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Been-Lon Chen, Yunfang Hu and Kazuo Mino

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Stabilization Effects of Taxation Rules in Small-Open Economies with Endogenous Growth*

Been-Lon Chen†, Yunfang Hu‡and Kazuo Mino§

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Abstract

This paper studies stabilization effects of nonlinear income taxation in small open economies with endogenous growth. We show that in the standard setting where domestic households freely lend to or borrow from foreign households under an exogenously given world interest rate, progressive taxation gives rise to equilibrium indeterminacy, while regressive taxation establishes equilibrium determinacy. These policy effects do not necessarily hold, either if the time discount rate is endogenously determined or if the world interest rate is elastic.

Keywords: Taxation Rules, Equilibrium Indeterminacy, Small-Open Economies, Endogenous Growth

JEL Classification: E62, O41

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†Institute of Economics, Academia Sinica, 128 Academia Road, Section 2, Taipei, Taiwan, bchen@econ.sinica.edu.tw

‡Graduate School of Economics, Kobe University. 2-1 Rokodai-cho, Nada-ku, Kobe 657-8501, Japan. E-mail: yhu@econ.kobe-u.ac.jp

§*Corresponding Author,* Faculty of Economics, Doshisha University, Karasuma Higashi-iru, Imadegawa, Sakyo-ku, Kyoto, 602-8580, kmino@mail.doshisha.ac.jp

1 Introduction

It has been known that progressive taxation contributes to stabilizing the one-sector real business cycle model with external increasing returns in the sense that it narrows the parameter space in which equilibrium intermediacy arises[1]. In contrast, regressive taxation enhances instability of the economy, because it widens the parameter space that generates equilibrium indeterminacy. Recently, Chen and Guo (2015 and 2016) revealed that those well-estabilised results fail to hold if the model economy allows endogenous growth. Chen and Guo (2015) introduce a nonlinear taxation rule à la Guo and Lansing (1998) into an AK growth model and show that progressive taxation generates equilibrium indeterminacy, while regressive taxation ensures equilibrium determinacy. Theese authors confirm that their finding still holds in an AK growth model with variable labor supply (Chen and Guo 2016). Their studies demonstrate that stabilization effects of taxation rules critically depend on the environment to which those policy rules are applied.

The purpose of this paper is to re-examine Guo and Chen's finding in the context of small-open economies. We construct small-open economy models with endogenous growth and explore the relation between taxation schedules and equilibrium determinacy in alternative settings. We first consider the standard model of small-open economies where domestic households freely lend to or borrow from foreign households under an exogenously given world interest rate. In this setting, the level of asset-capital ratio on the balanced growth path constitutes a continuum and a specific steady state is determined by the initial conditions if the equilibrium path is determinate. In this case, we find that the outcomes established in the closed economy model still hold in the small-open economy counterpart. Namely, if the taxation schedule is regressive, the equilibrium path is determinate and the steady state of the economy depends on the households' initial holdings of capital and financial asset. If the tax rule is progressive, then the equilibrium path is indeterminate, so that selection of the long run equilibrium of the economy may be affected not only by the initial conditions but also by sunspot shocks.

We then explore the models in which the steady-state level of asset-capital ratio is uniquely given regardless of the initial conditions. The first example we investigate is a model with

[1] Guo and Lansing (1998) is the first study that shows the stabilization effect of progressive taxation in the context of Benhabib and Framer's (1994) model.

endogenous time discount rate. We assume that the time discount rate of the household depends on the consumption share of income. Given this assumption, we find that the policy effects are the same as those in the standard model of small open economy, if the time discount rate increases with the consumption-income ratio (the case of increasing marginal impatience). In contrast, if the time discount rate decreases with the consumption-income ratio (the case of decreasing marginal impatience), then the stabilization effects of taxation schemes drastically change: the equilibrium path is determinate under progressive taxation, while the balanced growth path is totally unstable under regressive taxation. Therefore, in this case the stabilization effect of progressive taxation shown in the one-sector real business cycle model can be established in the small open economy with endogenous growth.

In the second example, we assume that the world interest rate increases with the debt-capital ratio of the home country. In this model we see that the stabilization effect of nonlinear income taxation would be close to those held in the one-sector real business cycle model of a closed economy. Our numerical examples with plausible parameter magnitudes show that progressive taxation realizes equilibrium determinacy and regressive taxation generates indeterminacy.

Our study is closely related to two issues in open-economy macroeconomics. First, our discussion deals with sunspot-driven fluctuations in small-open economies. In the existing literature, several authors have revealed that the small open economy versions of the real business cycle models with production externalities tend to yield equilibrium indeterminacy under weaker restrictions than in the corresponding closed economy model: see, for example, Weder (2001), Lahiri (2003), Meng and Velasco (2004) and Meng (2003)[2]. This is mainly because consumption smoothing can be perfect in the financially integrated world, which contributes to yielding self-fulfilling expectations. On the other hand, Meng (2014) shows that destabilizing effect of balanced budget rule pointed out by Schmitt-Grohé and Uribe (1997) will not arise in the small-open economy. In his model the time discount rate is endogenously determined and, hence, the steady state is independent of the initial conditions. Our evaluation of the stabilization effects of taxation schedules in small-open economies has the similar implication.

[2]Weder (2001), Lahiri (2001) and Meng and Velasxo (2004) study small-open economy, exogenous growth models, while Meng (2003) analyzes an endogenous growth model of small open economy.

Second, our study treats the way of 'closing' a small-open economy model. In their well-cited paper, Schmitt-Grohé and Uribe (2003) examine five alternative formulations that pin down the steady state equilibrium of the small-open economy under free capital mobility. They study the calibrated models under alternative formulations and compare their performances. Their main conclusion is that alternative formulations of closing a small open economy model do not yield significant differences from the quantitative perspective. Hence, they suggest that researchers may select a specific formulation based on computational convenience or on data availability. Our study means that Schmitt-Grohé and Uribe's (2003) conclusion may not hold, when the model involves the possibility of sunspot driven fluctuations. Our model examples show that the stabilization effects of income tax schedule critically depends on how the model is closed to pin down its steady state equilibrium. Therefore, specifications of preferences and financial market structure are relevant for evaluating stabilization effects of taxation rules in small open economies.

This paper is organized as follows. Next section summarizes the closed economy model of Chen and Guo (2015). This section also introduces adjustment costs of investment into the base model. Section 3 examines a small open economy based on the standard formulation. Sections 4 discusses the models with endogenous time discount and endogenous world interest rate. Section 5 concludes.

2 Closed Economy

2.1 Baseline Model

Based on Chen and Guo (2015), we first summarize stabilization effects of taxation rules in an endogenously growing economy in the simplest manner. Consider an AK growth model in which the aggregate production function is given by

$$Y_t = AK_t^{\alpha}\bar{K}_t^{1-\alpha}, \quad A > 0, \quad 0 < \alpha < 1,$$

where Y_t is the total output and K_t denotes the private capital and $\bar{K}_t$ represents external effects associated with the social average capital. In the representative agent setting, $\bar{K}_t = K_t$ holds in equilibrium, implying that the social production function is $Y_t = AK_t$ and the private

rate of return on capital is given by $r_t = \alpha A$. The representative household maximizes a discounted sum of utilities

$$U = \int_0^\infty e^{-\rho t} \log C_t dt, \quad \rho > 0 \tag{1}$$

subject to

$$\dot{K}_t = (1 - \tau_t) Y_t - C_t - \delta K_t, \quad K_0 = \text{given},$$

where C_t is consumption of the household and τ_t denotes the rate of income tax[3].

Following Guo and Lansing (1998), we assume that the fiscal authority adjusts the rate of income tax according to the following rule:

$$\tau_t = 1 - \eta \left(\frac{Y_t^*}{Y_t} \right)^\phi, \quad 0 < \eta < 1, \ \phi_0 < \ \phi < 1, \tag{2}$$

where Y_t^* denotes a reference level of income on the balanced growth path and ϕ_0 is given by

$$\phi_0 = \max \left\{ \frac{\eta - 1}{\eta}, \frac{\alpha - 1}{\alpha} \right\}. \tag{3}$$

The restriction on η means that when $Y_t = Y_t^*$ holds, the rate of average tax is in between 0 and 1. The condition on ϕ ensures that if $Y_{t'} = Y_t^*$, the after-tax income of the representative household increases with Y_t and that the after tax rate of return on the private capital decreases with K_t[4]. Under this policy rule, the marginal tax revenue given by

$$\frac{d}{dY_t} (\tau_t Y_t) = 1 - (1 - \phi) \eta \left(\frac{Y_t^*}{Y_t} \right)^\phi$$

is higher (lower) than the average tax revenue, τ_t, if $0 < \phi < 1$ $(-\phi_0 < \phi < 0)$. Thus taxation is progressive (regressive) if $0 < \phi < 1$ $(\phi_0 < \phi < 0)$.

We assume that when solving the optimization problem, the representative household takes sequences of the reference income and external effects of capital, $\left\{ Y_t^*, \bar{K}_t \right\}_{t=0}^\infty$, as given.

[3] No substantial change arises, if we use a more general CES utility function such that∞ $u(C_t) = C_t^{1-\sigma} / (1 - \sigma), \ \sigma > 0$.

[4] Note that the after-tax income is $(1 - \tau_t) Y_t = Y_t - \eta Y_t^{1-\phi} Y_t^{*\phi}$ and the after tax rate of return on private capital is given by $(1 - \tau_t) \alpha Y_t / K_t = \eta Y_t^{*\phi} A^{1-\phi} K_t^{\alpha(1-\phi)-1} \bar{K}_t^{(1-\alpha)(1-\phi:)}$.

The optimization conditions yield the Euler equation such that

$$\frac{\dot{C}_t}{C_t} = (1-\phi)\,\eta\left(\frac{Y_t^*}{Y_t}\right)^{\phi}\alpha A - \rho - \delta.$$

The transversality condition is given by $\lim_{t\to\infty} e^{-\rho t}\,(K_t/C_t) = 0$. Denoting the government consumption as G_t, the flow budget constraint for the government is

$$G_t = \tau_t Y_t = \left[1 - \eta\left(\frac{Y_t^*}{Y_t}\right)^{\phi}\right] Y_t.$$

Here, the government simply consumes its tax revenue and the level of G_t directly affects neither production activities nor household's felicity. The equilibrium condition for the final goods gives

$$\dot{K}_t = (1-\tau_t)\,Y_t - C_t - \delta K_t.$$

On the balanced growth path, it holds that

$$\frac{\dot{C}_t}{C_t} = \frac{\dot{K}_t}{K_t} = \frac{\dot{Y}_t}{Y_t} = \frac{Y_t^*}{Y_t^*} \equiv g,$$

where g denotes a common balanced growth rate which is endogenously determined. Define $z_t = C_t/K_t$ and $x_t = Y_t^*/Y_t$. Then the growth rates of capital and consumption are respectively given by

$$\frac{\dot{K}_t}{K_t} = \eta A x_t^{\phi} - z_t - \delta,$$

$$\frac{\dot{C}_t}{C_t} = (1-\phi)\,\eta\alpha A x_t^{\phi} - \rho - \delta.$$

Since $\dot{Y}_t^*/Y_t^* = g$, a complete dynamics system is as follows:

$$\frac{\dot{x}_t}{x_t} = g - \eta A x_t^{\phi} + z_t + \delta, \tag{4a}$$

$$\frac{\dot{z}_t}{z_t} = -\phi\eta\alpha A x_t^{\phi} + z_t - \rho. \tag{4b}$$

In the steady state where $\dot{z}_t = \dot{x}_t = 0$, the following conditions are fulfilled:

$$g - \eta A x^{\phi} + z + \delta = 0, \tag{5a}$$

$$-\phi\eta\alpha A x^{\phi} + z - \rho = 0. \tag{5b}$$

In the above, x and z respectively denote the steady state values of x_t and z_t.,Since these two equations involve three endogenous variables, x, z and g, we need to have an additional condition to determine the steady state. A natural condition is $Y_t = Y_t^*$ holds on the balanced growth path, so that the steady state level of $x = 1$. Then the steady state value of z is

$$z = \phi\eta A + \rho,$$

and the balanced growth rate is given by

$$g = (1 - \phi)\,\eta\alpha A - \rho - \delta.$$

The coefficient matrix of the above dynamic system linearized at $x = 1$ and z is

$$J_0 = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} -\phi\eta A & 1 \\ -\phi^2\eta\alpha A & 1 \end{bmatrix}.$$

We see that

$$\det J_0 = zA\eta\phi\,(\alpha\phi - 1)\,, \quad \text{trace } J_0 = z - \eta\phi A.$$

As a result, if $0 < \phi < 1$, then det $J < 0$, so that J has a one negative eigenvalue. If $\phi < 0$, then $\det J_0 > 0$ trace $J_0 > 0$, meaning that both eigenvalues of J have positive real parts. Since the initial level of the reference income, Y_0^*, is not predetermined even though its growth rate is fixed at g,both x_t and z_t are jump variables.

If J has no stable root, then $x_t = x = 1$ and $z_t = z$ for all $t \geq 0$. In this case, the initial levels of Y_t^* and C_t are respectively given by

$$Y_0^* = AK_0, \quad C_0 = (\phi\eta A + \rho)K_0.$$

In contrast, when J has one stable root, there is a unique converging path in (x_t, z_t) space. It is easy to confirm that this stable path has a positive slope and, hence, the relation between the equilibrium levels of x_t and z_t around the steady state is expressed as $x_t = \Phi z_t$, where Φ

is a positive constant. This means that the initial levels of Y_t and C_t satisfy

$$Y_0^* = \Phi A C_0. \tag{6}$$

Since the initial level of C_0 can take any value if the equilibrium is realized on the stable saddle path, the initial level of Y_0^* is not historically specified either. Consequently, in contrast to the neoclassical (exogenous) growth model where Y_t^* is fixed at the steady state level of output, progressive taxation generates sunspot-driven fluctuations, while the regressive taxation establishes determinacy of equilibrium. Such a conclusion is opposite to the result obtained in the standard one-sector real business cycle model[5].

To give an intuitive implication of the above result, suppose that the tax scheme is progressive ($0 < \phi < 1$) and that a positive sunspot shock raises the future income anticipated by the households. Hence, due to the income effect, the households increases their current consumption. Equation (6) means that such a rise in consumption increases the reference level of income Y^*, which depresses the rate of income tax under our taxation rule. A lower tax rate accelerates capital accumulation so that income will increase. Therefore, the intimal anticipated rise in future income can be self-fulfilled. If the tax rule is regressive ($\phi < 0$), the economy always stays on the balanced growth path. Thus the economy will not respond to an extrinsic sunspot shock. Figure 1 depicts the above intuition. In this figure the economy is assumed to stay on the balanced growth path denoted by Path A up untile time $\bar{t}\,(>0)$. Now suppose that a positive sunspot shock hits at $t = \bar{t}$. If taxation is regressive, such an extrinsic shock fails to affect the equilibrium path of the economy and, hence, the economy continues staying on Path A. However, if taxation is progressive, a positive sunspot shock raises $Y_{\bar{t}}^*$ up to $\bar{Y}_{\bar{t}}^*$ $\left(> Y_{\bar{t}}^*\right)$. As a result. the reference income Y_t^* starts to follow Path B if further shocks will not hit the economy afterwards. In this situation, the actual income Y_t follows Path C that converges to Path B.

[5] When the model economy does not allow endogenous growth, the reference level of income Y^* is the steady state level of Y_t which is fixed. As a result, the rate of income tax $\tau_t = 1 - \eta\left(\frac{Y^*}{Y_t}\right)^{\phi}$ increases (decreases) if $\phi > 0$ $(\phi < 0)$. Hence, an expansion of income caused by an optimistic sunspot shock raises the rate of income tax, under which the in expectaions caused by the sunspot shock will not be self-fulfilled, Such a stabilization effect of progressive tax may not hold in an endogenous growth environment where Y_t^* is also affected by sunspots.

Figure 1

2.2 Adjustment Costs of Investment

The small-open economy models discussed below assume the presence of adjustment costs of investment in order to avoid indeterminacy in the household's portfolio choice between financial assets and real capital. Hence, it is useful to study the behavior of the closed economy model with adjustment costs of investment before examining the open-economy models.

In the presence of adjustment costs of investment, the representative household maximizes (1) subject to

$$(1-\tau_t)Y_t = C_t + \left[I_t + \frac{\theta}{2}\left(\frac{I_t}{K_t}\right)^2 K_t\right], \quad \theta > 0, \tag{7}$$

$$\dot{K}_t = I_t - \delta K_t, \tag{8}$$

where τ_t is determined by (2). Here, the term $(\theta/2)\left(I_t/K_t\right)^2 K_t$ represents the adjustment costs of investment.

Set up the Hamilton function in such a way that

$$H_t = \ln C_t + q_t(I_t - \delta K_t) + \lambda_t\left[(1-\tau_t)AK_t^{\alpha}\bar{K}_t^{1-\alpha} - C_t - \left[I_t + \frac{\theta}{2}\left(\frac{I_t}{K_t}\right)^2 K_t\right]\right].$$

When $\bar{K}_t = K_t$, we have the following first-order conditions:

$$\frac{1}{C_t} = \lambda_t, \tag{9a}$$

$$q_t = \lambda_t\left[1 + \theta\frac{I_t}{K_t}\right], \tag{9b}$$

$$\dot{q}_t = (\rho+\delta)q_t - \lambda_t\left[\eta(1-\phi)\left(\frac{Y_t^*}{Y_t}\right)^{\phi}\alpha A + \frac{\theta}{2}\left(\frac{I_t}{K_t}\right)^2\right], \tag{9c}$$

together with the transversality condition: $\lim_{t\to\infty} e^{-\rho t}q_t K_t = 0$.

Now define $x_t = Y_t^*/Y_t$, and $v_t = q_t/\lambda_t$. From (7) , (9a) and (9b) , we obtain

$$\frac{C_t}{K_t} = \frac{1}{\lambda_t K_t} = \eta x_t^{\phi} A - \left[\frac{1}{\theta}(v_t - 1) + \frac{(v_t - 1)^2}{2\theta}\right] \equiv Z(x_t, v_t), \tag{10}$$

where

$$\text{sign } Z_x = \text{sign } \phi, \quad Z_v < 0.$$

In addition, (8) and (9c) are respectively written as

$$\frac{\dot{K}_t}{K_t} = \frac{1}{\theta}(v_t - 1) - \delta,$$

$$\frac{\dot{q}_t}{q_t} = \rho + \delta - \frac{1}{v_t}\left[\eta(1-\phi)x_t^{\phi}\alpha A + \frac{(v_t - 1)^2}{2\theta}\right] \equiv Q(x_t, v_t).$$

Notice that sign $Q_x(x_t, v_t) = -\text{sign } \phi$.

$$-\frac{\dot{\lambda}_t}{\lambda_t} = \frac{Z_x x}{Z}\left(g - \frac{\dot{K}_t}{K_t}\right) + \frac{Z_v v_t}{Z}\frac{\dot{v}_t}{v_t} + \frac{\dot{K}_t}{K_t}.$$

Since $\dot{Y}_t^*/Y_t^* = g$,the dynamic behavior of x_t is given by

$$\frac{\dot{x}_t}{x_t} = g - \frac{\dot{K}_t}{K_t} = g - \frac{1}{\theta}(v_t - 1) + \delta. \tag{11a}$$

Using (10) and $\dot{v}_{\cdot t}/v_t = \dot{q}_t/q_t - \dot{\lambda}_t/\lambda_t$, we obtain:

$$\frac{\dot{v}}{v} = \left[1 - \frac{Z_v v_t}{Z}\right]^{-1}\left\{Q(x_t, v_t) + \frac{1}{\theta}(v_t - 1) - \delta + \frac{Z_x x_t}{Z}\left(g - \frac{1}{\theta}(v_t - 1) + \delta\right)\right\}. \tag{11b}$$

Differential equations (11a) and (11b) constitute a complete dynamic system.

The steady state conditions that establish $\dot{v}_t = \dot{x}_t = 0$ and $x = 1$ are summarized as:

$$\rho + \frac{1}{\theta}(v - 1) + \frac{1}{v}\left[\eta(1-\phi)x^{\phi}\alpha A + \frac{(v-1)^2}{2\theta}\right] = 0. \tag{12}$$

Equation (12) has two solutions and we focus on the value of v that may give a positive balanced growth rate, $g = (1/\theta)(v - 1) - \delta$. Then the balanced growth rate can be expressed

as

$$g = \frac{1}{\theta}\left(\sqrt{1+(\theta\rho)^2+2\theta\eta(1-\phi)\alpha A}-(1+\theta\rho)\right).$$

Evaluating the coefficient matrix of the system linearized at the steady state, it is given by

$$J_1 = \begin{bmatrix} x & 0 \\ 0 & v\left(1-\frac{Z_v v}{Z}\right)^{-1} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\theta} \\ Q_x & Q_v+\frac{1}{\theta}-\frac{1}{\theta}\frac{Z_v}{Z} \end{bmatrix},$$

which leads to

$$\begin{aligned} \det J_1 &= \frac{vx}{\theta}\left[1-\frac{Z_v(1,v)\,v}{Z(1,v)}\right]^{-1} Q_x(1,v), \\ \text{trace } J_1 &= v\left(1-\frac{Z_v v}{Z}\right)^{-1}\left\{Q_v(1,v)+\frac{1}{\theta}-\frac{1}{\theta}\frac{Z_x(1,v)}{Z(1,v)}\right\}, \end{aligned}$$

where

$$Q_v = \frac{1}{v^2}\left[\eta(1-\phi)\alpha A+\frac{(v-1)^2}{2\theta}\right]+\frac{1}{v\theta}-\frac{1}{\theta}.$$

Remember that sign $Z_x = -\text{sign}\ \phi$ and sign $\det J_1 = \text{sign}\ Q_x = -\text{sign}\ \phi$. In addition, we see that if $\phi < 0$, then trace $J_1 > 0$. Those results show that, progressive taxation still gives rise to indeterminacy, whereas regressive taxation establishes determinacy. Consequently, adding the adjustment costs of investment to the baseline closed economy model will not alter the stabilization effects of taxation rules[6]. To sum up, we have found:

Proposition 1 *In the AK growth model with convex adjustment costs of investment, equilibrium indeterminacy (determinacy) holds under the progressive (regressive) taxation.*

3 Small-Open Economy

3.1 Baseline Model

We now open up the model economy with investment adjustment costs discussed above. The model is the standard one: domestic households freely lend to or borrow from foreign

[6]In the context of closed economy models, Herrendolf and Valentinyi (2003) and Kim (2003) reveal that indeterminacy needs stronger restrictions if the model economy involves convex adjustment costs of investment. On the other hand, Chin et al. (2012) show that the presence of investment adjustment costs causes indeterminacy in a small open economy model with endogenous growth. Since our simple AK growth model does not assume production externalities, indeterminacy arrises due to the distortionary taxation rule imposed by the fiscal authority.

households and international lending and borrowing are carried out by trading foreign bonds under a given world interest rate. The flow budget constraint for the households is

$$\dot{B}_t = (1-\tau_{y,t})Y_t + (1-\tau_b)\,RB_t - \left[\frac{I_t}{K_t} + \frac{\theta}{2}\left(\frac{I_t}{K_t}\right)^2\right]K_t - C_t, \quad \theta > 0, \tag{13}$$

where B_t denotes the stock of foreign bond (net asset position) held by the domestic households, R is a given world interest rate and τ_b denotes the fixed rate of tax on interest income. Here, we assume that the nonlinear taxation rule applies to the domestic income alone, The rate of tax on domestic income, $\tau_{y,t}$, follows (2) .

The household maximizes U in (1) subject to (13) and

$$\dot{K}_t = I_t - \delta K_t, \tag{14}$$

together with the initial condition on K_t and B_t as well as with the no-Ponzi-game condition:

$$\lim_{t\to\infty} e^{-(1-\tau_b)R}B_t \geq 0.$$

We set up the Hamiltonian function such that

$$H_t = \log C_t + \lambda_t\left[(1-\tau_t)AK_t^{\alpha}\bar{K}_t^{1-\alpha} + RB_t - \left[\frac{I_t}{K_t} + \frac{\theta}{2}\left(\frac{I_t}{K_t}\right)^2\right]K_t - C_t\right] + q_t\left(I_t - \delta K_t\right),$$

where λ_t and q_t respectively denote the implicit prices of B_t and K_r. Given $\bar{K}_t = K_t$, the optimization conditions include the following:

$$1/C_t = \lambda_t, \tag{15a}$$

$$\lambda_t\left(1 + \theta\frac{I_t}{K_t}\right) = q_t, \tag{15b}$$

$$\dot{\lambda}_t = \lambda_t\left[\rho - (1-\tau_b)\,R\right], \tag{15c}$$

$$\dot{q}_t = (\rho+\delta)\,q_t - \lambda_t\left[\eta\,(1-\phi)\left(\frac{Y_t^*}{Y_t}\right)^{\phi}\alpha A + \frac{\theta}{2}\left(\frac{I_t}{K_t}\right)^2\right], \tag{15d}$$

together with the transversality condition:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_t B_t = 0, \qquad \lim_{t\to\infty} e^{-\rho t} q_t K_t = 0.$$

Conditions $(15a)$ and $(15c)$ show that consumption changes at a constant rate of

$$\frac{\dot{C}_t}{C_t} = (1-\tau_b)\,R - \rho = g, \tag{16}$$

which gives the balanced growth rate of this economy.

Now define

$$v_t = q_t/\lambda_t, \quad x_t = Y_t^*/Y_t = Y_t^*/(AK_t), \quad b_t = B_t/K_t.$$

Since it holds that $\dot{Y}_t^*/Y_t^* = g$, we see that C_t and Y_t^* change at the same rate. Hence, the relation between these two variables is

$$C_t = \bar{\psi} Y_t^*, \tag{17}$$

where $\bar{\psi}$ is an undetermined positive constant. Using (13) through (17) as well as $\dot{v}_t/v_t = \dot{q}_t/q_t - \dot{\lambda}_t/\lambda_t$, $\dot{x}_t/x_t = g - \dot{K}_t/K_t$ and $\dot{b}_t/b_t = \dot{B}_t/B_t - \dot{K}_t/K_t$, we can derive a complete dynamic system with respect to v_t, x_t and b_t in the following manner:

$$\frac{\dot{x}_t}{x_t} = (1-\tau_b)\,R - \rho + \delta - \frac{1}{\theta}\left(v_t - 1\right), \tag{18a}$$

$$\frac{\dot{v}_t}{v_t} = (1-\tau_b)\,R + \delta - \frac{1}{v_t}\left[\eta\left(1-\phi\right)\left(x_t\right)^{\phi}\alpha A + \frac{1}{2\theta}\left(v_t - 1\right)^2\right], \tag{18b}$$

$$\frac{\dot{b}_t}{b_t} = \eta x_t^{\phi}\frac{A}{b_t} + (1-\tau_b)\,R - \frac{1}{b_t}\left[\frac{v_t-1}{\theta} + \frac{1}{2\theta}\left(v_t-1\right)^2\right] - \frac{\bar{\psi}Ax_t}{b_t} - \frac{v_t-1}{\theta} + \delta. \tag{18c}$$

Note that when deriving $(18c)$, we use (17) to obtain $z_t = C_t/K_t = \bar{\psi}Y_t^*/K_t = \bar{\psi}Ax_t$.

3.2 Balanced Growth Path and Equilibrium (In)determinacy

On the balanced growth path, it holds that

$$\frac{\dot{Y}_t}{Y_t} = \frac{Y_t^*}{Y_t^*} = \frac{\dot{K}_t}{K_t} = \frac{\dot{B}_t}{B_t} = \frac{\dot{C}_t}{C_t} = g = (1-\tau_b)\,R - \rho - \delta.$$

Letting x, v and b be the steady state levels of x_t, v_t and b_t, the steady state conditions that establish $\dot{x}_t = \dot{v}_t = \dot{b}_t = 0$ are:

$$([1-\tau_b)\,R+\delta]v-\left[\eta\,(1-\phi)\,x^{\phi}\alpha A+\frac{1}{2\theta}\,(v-1)^2\right]=0, \tag{19a}$$

$$\frac{1}{\theta}\,(v-1)=(1-\tau_b)\,R-\rho+\delta, \tag{19b}$$

$$\eta x^{\phi}A+\rho b-\left[\frac{v-1}{\theta}+\frac{1}{2\theta}\,(v-1)^2\right]-\bar{\psi}xA=0. \tag{19c}$$

A notable departure from the closed economy model in the previous section is that the rate of change in consumption is fixed at $g=(1-\tau_b)\,R-\rho$ even out of the balanced growth path. As a result, the above three conditions may determine x, v and b, implying that we cannot impose the consistency condition, $x=1$ $(Y_t=Y_t^*)$, in the steady state equilibrium. In fact, $x=1$ holds only when the magnitudes of parameters satisfy very specific conditions. Keeping this fact in mind, it is easy to confirm that once $\bar{\psi}$ is given, x, v and b are uniquely determined.

The dynamic system consisting of $(18a)$, $(18b)$ and $(18c)$ is block recursive. Namely, behaviors of x_t and v_t are independent of b_t. The coefficient matrix of the linealized subsystem of x_t and v_t is

$$J_2=\begin{bmatrix} 0 & -\frac{x}{\theta} \\ -\phi\eta\,(1-\phi)\,x^{\phi-1}\alpha A & \rho \end{bmatrix}.$$

Consequently, sign $\det J_2 = -\text{sign}\ \phi$ and trace $J_2=\rho>0$, which means that J_2 has one stable root if $0<\phi<1$,whereas it has no stable root if $\phi<0$.

Regresive Taxation

First, suppose that taxation is regressive so that J_2 has two eigenvalues with positive real parts. In this case $x_t=x$ and $v_t=v$ for all $t\geq 0$. In view of the no-Ponzi game restriction and transiversality conditions, we find that the intertemporal budget constraint for the household

is expressed as

$$B_0 + \int_0^{\infty} e^{-(1-\tau_b)Rt}(1-\tau_{y,t})Y_s dt$$
$$= \int_0^{\infty} e^{-(1-\tau_b)Rt} C_t dt \quad + \int_0^{\infty} e^{-(1-\tau_b)Rt} \left[\frac{1}{\theta}(v-1) + \frac{1}{2\theta}(v-1)^2\right] K_t dt. \tag{20}$$

Notice that if v_t is fixed at v, then it hods that $K_t = K_0 e^{gt}$ and that $g = (1-\tau_b)R - \rho$. Hence, (20) leads to

$$\frac{C_0}{K_0} = \rho b_0 + \left[\eta A x^{\phi} - \frac{1}{\theta}(v-1) - \frac{1}{2\theta}(x-1)^2\right], \tag{21}$$

which determines the initial level of consumption under a given level of B_0. This means that from (17), $\bar{\psi}$ is uniquely determined by

$$\bar{\psi} = \frac{C_0}{K_0 x A}. \tag{22}$$

From (19b), equation (18c) is written as

$$\dot{b}_t = \rho b_t + \eta A x^{\phi} - \left[\frac{v-1}{\theta} + \frac{1}{2\theta}(v-1)^2\right] - \bar{\psi} A x.$$

Using (21) and (22), we find that the above becomes

$$\dot{b}_t = \rho(b_t - b_0).$$

This means that the steady state condition, $\dot{b}_0 = 0$, is not fulfilled unless $b = b_0$. Therefore, B_t and K_t grow at the same rate of g from the outset, which demonstrates that the economy always stays on the balanced growth path. This outcome ensures equilibrium determinacy[7].

Progressive Taxstion

Next, consider the case of progressive taxation under which J_2 has one stable root. In this case, x_t is uniquely related to v_t on the stable saddle path. We express such a relation

[7] In the real business cycle model without endogenous growth, it is assumed that $(1-\tau_b)R = \rho$ in order to satisfy the transversality and feasibility conditions. Given this restriction, C_t stays constant over time: $C_t = \bar{C}$. The level of $\bar{C}$ is determined by the intertemporal budge constraint for the household and the steady state level of B_t is determined by the choice of $\bar{C}$ which depends on K_0 and B_0. In our endogenous growth environment, $(1-\tau_b)R$ exceeds ρ and C_t continues growing. However, the steady state value of B_t/K_t is not given without specifying C_0 even though the equilibrium path is determinate.

as

$$v_t = \xi\left(x_t\right). \tag{23}$$

Drawing the phase diagram of $(18a)$ and $(18b)$, we find that the stable path has a positive slope so that $\xi'\left(x_t\right) > 0$. Using (23), the dynamic system is reduced to

$$\dot{x}_t = x_t\left[\frac{1}{\theta}\left(\xi\left(x_t\right) - 1\right) - \delta\right], \tag{24a}$$

$$\dot{b}_t = \eta A x_t^{\phi} + \left[\left(1-\tau_b\right)R - \frac{\xi\left(x_t\right)-1}{\theta} + \delta\right] b_t - \left[\frac{v_t - 1}{\theta} + \frac{1}{2\theta}\left(\xi\left(x_t\right)-1\right)^2\right] - \bar{\psi} A \xi\left(x_t\right). \tag{24b}$$

Figure 2 depicts the phase diagrams of the dynamic system of $(24a)$ and $(24b)$ *under a given level of* $\bar{\psi}$. If $\bar{\psi}$ is fixed, then there is a unique stationary point where it holds that $\dot{x}_t = \dot{b}_t = 0$. As Figure 2 shows, the stationary equilibrium of the dynamic system is a saddle point and the stable saddle path is positively sloped. The relation between x_t and b_t on the stable saddle path is thus described as

$$x_t = \zeta\left(b_t; \bar{\psi}\right), \qquad \zeta_b\left(b_t; \bar{\psi}\right) > 0. \tag{25}$$

Notice that the stable saddle path depends on the level of $\bar{\psi}$.

Figure 2

To determine the magnitude of $\bar{\psi}$, we again use the intertemporal budget constraint for the household such that

$$B_0 + \int_0^{\infty} e^{-(1-\tau_b)Rt} \eta A x_t^{\phi} K_t dt = \frac{C_0}{\rho} + \int_0^{\infty} e^{-(1-\tau_b)Rt}\left[\frac{1}{\theta}\left(\xi\left(x_t\right)-1\right) + \frac{1}{2\theta}\left(\xi\left(x_t\right)-1\right)^2\right] K_t dt. \tag{26}$$

When deriving the above, we use $C_t = C_0 e^{gt}$ and $\left(1-\tau_b\right)R = g + \delta$. Since the capital stock follows

$$\dot{K}_t = K_t\left[\frac{1}{\theta}\left(\xi\left(x_t\right)-1\right) - \delta\right],$$

we obtain

$$K_t = K_0 \exp\left(\int_0^t \left[\frac{1}{\theta}\xi\left(x_s\right) - 1\right] ds\right).$$

Hence, once a sequence of $\{x_t\}_{t=0}^{\infty}$ is selected, the path of K_t is determined as well. Then the level of C_0 (so the level of $\bar{\psi}$) is determined by (21).

Now remember that x_t is a jump variable. If a sunspot sock hits at $t = 0$, the expectations of households may change and the initial value of x_t (so the initial level of v_t) will change. Such a change shifts the paths of x_t and K_t, which generates a changes in C_0 and $\bar{\psi}$ determined by (26) . Figure 3 shows an example. We should note that $\dot{x}_t = 0$ locus and $\bar{x}$ are independent of $\bar{\psi}$, meaning that a sunspot shock will not affect those. Suppose that the initial position of the economy is the steady state E_0 and that a negative sunspot shock lowers C_0 and $\bar{\psi}$. Then both $\dot{b}_t = 0$ locus and the stable saddle path shift upward. If no shock hits the economy afterwards, then the economy jumps to the new saddle path and follows it towards the new steady state E_1. From the steady state condition for $(24b)$, the steady state level of b is determined by

$$b = \frac{1}{\rho}\left[\frac{\xi(x) - 1}{\theta} + \frac{1}{2\theta}\left(\xi(x) - 1\right)^2 - \eta x^{\phi} A + \bar{\psi} x A\right].$$

Hence, a decrease in $\bar{\psi}$ reduces b.

Figure 3

In sum, the long-run level of asset position of the small country depends not only fundamentals but also on the expectations of households. In this sense, equilibrium indeterminacy holds under the progressive taxation rule:

Proposition 2 *In the standard model of small-open economy with free trade of goods and financial assets, the balanced growth path is locally indeterminate (determinate) if the taxation schedule is progressive (regressive).*

4 Alternative Settings

The small-open economy model treated in the previous section follows the standard formulation where the steady state of the home country constitutes a continuum. This means that when the equilibrium is determinate, the steady state levels of key macroeconomic variables depends on the initial levels of physical capital as well as financial assets held by the households. If indeterminacy prevails, then the selection of the steady state will be affected by a sunspot-driven expectations change. In this section we examine small-open economy models

in which the steady state of the economy is independent of the initial conditions.

4.1 Endogenous Time Discount Rate

One of the simple examples where a small open economy may have a unique steady state is to assume that the time discount rate of the household is endogenously determined. In the existing literature, there are two alternative formulations of endogenous time preference. One is the inward-looking time preference in which the rate of time discount depends on the level of private consumption. In this case the household perceives such a dependency of her patience on her own consumption. The other formulation is the outward-looking time preference where the time discount rate is a function of social average consumption. In this modelling, the rate of time preference of an individual consumer is affected by external effects generated by the average level of consumption in the economy at large. It has been shown that both formulations yield similar analytical results. In what follows, we assume the outward-looking time preference under which model manipulation is simpler than the inward-looking time preference.

If the representative household's time discount rate depends on the social level of consumption, the objective function of the household is:

$$U = \int_0^{\infty} \exp\left(-\int_0^t \rho\left(\frac{\bar{C}_s}{\bar{Y}_s}\right) ds\right) \log C_t dt,$$

where $\bar{C}_t$ and $\bar{Y}_t$ repectively denote the average consumption and domestic income in the economy at large. Since in the steady state of our economy, income and consumption continue growing, we assume that the the time preference is a function of the average consumption-income ratio rather than the absolute level of average consumption. If $\rho(.)$ increases with $\bar{C}_t/\bar{Y}_t$, the preferences exibit increasing (social) marginal impatience. If $\rho(.)$ is a decreasing function of $\bar{C}_t/\bar{Y}_t$, then preferences satisfy decreasing (social) marginal impatience. Ever since Usawa (1969), the increasing marginal impatience has been frequently assumed in the literature, mainly because it usually ensures saddle stability of dynamic macroeconomic models. However, it is often claimed that the increasing marginal impatience is counter intuitive: under this assumption, relatively rich consumers with high levels of income and consumption should be more impatient than relatively poor consumers who attain lower levels income

and consumption[8]. In fact, several authors have investigated optimal growth models under the assumption of decreasing marginal impatience: see, for example, Chang (2009) and Das (2003). In addition, Ikeda and Hirrose (2012a and 2102b.) and Kawagishi and Mino (2015) investigate dynamic trade models where households' preferences exhibit decreasing marginal impatience. In what follows we consider both increasing and decreasing marginal impatience, so that the sign of $\rho'(.)$ is not specified at this stage.

Given our specification of time preference, the household's optimization problem is:

$$\max \int_0^\infty \Psi_t \log C_t dt$$

subject to

$$\dot{B}_t = (1-\tau_b) RB_t + \eta \left(\frac{Y_t^*}{Y_t}\right)^\phi A K_t^\alpha \bar{K}_t^{1-\alpha} - \left(\frac{I_t}{K_t} + \frac{\theta}{2}\left(\frac{I_t}{K_t}\right)^2\right) K_t - C_t, \tag{27}$$

$$\dot{\Psi}_t = -\rho\left(\frac{\bar{C}_t}{\bar{Y}_t}\right)\Psi_t. \tag{28}$$

In this problem the representative households takes the sequences of $\left\{\bar{K}_t, \bar{C}_t, \bar{Y}_t, Y_t^*\right\}_{t=0}^\infty$ as given.

The Hamiltonian function is given by

$$\begin{aligned} H_T &= \Psi_t \log C_t + \lambda_t \left[(1-\tau_r) RB_t + \eta \left(\frac{Y_t^*}{Y_t}\right)^\phi A K_t^\alpha \bar{K}_t^{1-\alpha} - \left(\frac{I_t}{K_t} + \frac{\theta}{2}\left(\frac{I_t}{K_t}\right)^2\right) K_t - C_t\right] \\ &\quad + q_r (I_t - \delta K_t) - \mu_t \Psi \rho\left(\frac{\bar{C}_t}{\bar{Y}_t}\right). \end{aligned}$$

We find that the optimization conditions are:

$$\frac{\Psi_t}{C_t} = \lambda_t, \tag{29a}$$

$$\dot{\lambda}_t = -\lambda_t (1-\tau_r) R, \tag{29b}$$

$$\dot{q}_t = \delta q_t - \lambda_t \left[\eta (1-\phi)\left(\frac{Y_t^*}{Y_t}\right)^\phi \alpha A K_t^{\alpha-1} \bar{K}_t^{1-\alpha} + \frac{\theta}{2}\left(\frac{I_t}{K_t}\right)^2\right], \tag{29c}$$

[8] See, for example, Obstfeld (1990).

$$\dot{\mu}_t = -\log C_t + \mu_t \rho \left(\frac{\bar{C}_t}{\bar{Y}_t} \right). \tag{29d}$$

By use of $(29a)$, $(29b)$ and $(29d)$, we obtain

$$\frac{\dot{C}_t}{C_t} = (1 - \tau_b) R - \rho \left(\frac{\bar{C}_t}{\bar{Y}_t} \right).$$

In the representative agent economy, the equilibrium conditions require that

$$\bar{K}_t = K_t, \quad \bar{C}_t = C_t, \ \bar{Y}_t = Y_t \ \text{ for all } t \geq 0.$$

As a result, the optimal consumption follows

$$\frac{\dot{C}_t}{C_t} = (1 - \tau_r) R - \rho \left(\frac{z_t}{A} \right), \tag{30}$$

where $z_t = C_t / K_t$.

Keeping (30) in mind and using the same notations employed in the previous section, it is easy to see that the complete dynamic system can be summarized as four differential equations with respect to x_t, v_t, z_t and b_t. The equations of x_t and v_t are the same as $(18a)$ and $(18b)$, so that they constitute a complete system. Behaviors of $z_t \, (= C_t / K_t)$ and $b_t \, (= B_t / K_t)$ are respectively described by

$$\dot{z}_t = (1 - \tau_b) R - \rho \left(\frac{z_t}{A} \right) - \frac{1}{\theta} (v_t - 1) + \delta, \tag{31}$$

$$\dot{b}_t = \left[(1 - \tau_b) R + \delta - \frac{v_t - 1}{\theta} \right] b_t + \eta \alpha A x_t^{\phi} - z_t - \frac{v_t - 1}{\theta} - \frac{1}{2\theta} (v_t - 1)^2, \tag{32}$$

It is to be noted that in this model the rates of change in x_t and z_t differ from each other out of the balanced growth equilibrium.

The steady state conditions are given by the following:

$$\dot{x}_t = 0 \Longrightarrow g - \frac{1}{\theta} (v - 1) + \delta = 0,$$

$$\dot{v}_t = 0 \Longrightarrow [(1 - \tau_b) R + \delta] v - \eta (1 - \phi) \alpha A x^{\phi} - \frac{1}{2\theta} (v - 1)^2 = 0,$$

$$\dot{z}_t = 0 \Longrightarrow (1-\tau_b)\, R - \rho\left(\frac{z}{A}\right) - \frac{1}{\theta}\,(v-1) + \delta = 0,$$

$$\dot{b}_t = 0 \Longrightarrow \left[(1-\tau_b)\, R + \delta - \frac{v-1}{\theta}\right] b + \eta A x^{\phi} - z - \frac{v-1}{\theta} - \frac{1}{2\theta}\,(v-1)^2 = 0.$$

The above four equations have five endogenous variables, x, v, z, b and g. Therefore, unlike the previous model, we may set the consistency condition, $x = 1$, to determine the balanced growth rate, g. Give this condition, the above four equations become:

$$\frac{1}{\theta}\,(v-1) + \delta - g = 0, \tag{33a}$$

$$\left[(1-\tau_b)\, R + \delta\right] v - \eta\,(1-\phi)\,\alpha A - \frac{1}{2\theta}\,(v-1)^2 = 0, \tag{33b}$$

$$\eta(1-\phi)A - \rho\left(\frac{z}{A}\right) - \frac{1}{\theta}\,(v-1) + \delta = 0, \tag{33c}$$

$$\left[(1-\tau_b)\, R - \frac{v-1}{\theta} + \delta\right] b + \eta A - z - \frac{v-1}{\theta} - \frac{1}{2\theta}\,(v-1)^2 = 0. \tag{33d}$$

First, (33b) presents v. Again, there are two levels of v. We select one that may yield a positive level of balanced growth rate determined by (33a) . Then under a given level of v, (33c) gives z so that b is determined uniquely by (33d).

Regressive Taxation

First, suppose that taxation is regressive ($\phi < 0$) . Since the dynamic system of (x_t, v_t) is totally unstable under regressive taxation, x_t and v_t always take their steady state values ($x = 1$ and v), so that dynamic equations of z_t and b_t are expressed as

$$\dot{z}_t = z_t\left[(1-\tau_b)\, R - \rho\left(\frac{z_t}{A}\right) - \frac{1}{\theta}\,(v-1) + \delta\right], \tag{34a}$$

$$\dot{b}_t = \left[(1-\tau_b)\, R - g\right] b_t + \eta A - z_t - \frac{v-1}{\theta} - \frac{1}{2\theta}\,(v-1)^2\,. \tag{34b}$$

The coefficient matrix of this system is

$$J_3 = \begin{bmatrix} -\frac{z}{A}\rho'\left(\frac{z}{A}\right) & 0 \\ -1 & b\rho\left(\frac{z}{A}\right) \end{bmatrix},$$

which shows that

$$\text{sign } \det J_3 = -\text{sign } \rho'\left(\frac{\bar{z}}{A}\right).$$

As a result, if the preferences exhibit increasing marginal impatience (i.e. $\rho'(.) > 0$), then the system of (34a) and (34b) satisfies saddle point stability and, hence, there is a unique converging path under a given initial level of $b_0 = B_0/K_0$. On the other hand, if $\rho'(.) < 0$ (decreasing marginal impatience) , the steady state is totally unstable.

Intuition behind this result is simple. Note that if $\rho'(z_t/A) > 0$ (< 0), then dynamic equation (34a) exhibits self-stabilizing (destabilizing) behavior under a fixed level of v. Thus if $\rho'(z_t/A) < 0$, then z_t should always stay at its steady state level. However, if z_t is fixed, the behavior of b_t near the steady state is $\dot{b}_t = \rho(z/A)\, b_t \eta A - z - \frac{v-1}{\theta} - \frac{1}{2\theta}(v-1)$, so that b_t is completely unstable, that is, there is no feasible equilibrium path converging to the balanced growth equilibrium. In contrast, if $\rho'(z_t/A) < 0$, then z_t converges to its steady state level for any initial level of z_0, Therefore, the economy can select z_0 so as to make b_t converges to its steady state value.

Progressive Taxation

In the case of progressive taxation, dynamic system has a stable path converging to the stationary state of (34a) and (34b). As before, such a converging path is expressed as

$$v_t = \zeta(x_t), \quad \xi' > 0. \tag{35}$$

Hence, the complete dynamic system is now written in the following manner:

$$\dot{x}_t = x_t\left[g - \frac{1}{\theta}(\xi(x_t) - 1)\right], \tag{36a}$$

$$\dot{z}_t = z_t\left[(1-\tau_b)R - \rho\left(\frac{z_t}{A}\right) - \frac{1}{\theta}(\xi(x_t) - 1) + \delta\right], \tag{36b}$$

$$\dot{b}_t = \left[(1-\tau_b)R - \frac{\xi(x_t) - 1}{\theta} + \delta\right] b_t + \eta A x_t^{\phi} - z_t - \frac{\zeta(x_t) - 1}{\theta} - \frac{1}{2\theta}(\zeta(x_t) - 1)^2. \tag{36c}$$

The steady state values of x, v, z and b are the same as before. The linearized system

has the following coefficient matrix:

$$J_4 = \begin{bmatrix} -\frac{1}{\theta}\xi'(1) & 0 & 0 \\ -\frac{1}{\theta}\xi'(1) & -\frac{z}{A}\rho'\left(\frac{z}{A}\right) & 0 \\ \phi\eta A - \frac{\xi'(1)}{\theta}(b+1) - \frac{1}{\theta}(\xi(1)-1) & -1 & \rho\left(\frac{z}{A}\right)b \end{bmatrix}.$$

The characteristic roots of this matrix are:

$$-\frac{1}{\theta}\xi'(1), \quad -\frac{\bar{z}}{A}\rho'\left(\frac{z}{A}\right), \quad \rho\left(\frac{z}{A}\right)b.$$

Therefore, if $\rho'(.) > 0$, then J_4 has two stable roots, meaning that the economy is locally indeterminate. In contrast, if $\rho'(.) < 0$, then the system has one stable root, which means that local determinacy holds around the balanced growth path.

As shown by (35) ,when $x_t\,(= Y_t^*/Y_t)$ rises, the relative values between real capital and foreign bond increases, which accelarates capital accumulation. Furthermore, the own responses of each endogenous variables around the balanced growth path are:

$$\frac{\partial \dot{x}_t}{\partial x_t} = -\frac{1}{\theta}\xi'(1) < 0, \quad \text{sign}\,\frac{\partial \dot{z}_t}{\partial z_t} = -\text{sign}\,\rho'\left(\frac{z}{A}\right), \quad \frac{\partial \dot{b}_t}{\partial b_t} = \rho\left(\frac{z}{A}\right)b > 0.$$

Namely, x_t displays self-stabilizing behavior, while b_t shows self-destabilizing behavior near the balanced growth path. Now suppose that the economy initially stays on the balanced growth path and an optimistic sunspot shock raises the households' anticipated future income. Such an optimism will increase consumption, C_t, as well as the reference income, Y_t^*. If $\rho'(z_t/A) > 0$, the Euler equation (30) shows that the growth rate of consumption decreases so that the current consumption will rise. Hence, the initial anticipation can be self-fulfilled. In contrast, if $\rho'(z_t/A) < 0$, a rise in consumption depresses the growth rate of consumption and current consumption decreases, implying that the initial anticipation will not be realized.

More formally, when $\rho'(.) > 0$, the steady state of the sub-dynamic system constituted by (36a) and (36b) is a sink. Therefore, the paths of (x_t, z_t) are indeterminate because both x_t and z_t are jump variables. Although b_t governed by (36c) exhibits self-destabilizing behavior, there are infinite number of initial values of x_t and z_t that make b_t converge to its steady state level. By contrast, if $\rho'(z_t/A) < 0$, the subsystem consisting of (36a) and (36b) has a

converging path expressed as $z_t = \zeta(x_t)$ and $\zeta'(x_t) > 0$. As a result, a complete system is summarized as (36a) and t

$$\begin{aligned}\dot{b}_t &= \left[(1-\tau_b)R - \frac{\xi(x_t)-1}{\theta} + \delta\right] b_t + \eta A x_t^{\phi} - \zeta(x_t) \\ &\quad -\frac{\zeta(x_t)-1}{\theta} - \frac{1}{2\theta}\left(\zeta(x_t)-1\right)^2.\end{aligned}$$

This two-dimensional system exhibits saddle-point stability and, hence, under a given level of b_0, the initial level of x_t (so the initial values of x_t and v_t) is uniquely determined. This ensures determinacy of equilibrium.

To sum up, we have shown:

Proposition 3 *Suppose that the time discount rate depends on the social average of consumption-income ratio. Then if the time preference exhibits increasing marginal impatience, the balanced growth path is locally indeterminate (determinate) under progressive (regressive) taxation rule. If the time preference holds decreasing marginal impatience, then the balanced growth path is locally determinate (unstable) under progressive (regressive) taxation.*

4.2 Debt Elastic Interest Rate

We now explore the model in which the world interest rate is endogenously determined. Consider an open economy that owes a debt to the rest of the world. We define $D_t = -B_t > 0$ as the stock of debt of the representative household in the home country. The flow budget constraint for the household is now expressed as

$$\dot{D}_t = R\left(\frac{D_t}{K_t}\right) D_t + C_t + \left(\frac{I_t}{K_t} + \frac{\theta}{2}\left(\frac{I_t}{K_t}\right)^2\right) K_t - \eta\left(\frac{Y_t^*}{Y_t}\right)^{\phi} A K_t^{\alpha} \bar{K}_t^{1-\alpha}. \tag{38}$$

Here, the world financial markets charges a debt-elastic interest rate: a higher debt relative to capital raises the interest rate. We assume:

$$R(0) = \bar{R} > 0, \quad R'\left(\frac{D_t}{K_t}\right) > 0, \quad R''\left(\frac{D_t}{K_t}\right) \geq 0.$$

The Hamiltonian function in this model is specified as

$$\begin{aligned} H_t &= \log C_t + \lambda_t \left\{ R\left(\frac{D_t}{K_t}\right) D_t + C_t + \left[\frac{I_t}{K_t} + \frac{\theta}{2}\left(\frac{I_t}{K_t}\right)^2\right] K_t - \eta \left(\frac{Y_t^*}{Y_t}\right)^{\phi} A K_t^{\alpha} \bar{K}_t^{1-\alpha} \right\} \\ &\quad + q_t \left(I_t - \delta K_t\right), \end{aligned}$$

In the above, λ_t is the shadow value of debt that has a negative value. Given $\bar{K}_t = K_t$, the optimization conditions include the following:

$$\frac{1}{C_t} = -\lambda_t, \tag{39a}$$

$$-\lambda_t \left(1 + \theta \frac{I_t}{K_t}\right) = q_t \tag{39b}$$

$$\dot{\lambda}_t = \lambda_t \left[\rho - R'\left(\frac{D_t}{K_t}\right)\frac{D_t}{K_t} - R\left(\frac{D_t}{K_t}\right)\right], \tag{39c}$$

$$\dot{q}_t = \left(\rho + \delta\right) q_t - \lambda_t \left[-R'\left(\frac{D_t}{K_t}\right)\left(\frac{D_t}{K_t}\right)^2 + \eta\left(1-\phi\right)\alpha A \left(\frac{Y_t^*}{Y_t}\right)^{\phi} + \frac{\theta}{2}\left(\frac{I_t}{K_t}\right)^2\right], \tag{39d}$$

Let us define

$$x_t = Y_t^*/Y_t, \quad v_t = -q_t/\lambda_t, \quad z_t = C_t/K_t, \quad m_t = D_t/K_t.$$

By use of the constraints for the household's problem and the optimization conditions $(39a)$ through $(39d)$,we obtain a the following complete dynamics system::

$$\dot{x}_t = x_t \left[g - \frac{1}{\theta}\left(v_t - 1\right) + \delta\right], \tag{40a}$$

$$\dot{v}_t = \left[R'\left(m_t\right) m_t + R\left(m_t\right) - \rho\right] v_t - R'\left(m_t\right) m_t^2 + \eta\left(1-\phi\right)\alpha A x_t^{\phi} + \frac{1}{2\theta}\left(v_t - 1\right)^2, \tag{40b}$$

$$\dot{z}_t = z_t \left[R'\left(m_t\right) m_t + R\left(m_t\right) - \rho - \frac{1}{\theta}\left(v_t - 1\right) + \delta\right], \tag{40c}$$

$$\dot{m}_t = \left[R\left(m_t\right) - \frac{1}{\theta}\left(v_t - 1\right) + \delta\right] m_t + z_t + \frac{1}{\theta}\left(v_t - 1\right) + \frac{1}{2\theta}\left(v_t - 1\right)^2 - \eta A x_t^{\phi}. \tag{40d}$$

Since $x = 1$, the steady state values of v_t, x_t, d_t and g are determined by the following

conditions:

$$g = \frac{1}{\theta}(v-1) - \delta, \tag{41a}$$

$$\left[R'(m)\, m + R(m) - \rho\right] v - R'(m)\, m^2 + \eta\alpha A(1-\phi) + \frac{1}{2\theta}(v-1)^2 = 0, \tag{41b}$$

$$R'(m)\, m + R(m) - \rho - \frac{1}{\theta}(v-1) + \delta = 0, \tag{41c}$$

$$\left[R(m) - \frac{1}{\theta}(v-1) + \delta\right] m + z + \frac{1}{\theta}(v-1) + \frac{1}{2\theta}(v-1)^2 - \eta A = 0. \tag{41d}$$

Without specifying functional form of $R(m)$ and imposing restrictions on parameter values, we cannot confirm the existence of a unique steady state. In the following, we assume that there is a unique set of (g, v, x, m) that fulfill the above steady-state conditions.

The coefficient matrix of the dynamic system approximated at the steady state specified above is:

$$J_5 = \begin{bmatrix} 0 & -\frac{1}{\theta}v & 0 & 0 \\ \phi(1-\phi)\eta A & \delta + 2g & 0 & v(R'' + 2R') - (2R' + R''m^2) \\ 0 & -\frac{z}{\theta} & 0 & z(R''m + 2R') \\ -\phi A\eta & -\frac{d}{\theta} + \frac{2}{\theta} & 1 & \rho - R"(m)\, m \end{bmatrix}.$$

This leads to

$$\det J_5 = -\phi(1-\phi)\eta A z[R'(m)\, m + 2Rm].$$

Hence, we find:

$$\text{sign } \det J_5 = \text{sign } -\phi.$$

Observe that the trace of J_5 has a positive value, so that at least one of eigenvalues of J_5 is positive. Therefore, $\det J_5$ shows that the number of stable roots of J_5 is either one or three if $\phi > 0$. On the other hand, if $\phi < 0$, then the number of stable roots is either zero or two.

To conduct further investigation of stabilization effect of each tax schedule, we examine numerical examples. We assume a simple functional form of $R(d_t)$ such as

$$R(m_t) = \bar{R} + \chi m_t, \quad \chi > 0.$$

Then we set the baseline parameter values in the following manner:

$$\eta = 0.3,\ \phi = 0.5, \quad \theta = 0.03,\ \chi = 0.005,.\ \bar{R} = 0.04,\ \rho = 0.02,\ \delta = 0.1, \quad \alpha = 0.4, \quad A = 0.1.$$

In the baseline case, we assume that the tax schedule is progressive ($\phi = 0.5$) and the average tax rate is $\eta = 0.3$ when $\phi = 0$. Additionally, we set $\alpha = 0.4$ so that the degree of external effects associated with aggregate capital is $1 - \alpha = 0.6$. Thus condition (3) indicates that $\phi_0 = -2.33$. Other parameter values basically follow the model with debt-elastic interest rate discussed by Schmitt-Grohé and Uribe (2003). In this baseline case, we obtain a unique, feasible steady state in which the balanced growth rate is $g = 0.032$ and the steady-state level of debt-capital ratio is $m = 1.31$. Given our specification, we find that J_5 has one negative and one positive real eigenvalues as well as conjugate complex eigenvalues values with positive real parts. This means that in this example progressive taxation ensures determinacy of equilibrium.

We then change ϕ in the rage of $[-2.0,\ 0.9]$. We see that the balanced growth rate, g, as well as the steady state value of $m_t\ (= D_t/K_t)$ decrease with ϕ. For example, if $\phi = 0.9$, then $g = 0.021$ and $m = 1.21$. If ϕ is lowered to 0.1, then $g = 0.036$ and $m = 1.41$. Similarly, if $\phi = -0.5$, then $g = 0.043$ and $m = 1,43$. If $\phi = -2.0$, then $g = 0.056$ and $m = 1.72$. Therefore, a higher progressiveness of taxation depresses the balanced growth rate and lowers the debt-capital ratio. We evaluate the coefficient matrix based on each set of steady-state values of (m, v, x, z) .We find that J_5 always has one stable, real root for all $\phi \in [0, 0.9]$.On the other hand, J_5 has two stable roots for all $\phi \in [-2.0, 0)$. Consequently, as far as we use plausible parameter values, our numerical experiment suggests that progressive taxation serves as a stabilizer and regressive taxation destabilizes the economy in the sense that it allows sunspot-driven fluctuations. These results are similar to the policy implications obtained in the one-sector, closed economy model of real business cycles.

The summary of this subsection is as follows:

Proposition 4 *If the world interest rate is elastic to the debt-capital ratio of the home country, progressive (regressive) taxation may yield determinacy (indeterminacy) of equilibrium.*

5 Conclusion

This paper examines the dynamic effects of nonlinear taxation in small-open economies. We confirm that in the standard framework of the small-open economy with free capital mobility, Chen and Guo's (2015 and 2016) findings still hold: equilibrium indeterminacy emerges under the progressive taxation schedule, while determinacy holds under the regressive taxation schedule. In this situation, not only the transition path towards the balanced-growth equilibrium but also the steady-state values of key variables may be affected by sunspot-driven changes in expectations of agents if the fiscal authority employs a progressive tax scheme. We also demonstrate that such a destabilization effect of progressive taxation does not necessarily hold when the steady state of the small open economy is fixed regardless of its initial conditions. We first examine a model with endogenous time discount rate. We find that the destabilizing effect of progressive tax still holds if the preference structure satisfies increasing marginal impatience. In contrast, if preferences exhibit decreasing marginal impatience, then progressive taxation may serve as a stabilizer in the sense that it eliminates the possibility of equilibrium indeterminacy. In the second example, we examine a model with the debt-elastic world interest rate. In this model. our numerical examples with plausible parameter values reveal that progressive taxation yields determinacy and the regressive tax generates indeterminacy, which is similar to the policy implication in the contest of one-sector real business cycle model of a closed economy.

In this paper we have focused on small-open economies. A useful extension of our discussion is to examine the role of taxation rules in the context of a global economy. For example, using a two-country model, we can investigate whether or not the (de)stabilizing effects of taxation shown in the small open economies still hold in the global economy. Exploring this kind of problem would be insightful to understand how fiscal actions conducted by each country affect volatility of the financially integrated world[9].

[9] Hu and Mino (2013) discuss the relation between equilibrium indeterminacy and financial capital mobility in a two county model. Introducing nonlinear income tax into Hu and Mino (2013) would deserve further investigation.

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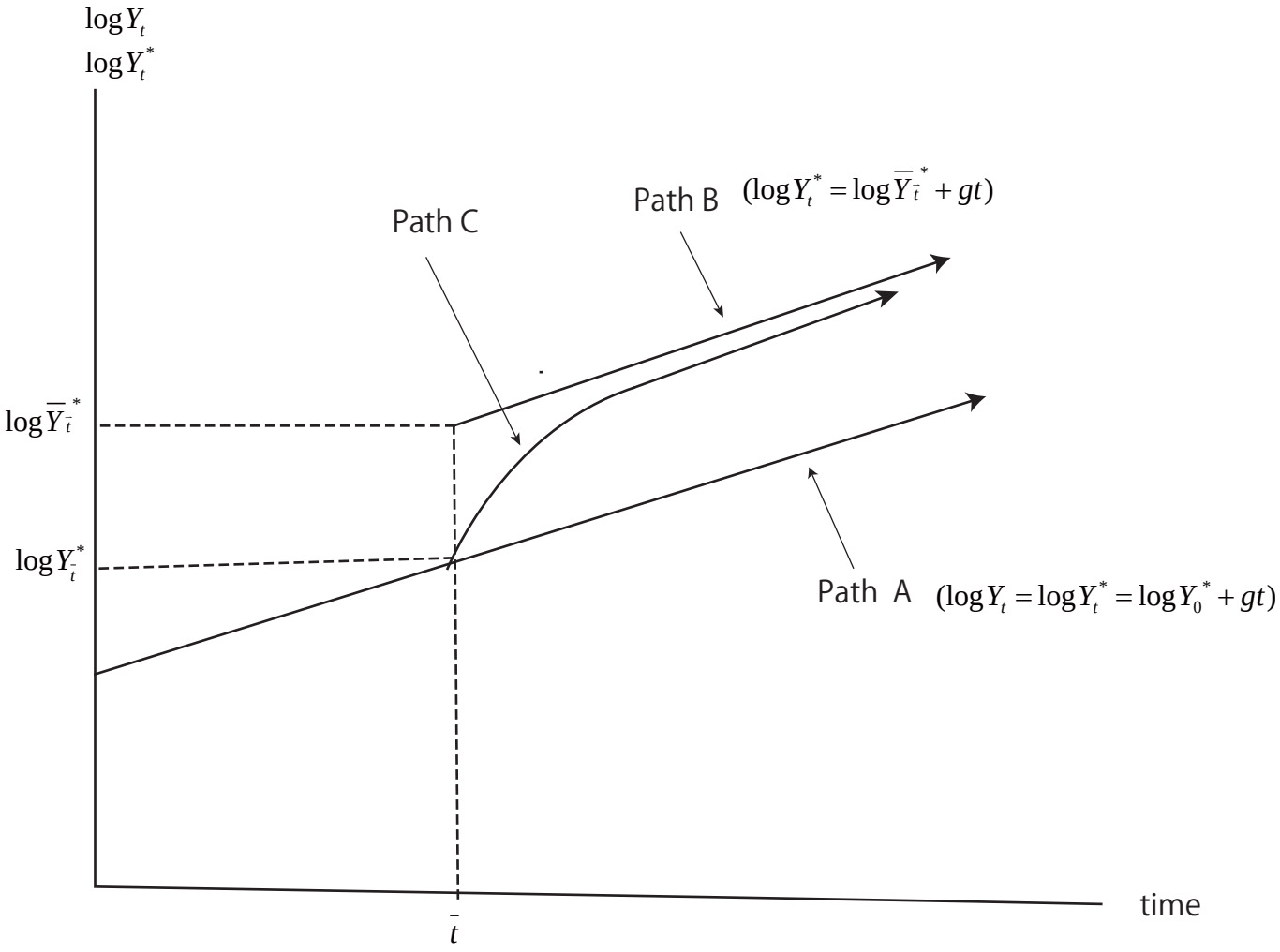
$\log Y_t$
$\log Y_t^*$
Path C
Path B $(\log Y_t^* = \log \overline{Y}_{\bar{t}}^* + gt)$
$\log \overline{Y}_{\bar{t}}^*$
$\log Y_{\bar{t}}^*$
Path A $(\log Y_t = \log Y_t^* = \log Y_0^* + gt)$
time
$\bar{t}$

Figure 1

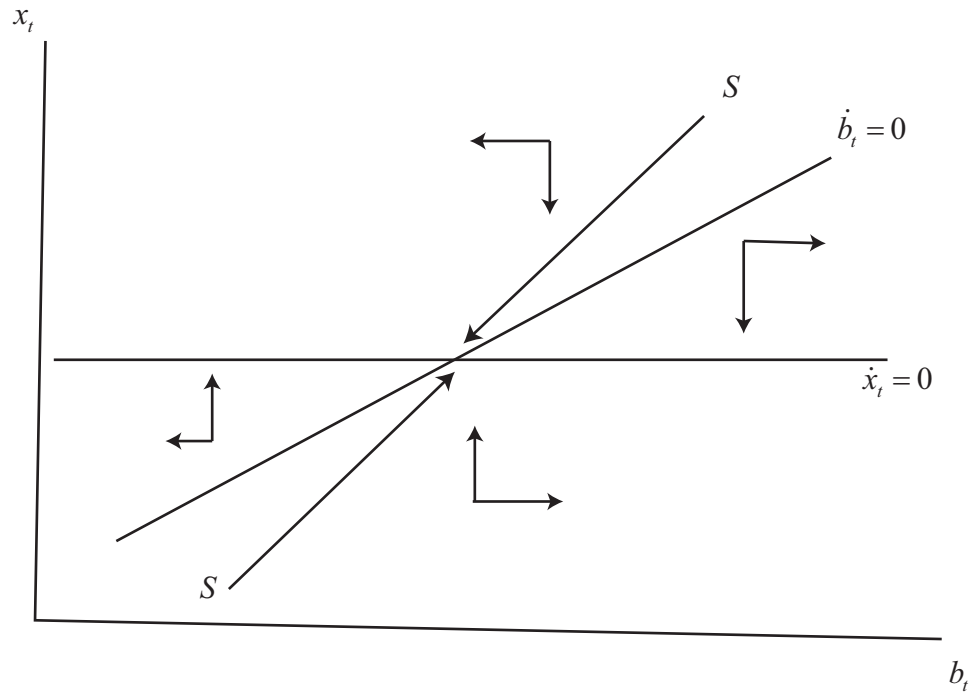

Figure 2

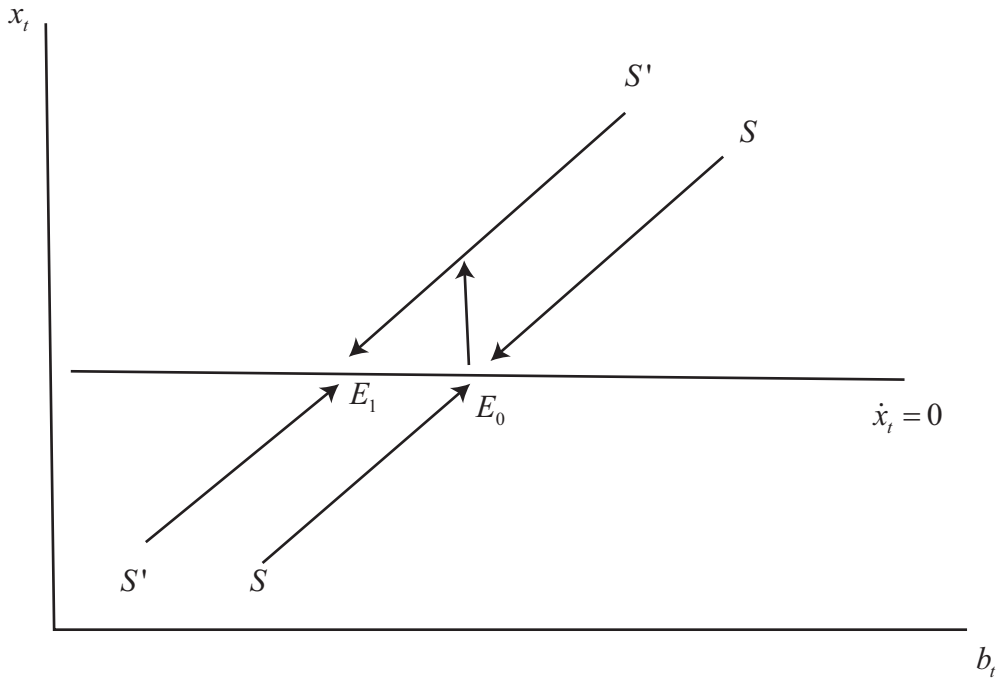

Figure 3