)]$ for all $\theta_{-i}^1 \in \prod_{N_{-i}^1} \Theta^1$,*

2. *the allocation rule satisfies (Mon-1), (Mon-Ma) (if $\beta > 0$), (Mon-Mb), (Mon-2), (Mon-22), (Cond-1M), (Cond-M1), (Cond-2M), and (ND) with*

$$d_i(\theta_{-i}^1) = \underline{\Pi}_i^1(\theta_{-i}^1) - E[\underline{\Pi}_i^2(\theta^2, \theta_{-i}^1)],$$

   and

3. *Truthful payoffs are given by*

$$\Pi_i^1((v,1), \theta_{-i}^1) = \underline{\Pi}_i^1(\theta_{-i}^1) + \max\{v - c_i^1(1, \theta_{-i}^1), 0\}, \tag{14}$$

$$\Pi_i^1((v,2), \theta_{-i}^1) = \underline{\Pi}_i^1(\theta_{-i}^1) + \int_0^v \alpha_i^2((s,2), \theta_{-i}^1)\mathrm{d}s, \tag{15}$$

$$\Pi_i^1((v,M), \theta_{-i}^1) = \underline{\Pi}_i^1(\theta_{-i}^1) + \max\Big\{\int_{c_i^1(M,\theta_{-i}^1)}^{v} \big((1-\beta)\alpha_i^2((s,M), \theta_{-i}^1) + \beta\big)\mathrm{d}s, 0\Big\}, \tag{16}$$

$$\Pi_j^2(v, \theta_{-j}^2, \theta^1) = \underline{\Pi}^2(\theta_{-j}^2, \theta^1) + \max\{v - c_j^2(\theta_{-j}^2, \theta^1), 0\}. \tag{17}$$

Proof. See Appendix.

A similar characterization of incentive compatibility is provided by Dizdar et al. (2011), Pai and Vohra (2013), and Mierendorff (2016). Our result is distinct from their characterizations in several respects. First, the equilibrium concept is an intermediate of ex-post and Bayesian equilibrium. Second, the demand types, which is the second private information, are not completely ordered, whereas in these studies the second private information represents demand quantities or private consumption deadlines, which are completely ordered. In addition, the preceding characterizations have relied on the assumption of so-called "single-minded" preferences. Our model is also a single-minded environment if multi-unit demand agents exhibit perfect complementarity. We allow $\beta > 0$ and agents are not single-minded, which makes characterization more complex.

According to Theorem 1, IR is characterized as follows.

Theorem 2 *A PEPIC mechanism is IR if and only if* $\underline{\Pi}_i^1(\theta_{-i}^1) \geq 0$ *and* $\underline{\Pi}_j^2(\theta_{-j}^2, \theta^1) \geq 0$ *for all* $i \in N^1$*, all* $j \in N^2$*, all* $\theta^1 \in \prod_{N^1} \Theta^1$*, and all* $\theta_{-j}^2 \in \prod_{N_{-j}^2} \Theta^2$.

Proof. It is immediate by Theorem 1, which implies Π_i^1 and Π_j^2 are increasing in agent's own valuation. ■

5 The Optimal Mechanism

Now we turn to the analysis of the optimal mechanism. From payoff equivalence formulas of Theorem 1, it is straightforward to transform the seller's objective function into the virtual surplus form. IR implies $\underline{\Pi}_i^1(\cdot) = \underline{\Pi}_j^2(\cdot) = 0$ at the optimum. The seller's revenue maximization problem is written as

$$\max E\Big[\sum_{k_i=1} a_i^1(\theta^1)\psi_1(v_i) + \sum_{k_i=M} a_i^1(\theta^1)\big((1-\beta)a_i^2(\theta^2,\theta^1)+\beta\big)\psi_M(v_i) + \sum_{k_i=2} a_j^2(\theta^2,\theta^1)\psi_2(v_j)\Big] \tag{18}$$

subject to (Mon-1), (Mon-2), (Mon-Ma), (Mon-Mb), (Mon-22), (Cond-1M), (Cond-M1), (Cond-2M), (ND), and the feasibility condition. Taking the standard approach, we first ignore all IC-related constraints and solve the relaxed problem. Then, we examine whether the relaxed solution satisfies each ignored condition or not.

By Assumption 1, the virtual valuation function has an inverse function ψ_k^{-1}. Let $r_k^* \equiv \psi_k^{-1}(0)$ be the valuation such that the virtual value is zero for each demand type k.

5.1 Single Agent

We consider a simple situation where at most one agent arrives in each period. The agent arriving at period 1 is named agent i, and the agent arriving at period 2 is named agent j, if any. We assume that agent i enters the mechanism with probability 1 and that agent j arrives with probability $q \in (0, 1]$. For convenience, denote $v_j \leq 0$ if agent j does not arrive. In the model of the single agent in each period, the (optimal) mechanism at period 1 is implemented ay an indirect mechanism using a menu of contracts: a "last-minute" price $p_1 = c_i^1(1)$ of object 1, a menu for contracts of single object 2, $\{(\alpha_2(v), p_2(v))\}$, and a menu of contracts for multiple objects, $\{(a_M^1(v), \alpha_M(v), p_M(v))\}$. For a contract $z_2(v) = (\alpha_2(v), p_2(v))$, $\alpha_2(v)$ indicates the probability of obtaining object 2 and $p_2(v)$ is the price of the contract.[8] Similarly, $\alpha_M(v)$ indicates the probability of obtaining object 2 for the agent with multi-unit demand and $p_M(v)$ is its price. The allocation of object 1 to a type-M agent is specified by a_M^1. At period 2, the mechanism is implemented by a posted price $p_j(\theta_i) = c_j^2(\theta_i)$. The feasibility condition requires

$$\alpha_k(v) + q \Pr\{v_j > p_j(v,k)\} \leq 1,$$

which is equivalent to

$$\alpha_k(v) \leq 1 - q + qF_2(p_j(v,k)) \tag{19}$$

for all $v \in [0, \bar{v}]$ and all $k \in \{2, M\}$.

5.1.1 The Relaxed Solution

Let us consider virtual surplus maximization given that agent i's demand type is known. First, suppose that agent i has demand type $k_i = 1$. Two agents demand different objects, so that the object is allocated if and only if each agent has a positive virtual value. Agent i obtains object 1 if and only if $\psi_1(v_i) \geq 0$. Agent j obtains object 2 if and only if $\psi_2(v_j) \geq 0$. Because of increasing virtual valuation, this allocation rule is implemented by simple posted prices: $p_1 = r_1^*$ to agent i and $p_j(\theta_i) = r_2^*$ to agent j.

Second, suppose that agent i has demand type $k_i = 2$. Given that agent j arrives at period 2, the virtual surplus maximization problem is written as

$$\max\{\psi_2(v_i), \psi_2(v_j), 0\},$$

which is the same as the optimal auction problem of object 2. The agent with the highest positive virtual valuation wins object 2. Remembering agents arrive sequentially, the allocation rule is implemented by the following mechanism: given agent i's type $\theta_i^1 = (v_i, 2)$, the allocation rule with respect to agent j is implemented by a posted price $p_j(\theta_i^1) = \max\{r_2^*, v_i\}$. When agent i makes a report, his allocation is not determined yet but he wins the object with probability

$$\Pr\{\psi_2(v_j) < \psi_2(v_i)\} = G(v_i) \equiv qF_2(v_i) + (1-q)$$

[8] Precisely, a contract specifies the full contingent-allocation plan $a_i^2(\cdot, v)$ and $\alpha_2(v) = E[a_i^2(v_j, v)]$. It is similar for α_M too.

for $v_i \geq r_2^*$. Hence, the contract for type $\theta_i^1 = (v_i, 2)$, where $v_i \geq r_2^*$, in the relaxed solution is given by $\alpha_2(v_i) = G(v_i)$ and

$$p_2(v_i) = G(v_i)v_i - \int_{r_2^*}^{v_i} G(s)\mathrm{d}s,$$

which is determined by payoff equivalence.

Third, suppose that agent i has demand type $k_i = M$. Given that he obtains object 1, the virtual surplus maximization at period 2 is similar to the previous case and written as

$$\max\{(1-\beta)\psi_M(v_i), \psi_2(v_j), 0\}.$$

Given agent i's type $\theta_i^1 = (v_i, M)$, the allocation rule with respect to agent j is implemented by a posted price $p_j(\theta_i^1) = \max\{r_2^*, \psi_2^{-1}\big((1-\beta)\psi_M(v_i)\big)\}$. At the time of agent i's reporting, agent i obtains object 2 with probability

$$\Pr\{\psi_2(v_j) < (1-\beta)\psi_M(v_i)\} = H(v_i) \equiv G\Big(\psi_2^{-1}\big((1-\beta)\psi_M(v_i)\big)\Big),$$

given $a_i^1 = 1$. It is clear that the mechanism designer allocates object 1 if and only if agent i has a positive virtual value. The contract for type (v_i, M) in the relaxed solution is given by $a_M^1(v_i) = 1$, $\alpha_M(v_i) = H(v_i)$, and

$$\begin{aligned} p_M(v_i) &= A_i((v_i, M), M)v_i - \int_{r_M^*}^{v_i} A_i((v_i, M), M)\mathrm{d}s \\ &= (1-\beta)\Big(H(v_i)v_i - \int_{r_M^*}^{v_i} H(s)\mathrm{d}s\Big) + \beta r_M^* \end{aligned}$$

for $v_i \geq r_M^*$.

5.1.2 Regularity

Let us examine whether the above relaxed solution satisfies the ignored conditions or not. First, by Assumption 1, it is clear that Value-Monotonicity condition for each demand type is satisfied: (Mon-1), (Mon-2), (Mon-Ma), (Mon-Mb), and (Mon-22) are satisfied. Second, (ND) is satisfied with equality. Suppose that agent i with $k_i = 2$ delays his entry. Because conditional distributions are identical, the optimal mechanism at period 2 is equivalent to a second-price auction with a reserve price r_2^*. The winning probability of agent i is $\tilde{\alpha}(v_i) = \Pr\{v_i \geq \max\{v_j, r_2^*\}\} = G(v_i)$, which is equivalent to that under the truthful entry.

Third, we consider (Cond-2M). It turns out that (Cond-2M) holds if the truthful payoff of demand type 2 is greater than that of demand type M.

Lemma 1 *Suppose that a mechanism satisfies all Value-Monotonicity conditions and payoff equivalence. Then, (Cond-2M) holds if* $\Pi_i^1(v,2) \geq \Pi_i^1(v,M)$ *for all* v.

Proof. See Appendix.

The remainder we need to check are (Cond-1M) and (Cond-M1); it turns out that they are not guaranteed under the current assumptions. The following theorem is immediate from the analysis thus far.

Theorem 3 *Consider the single-agent case. The relaxed solution is optimal if*

$$\beta r_M^* \leq r_1^* \leq \big((1-\beta)G(r_2^*) + \beta\big) r_M^* \tag{20}$$

and $\Pi_i^1(v,2) \geq \Pi_i^1(v,M)$ *for all* $v \in [0,\bar{v}]$. *The optimal menu of contracts at period 1 is* $p_1 = r_1^*$,

$$\begin{cases} \alpha_2(v) = G(v), \\ p_2(v) = G(v)v - \int_{r_2^*}^{v} G(s)\mathrm{d}s \end{cases} \tag{21}$$

for $v \geq r_2^*$, *and*

$$\begin{cases} a_M^1(v) = 1, \\ \alpha_M(v) = H(v), \\ p_M(v) = (1-\beta)\Big(H(v)v - \int_{r_M^*}^{v} H(s)\mathrm{d}s\Big) + \beta r_M^* \end{cases} \tag{22}$$

for $v \geq r_M^*$. *The optimal price at period 2 is*

$$p_j(\theta_i) = \begin{cases} r_2^* & \text{if } k_i = 1 \text{ or } \psi(\theta_i) < 0, \\ v_i & \text{if } k_i = 2 \text{ and } \psi(\theta_i) \geq 0, \\ (1-\beta)\psi_2^{-1}(\psi_M(v_i)) & \text{if } k_i = M \text{ and } \psi(\theta_i) \geq 0. \end{cases} \tag{23}$$

The optimal mechanism in Theorem 3 can be interpreted as a sequential sales mechanism. As we have seen in Section 2, the optimal allocation rule utilizes "overbooking" in the sense that even if agent i signs an "advance (contingent-) contract" on object 2, the seller sells the object to agent j too by a posted price. Even when agent i has a contract for object 2, there is no guarantee he will obtain it, indeed he may be denied at period 2. Due to the risk of losing the object, the price of advance agent i is discounted. The seller offers a menu of contracts, which she finely screens and price-discriminates the advance agent by using the risk of overbooking.

The same is true of contracts for multiple objects. Even at the consumption time of the first object, a multi-unit contract is discounted because the agent has a risk of being overbooked with respect to the future object.

In the presence of multiple objects with different periods, an incentive problem may arise for an agent with a unit demand for the object with the early deadline. At the consumption time of object 1, the last-minute price is relatively expensive because there is no risk of overbooking. However, because a multi-unit contract is discounted, an agent with single-unit demand may be better off purchasing a discounted multi-unit contract. The seller needs to design a menu of contracts so that such a non-monotonic price between single- and multi-unit contracts does not arise.

To have a primitive condition for the relaxed solution being optimal, we introduce hazard rate ordering of conditional distributions. Given the following Assumption 3, a sufficient condition for the relaxed solution being optimal is that the marginal value for object 2 is large for demand type M and that the last-minute buyer j at period 2 arrives with only a small probability. A similar assumption is considered by Dizdar et al. (2011), Pai and Vohra (2013), and Mierendorff (2016).

Assumption 3 For all v, $\lambda_M(v) < \lambda_1(v)$ and $\lambda_M(v) < \lambda_2(v)$.

Proposition 2 *Suppose Assumptions 1–3 hold. In the single-agent case, the relaxed solution is optimal if β is sufficiently small and the arrival rate of buyer j, q, is sufficiently small.*

Proof. See Appendix.

Hazard rate ordering and high complementarity for demand type M (i.e., a small β) guarantee the relaxed solution satisfies (Cond-M1) and (Cond-2M). Nevertheless, (Cond-1M) is not guaranteed because it critically depends on the distribution of the number of future agents.

5.1.3 When (Cond-1M) Is Binding

Condition (Cond-1M) is likely to be binding when the number of entrants at period 2, N^2, is expected to be large. The optimal mechanism under (Cond-1M) binding needs "ironing" of the last-minute price p_1 and the cheapest multi-unit contract price $p_M(c_i^1(M))$. In the following theorem, we focus on the case with $\beta = 0$ and $q = 1$.

Proposition 3 *Consider the single-agent case and suppose $\beta = 0$, $q = 1$, and $r_1^* > F_2(r_2^*)r_M^*$. Further suppose that in the optimal mechanism (Cond-2M) is not binding. Then, there exist $\bar{\alpha} \in (F_2(r_2^*), 1]$ and $c_i^1(M) \in (r_M^*, \bar{v}]$, and the optimal menu of contracts at period 1 is such that $p_1 = \bar{\alpha} c_i^1(M) < r_1^*$, $\alpha_2(v) = G(v)$ for $v \geq r_2^*$, and*

$$\begin{cases} a_M^1(v) = 1 \\ \alpha_M(v) = \max\{\bar{\alpha}, H(v)\} \end{cases} \tag{24}$$

for $v \geq c_i^1(M)$. The optimal price at period 2 is

$$p_j(\theta_i) = \begin{cases} r_2^* & \text{if } k_i = 1 \text{ or } \psi(\theta_i) < 0, \\ v_i & \text{if } k_i = 2 \text{ and } \psi(\theta_i) \geq 0, \\ F_2^{-1}(\bar{\alpha}) & \text{if } k_i = M \text{ and } v_i \in [c_i^1(M), H^{-1}(\bar{\alpha})], \\ \psi_2^{-1}(\psi_M(v_i)) & \text{if } k_i = M \text{ and } v_i > H^{-1}(\bar{\alpha}). \end{cases} \tag{25}$$

Proof. See Appendix.

The optimal mechanism in the case where (Cond-1M) is binding is understood as follows. Suppose $r_1^* > G(r_2^*)r_M^*$. Then, the optimal mechanism is determined so that

$$p_1 = \alpha_i^2(c_i^1(M), M)c_i^1(M). \tag{26}$$

To have (26), the cutoff value of demand type M should be increased:

$$c_i^1(M) > r_M^*. \tag{27}$$

For any given cutoff value $c_i^1(M) > r_M^*$, the probability of obtaining object 2 in the relaxed solution is given by $H(c_i^1(M)) = G(\psi_2^{-1}(\psi_M(c_i^1(M)))) > F_2(r_2^*)$. By (26), the probability will be increased more:

$$\bar{\alpha}_i^2(c_i(M), M) = \bar{\alpha} > H(c_i^1(M)). \tag{28}$$

In addition, because α_M must be increasing, we have

$$\alpha_M(v) = \max\{\bar{\alpha}, H(v)\} \tag{29}$$

for $v > c_i^1(M)$. That is, the optimal allocation at period 2 is distorted toward favoring demand type M when he has a relatively small virtual valuation. Finally, because the multi-unit contract generates allocative distortion at period 2, the last-minute price of object 1 is reduced:

$$p_1 < r_1^*. \tag{30}$$

When (Cond-1M) is binding, the seller faces a tradeoff between current and future profits. When the presence of a highly profitable agent in the future is expected, the seller wants to keep the option to allocate the object to the future agent. However, the option reduces the profitability of a current agent having multi-unit demand. When the agent evaluates the objects as complements, a high option value to the seller implies a low value of the current object for that agent. The seller may need to reduce the current price because it is bounded from above by the contract for the multi-unit demand agent.

When the seller faces the tradeoff, she optimally designs contracts for the agents with multi-unit demand in both advantageous and disadvantageous ways. First, as in (27), the seller excludes the agent with a low virtual value to keep the future option value high. Second, as in (28) and (29), the seller increases the probability of allocating object 2 to the agent to make the current object more valuable.

5.2 Multiple Agents

Let us examine the case where many agents arrive in each period. To simplify the description of the solution, let us introduce two dummy agents, named 0_1 and 0_2 and $0_1, 0_2 \in N^1$, each of whom has a type $\theta_{0_1}^1 = (r_1^*, 1)$ and $\theta_{0_2}^1 = (r_2^*, 2)$, respectively. In addition, the j-th highest order statistic of type-k virtual valuations, including the dummy 0_k, is denoted by $\psi_k^{(j)}$.

Consider $t = 2$ with any type profile θ^1 and an allocation a^1 at period 1. The relaxed virtual surplus maximization problem at $t = 2$ is written as

$$\max_{a^2} \sum_{k_i=M} a_i^2 a_i^1 (1-\beta)\psi_M(v_i) + \sum_{\{i \in \mathcal{N}^2 | k_i = 2\}} a_i^2 \psi_2(v_i) \tag{31}$$

subject to $\sum_i a_i^2 \leq 1$. Thus, the relaxed solution chooses the agent with the maximum positive virtual valuation among all type-2 agents and a type-M winner (multiplied by $(1-\beta)$) at period 1 if any. Agent i with demand type 2 obtains object 2 if

$$\psi_2(v_i) = \max\{a_{M_*}^1 (1-\beta)\psi_M^{(1)}, \psi_2^{(1)}\}, \tag{32}$$

where M_* denotes the demand-type-M agent's identity such that $\psi_M(v_{M_*}) = \psi_M^{(1)}$. Equivalently, given obtaining object 1, agent i with demand type M wins object 2 if his marginal virtual value for object 2 is greater than the maximum virtual valuation among demand type 2:

$$(1-\beta)\psi_M(v_i) \geq \psi_2^{(1)}. \tag{33}$$

This allocation rule is denoted by a^{2*}.

Consider $t = 1$ given a^{2*}. Let $\psi_2^{t,(j)}$ be the j-th highest order statistic of demand-type-2 virtual valuations among period-t agents; $\psi_2^{(1)} = \max\{\psi_2^{1,(1)}, \psi_2^{2,(1)}\}$. Let

$$\Psi(y) \equiv \max\{\psi_2^{2,(1)}, y\}$$

be the highest virtual value of demand type 2 given $\psi_2^{1,(1)} = y \geq 0$. The virtual surplus maximization at $t = 1$ is written by

$$\max\Big\{\psi_1^{(1)} + E[\Psi(\psi_2^{1,(1)})], E\big[\max\{\psi_M^{(1)}, \beta\psi_M^{(1)} + \Psi(\psi_2^{1,(1)})\}\big]\Big\}. \tag{34}$$

The highest type-1 agent wins if the first term is larger than the second term. In addition, the period-1 agent having $k_i = 2$ and $\psi_2(v_i) = \psi_2^{1,(1)} \geq 0$ obtains object 2 with a positive probability. When agent i is such an agent, he obtains object 2 with probability

$$\Pr\{\psi_2(v_i) > \max_{j \in N^2} \psi_2(v_j)\} = E[F_2^n(v_i)] \equiv G(v_i),$$

where expectation is taken over the number of period-2 agents $n = |N^2|$. Otherwise, the agent with demand type M having the highest positive virtual value is assigned object 1. The type-M agent i is assigned object 2 if $(1-\beta)\psi_M(v_i) > \psi_2^{(1)}$. It is clear that the type-M agent i is assigned object 1 only if $(1-\beta)\psi_M(v_i) > \psi_2^{1,(1)}$. Hence, the probability, given that an agent with $k_i = M$ obtains object 1, is denoted by

$$\Pr\{\max_{j \in N^2} \psi_2(v_j) < (1-\beta)\psi_M(v_i)\} = G\Big(\psi_2^{-1}\big((1-\beta)\psi_M(v_i)\big)\Big) \equiv H(v_i). \tag{35}$$

The derived allocation rule maximizes the virtual surplus and is denoted by a^*. The following theorem provides a condition for the allocation policy a^* being optimal.

Proposition 4 *The relaxed solution a^* is optimal and maximizes the seller's expected revenue if $\Pi_i^1((v_i, 2), \theta_{-i}^1) \geq \Pi_i^1((v_i, M), \theta_{-i}^1)$ for all $v_i \in [0, \bar{v}]$ and all $\theta_{-i}^1 \in \prod_{-i} \Theta^1$, and for all $x \geq r_M^*$ and all $y \geq 0$,*

$$\psi_1(\beta x) \leq E[\max\{\psi_M(x) - \Psi(y), \beta\psi_M(x)\}] \leq \psi_1\Big(x\big(\beta + (1-\beta)H(x)\big)\Big). \tag{36}$$

Proof. See Appendix.

To have a more primitive sufficient condition for the regularity, we impose some additional conditions on conditional distributions.

Assumption 4 The virtual valuation function ψ_M of demand type M is concave over $[r_M^*, \bar{v}]$.

Assumption 5 Inverse functions of virtual valuation functions, ψ_k^{-1}, satisfies

$$(\psi_1^{-1})'(y) \leq (\psi_M^{-1})'(y)$$

for all $y \geq 0$.

With these assumptions, including hazard rate ordering, a sufficient condition for the regularity (36) is equivalent to the single-agent case. Hence, the relaxed solution is optimal if the arrival rate of period-2 agents is sufficiently low.

Theorem 4 *Suppose that Assumptions 1–5 hold and that β is sufficiently small. The relaxed solution a^* is optimal if*

$$r_1^* \leq \big(\beta + (1-\beta)G(r_2^*)\big) r_M^*. \tag{37}$$

Proof. See Appendix.

Concave virtual valuation function is discussed by Mierendorff (2016) too. In Assumption 4, concavity of ψ_M only is imposed. If ψ_1 is also concave, we have another simple sufficient condition for the regularity.

Theorem 5 *Suppose that Assumptions 1–3 hold and that β is sufficiently small. Further suppose that both ψ_1 and ψ_M are concave in their non-negative ranges and that they satisfy $\psi_1'(x) \geq \psi_M'(x)$ for all $x \geq r_M^*$. Then, the relaxed solution a^* is optimal if (37) holds.*

Proof. Let $y = \psi_M(x)$ for arbitrary $x \geq r_M^*$. Because $\psi_1(x) \geq y$, $\tilde{x} \equiv \psi_1^{-1}(y) \leq x$ and we have $\psi_M'(x) \leq \psi_1'(x) \leq \psi_1'(\tilde{x})$. Hence, we have $(\psi_1^{-1})'(y) \leq (\psi_M^{-1})'(y)$. Therefore, the theorem holds immediately by Theorem 4. ■

6 Commitment Mechanisms

In this section we consider the case where complete contracts contingent on future events are not available but the seller has to determine the allocation of the future object to advance agent. Thus far, we have examined the fully optimal mechanism using contingent-contracts. In practice, however, it is often hard to implement a contingent-allocation rule because it is too complex. The seller may have to commit advance agents to allocate the future object. We consider *commitment mechanisms*, in which an allocation rule is restricted to satisfy

- for all $i \in N^1$ with $k_i = M$, $a_i^2(\theta^1, \theta^2) = a_i^1(\theta^1) \in \{0, 1\}$, and
- for all $i \in N^1$ with $k_i = 2$, $a_i^2(\theta^1, \theta^2)$ is independent of θ^2: i.e., $\alpha_i^2(\theta^1) \in \{0, 1\}$

in addition to Assumption 2.

Even when we focus on commitment mechanisms, the characterization of incentive compatibility does not change but is simplified. By definition of the cutoff value $c_i^1(M, \theta_{-i}^1)$, an agent i with $\theta_i^1 = (v_i, M)$ is allocated both objects if $v_i > c_i^1(M, \theta_{-i}^1)$. Similarly, we also define the cutoff value of demand type 2 as

$$c_i^1(2, \theta_{-i}^1) \equiv \inf\{v_i | a_i^2(\theta_i^1, \theta_{-i}^1) = 1\}. \tag{38}$$

Theorem 1 immediately provides the following result.

Corollary 1 *A commitment mechanism is PEPIC if and only if*

1. *there exist two functions* $\underline{\Pi}_i^1(\theta_{-i}^1)$ *and* $\underline{\Pi}_j^2(\theta_{-j}^2, \theta^1)$, *satisfying* $\underline{\Pi}_i^1(\theta_{-i}^1) \geq E[\underline{\Pi}_i^2(\theta^2, \theta_{-i}^1)]$ *for all* $\theta_{-i}^1 \in \prod_{N_{-i}^1} \Theta^1$,
2. *the allocation rule is weakly increasing in each agent's own valuation (i.e., (Mon-1), (Mon-Ma), (Mon-2), and (Mon-22) hold),*
3. *the associated cutoff values satisfy*

$$\beta c_i^1(M, \theta_{-i}^1) \leq c_i^1(1, \theta_{-i}^1) \leq c_i^1(M, \theta_{-i}^1), \tag{39}$$

$$c_i^1(2, \theta_{-i}^1) \leq c_i^1(M, \theta_{-i}^1), \tag{40}$$

$$\max\{v - c_i^1(2, \theta_{-i}^1), 0\} + d_i(\theta_{-i}^1) \geq \int_0^v \tilde{\alpha}_i^2(s, \theta_{-i}^1) \mathrm{d}s, \tag{41}$$

where

$$d_i(\theta^1_{-i}) = \underline{\Pi}^1_i(\theta^1_{-i}) - E[\underline{\Pi}^2_i(\theta^2, \theta^1_{-i})],$$

and

4. *Truthful payoffs are given by*

$$\Pi^1_i((v_i, k_i), \theta^1_{-i}) = \underline{\Pi}^1_i(\theta^1_{-i}) + \max\{v_i - c^1_i(k_i, \theta^1_{-i}), 0\}, \tag{42}$$

$$\Pi^2_j(v_j, \theta^2_{-j}, \theta^1) = \underline{\Pi}^2(\theta^2_{-j}, \theta^1) + \max\{v_j - c^2_j(\theta^2_{-j}, \theta^1), 0\}. \tag{43}$$

The optimal allocation policy is derived in the same manner as the fully optimal mechanism case. The optimal allocation rule in period 2 is exactly the same, so that we focus on period 1. Recall $\Psi(0) = E[\max\{\psi_2^{2,(1)}, 0\}]$. The maximum virtual surplus is written by

$$\max\{\psi_1^{(1)} + \max\{\psi_2^{1,(1)}, \Psi(0)\}, \psi_M^{(1)}\}. \tag{44}$$

The agent with the highest virtual value among demand type 1 is allocated object 1 if $\psi_1(v_i) > 0$ and $\psi_1(v_i) > \psi_M^{(1)} - \max\{\psi_2^{1,(1)}, \Psi(0)\}$. The agent with the highest virtual value among demand type 2 at period 1 is allocated object 2 if $\psi_2(v_i) \geq \Psi(0)$ and $\psi_2(v_i) > \psi_M^{(1)} - \psi_1^{(1)}$. The agent with the highest virtual value among demand type M is allocated both objects if $\psi_M(v_i) > \psi_1^{(1)} + \max\{\psi_2^{1,(1)}, \Psi(0)\}$. Otherwise, agents are not allocated either object. In period 2, the agent with the highest positive virtual value is allocated object 2 if it is still available. This allocation policy is denoted by $\hat{a}$.

The allocation policy $\hat{a}$ satisfies all the PEPIC conditions except for (41) if Assumption 3 holds. However, it turns out that the allocation policy $\hat{a}$ does not satisfy (41). Under $\hat{a}$, advance agents demanding object 2 need to compensate the option value of the period-2 agents and pay much. However, if no high-valued agent arrives at period 2, the object is sold with a lower price r_2^*. Expecting the possibility of this "fire-sale," an advance agent with demand type 2 has an incentive to wait and purchase later. Therefore, the constrained-optimal allocation policy $\hat{a}$ is implementable if demand-type-2 agents are not forward-looking but short-lived.

Proposition 5 *The allocation policy $\hat{a}$ does not satisfy (41) and is not implementable if agents with demand type 2 are forward-looking. It is implementable if Assumption 3 holds, $\beta > 0$ is sufficiently small, and if period-1 agents with $k_i = 2$ are short-lived and exit the mechanism when they are not allocated.*

Proof. See Appendix.

7 Conclusion

This paper considers a dynamic mechanism design in which the seller allocates heterogeneous objects over time and agents arrive in different points in time. Agents have private information about their desired objects and valuations. They may demand multiple objects and evaluate them as complements. The seller has a full commitment power, and a complex contingent-contract is available. We provide a necessary and sufficient condition for a mechanism being periodically ex-post incentive compatible. Myerson's (1981) canonical result is extended to a multi-dimensional type, and our characterization extends those of technically similar models by Dizdar et al. (2011), Pai and Vohra (2013), and Mierendorff (2016). The seller's expected revenue is transformed into virtual surplus form, and we provide a regularity condition such that the relaxed solution satisfies implementability conditions. The assumption of hazard rate ordering of conditional distributions is not sufficient for implementability. The seller may face a tradeoff between holding a high option profit raised from the future and posting a high price of a current object. The tradeoff does not arise when the probability of arrival of a new agent in the future is sufficiently small and the option profit is not large.

We have many open questions for future research. First of all, extension to a general T-period model is an important work but is beyond this study. One might wonder if in a ticket sales problem an agent with multi-unit demand may want to purchase multiple "single tickets" separately. This type of deviation, which is examined by Todo et al. (2011) and Deb and Said (2015), is not captured by the current model. Because in our model a contract for a multi-unit-demand agent is likely to be cheap, the incentive to signing multiple contracts might be limited.

A Proofs

A.1 Proof of Theorem 1

The cutoff values are defined as in the main text. We show the case where $N^1 = \{i\}$ and $|N^2| = \{j\}$. The proof for a general number of agents is the same.

Only If part. Suppose that a mechanism is PEPIC and $i \in N^1$. Value-Monotonicity conditions (Mon-1), (Mon-2), (Mon-Ma), (Mon-Mb), and (Mon-22) are straightforwardly implied by Proposition 1. Specifically, for each k_i, PEPIC requires

$$A_i(v_i, k_i, k_i)v_i - p_i^1(v_i, k_i) \geq A_i(\tilde{v}_i, k_i, k_i)v_i - p_i^1(\tilde{v}_i, k_i).$$

Hence,

$$(A_i(v_i, k_i, k_i) - A_i(\tilde{v}_i, k_i, k_i))v_i \geq p_i^1(v_i, k_i) - p_i^1(\tilde{v}_i, k_i).$$

Similarly, we have

$$(A_i(v_i, k_i, k_i) - A_i(\tilde{v}_i, k_i, k_i)\tilde{v}_i \leq p_i^1(v_i, k_i) - p_i^1(\tilde{v}_i, k_i).$$

Thus,

$$(A_i(v_i, k_i, k_i) - A_i(\tilde{v}_i, k_i, k_i))\tilde{v}_i \leq (A_i(v_i, k_i, k_i) - A_i(\tilde{v}_i, k_i, k_i))v_i.$$

Therefore, $\tilde{v}_i < v_i$ leads to $A_i(\tilde{v}_i, k_i, k_i) \leq A_i(v_i, k_i, k_i)$.

For demand type M, value-monotonicity requires

$$A_i(v_i, M, M) = a_i^1(v_i, M)\big((1-\beta)\alpha_i^2(v_i, M) + \beta\big)$$

is weakly increasing in v_i. If $\beta > 0$, $A_i > 0$ whenever $a_i^1(v_i, M) = 1$. Hence, $a_i^1(\cdot, M)$ is increasing. In contrast, if $\beta = 0$, we have $A_i(v_i, M, M) = a_i^1(v_i, M)\alpha_i^2(v_i, M) = \alpha_i^2(v_i, M)$ becuase $\alpha_i^2(v_i, M) = 0$ whenever $a_i^1(v_i, M) = 0$ by Assumption 2.3. Hence, (Mon-Ma) is not necessary in the case of $\beta = 0$.

By the envelope theorem (Milgrom and Segal, 2002), if $\Pi_i^t(v_i, k_i) \geq \pi_i^t((\tilde{v}_i, k_i), \theta_i^t)$ for all $\tilde{v}_i$, then

$$\frac{\partial \Pi_i^t(v, k_i)}{\partial v} = \frac{\partial \pi_i^t((\tilde{v}, k_i), (v, k_i))}{\partial v}\Big|_{\tilde{v}=v}$$

almost everywhere. Hence, we have

$$\Pi_i^1(v, 1) = \Pi_i^1(0, 1) + \max\{v - c_i^1(1), 0\}, \tag{45}$$

$$\Pi_j^2(v, \theta^1) = \Pi_i^2(0, \theta^1) + \max\{v - c_i^2(\theta^1), 0\}, \tag{46}$$

$$\Pi_i^1(v, 2) = \Pi_i^1(0, 2) + \int_0^v \alpha_i^2(s, 2)\mathrm{d}s \tag{47}$$

$$\Pi_i^1(v, M) = \Pi_i^1(0, M) + \int_{c_i^1(M)}^v \big((1-\beta)\alpha_i^2(s, M) + \beta\big)\mathrm{d}s. \tag{48}$$

Suppose $\Pi_i^1(0,k_i) < \Pi_i^1(0,k_i')$ for some pair (k_i,k_i'). Then, agent i with a type $\theta_i^1 = (0,k_i)$ is better off by deviating and reporting $\theta_i^{1'} = (0,k_i')$. This violates PEPIC. Therefore, $\Pi_i^1(0,k_i)$ is independent of k_i and $\Pi_i^1(0,k_i) = \underline{\Pi}_i^1$ for all $k_i \in \{1,2,M\}$.

Suppose that agent i with a type $(v_i,1)$ deviates and reports $(\tilde{v}_i,M)$ and that $v_i > c_i^1(1)$. By (48) and $A_i(\cdot,M,M)$ increasing, $p_i^1(\cdot,M)$ is non-decreasing. Hence, a demand-type-M payment is bounded from below by

$$\bar{p}_i^1(c_i^1(M),M) \equiv \inf\{p_i^1(v,M)|a_i^1(v,M)=1\} = \lim_{v\to c_i^1(M)+} p_i^1(v,M). \tag{49}$$

By (48), we have

$$\bar{p}_i^1(c_i^1(M),M) = \big((1-\beta)\bar{\alpha}_i^2(c_i^1(M),M)+\beta\big)c_i^1(M) - \underline{\Pi}_i^1.$$

Hence, incentive compatibility implies

$$\begin{aligned} \underline{\Pi}_i^1 + v_i - c_i^1(1) &\geq \underline{\Pi}_i^1 + v_i - \big((1-\beta)\bar{\alpha}_i^2(c_i^1(M),M)+\beta\big)c_i^1(M), \\ \therefore\ c_i^1(1) &\leq \big((1-\beta)\bar{\alpha}_i^2(c_i^1(M),M)+\beta\big)c_i^1(M). \end{aligned}$$

Conversely, suppose agent i with a type $(c_i^1(M),M)$ deviates and reports $(\tilde{v}_i,1)$. The associated payoff under such a deviation is $\underline{\Pi}_i^1 + \max\{\beta c_i^1(M) - c_i^1(1), 0\}$. Because the truthful payoff of type $(c_i^1(M),M)$ is $\underline{\Pi}_i^1$, PEPIC implies

$$\beta c_i^1(M) - c_i^1(1) \leq 0,$$

which is (Cond-M1).

Suppose that agent i with a type $(v_i,2)$ deviates and reports $(\tilde{v}_i,M)$. The associated payoff under such a deviation is

$$\alpha_i^2(\tilde{v}_i,M)v_i - p_i^1(\tilde{v}_i,M).$$

By PEPIC in valuation with demand type M, we have for every $s > c_i^1(M)$ and every $\tilde{v}_i > c_i^1(M)$,

$$\begin{aligned} \big((1-\beta)\alpha_i^2(s,M)+\beta\big)s - p_i^1(s,M) &\geq \big((1-\beta)\alpha_i^2(\tilde{v}_i,M)+\beta\big)s - p_i^1(\tilde{v}_i,M) \\ \therefore\ \alpha_i^2(s,M)(1-\beta)s - p_i^1(s,M) &\geq \alpha_i^2(\tilde{v}_i,M)(1-\beta)s - p_i^1(\tilde{v}_i,M). \end{aligned}$$

Hence, we have

$$\alpha_i^2(\tilde{v},M)v - p_i^1(\tilde{v},M) \leq \alpha_i^2\Big(\frac{v}{1-\beta},M\Big)v - p_i^1\Big(\frac{v}{1-\beta},M\Big) \tag{50}$$

for all $\tilde{v} > c_i^1(M)$ and all $v = (1-\beta)s > (1-\beta)c_i^1(M)$. Using revenue equivalence formula (48),

$$\alpha_i^2\Big(\frac{v}{1-\beta}, M\Big)v - p_i^1\Big(\frac{v}{1-\beta}, M\Big) = \underline{\Pi}_i^1 + \int_{(1-\beta)c_i^1(M)}^{v} \alpha_i^2\Big(\frac{s}{1-\beta}, M\Big)\mathrm{d}s - \beta c_i^1(M), \tag{51}$$

which is the optimal payoff under the deviation. Because $\alpha_i^2(v_i, M) = 0$ for all $v_i < c_i^1(M)$, PEPIC requires for all $v_i \geq (1-\beta)c_i^1(M)$,

$$\Pi_i^1(v_i, 2) \geq \underline{\Pi}_i^1 + \int_0^{v_i} \alpha_i^2\Big(\frac{s}{1-\beta}, M\Big)\mathrm{d}s - \beta c_i^1(M) \tag{52}$$

which implies (Cond-2M).

Finally, suppose that agent i with a type $\theta_i^1 = (v_i, 2)$ delays his arrival and reports $\hat{\theta}_i^2$ at period 2. By PEPIC, the optimal report at period 2 is truthful: $\hat{\theta}_i^2 = v_i$. Hence, PEPIC requires

$$\underline{\Pi}_i^1 + \int_0^v \alpha_i^2(s, 2)\mathrm{d}s \geq E[\underline{\Pi}_i^2(\theta^2, \theta_{-i}^1)] + \int_0^v \tilde{\alpha}_i^2(s, \theta_{-i}^1)\mathrm{d}s.$$

If part. By Proposition 1, we have incentive compatibility in valuation from (Mon-1), (Mon-2), (Mon-Ma), (Mon-Mb), (Mon-22), and payoff equivalence formulas.

Suppose that agent i has a type $\theta_i^1 = (v_i, 1)$ and misreports $(\tilde{v}_i, M)$. The most profitable deviation of agent i is such that he reports $(c_i^1(M)_+, M)$ because $a_i^1(v_i, M) = 1$ whenever $v_i > c_i^1(M)$ and $p_i^1(\cdot, M)$ is non-decreasing. By (16), $\bar{p}_i^1(c_i^1(M), M) = \big((1-\beta)\bar{\alpha}_i^2(c_i^1(M), M) + \beta\big)c_i^1(M) - \underline{\Pi}_i^1$. If $v_i > c_i^1(1)$, we have $v_i - c_i^1(1) + \underline{\Pi}_i^1 \geq v_i - \bar{p}_i^1(c_i^1(M), M)$ by (Cond-1M) and the deviation is not profitable. If $v_i \leq c_i^1(1)$, we have $v_i - \bar{p}_i^1(c_i^1(M), M) < \underline{\Pi}_i^1$ by (Cond-1M) and the deviation is not profitable. Deviations from an agent with demand-type-1 to type-2 or delaying reporting are obviously unprofitable.

Suppose that agent i has a type $\theta_i^1 = (v_i, M)$ and misreports $(\tilde{v}_i, 1)$. Because the truthful payoff is at least $\underline{\Pi}_i^1$, suppose $\beta v_i \geq c_i^1(1)$. (Cond-M1) implies $v_i \geq c_i^1(M)$ and

$$\begin{aligned}\Pi_i^1(v_i, M) &= \underline{\Pi}_i^1 + \int_{c_i^1(M)}^{v_i} \big((1-\beta)\alpha_i^2(s, M) + \beta\big)\mathrm{d}s \\ &\geq \underline{\Pi}_i^1 + \beta(v_i - c_i^1(M)) \\ &\geq \underline{\Pi}_i^1 + \beta v_i - c_i^1(1).\end{aligned}$$

Hence, such a deviation is not profitable. Deviation to demand type 2 is obviously unprofitable.

Suppose that agent i has a type $\theta_i^1 = (v_i, 2)$ and misreports $(\tilde{v}_i, M)$. When $v_i \geq (1-\beta)c_i^1(M)$, his expected payoff is

$$\alpha_i^2(\tilde{v}_i, M)v_i - p_i^1(\tilde{v}_i, M) \leq \underline{\Pi}_i^1 + \int_{(1-\beta)c_i^1(M)}^{v_i} \alpha_i^2\Big(\frac{s}{1-\beta}, M\Big)\mathrm{d}s - \beta c_i^1(M)$$
$$\leq \Pi_i^1(v_i, 2)$$

by (Cond-2M). Suppose $v_i < (1-\beta)c_i^1(M)$. We already have incentive compatibility in valuation with demand type M, so that it implies

$$(1-\beta)\alpha_i^2(\tilde{v}, M)s + \beta s - p_i^1(\tilde{v}, M) \leq \underline{\Pi}_i^1$$

for all $s < c_i^1(M)$ and all $\tilde{v}_i \geq c_i^1(M)$. Hence, for every $v_i = (1-\beta)s < (1-\beta)c_i^1(M)$,

$$\alpha_i^2(\tilde{v}, M)v_i - p_i^1(\tilde{v}, M) \leq \underline{\Pi}_i^1 - \beta\frac{v_i}{1-\beta} \leq \underline{\Pi}_i^1 \leq \Pi_i^1(v_i, 2).$$

Deviation of a type-2 to type-1 is obviously unprofitable.

Suppose a demand-type-2 agent at period 1 delays reporting. Because the mechanism at period 2 is ex-post incentive compatible, it is optimal to report truthfully at period 2. Hence, by (ND), it is unprofitable to delay reporting. ■

A.2 Proof of Lemma 1

Suppose that a mechanism satisfies Value-Monotonicity and payoff equivalence. This means that the mechanism is incentive compatible in valuation. The RHS of (Cond-2M) is

$$\begin{aligned}
&\int_{(1-\beta)c_i^1(M)}^{v} \alpha_i^2\Big(\frac{s}{1-\beta}, M\Big)\mathrm{d}s - \beta c_i^1(M) \\
=&(1-\beta)\int_{c_i^1(M)}^{\frac{v}{1-\beta}} \alpha_i^2(\tilde{s}, M)\mathrm{d}\tilde{s} - \beta c_i^1(M) \\
=&(1-\beta)\Big(\int_{c_i^1(M)}^{v} \alpha_i^2(s, M)\mathrm{d}s + \int_{v}^{\frac{v}{1-\beta}} \alpha_i^2(s, M)\mathrm{d}s\Big) - \beta c_i^1(M) \qquad (53) \\
\leq&(1-\beta)\int_{c_i^1(M)}^{v} \alpha_i^2(s, M)\mathrm{d}s + (1-\beta)\Big(\frac{v}{1-\beta} - v\Big) - \beta c_i^1(M) \\
=&\int_{c_i^1(M)}^{v} \big((1-\beta)\alpha_i^2(s, M) + \beta\big)\mathrm{d}s = \Pi_i^1(v, M).
\end{aligned}$$

The first equality follows by transformation of variable from s to $\tilde{s} = s/(1-\beta)$. The inequality comes from $\alpha_i^2(v, M) \leq 1$. Hence, (Cond-2M) holds if $\Pi_i^1(v, 2) \geq \Pi_i^1(v, M)$. ■

A.3 Proof of Proposition 2

With a sufficiently small $\beta > 0$, it is clear that $\beta r_M^* \leq r_1^*$ holds. Because $G(v) = qF_2(v) + (1-q)$ for $v \geq r_2^*$, $G(r_2^*)$ is close to 1 as q is sufficiently small. Assumption 3 implies

$$\psi_1(v) > \psi_M(v)$$

for all v. Hence, we have $r_1^* = \psi_1^{-1}(0) < \psi_M^{-1}(0) = r_M^*$. With a sufficiently small q, we have

$$r_1^* \leq \big((1-\beta)G(r_2^*) + \beta\big) r_M^*.$$

Assumption 3 also implies $G(v) > G(\psi_2^{-1}(\psi_M(v)))$ for all v. Taking a sufficiently small β, we have for every $v \geq r_M^*$,

$$\begin{aligned}
\alpha_2(v) = G(v) &\geq (1-\beta)G(\psi_2^{-1}(\psi_M(v))) + \beta \\
&> (1-\beta)G(\psi_2^{-1}((1-\beta)\psi_M(v))) + \beta \\
&= (1-\beta)H(v) + \beta.
\end{aligned}$$

In addition, we have $r_2^* < r_M^*$. Therefore, for every $v < r_M^*$, we have $\Pi_i^1(v,2) \geq 0 = \Pi_i^1(v,M)$, and for every $v \geq r_M^*$,

$$\Pi_i^1(v,2) = \int_{r_2^*}^{v} G(s)\mathrm{d}s > \int_{r_M^*}^{v} \big((1-\beta)H(s) + \beta\big)\mathrm{d}s = \Pi_i^1(v,M). \tag{54}$$

The relaxed solution is optimal by Theorem 3. ■

A.4 Proof of Proposition 3

Because $\beta = 0$, we have $r_2^* > r_M^*$ and $G(v) > H(v)$ for all v, so that (Cond-2M) is satisfied in the relaxed solution. In addition, (Cond-M1) is clearly satisfied.

The optimal mechanism at period 1 is specified by cutoff values, $c_i^1(1)$ and $c_i^1(M)$, and winning probability functions $\alpha_i^2(v,2)$ and $\alpha_i^2(v,M)$. Because incentive compatibility with respect to demand type 2 is satisfied, we determine $\alpha_i^2(v,2) = F_2(v)$ for all $v \geq r_2^*$. We can focus on the cases in which $k_i \in \{1, M\}$.

When $k_i = 1$, it is obvious that the optimal allocation rule is implemented by the posted price $p_j = r_2^*$. Suppose $k_i = M$ and $\alpha_i^2(v,M) = \bar{\alpha} \neq H(v)$ for some v.

Given this, the revenue maximization problem at period 2 is given by[9]

$$\begin{aligned} \max_{a^2} \int_{v_j} & a_j^2(v_i, v_j)\psi_2(v_j) f_2(v_j) \mathrm{d}v_j \\ \text{s.t.} \quad \int_{v_j} & a_i^2(v_i, v_j) f_2(v_j) \mathrm{d}v_j = \bar{\alpha}, \\ & a_i^2(v_i, v_j) + a_j^2(v_i, v_j) \leq 1. \end{aligned} \tag{55}$$

Hence, the constraint is replaced with

$$\int_{v_j} a_j^2(v_i, v_j) f_2(v_j) \mathrm{d}v_j \leq 1 - \bar{\alpha}. \tag{56}$$

When $\bar{\alpha} \geq F_2(r_2^*)$ (and $\bar{\alpha} > H(v)$), the solution of the maximization problem is

$$a_j^{2*}(v_i, v_j) = \begin{cases} 1 & \text{if } v_j > F_2^{-1}(\bar{\alpha}) \\ 0 & \text{if } v_j < F_2^{-1}(\bar{\alpha}) \end{cases}. \tag{57}$$

When $\bar{\alpha} < F_2(r_2^*)$, the solution is

$$a_j^{2*}(v_i, v_j) = \begin{cases} 1 & \text{if } v_j > r_2^* \\ 0 & \text{if } v_j < r_2^* \end{cases}, \tag{58}$$

as in the case of $k_i = 1$.

(Cond-1M) conditions $c_i^1(M)$ and $\bar{\alpha}_i^2(c_i^1(M), M)$ only. We have no constraint on $\alpha_i^2(v, M)$ for $v > c_i^1(M)$, except for α_i^2 being increasing. Suppose that $c_i^1(M) \geq r_M^*$ and that $\alpha_i^2(c_i^1(M), M) = \bar{\alpha} > H(c_i^1(M))$. Then, by the optimal allocation rule at period 2 and the virtual value maximization, it is clear that the optimal allocation policy is determined by

$$\alpha_i^2(v, M) = \max\{\bar{\alpha}, H(v)\}$$

for $v > c_i^1(M)$.

Given the above properties, the control variables in the revenue maximization problem is a tuple $(c_i^1(1), c_i^1(M), \bar{\alpha})$. The Lagrange function for the revenue maxi-

[9] By fixing $k_i = M$, $a_i^2(\theta_i, v_j)$ is replaced with $a_i^2(v_i, v_j)$ for simplicity.

mization problem is written as follows[10]:

$$\begin{aligned}\max_{c_1,c_M,\bar{\alpha}} \quad & f(1)\int_{c_1}^{\bar{v}}\psi_1(v)f_1(v)\mathrm{d}v + f(M)F_M(c_M)\int_{r_2^*}^{\bar{v}}\psi_2(v_j)f_2(v_j)\mathrm{d}v_j \\ & +f(M)\Big[\int_{c_M}^{v(\bar{\alpha})}\bar{\alpha}\psi_M(v)f_M(v)\mathrm{d}v + \int_{F_2^{-1}(\bar{\alpha})}^{\bar{v}}\psi_2(v_j)f_2(v_j)\mathrm{d}v_j(F_M(v(\bar{\alpha}))-F_M(c_M))\Big] \\ & +f(M)\int_{v(\bar{\alpha})}^{\bar{v}}\int_{v_j}[a_i^2\psi_M(v_i)+a_j^2\psi_2(v_j)]f_M(v_i)f_2(v_j)\mathrm{d}v_j\mathrm{d}v_i + \mu(\bar{\alpha}c_M - c_1) \\ \text{s.t. } & a_i^2 + a_j^2 \leq 1, \\ & \mu(\bar{\alpha}c_M - c_1) = 0,\end{aligned} \tag{59}$$

where $c_1 = c_i^1(1)$, $c_M = c_i^1(M)$, $v(\bar{\alpha}) = H^{-1}(\bar{\alpha})$, and μ is a Lagrange multiplier with respect to (Cond-1M). Taking the first-order conditions, we have the following equations:

$$\frac{\partial L}{\partial c_1} = 0 : f(c_1, 1)\psi_1(c_1) + \mu = 0, \tag{60}$$

$$\frac{\partial L}{\partial c_M} = 0 : f(c_M, M)\Big[\int_{r_2^*}^{F_2^{-1}(\bar{\alpha})}\psi_2(v_j)f_2(v_j)\mathrm{d}v_j - \bar{\alpha}\psi_M(c_M)\Big] + \mu\bar{\alpha} = 0, \tag{61}$$

$$\frac{\partial L}{\partial \bar{\alpha}} = 0 : f(M)\int_{c_M}^{v(\bar{\alpha})}\big[\psi_M(v_i) - \psi_M(v(\bar{\alpha}))\big]f_M(v_i)\mathrm{d}v_i + \mu c_M = 0. \tag{62}$$

Equation (61) is rewritten as

$$f(c_M, M)\Big[\int_{r_2^*}^{F_2^{-1}(\bar{\alpha})}[\psi_2(v_j) - \psi_M(c_M)]f_2(v_j)\mathrm{d}v_j - \int_0^{r_2^*}\psi_M(c_M)f_2(v_j)\mathrm{d}v_j\Big] + \mu\bar{\alpha} = 0. \tag{63}$$

Because $r_1^* > F_2(r_2^*)r_M^*$, condition (Cond-1M) is binding: $\mu > 0$ and $\bar{\alpha}c_i^1(M) = c_i^1(1)$. Then, we have $c_i^1(1) = \psi_1^{-1}(-\mu/f(c_i^1(1), 1)) < r_1^*$. To have (61), we need $\psi_M(c_M) > 0$, which implies $c_i^1(M) > r_M^*$. To have (62), we need $v(\bar{\alpha}) > c_M$, which implies $\bar{\alpha} > H(c_i^1(M))$. ■

A.5 Proof of Proposition 4

By the standard regularity Assumption 1, it is clear that Value-Monotonicity for each k_i is satisfied. In particular, when agent i has demand type 2 and $\psi_2(v_i) = \psi_2^{1,(1)}$,

[10] As we noted, we can ignore the cases in which $k_i = 2$.

and when another agent with demand type 1 (who may be a dummy) obtains object 1, agent i obtains object 2 with probability

$$\Pr\{\psi_2(v_i) > \psi_2^{2,(1)}\} = G(v_i)$$

by Assumption 1 and it is increasing in v_i. Given that agent i with demand type M is assigned object 1, he obtains object 2 too with probability

$$\Pr\{(1-\beta)\psi_M(v_i) > \psi_2^{2,(1)}\} = G\big(\psi_2^{-1}((1-\beta)\psi_M(v_i)\big) = H(v_i),$$

which is increasing in v_i.

Consider condition (ND). We have $\underline{\Pi}_i^1(\theta_{-i}^1) = \underline{\Pi}_i^2(\theta^2, \theta_{-i}^1) = 0$ for all θ_{-i}^1 and all θ^2. Suppose that agent i with $\theta_i^1 = (v_i, 2)$ has a positive probability of obtaining object 2 under θ_{-i}^1. Then, the probability of obtaining the object is $G(v_i)$, and the probability of obtaining the object in the case of strategic delay is the same. What we will verify is that given any θ_{-i}^1 and θ^2, $a_i^{2*}((v_i, \theta^2), \theta_{-i}^1) = 1$ always implies $a_i^{2*}(\theta^2, ((v_i, 2), \theta_{-i}^1)) = 1$. Indeed, suppose $a_i^{2*}((v_i, \theta^2), \theta_{-i}^1) = 1$. Suppose first that object 1 is allocated to agent j with demand type 1 (or a dummy). Then, agent i obtains object 2 because

$$v_i \geq \max_{m \in \mathcal{N}^2 : k_m = 2} v_m.$$

Hence, when agent i arrives at period 1 and reports $(v_i, 2)$, it is clear that another agent j obtains object 1 and so that agent i obtains object 2. Suppose second that object 1 is allocated to an agent j with demand type M. Then, agent i obtains object 2 because

$$v_i \geq \max\{\psi_2^{-1}((1-\beta)\psi_M(v_j), \max_{m \in \mathcal{N}^2 : k_m = 2} v_m\}.$$

Thus, we have $\psi_2(v_i) \geq (1-\beta)\psi_M(v_j)$, so that agent i still obtains the object when he arrives at period 1 and reports $(v_i, 2)$. Therefore, we conclude that for any θ_{-i}^1, $\alpha_i^2((v_i, 2), \theta_{-i}^1) \geq \tilde{\alpha}_i^2(v_i, \theta_{-i}^1)$, which implies (ND).

By Lemma 1, Condition (Cond-2M) holds when $\Pi_i^1((v_i, 2), \theta_{-i}^1) \geq \Pi_i^1((v_i, M), \theta_{-i}^1)$.

Hence, the relaxed solution is optimal if it satisfies (Cond-M1) and (Cond-1M). Condition (Cond-M1) is given by

$$\beta c_i^1(M) \leq c_i^1(1) \;\Leftrightarrow\; \psi_1(\beta c_i^1(M)) \leq \psi_1(c_i^1(1)).$$

Given the others' type profile, agent i's cutoff values, $c_i^1(1)$ and $c_i^1(M)$, satisfy

$$\psi_1(c_i^1(1)) + E[\Psi(z)] = E[\max\{\psi_M(c_i^1(M)), \beta\psi_M(c_i^1(M)) + \Psi(z)\}]$$

with $z = \psi_2^{1,(1)}$. Hence,

$$
\begin{aligned}
\psi_1(c_i^1(1)) &= E[\max\{\psi_M(c_i^1(M)), \beta\psi_M(c_i^1(M)) + \Psi(z)\} - \Psi(z)] \\
&= E[\max\{\psi_M(c_i^1(M)) - \Psi(z), \beta\psi_M(c_i^1(M))\}].
\end{aligned}
\tag{64}
$$

Therefore, (Cond-M1) and (Cond-1M) hold if (36) holds. ■

A.6 Proof of Theorem 4

By taking a sufficiently small β, we have for $x \geq r_M^*$ and $y \geq 0$,

$$
\psi_1(\beta x) \leq 0 \leq E[\max\{\psi_M(x) - \Psi(y), \beta\psi_M(x)\}].
$$

Let $A(x) \equiv \beta + (1-\beta)H(x)$. $A(x)$ is increasing, and by $H(r_M^*) = G(r_2^*)$, we have $\beta + (1-\beta)G(r_2^*) \leq A(x) \leq 1$. Fix arbitrary $x > r_M^*$. By concavity of ψ_M, we have

$$
A(x)\psi_M(x) = A(x)\psi_M(x) + (1-A(x))\psi_M(r_M^*) \leq \psi_M\big(A(x)x + (1-A(x))r_M^*\big). \tag{65}
$$

Then there exists $\tilde{x} \in (r_M^*, A(x)x + (1-A(x))r_M^*]$ such that $\psi_M(\tilde{x}) = A(x)\psi_M(x)$. Note that

$$
A(x)x \geq \tilde{x} - (1-A(x))r_M^* \geq \tilde{x} - (1-A(r_M^*))r_M^*. \tag{66}
$$

By Assumption 5, we have

$$
\psi_M^{-1}(y) - \psi_1^{-1}(y) \geq \psi_M^{-1}(0) - \psi_1^{-1}(0)
$$

for all $y > 0$. By substituting $y = A(x)\psi_M(x)$,

$$
\tilde{x} - \psi_1^{-1}\big(A(x)\psi_M(x)\big) \geq r_M^* - r_1^* \geq (1-A(r_M^*))r_M^*.
$$

The latter inequality follows by (37). Hence, by (66),

$$
\psi_1^{-1}\big(A(x)\psi_M(x)\big) \leq \tilde{x} - (1-A(r_M^*))r_M^* \leq A(x)x.
$$

Therefore, we have

$$
\begin{aligned}
E[\max\{\psi_M(x) - \Psi(y), \beta\psi_M(x)\}] &\leq E[\max\{\psi_M(x) - \Psi(0), \beta\psi_M(x)\}] \\
&\leq H(x)\psi_M(x) + (1-H(x))\beta\psi_M(x) \\
&= A(x)\psi_M(x) \\
&\leq \psi_1(xA(x)).
\end{aligned}
\tag{67}
$$

The remainder we need to show is $\Pi_i^1((v,2),\theta_{-i}^1) \geq \Pi_i^1((v,M),\theta_{-i}^1)$. Fix arbitrary θ_{-i}^1. Consider any $v_i > c_i^1(M,\theta_{-i}^1)$. By the definition of a^*, it is obvious that under a profile of types $((v_i,2),\theta_{-i}^1)$, agent i has a positive probability that object 2 is allocated to i.[11] Note that by hazard rate ordering, $v > \psi_2^{-1}(\psi_M(v))$ for all v. When β is sufficiently small, the probability of obtaining the object for the type $(v_i,2)$ is

$$\begin{aligned} G(v_i) &\geq \beta + (1-\beta)G\big(\psi_2^{-1}(\psi_M(v_i))\big) \\ &> \beta + (1-\beta)G\Big(\psi_2^{-1}\big((1-\beta)\psi_M(v_i)\big)\Big) \qquad (68) \\ &= \beta + (1-\beta)H(v_i). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \Pi_i^1((v_i,2),\theta_{-i}^1) = \int_0^{v_i} \alpha_i^2((s,2),\theta_{-i}^1)\mathrm{d}s &> \int_{c_i^1(M,\theta_{-i}^1)}^{v_i} \big(\beta + (1-\beta)H(s)\big)\mathrm{d}s \\ &= \Pi_i^1((v_i,M),\theta_{-i}^1). \end{aligned}$$

Thus, by Proposition 4, the relaxed solution a^* is optimal. ■

A.7 Proof of Proposition 5

Consider $\hat{a}$ and the associated cutoff values $c_i^1(k_i,\theta_{-i}^1)$. Suppose that $\{i \in N^1 | k_i = M\} = \emptyset$ and $\{i \in N^1 | k_i = 2\} = \{i\}$. Then, we have the cutoff value of agent i, $c_i^1(2,\theta_{-i}^1) = \psi_2^{-1}(\Psi(0)) > r_2^*$.

If agent i has $v_i < c_i^1(2,\theta_{-i}^1)$, then he is not allocated under $\hat{a}$. If agent i has $v_i \in (r_2^*, c_i^1(2,\theta_{-i}^1)]$ and delays his entry, he is allocated object 2 if $v_j < v_i$ for all $j \in N^2$. Such probability is strictly positive: $\tilde{\alpha}_I^2(v_i,\theta_{-i}^1) > 0$. Hence, we have for every $v_i \in (r_2^*, c_i^1(2,\theta_{-i}^1)]$

$$\int_{r_2^*}^{v_i} \tilde{\alpha}_i^2(s,\theta_{-i}^1)\mathrm{d}s > 0,$$

which violates (41).

If agents with demand type 2 are short-lived, we can ignore the violation of (41). What we need to show is to verify (39) and (40). For any θ_{-i}^1, it is clear that if agent i with $\theta_i^1 = (v_i,M)$ is allocated both objects, he is allocated object 1 under $\tilde{\theta}^1 = ((v_i,1),\theta_{-i}^1)$. Similarly, it is clear that if agent i with $\theta_i^1 = (v_i,M)$ is allocated both objects, he is allocated object 2 under $\tilde{\theta}^1 = ((v_i,2),\theta_{-i}^1)$. Therefore, we have $c_i^1(1,\theta_{-i}^1) \leq c_i^1(M,\theta_{-i}^1)$ and $c_i^1(2,\theta_{-i}^1) \leq c_i^1(M,\theta_{-i}^1)$. Therefore both (39) and (40) are satisfied if β is sufficiently small. ■

[11] Precisely, we have $\psi_2(v_i) = \psi_2^{1,(1)}$ and $\psi_2(v_i) \geq \psi_M^{(1)}$.

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