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An evolutionary approach to social choice problems with q-quota rules *

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Abstract This paper considers a dynamic process of n-person social choice problems under q-majority where a status-quo policy is challenged by an opposing policy drawn randomly in each period. The opposing policy becomes the next status-quo if it receives at least q votes. We characterize stochastically stable policies under a boundedly rational choice rule of voters. Under the best response rule with mutations, a Condorcet winner is stochastically stable for all q-quota rules, and uniquely so if q is greater than the min-max quota. Under the logit choice rule, the Borda winner is stochastically stable under the unanimity rule. Our evolutionary approach provides a dynamic foundation of the mini-max policies in multidimensional choice problems with Euclidean preferences.

Keywords: Stochastic stability; Social choice; Voting; Condorcet winner.
JEL Classification Numbers: C71, C73, D71.

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1 Introduction

A social choice in a democratic society is viewed as a dynamical system in which a status-quo policy is repeatedly challenged by an alternative one. For example, a ruling party is challenged by an opposition party in an election, and if the ruling party loses, then a policy proposed by the opposition party becomes a new status-quo. This political process is repeated. A central problem in the society is what the long-run outcome of a dynamical political process is under various voting rules. The aim of this paper is to consider a dynamic social choice problem by applying the stochastic evolutionary game theory developed by Foster and Young (1990), Kandori et al. (1993) and Young (1993).

Our evolutionary approach to a social choice problem is contrasted with the traditional normative one in several aspects. First, while the traditional approach searches for socially desirable choices defined by a set of axioms, the evolutionary one investigates dynamically stable choices in a process of collective decision making. Second, as an ideal model of social agents, the traditional approach presumes that voters are perfectly rational in the sense that they do not make any decision error under their own consistent preferences. The evolutionary approach deals with boundedly rational voters who are myopic and may make various kinds of error in judgment, evaluation and decision. Third, whereas the basic result in the traditional approach is Arrow (1954)'s impossibility theorem, an impossibility result does not play a major role in the evolutionary one since the long-run equilibrium outcome satisfying stochastic stability exists under a weak condition. Lastly, while the traditional approach postulates that a social choice is implemented by a third-party such as a social planner, the evolutionary one has the view that a social choice is implemented through a dynamic political process. This difference has an important implication to our analysis. Since the social planner does not know voters' true preferences, she needs to let them report their own preferences. In the traditional framework, the property of strategy-proofness for a social choice function that requires truth-reporting to be optimal for voters is crucial. In contrast, the property is irrelevant in our evolutionary approach without a social planner. Moreover, the evolutionary approach does not assume that voters know others' true preferences, in contrast to the framework of other positive theories of social choice based on rational game theory.

In this paper, we consider a dynamic social choice problem where voters make a collective choice repeatedly over a finite set of policies. In each period, a challenging policy is randomly chosen given a status-quo policy, and voters choose between them under a majority voting rule with q-quota. If the alternative wins at least q votes, then it is chosen and it will become a status-quo policy in the next period. Otherwise, the status-quo remains effective. As for a choice rule of voters, we assume the following. In an unperturbed process, voters make optimal choices according to their preferences. In a perturbed process, they may make mistakes to choose suboptimal policies with small probability. Given a stochastic process for challenging policies and a stochastic choice rule of voters as well as a voting quota q, the dynamic voting process sketched above can be formulated into a Markov chain with finite states (policies). The aim of our analysis is to characterize stochastically stable states of the Markov chain of voting. The notion of a stochastically stable state is now standard, and it describes the long-run equilibrium of the process that can be observed with positive frequency in the long-run as stochastic noise (error probabilities) vanishes. It is known that a stochastically stable state exists under a general choice rule satisfying a regularity condition. There have been recent developments of the literature of stochastic stability on cooperative games, e.g. Newton (2012) and Sawa (2014). Stochastically stable states are characterized via a similar technique to Sawa (2014).

The main results are summarized as follows. Let n be the number of voters. We first prove that a Condorcet winner (an alternative which beats all others under simple majority) is stochastically stable under a general choice rule if the voting rule is either super majority or simple majority where the voting quota q has an upper limit being equal to $n - \overline{n}$.[1] We then focus the analysis on two well-known choice rules, best response with mutations (BRM) and logit choice. Under BRM, voters may choose a suboptimal policy with a small probability independently and uniformly. Under the logit choice rule, the probability that voters may make a mistake is governed by a logit function. Under BRM, we prove that the Condorcet winner (if any) is stochastically stable for every voting quota from one (dictator rule) to n (unanimity). Furthermore, it is uniquely so if a quota q is larger than the min-max one $\overline{n}$. In a social choice problem without a Condorcet winner where the celebrated "voting paradox" occurs, a stochastically stable policy for all (super) majority rules belongs to the top cycle under $\underline{q}$ quota.[2] Under the logit choice, stochastic stability selects the Borda winner that maximizes the total score when voters rank all alternatives by the linear score from one to n according to their preferences. Finally, we analyze multidimensional choice problems with Euclidean preferences over a convex set of alternatives where a Condorcet winner does not exist almost surely. When $q > \overline{n}$, we show that the set of stochastically stable alternatives under BRM is approximately within that of min-max alternatives as the policy space is discretized sufficiently fine. When $q \leq \overline{n}$, the (unperturbed) dynamic process of voting has a unique recurrent class. Under the logit choice rule, a stochastic stable alternative is close to an alternative called the

[1] $\overline{n}$ is the min-max quota. See Definition 2.
[2] $\underline{q} = n/2$ for even n and $\underline{q} = (n+1)/2$ for odd n.

geometric median.

The evolutionary approach in this paper gives a new insight to an old debate concerning a Condorcet winner (Condorcet, 1785) and a Borda winner (Borda, 1781). There exist two strands of works in the literature. In the normative works, several axioms that each solution satisfies are investigated comparatively. The book of Moulin (1988) is a classic on the subject. In the positive works, a class of sequential voting models that implement these solutions have been proposed. See McKelvey and Niemi (1978), Dutta et al. (2002) and Bag et al. (2009) among others. In contrast to the literature, our evolutionary result shows that both the Condorcet winner and the Borda winner can emerge as a long-run equilibrium of the same dynamic process of voting, depending on a behavioral mode of boundedly rational voters.

Like this paper, several works in the positive social choice theory present dynamic voting models with sequential structure. It might be useful to clarify differences between their works and ours in modeling and analytical methods. Most sequential voting models formulate a class of elimination processes where one alternative is eliminated in each round. The remaining alternative in the final round is selected. Thus, they can be interpreted best as models of one-shot election composed by several steps of voting. In contrast, our dynamic model formulates a political process where an election takes place in real time repeatedly between a ruling party and an opposing party. As for the analytical method, previous studies are based on traditional game theory assuming rational players with perfect foresight and complete information. As typically observed in the rational equilibrium approach, their results tend to be sensitive to an extensive form of voting games. Our analysis is based on stochastic evolutionary game theory. We consider boundedly rational players with limited knowledge on other players. Our result does not depend on procedural details regarding the selection of a challenging policy.

The dynamic voting model of Kramer (1977) is closely related to ours in character. In both models, two parties compete for votes by advocating particular policies, and the policy of a party who wins a majority will become the status-quo in the next period. It, however, should be noted that the dynamic processes of the two models are differ in one critical aspect. The process of Kramer (1977) is deterministic in that the opposing party is assumed to choose a policy that maximizes votes. When a Condorcet winner does not exist, this vote-maximizing process has the property that the incumbent's policy is always defeated and that the two parties will alternate in office. Kramer (1977) proves that the vote-maximizing process approaches the min-max set of policies in multidimensional choice problems. A convergence result is not obtained. Our stochastic model provides a more general framework for dynamic collective choice problems; It can be applied not only to a policy competition as in Kramer (1977) but also to stochastic dynamic settings where a group of voters (legislators) make decisions over periods.

In the evolutionary approach, we consider boundedly rational voters who make myopic decisions. This behavioural assumption of voters is relevant in many situations where voters' foresight is limited by several factors such as impatience, limited information and reasoning inability. There are other works which consider dynamic voting games with patient players (Gomes and Jehiel (2005), Roberts (2007) and Bernheim and Slavov (2009)).[3] A common observation in these works is that a Condorcet winner is not always selected due to voters' strategic incentive in intertemporal settings.

The rest of the paper is organized as follows. Section 2 presents several basic notions for a social choice game. A dynamic process of voting is presented. Section 3 defines the notion of stochastic stability and gives general properties of the existence and the computation algorithm for it. Section 4 characterizes stochastically stable policies under BRM for various voting quotas. Section 5 characterizes a stochastically stable policy under logit choice for the unanimity rule. Section 6 analyzes multidimensional choice problems. Section 7 discusses the evolutionary approach in terms of a social choice correspondence. Section 8 concludes. Appendix includes proofs.

2 Model

2.1 Static setting

A social choice game, G, is given by a tuple of $(\mathcal{A}, N, \{u_i\}_{i \in N}, q)$. $\mathcal{A}$ denotes a finite set of alternatives. $N = \{1, \ldots, n\}$ is the set of players who jointly choose an alternative in $\mathcal{A}$. Let $u_i : \mathcal{A} \to \mathbb{R}$ be player i's utility function. We assume that preferences are strict, i.e., $u_i(a') \neq u_i(a)$ for all $i \in N$ and $a, a' \in \mathcal{A}$, $a \neq a'$. $q \in \{1, \ldots, n\}$ denotes a quota, which is the minimum number of votes required for an alternative to be implemented.

A social choice is made by the following rule. Consider that the status-quo policy is given by some $a \in \mathcal{A}$. An alternative $a' \in \mathcal{A}$ is randomly made.[4] All players vote simultaneously between the two policies, a and a'. The alternative a' is chosen if at least q players vote for it. The payoff of player i is given by $u_i(a')$ if a' is chosen, and by $u_i(a)$ otherwise. This game is sometimes referred to as a q-quota game. When $q = n$, the rule is unanimity. When $q = (n+1)/2$ for odd n or $q = 1 + n/2$ for even n, the rule is the

[3] Roberts (2007) considers a repeated game where in each period the status-quo policy is challenged by an alternative chosen randomly according to the uniform distribution.

[4] We shall discuss some examples of collective decision problems in the next subsection.

(exact) majority.

Let us define

$$N(a,a') = \{i \in N \,|u_i(a') > u_i(a)\}, \qquad n(a,a') = |N(a,a')|.$$

$N(a,a')$ is the set of players who prefer a' to a, and $n(a,a')$ is the number of such players. Note that the strict preference assumption implies that $n(a,a') + n(a',a) = n$ for $a \neq a'$. Also let

$$\overline{n}(a) = \max_{a' \in \mathcal{A}\setminus\{a\}} n(a,a').$$

$\overline{n}(a)$ is the maximum number of voters who prefer some alternative to a. If $\overline{n}(a) < q$, then a is unbeatable by any $a' \neq a$ under q-quota rule. We also let

$$\underline{q} = \lfloor (n+1)/2 \rfloor,$$

where $\lfloor x \rfloor$ is the largest integer not greater than x. $\underline{q}$ is equal to $(n+1)/2$ for odd n, and to $n/2$ for even n. $\underline{q}$ is the maximum number of q-quota under which at least either alternative wins q votes in a pairwise voting, i.e., either $n(a,a') \geq q$ or $n(a',a) \geq q$ holds for any pair $a, a' \in \mathcal{A}, a \neq a'$. Note that $\underline{q}$ is the majority for odd n, and $\underline{q}+1$ is so for even n.

We define a couple of solution concepts of a social choice problem.

Definition 1 (Condorcet winner)**.** *A* Condorcet winner *is an alternative which defeats any other alternative under majority rule. That is, $a \in \mathcal{A}$ is a* Condorcet winner *if $n(a',a) > n/2$ for all $a' \neq a$.*

By definition, a Condorcet winner is necessarily unique if it exists. Under a strict preference, it holds that $a \in \mathcal{A}$ is the Condorcet winner if and only if $\overline{n}(a) < n/2$.

The well-known "voting paradox" implies that the Condorcet winner does not necessarily exist. If it exists, the Condorcet winner has the property that it minimizes $\overline{n}(a)$ for all alternatives a. In view of this fact, we introduce the following notion weaker than the Condorcet winner (Kramer (1977) and Caplin and Nalebuff (1988)).

Definition 2 (Min-max alternative)**.** *An alternative $a^* \in \mathcal{A}$ is called a* min-max alternative *if it is a solution of* $\min_{a \in \mathcal{A}} \overline{n}(a)$*. The set $\mathcal{A}^*$ of min-max alternatives is called the* min-max set*. $\overline{n}$ is called the* min-max quota *if $\overline{n} = \min_{a \in \mathcal{A}} \overline{n}(a)$.*

Note that $\overline{n} \leq n-1$. If $\overline{n} < n/2$, then a Condorcet winner exists and it is a unique min-max alternative. For a min-max alternative a^*, $n(a^*,a) \leq \overline{n}$ for all $a \neq a^*$. This means that the min-max alternative a^* is not defeated by any other alternative under q-quota rule with $q > \overline{n}$.[5] An alternative a is said to be *unbeatable under q-quota rule* if it does not lose q votes against any other, i.e. $n(a, a') < q$ for any $a \neq a'$. A min-max alternative is unbeatable under $(\overline{n}+1)$-quota. This implies that the min-max alternatives of game G are in its core if the quota q is greater than $\overline{n}$.

Definition 3 (Core)**.** *The* core *of game G under q-quota rule is the (possibly empty) set of unbeatable alternatives under q-quota.*

The other generalization of a Condorcet winner is that of a top cycle. A top cycle is the set of alternatives of which each member defeats every other alternative in $\mathcal{A}$ directly or indirectly. For $L \geq 2$, an alternative $a_L \in \mathcal{A}$ is said to defeat $a_1 \in \mathcal{A}$ indirectly under q-quota rule, denoted by $a_L \succeq_q^* a_1$, if there exists a sequence $\{a_1, \ldots, a_{L-1}\} \subseteq \mathcal{A}$ such that $n(a_i, a_{i+1}) \geq q$ for all $i \in \{1, \ldots, L-1\}$. If $L = 2$, then we say that a_L directly defeats a_1 under q-quota rule. We define a top cycle with respect to q-quota rule as follows.

Definition 4 (Top cycle)**.** *The* top cycle *with respect to q, $\mathcal{TC}_q$, is defined by $\mathcal{TC}_q = \{a \in \mathcal{A} : \forall a' \in \mathcal{A},\ a' \neq a,\ a \succeq_q^* a'\}$.*

The definitions imply three interesting properties of the two sets: the core and the top cycle. First, they coincide if either of them contains unique alternative. Second, if the top cycle exists and contains more than one alternative, then the core is empty. Third, the top cycle is empty if the core does so. To see the second property, observe that for any pair a, a' contained in the top cycle, a' must be defeated by some alternative in order for $a \succeq_q^* a'$ to hold. Then, a' is not in the core. Since this argument applies to all alternatives in the top cycle, the core must be empty. Similarly to that, we can show the third one.

Obviously, the Condorcet winner (if any) is the top cycle under majority rule. Note that $\mathcal{TC}_q$ is always nonempty for $q \leq \underline{q}$, while it may be empty for $q > \underline{q}$. The top cycle with respect to $\underline{q}$ will play an important role in characterizing stochastically stable alternatives under the best response choice rule with mutations.

The Condorcet winner, the core and the min-max set are included in the set of Pareto efficient alternatives. The efficiency of a top cycle crucially depends on the Nakamura number of a social choice game (Nakamura (1979)). The Nakamura number ν with q-quota rule is defined by $\nu = \lceil n/(n-q) \rceil$ for $q < n$ and $\nu = +\infty$ for $q = n$, where $\lceil x \rceil$ is the

[5] Caplin and Nalebuff (1988) consider the min-max majority rule when the alternative set A is a compact set of the h-dimensional Euclidean space R^h and voters have Euclidean preferences. When voters' ideal points distribute uniformly over its convex support in A, they show that the portion of the min-max quota is not greater than $1 - (\frac{h}{h+1})^h$, converging to 0.632 as dimension h goes to infinity. The min-max majority is less than 64%.

smallest integer greater than x. A top cycle might include Pareto-dominated alternatives if $|\mathcal{A}| > \nu$. Note that the Nakamura number is originally defined for a game without veto players (Nakamura (1979)). We here extend it to be infinity for the unanimous voting game where every player has a veto.

Proposition 1. *Suppose that $\mathcal{A}$, N and q are given. If $|\mathcal{A}| < \nu$, then the core is nonempty for all profiles of preferences. If $|\mathcal{A}| = \nu$, then the core is nonempty or a top cycle with respect to q exists for all profiles of preferences. Furthermore, if a top cycle exists, then it does not include any Pareto-dominated alternative. If $|\mathcal{A}| > \nu$, then there exists a profile of preferences $\{u_i\}_{i \in N}$ with which the social choice game has a top cycle with respect to q including Pareto-dominated alternatives.*

Finally, we introduce the Borda winner which is based on a scoring method of voting. Scoring methods take into account the ranking of each alternative in the players' preferences. The Borda rule is one of the most popular scoring rules, in which each alternative is assigned points linearly increasing with the rank.

Definition 5 (Borda winner)**.** *Each player ranks alternatives in order of her preferences. The rankings are converted into points; an alternative receives one point for being ranked last, two for being next-to-last, and so on, up to $|\mathcal{A}|$ points for being ranked first. An alternative which receives the highest total score is called a* Borda winner.

2.2 Dynamic process

We consider a dynamic process in which players recurrently play a social choice game. Let $a^t \in \mathcal{A}$ be the status-quo policy in period t. A proposal $a' \in \mathcal{A}$ against the status-quo a^t is randomly made.[6] All players vote simultaneously between a' and a^t. The proposal a' is chosen if at least q players vote for it. Then, the status-quo in the next period will be $a^{t+1} = a'$, and $a^{t+1} = a^t$ otherwise. For $q \leq n/2$, it may happen that both policies a' and a^t obtain at least q votes. We remark that proposal a' has the priority for such cases. Every player is assumed to be boundedly rational; she typically chooses an optimal policy but occasionally does a suboptimal one due to stochastic noise.

The formal model above of a social choice game with random alternatives can be explained by the following examples of collective decision making. Legislatures vote on a bill proposed by a committee. They do not know in advance which bill is proposed by the committee, and anticipate it in a probabilistic manner. Another example is a political

[6]A probability distribution over proposals can be arbitrary. The result of the paper is not affected in any critical way by it as long as every proposal may be made with positive probability.

process in which a player is randomly selected as an agenda-setter according to a formal rule.[7] A political party may be selected as a formateur in government formation with probability proportional to the number of its seats. A random dictatorship is a particular case of the random proposer rule when $q = 1$.

Let $p_{a,a'}$ be probability with which a proposal $a' \in \mathcal{A}$ may be made against the current policy a. We assume that $p_{a,a'} > 0$ for every a and a' in $\mathcal{A}$. Let N_k be the set of subsets of N with size k. When the process is not perturbed by stochastic noise, the transition probability from a to $a' \neq a$ is given by

$$P^{0,q}_{a,a'} = p_{a,a'} \sum_{k \geq q}^{n} \sum_{J \in N_k} \underbrace{\prod_{j \in J} \mathbb{1}\{u_j(a') > u_j(a)\}}_{\substack{\text{All members} \\ \text{in } J \text{ accept.}}} \underbrace{\prod_{h \in N \setminus J} \mathbb{1}\{u_h(a') < u_h(a)\}}_{\substack{\text{All members} \\ \text{in } N \setminus J \text{ reject.}}}, \tag{1}$$

where $\mathbb{1}\{x > y\}$ is 1 if $x > y$ and 0 otherwise. The transition probability from a to a, i.e., the status quo remains, is given by $P^{0,q}_{a,a} = 1 - \sum_{a' \neq a} P^{0,q}_{a,a'}$. Note that the product of the last two products of Equation (1) becomes one if and only if $J = N(a, a')$. So, it is reduced to a simple form:

$$P^{0,q}_{a,a'} = p_{a,a'} \mathbb{1}\{n(a, a') \geq q\}. \tag{2}$$

Eq. (1) defines a Markov chain $P^{0,q} = (P^{0,q}_{a,a'})$ with finite states $\mathcal{A}$. We call $P^{0,q}$ an unperturbed process with q-quota rule.

We say that a set of alternatives is a recurrent class of the unperturbed process $P^{0,q}$ if the process never escapes from the set once it is reached, and all alternatives in the set are visited infinitely many times. It is formally defined below.

Definition 6. *Let $\theta \subseteq \mathcal{A}$. θ is a* recurrent class *of the unperturbed process $P^{0,q}$ with q-quota rule if it satisfies the following two conditions.*

1. *For all $a_1, a_k \in \theta$, there exists a sequence of alternatives $\{a_1, \ldots, a_k\} \subseteq \theta$ such that $P^{0,q}_{a_i,a_{i+1}} > 0$ for all $i \in \{1, \ldots, k-1\}$.*
2. *For all $a \in \theta$ and $a' \in \mathcal{A} \setminus \theta$, $P^{0,q}_{a,a'} = 0$.*

Let $\Theta(q)$ denote the set of recurrent classes of the unperturbed process with quota q. Two lemmas below shows a link between a top cycle and a recurrent class. Lemma 1 shows that if a top cycle exists under q-quota rule, then it must be a unique recurrent class

[7] The model can be interpreted as a model of random proposers. Let $p_{i,a,a'}$ be the probability that i becomes a proposer and proposes a'. Then, we let $p_{a,a'} = \sum_i p_{i,a,a'}$.

of the unperturbed process $P^{0,q}$. Lemma 2 says that the recurrent class is unique if $q \leq \underline{q}$.[8] Thus, the top cycle coincides with the recurrent class for $q \leq \underline{q}$.

Lemma 1 (Top cycle and recurrent class). *A top cycle exists in the game if and only if the unperturbed process has a unique recurrent class. For such cases, those two sets coincide.*

Proof. Suppose that there is a unique recurrent class θ. The definition implies that $a \succeq_q^* a'$ for all $a, a' \in \theta$. For all $a_1 \notin \theta$, there must exist a sequence $\{a_1, \dots, a_L\} \in \mathcal{A}$ with $a_L \in \theta$ such that $P^{0,q}_{a_i,a_{i+1}} > 0$ for all $i \in \{1, \dots, L-1\}$. Otherwise, the process starting from a_1 will not reach θ for all time, which implies another recurrent class in $\mathcal{A} \setminus \theta$, i.e. a contradiction. Thus, if $a \in \theta$, then $a \succeq_q^* a'$ for $a' \neq a$. By definition, alternatives in θ constitute a top cycle. We can prove the converse similarly. □

Lemma 2. *For $q \leq \underline{q}$, the unperturbed process has a unique recurrent class.*

Proof. By way of contradiction, suppose that the process has two recurrent classes, $\mathcal{A}_1$ and $\mathcal{A}_2$. By definition, $\mathcal{A}_1 \cap \mathcal{A}_2 = \varnothing$. Choose $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$. Strict preferences imply that at least $\underline{q}$ players prefer a_1 to a_2 or a_2 to a_1. Then, it follows from $q \leq \underline{q}$ that either $P^{0,q}_{a_1,a_2} > 0$ or $P^{0,q}_{a_2,a_1} > 0$ holds. It contradicts that $\mathcal{A}_1$ and $\mathcal{A}_2$ are recurrent classes. □

We now assume that every player's choice is perturbed due to stochastic noise. For a real value $\eta > 0$, the transition probability from a to $a' \neq a$ is given by

$$P^{\eta,q}_{a,a'} = p_{a,a'} \sum_{k \geq q}^{n} \sum_{J \in N_k} \underbrace{\prod_{j \in J} \Psi_j^\eta(a, a')}_{\substack{\text{All members} \\ \text{in } J \text{ accept.}}} \underbrace{\prod_{h \in N \setminus J} (1 - \Psi_h^\eta(a, a'))}_{\substack{\text{All members} \\ \text{in } N \setminus J \text{ reject.}}}, \tag{3}$$

where $\Psi_j^\eta(a, a')$ is the probability that player j may vote for proposal a' given the status quo a. The probability of her voting for the status quo a is given by $1 - \Psi_j^\eta(a, a')$. The probability of staying in the status quo is given by $P^{\eta,q}_{a,a} = 1 - \sum_{a' \neq a} P^{\eta,q}_{a,a'}$. We call the Markov chain $P^{\eta,q} = (P^{\eta,q}_{a,a'})$ a perturbed process with q-quota rule. The stochastic choice rule of every player satisfies the following regularity conditions.[9]

Definition 7. *A choice rule Ψ_j^η is regular if it satisfies the following conditions* (i)–(iv).

(i) *$\Psi_j^\eta(a, a')$ varies continuously in η for all $j \in N$ and all $a, a' \in \mathcal{A}$.*

[8] Recall that $\underline{q} = n/2$ for even n and $\underline{q} = (n+1)/2$ for odd n.

[9] A version of our results holds for choice rules satisfying (i)–(iii) but (iv). See Remark 1.

(ii) *As η approaches zero, the limit of $\Psi_j^\eta(a, a')$ satisfies that*[10]

$$\lim_{\eta \to 0} \Psi_j^\eta(a, a') = \begin{cases} 1 & \text{if } u_j(a') > u_j(a), \\ 0 & \text{if } u_j(a') < u_j(a), \\ \xi \in [0,1] & \text{if } u_j(a') = u_j(a), \end{cases} \qquad \forall j \in N, a, a' \in \mathcal{A}. \tag{4}$$

(iii) *As η approaches zero, the limit of $\eta \log \Psi_j^\eta(\cdot, \cdot)$ exists and satisfies that for some $\kappa_j(a, a') > 0$,*

$$-\lim_{\eta \to 0} \eta \log \Psi_j^\eta(a, a') = \begin{cases} 0 & \text{if } u_j(a') \geq u_j(a), \\ \kappa_j(a, a') & \text{if } u_j(a') < u_j(a). \end{cases}$$

(iv) *For (a, a') with $u_j(a') < u_j(a)$, there exists some $\kappa_j(a, a') > 0$ such that*

$$-\lim_{\eta \to 0} \frac{\Psi_j^\eta(a, a')}{\exp(-\eta^{-1}\kappa_j(a, a'))} > 0.$$

Many noisy best response rules studied in the literature, including best response with mutations and logit choice, satisfy the above regularity conditions. The real value η is interpreted as a stochastic noise level. The second condition implies that, as η approaches zero, the player's choice rule can be arbitrarily close to the optimal one. The last two put restrictions on how fast the probability of choosing suboptimal choices vanishes as η approaches zero. Roughly speaking, the probability of choosing a suboptimal choice will vanish at a constant rate. The condition (iii) rules out oscillating choice rules, e.g., $\Psi_j^\eta(a, a') = \exp(-\eta^{-1}(\kappa_j(a, a') + \sin(1/\eta)))$, while (iv) rules out choice rules where the power of the leading term is not linear in η^{-1}, e.g., $\Psi_j^\eta(a, a') = \exp(-\eta^{-1}(\kappa_j(a, a') + \sqrt{\eta}))$.

Example 1 (Best response with mutations)**.** Consider the best response with mutations rule as in Kandori et al. (1993) and Young (1993), in which a player may make a subopti-

[10] The limit value ξ of $\Psi_j^\eta(a, a')$ for $u_j(a') = u_j(a)$ does not affect the result of the paper. By the strict preference assumption, $u_j(a') = u_j(a)$ implies $a' = a$. Thus, the state will remain the same after voting, independent of the value of ξ.

mal choice with probability $\varepsilon > 0$. The choice rule Ψ_j^η is given as

$$\Psi_j^\eta(a,a') = \begin{cases} 1-\varepsilon & \text{if } u_j(a') > u_j(a), \\ \varepsilon & \text{if } u_j(a') < u_j(a), \\ \frac{1}{2} & \text{if } u_j(a') = u_j(a), \end{cases} \qquad \forall j \in N, a, a' \in \mathcal{A}, \tag{5}$$

where $\varepsilon = \exp(-\eta^{-1})$. Each player votes for the policy she prefers with probability $1-\varepsilon$ and votes for a suboptimal one with ε.

Example 2 (Logit choice)**.** Following Blume (1993), suppose that players employ the logit choice rule with noise level $\eta > 0$:

$$\Psi_j^\eta(a,a') = \frac{\exp(\eta^{-1}u_j(a'))}{\exp(\eta^{-1}u_j(a)) + \exp(\eta^{-1}u_j(a'))}.$$

The logit choice rule can be derived from a random utility model in which the utility for each alternative is perturbed by i.i.d. random variables with the Gumbel distribution. The distribution is bell-shaped and is similar to a normal distribution. This choice rule gives us a model which is amenable to computation and approximates a random utility model with normally distributed noise.[11]

3 Stochastic stability under regular choice rules

We present the standard results of stochastic evolutionary game theory which can be applied to the dynamic process of a social choice game described in the last section. All results in this section hold for all regular choice rules unless otherwise stated. In what follows, we shall call sometimes a policy in $\mathcal{A}$ a state of the process, if no confusion arises.

The perturbed Markov chain $P^{\eta,q} = \{P^{\eta,q}_{a,a'}\}_{a,a' \in \mathcal{A}}$ is irreducible and aperiodic for $\eta > 0$, and so admits a unique stationary distribution, denoted by π^q_η. Let $\pi^q_\eta(a)$ denote the probability that π^q_η places on state $a \in \mathcal{A}$.[12] Players' behavior is asymptotically summarized by π^q_η because of two properties; $\pi^q_\eta(a)$ represents the fraction of time in which state a is observed over a long time horizon, and it is also the probability that a will be observed at any sufficiently large time t. We say that state a is stochastically stable if it is in the support of the limiting stationary distribution as η approaches zero.

[11]See McFadden (1976) for the logit choice model and its applications in economics.

[12]With abuse of notations, we write $\pi^q_\eta(\mathcal{A}_1) = \sum_{a \in \mathcal{A}_1} \pi^q_\eta(a)$ for a subset $\mathcal{A}_1 \subseteq \mathcal{A}$, and $\pi^q_\eta(\Theta(q)) = \sum_{\mathcal{A}_1 \in \Theta(q)} \sum_{a \in \mathcal{A}_1} \pi^q_\eta(a)$ for the collection of recurrent classes $\Theta(q)$.

Definition 8. *A state $a \in \mathcal{A}$ is* stochastically stable under q-quota rule *if* $\lim_{\eta \to 0} \pi^q_\eta(a) > 0$.

We next define a notion of transition cost from one alternative to another, which measures the unlikeliness of the transition. Define the cost of player i's accepting alternative a' given the status quo a as,

$$c_i(a, a') = -\lim_{\eta \to 0} \eta \log \Psi^\eta_i(a, a').$$

The cost is the exponential rate of decay of the choice probability as η approaches zero.[13] Roughly speaking, it represents the unlikeliness of player i's agreeing to a switch from a to a'. Property (iii) in Definition 2.7 implies that $c_i(a, a')$ is zero if $u_i(a') > u_i(a)$. Namely, player i accepts the switch with zero cost if she prefers a' to a. Further, define the transition cost from a to a' under q-quota rule as

$$c^q_{aa'} = \min_{J \in N_q} \sum_{i \in J} c_i(a, a')$$

where N_q is the class of subsets J of N with size q. Recall that $N(a, a')$ denote the set of players who prefer a' to a and that $n(a, a')$ is the cardinality of $N(a, a')$. Since $c_i(a, a') = 0$ for every $i \in N(a, a')$, we can rewrite the cost $c^q_{aa'}$ as follows.

$$c^q_{aa'} = \begin{cases} 0 & \text{if } q \leq n(a, a'), \\ \min_{J \subset N(a',a), |J| = q - n(a,a')} \sum_{i \in J} c_i(a, a') & \text{otherwise.} \end{cases} \tag{6}$$

Namely, the transition cost $c^q_{aa'}$ from the status quo a to an alternative a' under q-quota rule is the smallest sum of transition costs for q players whose acceptance is needed for proposal a' to be chosen. Some of such players may not prefer a' to the status quo a, but they may accept proposal a' "by mistake". If this is the case, then the transition cost $c^q_{aa'}$ will be positive. If there exists at least q players who prefer a' to a, then the transition cost from a to a' is zero.

Example 3 (BRM)**.** Suppose that the players' choice rule is given by best response with mutations in (5). When $u_i(a') < u_i(a)$, the cost of a player's choosing a' against a by mistake is given by $c_i(a, a') = \lim_{\eta \to 0} -\eta \log \varepsilon = \lim_{\eta \to 0} -\eta \log \exp(-\eta^{-1}) = 1$. The transition cost from a to a' is given by

$$c^q_{aa'} = \max\{q - n(a, a'), 0\}. \tag{7}$$

[13] This expression of the costs of transitions follows Section 12.A.5 of Sandholm (2010).

In words, $c^q_{aa'}$ is the minimum number of mistakes (mutations) by which the social choice is potentially switched from a to a' under q-quota rule.

For two alternatives a and a', we denote the transition from a to a' by notation (a, a'). We call a set of transitions $\{(a_1, a_2), (a_2, a_3), \ldots, (a_{L-1}, a_L)\}$ a *path* from a_1 to a_L on $\mathcal{A}$ if $a_i \neq a_j$ for all $i \neq j$. Note that any transition (a_i, a_{i+1}) may occur with a positive probability under a perturbed process $P^{\eta,q}$. For $a \in \mathcal{A}$, we call a set of transitions, denoted by τ_a, an *a-tree* if there exists a unique path from a' to a for all $a' \in \mathcal{A}$ with $a' \neq a$. Let Υ_a denote the set of all a-trees. Given an a-tree τ_a and quota q, we define the cost of a-tree τ_a as

$$c_q(\tau_a) = \sum_{(v,w)\in\tau_a} c^q_{vw}. \tag{8}$$

We define $c^*_q(a)$ as the lowest cost among all a-trees, and define c^*_q as the minimum of $c^*_q(a)$ among all alternatives a in $\mathcal{A}$. That is,

$$c^*_q(a) = \min_{\tau_a\in\Upsilon_a} c_q(\tau_a), \qquad\qquad m^*_q = \min_{a\in\mathcal{A}} c^*_q(a). \tag{9}$$

Define

$$M_q = \left\{ a \in \mathcal{A} : c^*_q(a) = m^*_q \right\}. \tag{10}$$

The following theorem is our version of Theorem 1 of Kandori et al. (1993).

Theorem 1. *An alternative $a \in \mathcal{A}$ is stochastically stable if and only if $a \in M_q$.*

The theorem states that an alternative is stochastically stable if and only if a tree with itself as the root has the minimum cost among all trees. This characterization leads to two useful results about stochastic stability. Firstly, a stochastically stable alternative $a \in \mathcal{A}$ under q-quota rule must belong to some recurrent class of the unperturbed process. If a is not in any recurrent class, then there exists a path from a to some recurrent class with zero cost. The cost of any a-tree must be greater than that of b-tree for every alternative b in the recurrent class. Secondly, if a recurrent class of the unperturbed process includes a stochastically stable alternative, then all alternatives in it are also stochastically stable.

Remark 1. A choice rule is said to be weakly regular if it satisfies conditions (i)–(iii) in Definition 7. The probit choice rule is not regular but weakly regular (Myatt and Wallace (2003) and Dokumacı and Sandholm (2011)). For a weakly regular choice rule, it is known that if a state a is stochastically stable, then $a \in M_q$. Thus it holds that $\lim_{\eta\to 0} \pi^q_\eta(M_q) = 1$. See Sandholm (2010).

The next theorem identifies the stochastic stability of the Condorcet winner under q-quota rules.

Theorem 2 (Stochastic stability of a Condorcet winner)**.** *Let $\overline{n}$ be the min-max quota. Then, the Condorcet winner $a \in \mathcal{A}$ is stochastically stable for every $q \leq n - \overline{n}$. Moreover, a is uniquely so for every q with $\overline{n} < q \leq n - \overline{n}$.*

Proof. Suppose that a is the Condorcet winner. By definition, it holds that for every $a' \neq a$, $n(a', a) > n/2$ and thus $n(a, a') < n/2$. This implies that $\overline{n}(a) < n/2$. Since $\overline{n}(a') > n/2$ for every $a' \neq a$, it can be seen that $\overline{n} = \overline{n}(a) < n/2$. First, assume that $q \leq n - \overline{n}$. Then, for every $a' \neq a$,

$$q \leq n - \overline{n}(a) \leq n - n(a, a') = n(a', a).$$

The last equality comes from the assumption of strict preference. It follows from (6) that $c^q_{a'a} = 0$ for all $a' \neq a$ and thus that $c^*_q(a) = 0$. Since $c^*_q(a')$ is non-negative for all a', $c^*_q(a)$ must be the minimum among all alternatives. Thus, a is stochastically stable for $q \leq n - \overline{n}$. Finally, assume that $\overline{n} < q \leq n - \overline{n}$. Observe that $q > \overline{n}(a) \geq n(a, a')$ for every $a' \neq a$. (6) implies that $c^q_{aa'} > 0$, and thus $c^*_q(a') > 0$, for all $a' \neq a$. Since the Condorcet winner a satisfies $c^*_q(a) = 0$, Theorem 3.2 implies that a is uniquely stochastically stable for every q with $\overline{n} < q \leq n - \overline{n}$. □

The stochastic stability of the Condorcet winner can be explained intuitively as follows. Since the Condorcet winner defeats every other alternative under majority rule, there always exist at least $n/2$ voters who prefer the Condorcet winner to another. This means that the process can transit to the Condorcet winner with zero cost from every other alternative for quota q less than $n/2$. Thus, it is straightforward to see that the Condorcet winner is stochastically stable under q-quota rule if q is less than $n/2$. Actually, the same property holds for $q \leq n - \overline{n}$ where $\overline{n}$ is the min-max quota. If $\overline{n} < q \leq n - \overline{n}$, then for every alternative $a' \neq a$, the process needs at least one voter who accepts proposal a' by mistake so that the transition from a to a' occurs. The transition cost for every a'-tree is positive and thus the Condorcet winner is uniquely stochastically stable for $\overline{n} < q \leq n - \overline{n}$. Theorem 2 implies that stochastic stability of the dynamic process for social choice games with q-quota rule is in favor of a Condorcet winner if the quota q is greater than the min-max quota $\overline{n}$, less than $n - \overline{n}$, for all regular choice rules including BRM and logit choice.

The upper limit of the quota, $n - \overline{n}$, is tight in the sense that there exists a game with a regular choice rule in which a Condorcet winner is not stochastically stable for all $q > n - \overline{n}$. Example 7 in Section 5 offers such a game, where the Borda winner is stochastically stable for all $q > n - \overline{n}$ under the logit choice rule.

Pareto efficiency of stochastically stable alternatives is closely related to the Nakamura number of the considered social choice game (Nakamura (1979)). If the cardinality of the set of alternatives is at most the Nakamura number, then the Pareto efficiency of a stochastically stable alternative is guaranteed. Otherwise, there exists a profile of preferences under which some Pareto-dominated alternative is stochastically stable. As the Nakamura number is infinity under the unanimity, all stochastically stable alternatives with $q = n$ are Pareto efficient. The next corollary is implied by Proposition 1.

Corollary 1. *Suppose that $\mathcal{A}$, N and q are given. If $|\mathcal{A}| \leq \nu$, then all stochastically stable alternatives are Pareto efficient for any social choice game with $(\mathcal{A}, N, q)$. Furthermore, all stochastically stable alternatives are in the core if the core is nonempty, and a set of stochastically stable alternatives coincide with the top cycle if it is empty. If $|\mathcal{A}| > \nu$, then there exists a profile of preferences under which Pareto-dominated alternatives are stochastically stable under any regular choice rule.*

For a general social choice problem without a Condorcet winner, we present an algorithm useful for computing stochastically stable alternatives.

An *abstract graph* is represented by a tuple (V, E, c) where V is the set of nodes, $E \subseteq V \times V$ is the set of directed edges, and c is a real-valued function defined on E. In what follows, we consider a complete graph V, that is, $E = V \times V$. For every two nodes v and w in V, a directed pair (v, w) is interpreted as an edge from node v to node w. The real value $c(v, w)$ means the cost for moving from v to w.

We define a recurrent class for a graph (V, E, c). Let $\mathcal{D}(v, w, E)$ be the set of paths from v to w on E.[14] Let $\tilde{c}(v, w)$ be the minimum cost of paths from v to w, i.e.,

$$\tilde{c}(v, w) = \min_{d \in \mathcal{D}(v,w,E)} \sum_{e \in d} c(e), \qquad \forall v, w \in V. \tag{11}$$

A subset $\mathbf{v} \subseteq V$ is said to be *V-recurrent* if it satisfies the two conditions: (i) $\tilde{c}(v, w) > 0$ for all $v \in \mathbf{v}$ and $w \in V \setminus \mathbf{v}$, and (ii) $\tilde{c}(v, \hat{v}) = 0$ for all $v, \hat{v} \in \mathbf{v}$.

Let $\mathbf{V}$ be a family of subsets of nodes in V. For $\mathbf{V}$, let $\Lambda(\mathbf{V}) = \{v \in \mathbf{v} : \mathbf{v} \in \mathbf{V}\}$. In words, $\Lambda(\mathbf{V})$ is the set of nodes in V which belongs to some subset $\mathbf{v}$ in $\mathbf{V}$.

Theorem 3. *Let $q > \underline{q}$ where $\underline{q}$ is the largest integer not greater than $(n+1)/2$. A state $a \in \mathcal{A}$ is stochastically stable under q-quota rule if and only if $a \in \Lambda^{\bar{i}}(V^{\bar{i}})$ where $V^{\bar{i}}$ is obtained by the algorithm below.*[15]

[14] A path from v to w on E is a sequence of edges starting with v and ending with w such that it connects nodes in E which are all distinct.

[15] For $i = 1, 2, \cdots, \Lambda^i$ means the i-times repetition of the operator Λ. For example, if $V^2 = \{\{\{a,b\},\{c\}\},\{\{d\}\}\}$, then $\Lambda^2(V^2) = \Lambda(\Lambda(V^2)) = \Lambda(\{\{a,b\},\{c\},\{d\}\}) = \{a,b,c,d\}$.

Step 0. For $i = 0$, let $V^0 = \mathcal{A}$, $E^0 = \mathcal{A} \times \mathcal{A}$, and $c_0(a,b) = c^q_{ab}$ for all $(a,b) \in E^0$.

Repeat steps 1–2 until V^{i+1} becomes a singleton. When V^{i+1} does so, let $\bar{i} = i+1$ and stop.

Step 1. (contraction) For the complete graph (V^i, E^i, c_i), compute the real-valued function $\tilde{c}_i$ on E^i by letting $E = E^i$ and $V = V^i$ in (11), and identify V^i-recurrent sets. Let V^{i+1} denote the family of all V^i-recurrent sets. Let $E^{i+1} = V^{i+1} \times V^{i+1}$.

Step 2. (cost update) Let

$$c_{i+1}(\mathbf{v}, \mathbf{w}) = \tilde{c}_i(v, w) - \tilde{c}^*_i \quad \forall (\mathbf{v}, \mathbf{w}) \in E^{i+1},$$

where $v \in \mathbf{v}$, $w \in \mathbf{w}$, and[16]

$$\tilde{c}^*_i = \min \left\{ \tilde{c}_i(v,w) : v \in \mathbf{v}, w \in \mathbf{w} \text{ for } (\mathbf{v}, \mathbf{w}) \in E^{i+1} \text{ with } \mathbf{v} \neq \mathbf{w} \right\}.$$

Increase step i by 1, i.e., $i = i + 1$. Go to Step 1.

Remark 2. The theorem is based on the Chu-Liu /Edmonds algorithm in graph theory. It is proposed by Chu and Liu (1965) and Edmonds (1967) to generate a minimum-cost spanning tree for a prescribed root. A difference between their algorithm and ours is that we consider all nodes in a complete graph as potential roots and identify a node whose spanning tree has the lowest cost among all nodes. A few studies have developed methods based on the algorithm with a different focus. A closely related one is the method used in Tröger (2002) which is focused on the BRM choice rule. Each iteration subtracts one from the cost in his model, which is equivalent to setting $c_{i+1}(\mathbf{v}, \mathbf{w}) = \tilde{c}_i(v, w) - 1$ in step 2 of ours. We improved the algorithm in two aspects. It is generalized to all regular choice rules, and fewer iterations are required thanks to subtracting $\tilde{c}^*_i$ from the cost in step 2.

To conclude this section, we summarize the stochastic stable alternative in a social choice problem with quota rules. The results for the BRM dynamic and the logit dynamic will be proved in the next sections.

[16] $c_{i+1}(\mathbf{v}, \mathbf{w})$ does not depend on the choice of v and w due to condition (ii) of V^i-recurrent sets.

quotas	dynamics	stochastic stability
$q \leq \underline{q}$	general	A unique recurrent class is stochastically stable.
$q > \underline{q}$	general	A Condorcet winner is stochastically stable if $q \leq n - \overline{n}$.
	BRM	A Condorcet winner is uniquely stochastically stable if $q > \overline{n}$. If a Condorcet winner does not exist, the alternative minimizing the cost in the top cycle with respect to $\underline{q}$ is stochastically stable.
	logit	A Borda winner is stochastically stable if $q = n$.

4 Best response with mutations (BRM)

We consider stochastic stability of BRM. We first introduce a class of trees on $\mathcal{A}$ which is useful for our analysis. Recall that $\underline{q}$ is the largest integer not greater than $(n+1)/2$.

Definition 9. *For $a \in \mathcal{A}$, an a-tree, say τ_a, is called a majority tree if $n(a', a'') \geq \underline{q}$ for all $(a', a'') \in \tau_a$.*

A majority tree is a tree in which for every transition there exists at least $\underline{q}$ players who prefer it. The next lemma shows that the existence of a majority tree is a necessary condition of stochastic stability in BRM if quota is larger than or equal to $\underline{q}$.

Lemma 3. *Let $q \geq \underline{q}$. Every stochastically stable state $a \in \mathcal{A}$ under q-quota rule must have a majority a-tree.*

Proof. Fix $q \geq \underline{q}$. Assume that $a \in \mathcal{A}$ is stochastically stable under q. By way of contradiction, suppose that a has no majority a-tree. Then, for every a-tree τ_a, there exists some transition $(a', a'') \in \tau_a$ such that $n(a', a'') < \underline{q}$. If $n(a', a) \geq \underline{q}$ for all such transitions (a', a'') in τ_a, then one can construct a majority a-tree by replacing all such (a', a'') with (a', a). This contradicts the supposition. If there exists some transition $(a', a'') \in \tau_a$ such that $n(a', a'') < \underline{q}$ and $n(a', a) < \underline{q}$, then remove (a', a'') from τ_a and add (a, a') to it. The resulting set of edges must be an a'-tree, denoted by $\tau_{a'}$. From (7), its cost is given by

$$c_q(\tau_{a'}) = c_q(\tau_a) - (q - n(a', a'')) + \max\{q - n(a, a'), 0\}.$$

Note that $q - n(a', a'') > 0$ since $q \geq \underline{q} > n(a', a'')$. Observe that

$$q - n(a', a'') > q - \underline{q} \geq q - n(a, a').$$

The last inequality comes from that $n(a', a) < \underline{q}$ implies that $n(a, a') \geq \underline{q}$. Then, we have $c_q(\tau_{a'}) < c_q(\tau_a)$. This contradicts that a is stochastically stable under q. $\square$

We now prove the theorem that the Condorcet winner is stochastically stable for every quota $q \in \{1, \dots, n\}$ in BRM.

Theorem 4 (Stochastic stability of Condorcet winner under BRM)**.** *Suppose that players employ BRM. For every $q \in \{1, \dots, n\}$, the Condorcet winner is stochastically stable under q-quota rule. Furthermore, it is uniquely stochastically stable under q-quota rule for all $q > \overline{n}$.*

Proof of Theorem 4. Consider that the Condorcet winner a exists. This implies that $\overline{n}(a) = \overline{n} < n/2$. First, suppose that $q \leq \overline{n}$. Since $q \leq \overline{n} < n - \overline{n}$, it follows from Theorem 2 that a is stochastically stable. Next, suppose that $q > \overline{n}$. For every $a' \neq a$, let $\tau_{a'}$ be an arbitrary a'-tree. Let $(a, a'') \in \tau_{a'}$. Since a is the Condorcet winner, it holds that $n(a, a'') < \underline{q}$. This means that $\tau_{a'}$ is not a majority tree. Since the choice of $\tau_{a'}$ is arbitrary, Lemma 3 implies that a' is not stochastically stable for q. Since Theorem 1 guarantees the existence of a stochastically stable state, it must be true that a is uniquely stochastically stable for every $q > \underline{q}$. □

Theorem 4 shows that the stochastic stability of the Condorcet winner is universal in BRM, namely it is stochastically stable for every quota $q \in \{1, \dots, n\}$. This is in contrast to Theorem 2 for a general class of regular dynamics which imposes an upper limit $n - \overline{n}$ of quota rules for the Condorcet winner to be stochastically stable. When a quota q exceeds the upper limit, a tree with the Condorcet winner as the root has a positive cost. However, its cost is always the lowest among all trees under BRM.

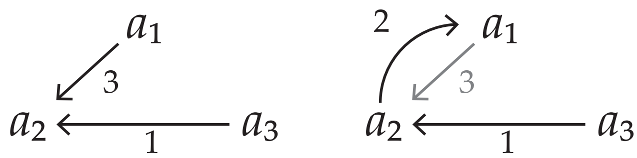
Example 4. Consider a social choice problem with 5 voters and 3 alternatives. Preferences are given by the left matrix below. They are summarized by the right matrix, in which each entry gives the total number of voters who prefer the row alternative to the column alternative. For example, 3 voters prefer a_1 to a_2, and 2 voters prefer a_2 to a_1.

Preferences	# of players
$a_1 \succ a_2 \succ a_3$	2
$a_2 \succ a_1 \succ a_3$	2
$a_3 \succ a_1 \succ a_2$	1

	a_1	a_2	a_3
a_1	–	3	4
a_2	2	–	4
a_3	1	1	–

Obviously, the Condorcet winner is a_1 since majority is 3. It can be seen that the set of stochastically stable alternatives for each q is given by

$$\mathcal{M}_q = \begin{cases} \{a_1, a_2, a_3\} & \text{for } q = 1, \\ \{a_1, a_2\} & \text{for } q = 2, \\ \{a_1\} & \text{for } q \geq 3. \end{cases}$$



(a) a_2-tree

(b) a_1-tree with a smaller cost

Figure 1: A non-majority a_2-tree and a majority a_1-tree

The Condorcet winner a_1 is stochastically stable for all q and uniquely so for $q \geq 3$. We sketch the proof of the result for $q = 5$. In Figure 1, an arrow from a_j to a_h means the transition from a_j to a_h, and a number adjacent to the arrow denotes its cost. A minimum-cost a_2-tree is shown in Figure 1(a). It is not a majority tree because the edge (a_1, a_2) has $n(a_1, a_2) = 2 < \underline{q} = 3$. As shown in Figure 1(b), replacing (a_1, a_2) with (a_2, a_1), we construct a_1-tree with a smaller cost than that of the a_2-tree. In fact, that a_1-tree is a majority tree.

4.1 Stochastic stability and the top cycle with $\underline{q}$-quota

We now consider a social choice problem with no Condorcet winner. Lemma 2 shows that every social choice problem has a unique recurrent class, equal to the top cycle, if quota q is not greater than $\underline{q}$. Then, all alternatives in the top cycle are stochastically stable. In what follows, we will investigate what alternatives are stochastically stable under super-majority rules whose quotas are larger than $\underline{q}$.

Our key lemma 4 will show that the top cycle with respect to $\underline{q}$-quota plays an important role to identify stochastically stable alternatives even under super-majority rules. Based on that result, Theorem 5 will offer an algorithmic characterization of stochastically stable alternatives for super-majority rules.

Let $\mathcal{A}_{\underline{q}}$ be the top cycle with respect to $\underline{q}$, i.e. a unique recurrent class with respect to $\underline{q}$. We consider the set of all trees over $\mathcal{A}_{\underline{q}}$ and show that only those trees matter for stochastic stability under every q-quota rule where $q > \underline{q}$. We first introduce several notations. Let τ_a be an a-tree over $\mathcal{A}$. We say that τ_a has an a-subtree over a subset $\mathcal{A}' \subset \mathcal{A}$ if, for all $a_1 \in \mathcal{A}'$, there exists a path $\{a_1, \ldots, a_k\} \subseteq \tau_a$ such that $a_k = a$ and $a_i \in \mathcal{A}'$ for all $i \in \{1, \ldots, k\}$. Let $\Upsilon_a(\mathcal{A}')$ denote the set of a-subtrees over $\mathcal{A}'$, and define

$$c^*_{q,\mathcal{A}_{\underline{q}}}(a) = \min_{\tau^\sharp_a \in \Upsilon_a(\mathcal{A}_{\underline{q}})} c_q(\tau^\sharp_a), \quad c^*_{q,\mathcal{A}_{\underline{q}}} = \min_{a \in \mathcal{A}_{\underline{q}}} c^*_{q,\mathcal{A}_{\underline{q}}}(a),$$

$$M_q(\mathcal{A}_{\underline{q}}) = \{a \in \mathcal{A}_{\underline{q}} : c^*_{q,\mathcal{A}_{\underline{q}}}(a) = c^*_{q,\mathcal{A}_{\underline{q}}}\}.$$

Lemma 4. $\lim_{\varepsilon\to 0} \pi^q_\varepsilon(a) > 0$ *for* $q > \underline{q}$ *if and only if* $a \in M_q(\mathcal{A}_{\underline{q}})$.

Lemma 4 shows that an alternative is stochastically stable if and only if it has a subtree on the top cycle $\mathcal{A}_{\underline{q}}$ which minimizes the sum of transition costs on the top cycle. This means that to identify stochastically stable alternatives, we can restrict our analysis to transitions in the top cycle under $\underline{q}$-quota rule. Using this observation, we can simplify the algorithm to compute stochastically stable alternatives for BRM in Theorem 3 as follows.

Theorem 5. $a \in \mathcal{A}$ *is stochastically stable for* $q > \underline{q}$ *if and only if* $a \in \Lambda^{\bar{i}}(V^{\bar{i}})$ *where* $V^{\bar{i}}$ *is obtained by the algorithm below.*

Step 0. Let $\mathcal{A}_{\underline{q}}$ *be a unique recurrent class for* $\underline{q}$*. Let* $\hat{E} = \{(a,b) \in \mathcal{A}_{\underline{q}} \times \mathcal{A}_{\underline{q}} : n(a,b) < \underline{q}\}$*. Let* $V^0 = \mathcal{A}_{\underline{q}}$*,* $E^0 = (\mathcal{A}_{\underline{q}} \times \mathcal{A}_{\underline{q}}) \setminus \hat{E}$*, and* $c_0(a,b) = c^q_{ab}$ *for all* $(a,b) \in E^0$*.*[17] *Let* $i = 0$*. Repeat steps 1–2 until* V^{i+1} *becomes a singleton. When* V^{i+1} *does so, let* $\bar{i} = i+1$ *and stop.*

Steps 1–2 The steps are the same as in Theorem 3.[18]

The next example illustrates how the algorithm in Theorem 5 works.

Example 5. Consider a social choice problem with 25 voters and 5 alternatives. Preferences are depicted by the left matrix in Table 1. Each entry of the right matrix gives the total number of voters who prefer a row alternative to a column alternative. Note that majority is 13, i.e. $\underline{q} = 13$. We apply Theorem 5 to compute a stochastically stable alternative with respect to $q = 14$ and $q = 25$.

Observe that a_5 is beaten by any other alternative under majority rule. In Step 0, the unique recurrent class for $\underline{q}$ is given as $\mathcal{A}_{\underline{q}} = \{a_1, a_2, a_3, a_4\} = V^0$. In Step 1, we consider transition costs among alternatives in V^0. Figure 2 shows cost $c_i(\cdot,\cdot)$ on $V^i \times V^i$ for $q = 14$ with $i = 0$ and $q = 25$ with $i = 0, 1$. An arrow from a_j to a_h in the figure means the transition from a_j to a_h, and a number adjacent to the arrow denotes the cost $c_i(a_j, a_h)$. Arrows and numbers in grey represent transitions for which $n(\cdot,\cdot) < \underline{q}$. As suggested by Step 1, the transitions in grey can be ignored in the subsequent computation.

For $q = 14$, it is easy to see that a_1 is uniquely stochastically stable. Figure 2(a) shows that $\tilde{c}_0(a_j, a_1) = 0$, $\tilde{c}_0(a_1, a_j) = 1$ for all $j \in \{2,3,4\}$. Note that $\tilde{c}_0(a_2, a_1) = 0$ due to that $c_0(a_2, a_4) + c_0(a_4, a_1) = 0$. Thus, $V^1 = \{a_1\}$ is uniquely V^0-recurrent.

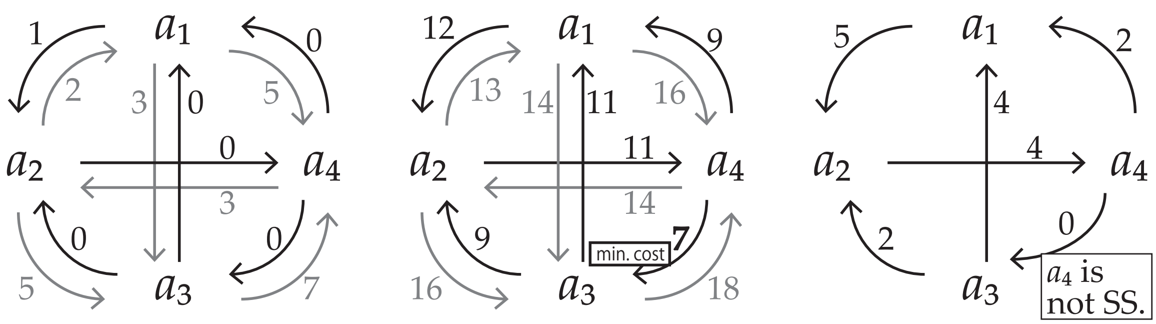
[17] We can omit edges with a cost greater than $q - \underline{q}$ because such edges are never used in the minimum cost tree.

[18] Although E^0 is a strict subset of $V^0 \times V^0$, the definition of a V^0-recurrent set does not differ from that in Theorem 3.

Table 1: Preference and voting matrices of 5 alternatives with 25 voters

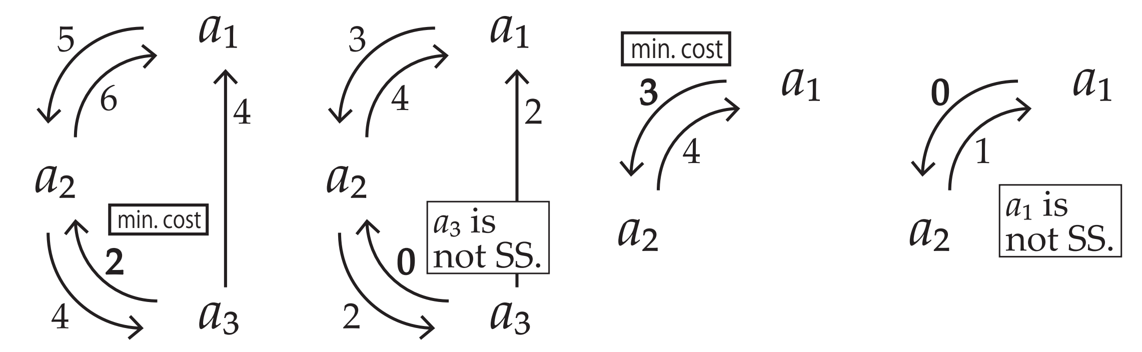
Preferences	# of players
$a_1 \succ a_3 \succ a_4 \succ a_2 \succ a_5$	5
$a_2 \succ a_1 \succ a_3 \succ a_4 \succ a_5$	3
$a_3 \succ a_1 \succ a_4 \succ a_2 \succ a_5$	1
$a_3 \succ a_2 \succ a_1 \succ a_4 \succ a_5$	2
$a_3 \succ a_4 \succ a_2 \succ a_1 \succ a_5$	1
$a_4 \succ a_2 \succ a_3 \succ a_1 \succ a_5$	1
$a_5 \succ a_1 \succ a_4 \succ a_2 \succ a_3$	6
$a_5 \succ a_2 \succ a_3 \succ a_4 \succ a_1$	6

	a_1	a_2	a_3	a_4	a_5
a_1	—	12	14	16	13
a_2	13	—	16	11	13
a_3	11	9	—	18	13
a_4	9	14	7	—	13
a_5	12	12	12	12	—



(a) $q = 14, i = 0$ (b) $q = 25, i = 0$, Step 1 (c) $q = 25, i = 0$, Step 2

Figure 2: Costs $c_i(\cdot,\cdot)$ for $q = 14$ and $q = 25$



(a) $q = 25, i = 1$, Step 1 ($\tilde{c}_i(\cdot,\cdot)$ on V^2) (b) $q = 25, i = 1$, Step 2 (c) $q = 25, i = 2$, Step 1 ($\tilde{c}_i(\cdot,\cdot)$ on V^3) (d) $q = 25, i = 2$, Step 2

Figure 3: Costs $c_i(\cdot,\cdot)$ for $q = 25$

For $q = 25$, Figure 2(b) shows $c_{\cdot\cdot}$ for 0th iteration ($i = 0$). Observe that $\tilde{c}_0(a_4, a_3) = 7$ is the minimum among all $\tilde{c}_0(\cdot,\cdot)$. In Step 2, we let $c_1(a_i, a_j) = \tilde{c}_0(a_i, a_j) - 7$. Those $c_1(\cdot,\cdot)$

are depicted in Figure 2(c). Note that $c_1(a_4, a_3) = \tilde{c}_0(a_4, a_3) - 7 = 0$, which implies that a_4 is not V^1-recurrent, and thus is not stochastically stable (SS).

Figure 3(a) shows V^2, or all V^1-recurrent sets, and $\tilde{c}_1$. Since a_4 is dropped, V^2 has three elements, a_1, a_2 and a_3.[19] Now, $\tilde{c}_1(a_3, a_2) = 2$ is the minimum cost among all $\tilde{c}_1(x, y)$. As shown in 3(b), we let $c_2(a_i, a_j) = \tilde{c}_1(a_i, a_j) - 2$. Since $c_2(a_3, a_2) = 0$, a_3 is not V^2-recurrent and thus is not stochastically stable. Iterating the process, we obtain $\tilde{c}_3(a_1, a_2) = 0$ in Step 1 of $i = 3$. The algorithm ends at $\bar{i} = 4$ and yields that $V^4 = \{\{\cdots\{a_2\}\ldots\}\}$. Thus, a_2 is uniquely stochastically stable for $q = 25$.

Remark 3. There is no monotonic relation between the sets of stochastically stable alternatives over q for $q > \underline{q}$. In Example 5, a_1 is stochastically stable for $q = 14$ while a_2 is so for $q = 25$.

5 Logit choice

In the last section, we have characterized the stochastically stable alternatives of BRM. Since behavior of boundedly rational players are diverse and it cannot be described by a unique choice model, it is important to examine how a difference in choice rules affects the set of stochastically stable alternatives. The logit choice is an alternative regular choice rule that is well studied in the literature of stochastic evolutionary theory.

For two alternatives a and a', the transition probability from a to a' under the logit choice rule is given by Equation (3) where every player j's choice rule Ψ_j^η is equal to the logit function

$$\Psi_j^\eta(a, a') = \frac{\exp(\eta^{-1}u_j(a'))}{\exp(\eta^{-1}u_j(a')) + \exp(\eta^{-1}u_j(a))}, \tag{12}$$

where $\eta > 0$. Note that Ψ_j^η satisfies the four assumptions in Definition 7. In what follows, let $P_{a,a'}^{\eta,q}$ denote the transition probability with $\Psi_j^\eta(a, a')$ above.

We define the stochastic stability for the logit choice rule in the same manner as Definition 8.

Definition 10. *A state a is* stochastically stable with q-quota rule *under the logit choice rule if* $\lim_{\eta\to 0} \pi_\eta^q(a) > 0$ *with $\Psi_j^\eta(a, a')$ given by (12).*

[19]More precisely, V^2 has $\{\{a_i\}\}$ (2 pairs of parentheses) for $i \in \{1, 2, 3\}$.

It follows from (6) that the cost of transition (a, a') under the logit choice is given as

$$c^q_{aa'} = \min_{J \in N_q} \sum_{i \in J} \max\{u_i(a) - u_i(a'), 0\}. \tag{13}$$

where N_k denotes the set of subsets of N with size k. $c^q_{aa'}$ represents the unlikeliness of the transition (a, a'). For the logit choice, it is known that the unlikeliness of a player i's choice is given by[20]

$$c_i(a, a') = -\lim_{\eta \to 0} \eta \log \Psi^\eta_i(a, a') = \max\{u_i(a) - u_i(a'), 0\}.$$

Since affirmative votes of any group of q players can make the transition, we take the minimum unlikeliness over all possible cases, i.e. all subsets of players with size q, as shown in (13).

For stochastic stability analysis, we define several notions such as c_q, c^*_q, m^*_q and M^{logit}_q according to Equations (8)–(10). The following theorem for logit choice immediately follows from Theorem 1.

Theorem 6. *Suppose that players employ the logit choice rule. Then, an alternative $a \in A$ is stochastically stable under q-quota rule if and only if $a \in M^{logit}_q$.*

For the unanimous rule, the set of stochastically stable alternatives under the logit choice has the property of maximizing the sum of players' utility functions. Thus, it is in favor of the utilitarian social welfare function.[21]

Proposition 2. *Let $\mathcal{P}(a) \equiv \sum_{i \in N} u_i(a)$. An alternative $a^* \in A$ is stochastically stable under the logit choice rule with the unanimous rule $(q = n)$ if and only if it maximizes $\mathcal{P}(a)$ among all alternatives.*

Proof of Proposition 2. Our setting for $q = n$ is similar to a unanimous game studied in Sawa (2014). The proof below is similar to the one there.

We first observe that, for all $a, a' \in \mathcal{A}$,

$$\begin{aligned}\mathcal{P}(a) - \mathcal{P}(a') &= \sum_{i \in N} (u_i(a) - u_i(a')) \\ &= \sum_{i \in N} \max\left\{u_i(a) - u_i(a'), 0\right\} - \sum_{j \in N} \max\left\{u_j(a') - u_i(a), 0\right\} = c^n_{aa'} - c^n_{a'a}.\end{aligned} \tag{14}$$

[20]See Alós-Ferrer and Netzer (2010) for example.

[21]There is an interesting similarity to Kandori et al. (2008), who study exchange economies. They show that allocations maximizing the sum of utility functions are stochastically stable under the logit choice.

Next, we will prove the claim that an alternative a^* minimizes $c_n^*(\cdot)$ over A if and only if it maximizes $P(\cdot)$ over A. Then, the theorem follows from Theorem 6.

To prove the claim above, it is sufficient to show that for every a_1 and a_k in A, $\mathcal{P}(a_1) \geq \mathcal{P}(a_k)$ if and only if $c_n^*(a_1) \leq c_n^*(a_k)$. Let τ_k^* be an a_k-tree such that $c_n(\tau_k^*) = c_n^*(a_k)$. Let $d = \{(a_1, a_2), \ldots, (a_{k-1}, a_k)\}$ be a path from a_1 to a_k in the tree τ_k^*. By the property of a tree, such a path exists and is unique. We construct an a_1-tree, denoted by τ_1, from τ_k^* by reversing the directions of all edges on the path d and keeping all other edges in τ_k^*. Formally, let τ_1 be such that

$$\tau_1 \ni \begin{cases} (a', a'') & \text{if } (a', a'') \in \tau_k^* \setminus d, \\ (a'', a') & \text{if } (a', a'') \in d. \end{cases}$$

Observe that

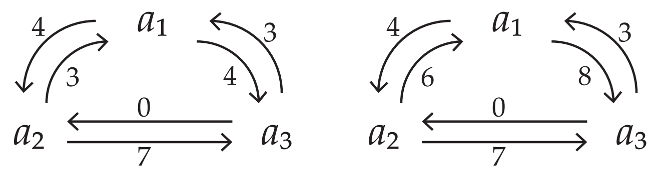
$$\begin{aligned} c_n^*(a_1) \leq c_n(\tau_1) &= c_n(\tau_k^*) + \sum_{(a_i, a_{i+1}) \in d} \left(c^n_{a_{i+1} a_i} - c^n_{a_i a_{i+1}} \right) \\ &= c_n(\tau_k^*) + \sum_{(a_i, a_{i+1}) \in d} \left(\mathcal{P}(a_{i+1}) - \mathcal{P}(a_i) \right) = c_n^*(a_k) + \mathcal{P}(a_k) - \mathcal{P}(a_1). \quad (15) \end{aligned}$$

We use Equation (14) in the second equality. Then, we have that $c_n^*(a_1) - c_n^*(a_k) \leq \mathcal{P}(a_k) - \mathcal{P}(a_1)$. It is easy to see that $\mathcal{P}(a_1) \geq \mathcal{P}(a_k)$ implies $c_n^*(a_1) \leq c_n^*(a_k)$.

Finally, since a_1 and a_k are chosen arbitrarily, interchanging a_1 and a_k in (15) yields $c_n^*(a_k) - c_n^*(a_1) \leq \mathcal{P}(a_1) - \mathcal{P}(a_k)$, which shows that $c_n^*(a_1) \leq c_n^*(a_k)$ implies $\mathcal{P}(a_1) \geq \mathcal{P}(a_k)$. □

Proposition 2 shows that a stochastically stable alternative of the logit choice rule under the unanimity is the maximizer of a potential function $\mathcal{P}(a)$ which is equal to the sum of all players' utilities. The key result for the proposition is Equation (14) which shows that the difference of potential values for two alternatives is equal to that of the costs assigned for two directed edges connecting them. If one alternative a has the potential higher than another alternative a', then moving from a to a' is more costly than moving from a' to a. This property implies that the minimum cost of all a-trees is smaller than that of all a'-trees.

The potential function of alternatives gives us various scoring methods in voting. Specifically, when players' preferences assign points to each alternative linearly increasing with their ranking, the potential maximizer selects a Borda winner which obtains the highest total score among all alternatives. Theorem 6 implies the following result.



(a) BRM (b) Logit

Figure 4: Graphs with transition costs under two dynamics with $q = n$

Corollary 2. *Suppose that every player ranks all alternatives by points in increment of one according to his preference order, that is, $u_i(a) \in \{1, 2, \ldots, |\mathcal{A}|\}$ for all $i \in N$ and $a \in \mathcal{A}$. Then, the set of stochastically stable alternatives of the logit choice under the unanimity coincides with the set of Borda winners.*

The next examples illustrate how stochastically stable alternatives differ for BRM and logit choice rules.

Example 6. Suppose that $N = \{1, \ldots, 7\}$ and $\mathcal{A} = \{a_1, a_2, a_3\}$. There are two types of players. Their payoffs are given by the following matrix on the left. For example, the first column of the table indicates that there exists four players of type 1, who have utility with $u(a_1) = 3$, $u(a_2) = 2$ and $u(a_3) = 1$. The right table is the voting matrix corresponding to the payoff matrix.

	type 1 (t_1) 4 players	type 2 (t_2) 3 players
a_1	3	1
a_2	2	3
a_3	1	2

	a_1	a_2	a_3
a_1	–	4	4
a_2	3	–	7
a_3	3	0	–

The total scores of a_1, a_2 and a_3 are equal to 15, 17, and 10, respectively. Thus, a unique Borda winner is a_2. Corollary 2 implies that a_2 is a unique stochastically stable alternative of the logit choice rule under the unanimity. The Borda winner a_2, however, is not equal to the Condorcet winner. Observe that majority $\underline{q}$ is 4. Since type t_1 voters compose a majority group, a_1 is the Condorcet winner. For an alternative a, recall that $\overline{n}(a)$ is the maximal number of voters against a. It follows from the table that $\overline{n}(a_1) = 3, \overline{n}(a_2) = 4$ and $\overline{n}(a_3) = 7$. Thus, the min-max quota $\overline{n}$ is 3. Theorem 4 implies that the Condorcet winner a_1 is a unique stochastically stable alternative of BRM for $q > 3$.

The difference of stochastic stability in BRM and logit choice can be explained by two graphs over three alternatives in Figure 4. The costs of each edge in BRM and logit choice rules are computed according to (7) and (13), respectively. It can be shown that BRM under unanimity has the smallest cost with the a_1-tree involving the path (a_3, a_2, a_1), and that the logit choice has the smallest cost with the a_2-tree involving edges (a_1, a_2) and (a_3, a_2). The analysis reveals that the cost moving from the Borda winner to the Condorcet winner critically affects the stochastic stability of BRM and the logit choice. Under BRM, the cost from a_2 to a_1 is three since mistakes of three t_2 voters are needed for a_1 to defeat a_2 under the unanimity. On the other hand, under the logit choice rule, the cost from a_2 to a_1 is six since mistakes of three t_2 voters are weighted doubly, reflecting their preference to rank a_1 the worst.

Example 7. Suppose that $N = \{1, \ldots, 7\}$ and $\mathcal{A} = \{a_1, \ldots, a_4\}$. There are two types of players whose payoffs are given by the matrix on the left. The corresponding voting matrix is given on the right.

	type 1 (t_1) 4 players	type 2 (t_2) 3 players
a_1	4	1
a_2	3	4
a_3	2	3
a_4	1	2

	a_1	a_2	a_3	a_4
a_1	–	4	4	4
a_2	3	–	7	7
a_3	3	0	–	7
a_4	3	0	0	–

This example shows the tightness of the upper limit of Theorem 2. Observe that a_1 is the Condorcet winner and that a_2 is the Borda winner. As Theorem 2 implies, a_1 is stochastically stable for all $q \leq \underline{q} = 4$ under all regular choice rules. For $q > 4$, a_2 is uniquely stochastically stable under the logit choice, and a_1 is no longer so.

Observe that a cost-minimizing tree for a_1 is the tree involving the path (a_3, a_2, a_1) and the edge (a_4, a_2). That for a_2 is the tree involving edges (a_1, a_2), (a_3, a_2) and (a_4, a_2). For $q > 4$, the cost of the former tree is $3(q-4)$ while that of the latter is $q-3$ which is strictly smaller than the former's. The Borda winner a_2 is stochastically stable under the logit choice rule for $q > 4$.

The next proposition characterizes a stochastically stable alternative of the logit choice under a general quota q. Recall that $\overline{n}(a) = \max_{a' \in \mathcal{A} \setminus \{a\}} n(a, a')$ for $a \in \mathcal{A}$.

Proposition 3. *Suppose that players employ the logit choice rule, and that* $u_i(a) \in \{1, 2, \ldots, |\mathcal{A}|\}$ *for all* $i \in N$ *and* $a \in \mathcal{A}$. *If there exists* $a \in \mathcal{A}$ *such that* $\overline{n}(a) < n/|\mathcal{A}|$, *then* a *is stochastically stable for all* $q \in \{1, \ldots, n\}$. *Moreover,* a *is uniquely so for all* $q \geq \overline{n}(a)$.

Proof. Suppose that $\overline{n}(a) < n/|\mathcal{A}|$. Since $\overline{n}(a) < n/2$, a must be the Condorcet winner with $\overline{n}(a) = \overline{n}$. Then, the claim for $q \leq n - \overline{n}$ is immediate from Theorem 2.

Consider the case that $q > n - \overline{n}$. By way of contradiction, suppose that some $a' \neq a$ is stochastically stable under q. Let $\tau^*_{a'}$ be an a'-tree which minimizes $c_q(\tau_{a'})$. For $\tau^*_{a'}$, apply two operations: (i) remove an edge emanating from a, say edge (a, a''), and (ii) add edge (a', a). Thereafter, the resulting set of edges must be an a-tree, say τ_a. Observe that

$$\begin{aligned}
&c^q_{aa''} \geq q - n(a, a'') \geq q - \overline{n}(a) > q - n/|\mathcal{A}|, \\
&c^q_{a'a} \leq (q - n(a', a))(|\mathcal{A}| - 1) \leq (q - (n - \overline{n}(a)))(|\mathcal{A}| - 1) < \left(q - \frac{|\mathcal{A}| - 1}{|\mathcal{A}|} n\right)(|\mathcal{A}| - 1).
\end{aligned}$$

The first inequality in the first expression comes from that, for transition (a, a''), there must be at least $q - n(a, a'')$ players who make a mistake and that one mistake costs at least 1. The first inequality in the second comes from that $q - n(a', a)$ players making a mistake are enough for transition (a', a), and that one mistake costs at most $|\mathcal{A}| - 1$.

Notice that

$$q - n/|\mathcal{A}| \geq \left(q - \frac{|\mathcal{A}| - 1}{|\mathcal{A}|} n\right)(|\mathcal{A}| - 1) \qquad \forall q \leq n.$$

This implies that $c^q_{aa''} > c^q_{a'a}$ for all $q \in \{\frac{|\mathcal{A}|-1}{|\mathcal{A}|} n, \ldots, n\}$. Then, observe that

$$c_q(\tau_a) = c_q(\tau^*_{a'}) - c^q_{aa''} + c^q_{a'a} < c_q(\tau^*_{a'}).$$

This contradicts that a' is stochastically stable. Thus, no alternative $a' \neq a$ is stochastically stable. By the existence of a stochastically stable alternative, it must be that a is stochastically stable. □

Remark 4. Proposition 3 offers a sufficient condition under which the Borda winner is stochastically stable for all q-quota rules under the logit choice, since an alternative $a \in \mathcal{A}$ which satisfies $\overline{n}(a) < n/|\mathcal{A}|$ is a unique Borda winner. To see this, choose $a' \neq a$. By definition of $\overline{n}(\cdot)$, at least $(n - \overline{n}(a))$ of players prefer a to a' and at most $\overline{n}(a)$ players prefer a' to a. For the players who prefer a, the sum of the points for a exceeds the sum for a' by at least $n - \overline{n}(a)$. For those who prefer a', the sum for a' exceeds that for a by at most $\overline{n}(a)(|\mathcal{A}| - 1)$. Observe that $(n - \overline{n}(a)) - \overline{n}(a)(|\mathcal{A}| - 1) = n - \overline{n}(a)|\mathcal{A}| > 0$.

Remark 5. There is an interesting connection with a finding in static settings. Our condition, $\overline{n}(a) < n/|\mathcal{A}|$, implies that a is a Condorcet winner which wins at least $\frac{|\mathcal{A}|-1}{|\mathcal{A}|} n$ votes against any other alternative. Baharad and Nitzan (2003) show that if such a Condorcet winner exists, then it is also a Borda winner. Thus, our condition is equivalent to a sufficient condition under which two winners coincide.

6 Application: Multidimensional choice problem

6.1 Best response with mutations

In this section, we apply the stochastic stability theory to a multidimensional choice problem where the set of alternatives is a subset of the h-dimensional Euclidean space $\mathbb{R}^h$. For an alternative $a \in \mathbb{R}^h$ and $i = 1, \cdots, h$, the i-th coordinate of a represents its position on the i-th issue. Every voter has an Euclidean preference over alternatives. There is an ideal outcome that the voter prefers the most, and alternatives that are further away from the ideal outcome are less preferred. The Euclidean preference generalizes the "single-peaked" preference of Black (1948).

Spacial models of collective choice have been intensively studied in the literature. In one dimensional case $(h = 1)$, Black (1948) demonstrates the celebrated "median-voter theorem" that the median voter's optimum is the Condorcet winner. Theorem 4 is applicable to one dimensional choice problem with single-peaked preferences. However, a multidimensional choice problem is very different from the unidimensional one. When the set of alternatives has two or more dimensions, the set of alternatives unbeatable under majority rule (the majority rule core) is empty without strong assumptions of symmetric preferences (see Plott 1967, Tullock 1967 and Davis et al. 1972 for early studies). Furthermore, McKelvey (1976, 1979) shows that when the core is empty, any one alternative can be reached from any other through a process of pairwise majority comparisons. In multidimensional choice problems, the traditional theory of core under majority rule is not sufficient to provide useful predictions on collective choice. To avoid this kind of Condorcet paradox phenomenon, Simpson (1969) and Kramer (1977) propose the min-max majority rule that is a minimal rule which guarantees the existence of unbeatable alternatives. The min-max set is the set of alternatives which are unbeatable under the min-max majority. In what follows, we will present a dynamic foundation of the min-max set in the framework of stochastic evolutionary game theory.

Let $\mathcal{A}^0 \subset \mathbb{R}^h$ be a bounded convex set. We assume that players rank alternatives according to the Euclidean distance from their most preferred ones.

Assumption 1 (Euclidean preferences)**.** *For $i \in N$, there exists an ideal point, denoted by $s_i \in \mathbb{R}^h$, such that player i's utility function u_i satisfies*

$$u_i(a) > u_i(a') \quad \textit{iff} \quad d(a, s_i) < d(a', s_i),$$

where $d(a, b)$ denotes the Euclidean distance for $a, b \in \mathbb{R}^h$, i.e. $d(a, b) = (\sum_{1 \le j \le h}(a_j - b_j)^2)^{1/2}$.

We introduce several notations. Let $n^* = \min_{r\in\mathbb{R}^h}\max_{r'\in\mathbb{R}^h} n(r,r')$, i.e. the min-max quota on $\mathbb{R}^h$. We note that the min-max quota $\overline{n}$ in Definition 2.2 is defined over a finite set $\mathcal{A}$ of alternatives. We will consider the min-max quota $\overline{n}$ to be defined in $\mathcal{A}^\delta$, a finite approximation of $\mathcal{A}^0$. For two alternatives a and a' in $\mathbb{R}^h$ and a group $J \subseteq N$ of players, we say that a' dominates a via J if all players in J prefer a' to a. The *undominated set* for J, denoted by $\mathfrak{C}(J)$, is defined as the set of alternatives in $\mathbb{R}^h$ which are undominated via J, i.e., $\mathfrak{C}(J) = \{a \in \mathbb{R}^h : \nexists a' \in \mathbb{R}^h,\ u_i(a') > u_i(a)\ \forall i \in J\}$. Let $\mathfrak{C}^* = \bigcap_{J\in N_{n^*+1}} \mathfrak{C}(J)$, i.e., the intersection of undominated sets for all coalitions of sizes greater than the min-max quota n^*. We call $\mathfrak{C}^*$ the *min-max core*. Note that $\mathfrak{C}^*$ always exists.[22]

To apply the stochastic stability theory, we consider a finite approximation of the alternative set $\mathcal{A}^0$. Let $\mathcal{A}^\delta \subset \mathcal{A}^0$ be a finite approximation of $\mathcal{A}^0$ with maximum distance δ, i.e., for every $r \in \mathcal{A}^0$, there exists some $a \in \mathcal{A}^\delta$ such that $d(a,r) < \delta$. We will first analyze stochastically stable alternatives over a finite set $\mathcal{A}^\delta$, and will characterize their limits as the approximation δ converges to zero.

In what follows, we assume that (i) $\mathfrak{C}(J) \subset \mathcal{A}^0$ for all $J \subseteq N$ and (ii) $\mathcal{A}^\delta \cap \mathfrak{C}^* \neq \varnothing$. The first assumption is satisfied if the alternative set $\mathcal{A}^0$ is large enough. It ensures that the undominated sets and the min-max quota defined over $\mathcal{A}^0$ coincides with those defined over $\mathbb{R}^h$. The second assumption bites only for the case that the min-max core $\mathfrak{C}^*$ has measure zero in $\mathbb{R}^h$. For $\delta > 0$, define the min-max quota and the min-max set for the alternative set $\mathcal{A}^\delta$ to be $\overline{n}^\delta = \min_{a\in\mathcal{A}^\delta}\max_{a'\in\mathcal{A}^\delta} n(a,a')$ and $\mathcal{A}^{*,\delta} = \{a \in \mathcal{A}^\delta : \max_{a'\in\mathcal{A}^\delta} n(a,a') = \overline{n}^\delta\}$, respectively. Let $\mathfrak{C}^{*,\sigma}$ denote a σ-neighborhood of $\mathfrak{C}^*$, i.e. $\mathfrak{C}^{*,\sigma} = \{r \in \mathbb{R}^h : \inf_{r'\in\mathfrak{C}^*} d(r,r') < \sigma\}$. The following lemma describes their limiting properties as the approximation δ goes to zero. It implies that $\mathcal{A}^{*,\delta}$ is included in the neighborhood of the min-max core for sufficiently small δ.

Lemma 5. (i) $\lim_{\delta\to 0}\overline{n}^\delta = n^*$. (ii) *Fix* $\sigma > 0$. $\mathcal{A}^{*,\delta} \subset \mathfrak{C}^{*,\sigma}$ *for all sufficiently small* δ.

Lemma 6 below shows that any pair of alternatives in an open ball of an arbitrary size can be connected via a sequence of pairwise voting under q-quota rule if $q \leq \overline{n}$ and $\mathcal{A}^0$ is sufficiently large.

Lemma 6. *For any* $\rho > 0$, *let* $B(\rho) = \{r \in \mathbb{R}^h : \|r\| < \rho\}$ *and* $B^\delta(\rho) = \{a \in \mathcal{A}^\delta : \|a\| < \rho\}$. *Suppose that* $q \leq \overline{n}$ *and* $B(5\rho) \subset \mathcal{A}^0$ *where* ρ *is large enough that* $s_i \in B(\rho)$ *for all* $i \in N$. *Then, for all* $a_1, a_L \in B^\delta(\rho)$, *there exists a sequence* $\{a_1, a_2, \ldots, a_L\} \subset \mathcal{A}^\delta$ *such that*

$$|\{i \in N : u_i(a_{j+1}) > u_i(a_j)\}| \geq q \qquad \forall j \in \{1,\ldots,L-1\}.$$

[22] Let $r^* \in \mathbb{R}^h$ satisfy $n^* = \max_{r\in\mathbb{R}^h} n(r^*,r)$. By definition of n^*, r^* is not beaten by any other alternative under any q-quota rule if $q \geq n^*+1$. This means that $r^* \in \mathfrak{C}(J)$ for any J with $|J| \geq n^*+1$. That is, $r^* \in \mathfrak{C}^*$.

for any sufficiently small δ.

McKelvey (1976) shows a similar result to Lemma 6 for the case of infinite state space and majority rule. We extend it to the case of finite state space and super-majority q-rules with $q \leq \overline{n}$. Austen-Smith and Banks (1999) show that for q-quota less then the min-max quota, all alternatives belong to the same cycle set, called the 'weak' top cycle, if transitions are made according to weak preferences, that is, transitions between two alternatives are feasible even when neither wins q votes against each other. Under the Euclidean preferences, the lemma strengthens their results in terms of strict preference paths.

In the transition process described in Lemma 6, a new alternative wins against a status-quo alternative under q-quota rule if $q \leq \overline{n}$. There exists at least q voters who prefer the transition to the status-quo. We next consider a particular process called the "vote-maximizing" process (Kramer (1977)), under which the status-quo alternative transits to the alternative which maximizes votes against it. Let $\overline{Q}(a) = \{a' \in \mathcal{A}^{\delta} : n(a,a') = \max_{a'' \in \mathcal{A}^{\delta}} n(a,a'')\}$. $\overline{Q}(a)$ is the set of alternatives which can transit from a under the vote-maximizing process. In other words, $\overline{Q}(a)$ denotes the set of least-cost deviations from a. The following lemma shows that the vote maximizing process necessarily leads to the min-max set.

Lemma 7. *Fix sufficiently small $\delta > 0$ such that $\overline{n}^{\delta} = n^*$. For every $a_1 \notin \mathcal{A}^{*,\delta}$, there exists a sequence $\{a_1, a_2, \ldots, a_L\} \subset \mathcal{A}^{\delta}$ with $a_L \in \mathcal{A}^{*,\delta}$ such that*

$$a_{i+1} \in \overline{Q}(a_i) \qquad\qquad \forall i = 1, \ldots, L-1.$$

Kramer (1977) proves a result similar to Lemma 7 for the infinite alternative set being $\mathbb{R}^h$. Specifically, he shows that on any vote-maximizing trajectory the distance to the min-max set must be monotonically decreasing. Due to the continuum of alternatives, it may be the case that the process does not reach the min-max set but only approaches it. Following the technique used in Kramer (1977), we show that the vote-maximizing trajectory converges to the min-max set for the finite alternative set $\mathcal{A}^{\delta}$.

The vote-maximizing process involves the following ad hoc assumptions. The opposing party chooses a policy maximizing votes, irrespective of its own preference. It is assumed that the voting quota is fixed to simple majority and there is no Condorcet winner. On the process, the ruling party always loses the election. The status-quo policy is changed to a challenging one, excluding the possibility that the status-quo policy remains. Our stochastic model of voting process can avoid these assumptions. A voting quota is fixed in the process but is not restricted to simple majority. A challenging policy is chosen through a stochastic process. The status-quo policy may remain with a positive probability. We will show that the set of stochastically stable alternatives is included in the min-max set in the limit that the approximation δ converges to zero.

The main theorem characterizes the set of stochastically stable alternatives for majority and super-majority rules under BRM.

Theorem 7. *Suppose that players employ the BRM choice rule and have Euclidean preferences. Let $B(\rho) = \{r \in \mathbb{R}^h : \|r\| < \rho\}$ and $B^\delta(\rho) = \{a \in \mathcal{A}^\delta : \|a\| < \rho\}$. Suppose that the alternative set $\mathcal{A}^0$ contains an open ball $B(5\rho)$ where ρ is large enough that $s_i \in B(\rho)$ for all $i \in N$.*
(i) *For $q \leq \overline{n}$, the unperturbed dynamic under BRM with state space $\mathcal{A}^\delta$ has a unique recurrent class for sufficiently small δ, and the recurrent class includes an open ball $B^\delta(\rho)$.*
(ii) *For $q > \overline{n}$, $\lim_{\delta\to 0}\lim_{\eta\to 0}\pi^q_\eta(\mathcal{A}^{*,\delta}) = 1$.*

Proof. (i) : Suppose that $q \leq \overline{n}$. Lemma 6 implies that all alternatives in $B^\delta(\rho)$ are connected via zero-cost transitions for any sufficiently small δ. Thus, those alternatives must be in one recurrent class, say $A \subset \mathcal{A}^\delta$. We will show that there is no recurrent class than A. Pick $a' \in \mathcal{A}^\delta \setminus A$. Note that $a' \notin \mathfrak{C}(N)$ since $s_i \in B(\rho)$ implies that $\mathfrak{C}(N) \subseteq B^\delta(\rho) \subseteq A$. Let $a^* \in \operatorname{argmin}_{r\in\mathfrak{C}(N)} d(a', r)$. Observe that $u_i(a^*) > u_i(a')$ for all $i \in N$. By the continuity of d, for sufficiently small δ, there exists $\hat{a}^* \in A$ with $d(a^*, \hat{a}^*) < \delta$ such that $u_i(\hat{a}^*) > u_i(a')$ for all $i \in N$. The cost of the transition from a' to $\hat{a}^*$ is zero. Since the cost must be positive for transitions between two recurrent classes, a' cannot be in any recurrent class. This proves that the recurrent class is unique and includes the open ball $B^\delta(\rho)$.

(ii) : Suppose that $q > \overline{n}$. Also suppose that δ is small enough that $\overline{n}^\delta = n^*$. By a way of contradiction, assume that there exists a stochastically stable alternative $a_1 \notin \mathcal{A}^{*,\delta}$. Let τ_1 denote the minimum cost spanning tree rooted at a_1. We will show that the minimum cost spanning tree rooted at some $a_L \in \mathcal{A}^{*,\delta}$ has a strictly smaller cost than that of a_1.

Lemma 7 implies that there exists a sequence $\{a_1, a_2, \ldots, a_L\} \subseteq \mathcal{A}^\delta$ with $a_L \in \mathcal{A}^{*,\delta}$ such that $a_{i+1} \in \overline{Q}(a_i)$ for all $i \in \{1, \ldots, L-1\}$. Construct a path of edges $\{(a_1, a_2), \ldots, (a_{L-1}, a_L)\}$. Add these edges to τ_1, replacing the existing edges exiting $a_2, \ldots, a_{L-1}$. Remove the edge exiting a_L. The resulting set of edges must be an a_L-tree, denoted by τ_L. Then, observe that

$$
\begin{aligned}
c_q(\tau_L) &\leq c_q(\tau_1) + \sum_{i=1}^{L-1} \max\{q - n(a_i, a_{i+1}), 0\} - \sum_{i=2}^{L} \max\{q - \overline{n}(a_i), 0\} \\
&= c_q(\tau_1) + \max\{q - n(a_1, a_2), 0\} - \max\{q - \overline{n}(a_L), 0\} \qquad (16) \\
&< c_q(\tau_1) = c^*_q.
\end{aligned}
$$

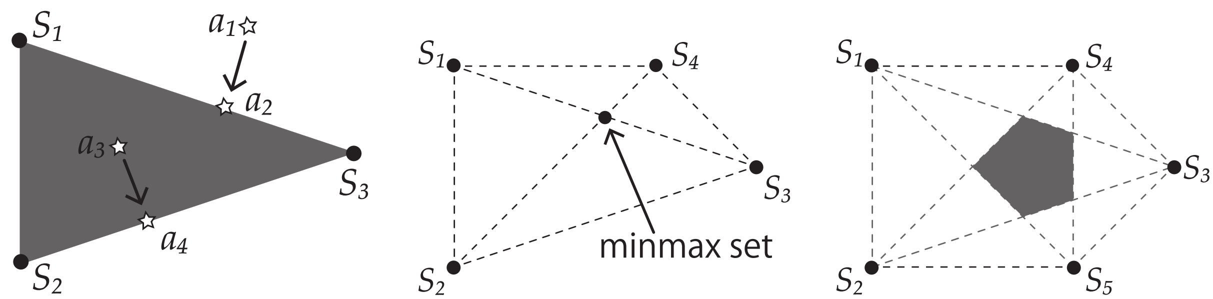
The last term of the first inequality represents the cost reduction by removing edges existing $a_2, \ldots, a_{L-1}$. The weak inequality holds because $\max\{q - \overline{n}(a_i), 0\}$ is the smallest possible cost of an edge exiting a_i. The second equality comes from that $n(a_i, a_{i+1}) = \overline{n}(a_i)$ due to the definition of the sequence $\{a_1, \ldots, a_L\}$. The last inequality holds as $a_1 \notin \mathcal{A}^{*,\delta}$ implies that $n(a_1, a_2) > \overline{n} = \overline{n}(a_L)$. For $q > \overline{n}$, it holds that $\max\{q - n(a_1, a_2), 0\} < \max\{q - \overline{n}(a_L), 0\}$. Thus,$\tau_L$ has a strictly smaller cost than τ_1, which contradicts that a_1 is stochastically stable. No $a_1 \notin \mathcal{A}^{*,\delta}$ can be stochastically stable. □

The theorem shows the following properties of stochastic stability under BRM in multidimensional choice problems. The set of stochastically stable alternatives differs, depending on whether $q \leq \overline{n}$ or not. When $q \leq \overline{n}$, the recurrent class is unique and includes alternatives within an open ball with radius ρ provided that the alternative set $\mathcal{A}^0$ includes a ball with radius 5ρ. Similarly to McKelvey (1976), we observe the intransitivity; all alternatives inside the open ball can be connected via zero-cost transitions. Thus, any alternative a can be stochastically stable by taking the state space large enough that $\|a\| < \|\rho\|$. Even for some super-majority rules, i.e., $\underline{q} < q \leq \overline{n}$, all alternatives inside the ball with ρ are still in the same recurrent class and the intransitivity remains. When $q > \overline{n}$, the intransitivity is drastically mitigated. Every stochastically stable alternative belongs to the min-max set.

The next example offers an illustration of the min-max sets.

Example 8 (min-max sets)**.** Figure 5(a) illustrates the min-max set, which is the portion with a grey shade, for a game with three players and the alternative space being on $\mathbb{R}^2$. The players' ideals points are depicted by s_1, s_2 and s_3 respectively. Note that $n^* = 2$ for this game and that the min-max set coincides with the convex hull of the ideal points. For any alternative outside the min-max set, there exists an alternative which $n^* + 1$ players strictly prefer. For example, for a_1 in Figure 5(a), the three players will strictly prefer a_2 to it. While, for an alternative within the min-max set, at most n^* players will vote for any move from it. For a move from a_3 to a_4 depicted in Figure 5(a), only s_2 and s_3 players will vote for that move.

Figures 5(b) and 5(c) show the min-max sets for settings with four and five players respectively. Note that $n^* = 2$ for the four-player setting, while $n^* = 3$ for the five-player one. Those min-max sets are within the convex hull of the ideal points and are the intersection of undominated sets for coalitions with size $n^* + 1$. For the four-player setting, that intersection for coalitions with size 3 is the point where the diagonals intersect. For $q > n^*$, the stochastically stable alternatives belong to those min-max sets.



(a) 3 players (b) 4 players (c) 5 players

Figure 5: The min-max sets

6.2 Logit choice

As for the logit choice rule, we characterize stochastically stable alternatives under unanimous rule. Those alternatives have an interesting geometric interpretation.

Proposition 4 (logit choice and geometric median)**.** *Suppose that every player i's utility function is given by $u_i(a) = -d(a, s_i)$, and that $q = n$. Then, an alternative is stochastically stable under the logit choice rule if and only if it minimizes the sum of the distances to ideal points $\{s_i\}_{i \in N}$, that is,*

$$M_n^{logit} = \operatorname*{argmin}_{a \in \mathcal{A}^\delta} \sum_{i \in N} d(s_i, a).$$

Proof. The proposition can be proved by Proposition 2 since

$$M_n^{logit} = \operatorname*{argmax}_{a \in \mathcal{A}^\delta} \sum_{i \in N} u_i(a) \Leftrightarrow M_n^{logit} = \operatorname*{argmin}_{a \in \mathcal{A}^\delta} \sum_{i \in N} d(s_i, a).$$

□

The proposition shows that the stochastically stable alternative of the logit choice rule in multidimensional choice problems is closely related to the *geometric median* (also called *L_1-median*) of a set of points, which minimizes the sum of the distances from points. It is considered as a solution for facility locations problems where one needs to find a point that minimizes the sum of distances from destination sites. As δ approaches zero, the stochastically stable alternatives under logit choice can be arbitrarily close to the geometric median of the ideal points.

Finally, for the unidimensional space ($h = 1$), we show that the prediction of stochastic stability under the logit choice rule approaches the median for all majority and supermajority rules in the limit of small δ. Thus, the prediction will coincide with the one under BRM. Without loss of generality, let $\mathcal{A}^0 = [0, 1]$. Also let $\mathcal{A}^\delta = \{0, \delta, 2\delta, \ldots, 1\}$ and

$s_1 \leq \ldots \leq s_n$. We assume odd n so that a Condorcet winner always exists. Let a^* and a^{**} be such that $a^* = a^{**} = s_{\underline{q}}$ if $s_{\underline{q}} \in \mathcal{A}^\delta$ and $a^* < s_{\underline{q}} < a^* + \delta = a^{**}$ otherwise. a^* and a^{**} are the two closest alternatives to the median player, and the closest one is the Condorcet winner.

Proposition 5. *Suppose that $h = 1$. The set of stochastically stable alternatives of the logit choice for all $q \geq \underline{q}$ is characterized by*

$$M_q^{logit} \subseteq \{a^*, a^{**}\}, \qquad \lim_{\delta \to 0} M_q^{logit} = \{s_{\underline{q}}\}.$$

The proposition implies that an alternative is stochastically stable only if it is the best alternative or the second best one for the median voter. It is known that the optimum alternative for the median voter is a Condorcet winner (Black, 1948). The prediction of the logit choice may differ from the Condorcet winner, but the difference is at most δ. In the limit of small δ, the prediction will coincide with the median's ideal point for all majority and super-majority rules. Theorem 4 and Proposition 5 imply that all the predictions of BRM, the logit choice and the median voter theorem will coincide in the limit of δ for the unidimensional space.

7 Discussion

7.1 Evolutionary social choice correspondence

The evolutionary approach to a social choice problem has a different perspective from the traditional normative one in that it investigates long-run equilibrium (stochastically stable outcomes) of a dynamic voting process. Nevertheless, it is important to see whether or not the stochastically stable choices studied in the paper satisfy some desirable properties considered in the normative theory of social choice.

For a social choice game $G = (\mathcal{A}, N, \{u_i\}_{i \in N}, q)$ and a choice rule Ψ of players, let S be the set of stochastically stable alternatives under Ψ (Definition 7). Fixing all elements except players' utility functions $u = \{u_i\}_{i \in N}$, we write $S = S(u)$ as a multi-valued mapping of u. Then, $S = S(u)$ can be regarded as a social choice correspondence, and we call it an evolutionary social choice correspondence under quota q and a choice rule Ψ. For clarity of discussion, we consider BRM under the unanimity rule($q = n$), and examine what kinds of desirable properties the evolutionary social choice correspondence $S(u)$ satisfy.[23]

[23]A similar evolutionary social choice correspondence can be defined under the logit choice. Such a

We list several desirable properties of social choice correspondence.[24] The following definitions are due to Moulin (1988).

Definition 11 (Pareto optimality (P))**.** *If candidate a is unanimously preferred to candidate b, then b should not be elected.*

Definition 12 (Anonymity (A))**.** *The name of voters does not matter: If two voters exchange their votes, the outcome of the election is not affected.*

Definition 13 (Neutrality (N))**.** *The name of candidates does not matter: If we exchange two candidates a and b in the ordering of every voter, then the outcome of the election changes accordingly (if a was previously elected then b now is and vice-versa; if some x different from a and b was elected, it still is).*

Definition 14 (Monotonicity (M))**.** *Suppose a is among the winners at a given profile and that the profile changes only inasmuch as the ranking of a improves, the relative comparison of any other pair of candidates by any voter being unaffected. Then a is still among the winners at the new profile.*

Definition 15 (Smith's consistency (S))**.** *If the set $\mathcal{A}$ of candidates splits into two disjoint subsets B_1, B_2 and every $b_1 \in B_1$ beats (by a strict majority) every $b_2 \in B_2$, then an outcome from B_1 should be elected.*

The last property is a property of robustness against strategic nomination of alternatives due to Tideman (1987). This requires a social choice correspondence to be robust against manipulations of adding new alternatives, called clones, which are almost identical to an existing one. Formally, a subset of $\mathcal{A}$ is a set of clones if no player ranks any candidate outside the set between any candidates that are in the set.

Definition 16 (Independence of clones (IC))**.**

1. *A candidate that is a member of a set of clones wins if and only if some member of that set of clones wins after a member of the set is eliminated from the ballot.*

2. *A candidate that is not a member of a set of clones wins if and only if that candidate wins after any clone is eliminated from the ballot.*

The next theorem shows that our evolutionary approach provides a social choice correspondence satisfying all desirable properties listed above.

correspondence selects Borda winners as shown in Corollary 2. We restrict our discussion to the BRM since properties satisfied by a Borda winner are known. See Section 9 of Moulin (1988).

[24] These properties can be defined formally in terms of a social choice correspondence.

Theorem 8. *The evolutionary social choice correspondence $S(u)$ generated by BRM and unanimity satisfies properties (P), (A), (N), (M), (S), and (IC).*

An intuitive explanation of its proof is the following. The first three properties will be satisfied by the evolutionary social choice correspondence under the unanimous rule and any regular choice rule, since the proof requires the results up to Section 3. (P) is implied by Corollary 1. Recall that Nakamura number is infinity for unanimous rule. Corollary 1 ensures that our correspondence will selects Pareto efficient ones for all profiles of preferences and all sets of alternatives. (A) is immediate from q-quota rules. (N) is due to two properties of our model. First, the choice rule is independent from the name of alternatives. Second, the set of stochastically stable alternatives is independent from the initial states. Even though the static setting is in favor of the status quo alternative for $q > \underline{q}$, the effect of favoring the status quo will be canceled out. This is because every alternative will repeat to become a status quo or a challenger in the dynamics.

The other three properties are specific to the correspondence under BRM. The changed profile in (M) makes weakly smaller the costs of inbound edges toward a, and makes weakly greater the costs of outbound edges from a. Then, the profile may never weaken the stability of a. For (S), observe that B_1 coincides with the top cycle with respect to $\underline{q}$ for odd n. Then, it is satisfied due to Lemma 4. In the proof, we show that the two sets also coincide for even n. For (IC), observe that all clones have the same costs of edges inbound from and outbound toward non-cloned alternatives. Only the costs of edges between the clones may differ. This suggests that if a member of the clones is more likely (or unlikely) than non-cloned one in the dynamic, then all the clones are more likely (or unlikely) there.

Remark 6. The evolutionary social choice correspondence under BRM is Condorcet consistent, i.e., a Condorcet winner will be selected if it exists. Properties which are not satisfied by any Condorcet consistent rule, e.g. *participation*, are not satisfied by our correspondence either. Closely related voting rules to ours would be those that satisfy all the properties in Theorem 8. Such rules are the 'ranked pairs' system (Tideman, 1987) and the Schulze method (Schulze, 2011), for example.

8 Conclusion

We have presented an evolutionary approach to social choice problems with q-majority rules. Unlike the traditional normative approach, we have considered the long-run equilibrium of a dynamic political process where a status-quo policy is repeatedly challenged by an opposing policy drawn randomly. By employing the stochastic evolutionary game theory, we have shown that a Condorcet winner is stochastically stable for all q-majority rules under the best response choice rule with mutations. In contrast, the Borda winner is stochastically stable under the logit-choice rule for unanimous voting. The result gives an evolutionary insight to an old debate concerning Condorcet and Borda. We also apply the stochastic stability theory to multidimensional choice problems where a Condorcet winner does not exist almost surely, and provide a dynamic foundation of the min-max policies proposed in the literature. Finally, from a normative point of view, we have discussed several desirable properties of the social choice correspondence generated by the stochastically stable outcomes.

A Appendix

A.1 Proofs

We show the proofs relegated to the Appendix.

Section 2

Proof of Proposition 1. For $|\mathcal{A}| < \nu$: the claim for $q < n$ is immediate from Theorem 2.5 of Nakamura (1979). For $q = n$, observe that the grand coalition is unique winning coalition, that is, every player is a veto player in the game. Then, the claim is also implied by that theorem.

For $|\mathcal{A}| = \nu$: We show that if the core is empty, then the claimed top cycle exists. If the core is empty, then there exists a cycle, $\{a_1, \ldots, a_p\}$, where $n(a_i, a_{i+1}) \geq q$ for all $i \in \{1, \ldots, p\}$ with $a_{p+1} = a_1$. Theorem 2.5 of Nakamura (1979) implies that any subset $A \subset \mathcal{A}$ such that $|A| < \nu$ cannot include such a cycle. Then, the cycle must include all alternatives. Thus, $\mathcal{A}$ is equal to the top cycle $\{a_1, \ldots, a_p\}$ with $p = |\mathcal{A}| = \nu$.

Let $J_i \subset N$ denote a set of players who vote against the transition (a_i, a_{i+1}). Note that $|J_i| \leq n - q$. Then, we have that

$$|J_1 \cup \ldots \cup J_{p-1}| \leq (n-q)(p-1) = (n-q)(\nu-1) < n.$$

This implies that there must be some player $i \in N$ who does not vote against any transition, that is, $i \in N(a_1, a_2) \cap \ldots \cap N(a_{p-1}, a_p)$. Then, we have

$$u_i(a_1) < u_i(a_2) < \ldots < u_i(a_p).$$

This implies that a_p is not Pareto-dominated. Since the choice of the order of alternatives is arbitrary, we can show that all alternatives are not Pareto-dominated.

For $|\mathcal{A}| > \nu$: We construct a profile of preferences under which the top cycle includes a Pareto-dominated alternative. Let $x = n - q$. Note that $n \geq (\nu - 1)x \geq q$.[25] Choose $\nu + 1$ alternatives, say $\{a_0, \ldots, a_\nu\} = A \subseteq \mathcal{A}$. Suppose that all of the players strictly prefer any alternative in A to any not in A, and that the players' preferences for alternatives in A are given as below.

$$\begin{array}{rl}
x \text{ players have preferences:} & a_1 \succ a_0 \succ a_\nu \succ a_{\nu-1} \succ \ldots \succ a_4 \succ a_3 \succ a_2 \\
x \text{ players have preferences:} & a_2 \succ a_1 \succ a_0 \succ a_\nu \succ a_{\nu-1} \succ \ldots \succ a_4 \succ a_3 \\
x \text{ players have preferences:} & a_3 \succ a_2 \succ a_1 \succ a_0 \succ a_\nu \succ a_{\nu-1} \succ \ldots \succ a_4 \\
& \vdots \\
x \text{ players have preferences:} & a_{\nu-1} \succ a_{\nu-2} \succ \ldots \succ a_2 \succ a_1 \succ a_0 \succ a_\nu \\
n - (\nu-1)x \text{ players have preferences:} & a_\nu \succ a_{\nu-1} \succ a_{\nu-2} \succ \ldots \succ a_2 \succ a_1 \succ a_0
\end{array}$$

Observe that all players prefer a_1 to a_0, and that $n - x$ players prefer a_{i+1} to a_i for all $i \in \{1, \ldots, \nu - 1\}$. Finally, observe that $(\nu - 1)x$ players prefer a_0 to a_ν. This implies that all a_0 to a_ν are in the top cycle with respect to q. The proof is complete by observing that a_0 is Pareto-dominated by a_1. $\square$

Section 3

Proof of Theorem 1. For $k \geq q$, define

$$N_k^* = \left\{ J \in N_k : \sum_{i \in J} c_i(a, a') + \sum_{j \in N \setminus J} c_j(a', a) = c_{aa'}^q \right\}.$$

N_k^* is a set of subsets of N with size k which can induce the minimum cost move from a to a'. Then, the transition probability can be written as

$$\begin{aligned}
P_{a,a'}^{\eta,q} = & \sum_{i \in N} p_{i,a,a'} \sum_{k \geq q}^{n} \underbrace{\sum_{J \in N_k^*} \prod_{j \in J} \Psi_j^\eta(a, a') \prod_{h \in N \setminus J} (1 - \Psi_h^\eta(a, a'))}_{\text{The exponential decay rate is equal to } c_{aa'}^q.} \\
& + \sum_{i \in N} p_{i,a,a'} \sum_{k \geq q}^{n} \underbrace{\sum_{J \in N_k \setminus N_k^*} \prod_{j \in J} \Psi_j^\eta(a, a') \prod_{h \in N \setminus J} (1 - \Psi_h^\eta(a, a'))}_{\text{The exponential decay rate is greater than } c_{aa'}^q.}
\end{aligned} \quad (17)$$

[25]To see this, observer that $\nu \geq n/(n-q) \geq \nu - 1$. $\nu \geq n/(n-q)$ implies that $\nu - 1 \geq q/(n-q)$.

Let

$$d_{a,a'}^{\eta,q} = \exp(\eta^{-1} c_{aa'}^{q}) \sum_{i \in N} p_{i,a,a'} \sum_{k \geq q}^{n} \sum_{J \in N_k^*} \prod_{j \in J} \Psi_j^{\eta}(a,a') \prod_{h \in N \setminus J} (1 - \Psi_h^{\eta}(a,a')).$$

Note that $\lim_{\eta \to 0} d_{a,a'}^{\eta,q} > 0$ due to the property (iv) of the regular choice rules. Equation (17) can be written as

$$P_{a,a'}^{\eta,q} = d_{a,a'}^{\eta,q} \exp(-\eta^{-1} c_{aa'}^{q}) + o(\exp(-\eta^{-1} c_{aa'}^{q})),$$

where $o(x)$ denotes a function such that $o(x)/x$ approaches zero as x approaches zero. Let

$$D^{\eta,q}(\tau_a) = \prod_{(v,w) \in \tau_a} d_{v,w}^{\eta,q}, \qquad \alpha^{\eta,q}(a) = \sum_{\tau_a : c_q(\tau_a) = c_q^*(a)} D^{\eta,q}(\tau_a). \tag{18}$$

According to the well known result of Freidlin and Wentzell (1998), the stationary distribution on $a \in \mathcal{A}$ is written as

$$\pi_\eta^q(a) = \frac{\alpha^{\eta,q}(a) \exp(-\eta^{-1} c_q^*(a)) + o\left(\exp(-\eta^{-1} c_q^*(a))\right)}{\sum_{b \in \mathcal{A}} \alpha^{\eta,q}(b) \exp(-\eta^{-1} c_q^*(b)) + o\left(\exp(-\eta^{-1} c_q^*(b))\right)}. \tag{19}$$

Note that $\pi_\eta^q(a)$ approaches zero as η approaches zero unless $c_q^*(a) = c_q^*$. Let $\alpha^q(a) = \lim_{\eta \to 0} \alpha^{\eta,q}(a)$. We have the following stationary distribution in the limit.

$$\pi_0^q(a) \equiv \lim_{\eta \to 0} \pi_\eta^q(a) = \begin{cases} \frac{\alpha^q(a)}{\sum_{b \in M(q)} \alpha^q(b)} & \text{if } a \in M_q. \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Thanks to the property (iv) of the regular choice rules, the RHS is strictly positive for $a \in M_q$.[26] □

Proof of Corollary 1. The first part ($|\mathcal{A}| \leq \nu$): We consider two cases: (i) the core is nonempty and (ii) it is empty. For (i), we show that there is a path from any alternative to the core with zero cost. Suppose a contrary, that is, such a path does not exist for some a_1 which is not in the core. Take a sequence of alternatives $\{a_1, a_2, \ldots, a_p\} \subseteq \mathcal{A}$ such that $n(a_i, a_{i+1}) \geq q$ for every $i = 1, \ldots, p-1$ where p is the first number such that $n(a_p, a_j) \geq q$ for some $j < p$. Since any path to the core does not exist, such a cyclic

[26] For weakly regular choice rules, $\lim_{\eta \to 0} \pi_\eta^q(a) = 0$ may hold even for $a \in M_q$.

sequence must exist. Observe that $|\{a_j, \ldots, a_p\}| < \nu$. However, this contradicts Theorem 3.1 of Nakamura (1979), which shows that such a sequence must be acyclic if the cardinality of the sequence is less than ν. Thus, there is a path from any alternative a_1 outside the core to the core with zero cost. The cost of the minimum cost spanning tree of the alternatives in the core must be smaller than that of a_1.

For (ii), Proposition 1 implies that $|\mathcal{A}| = \nu$, and that there exists a top cycle including no Pareto-dominated alternatives. Due to the definition, there exists a sequence with zero cost from any alternative in $\mathcal{A}$ to any alternative in the top cycle under q-quota rule. Lemma 1 implies that the top cycle coincides with the unique recurrent class. Then, the cost of escaping from the top cycle is positive. Thus, the minimum cost spanning tree of any alternative in the top cycle must have a smaller cost than that of those outside the top cycle has. And the cost is identical among all alternatives in the top cycle.

The second part ($|\mathcal{A}| > \nu$): For the latter claim, consider the preference profile given in the proof of Proposition 1 for the case $|\mathcal{A}| > \nu$. Observe that the top cycle constitutes a unique recurrent class in the associated unperturbed process. The claim follows. □

Proof of Theorem 3. In the proof, a notational convention is that letters in normal fonts (e.g. v, w, x, y) denote nodes in V^i and those in bold fonts (e.g. $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$) denote nodes in V^{i+1}.

We make a couple of definitions first. Let $\Upsilon_i(v)$ denote a set of v-trees on V^i for $v \in V^i$. Define the cost of a tree $\tau_v \in \Upsilon_i(v)$, and the cost of v as

$$c_i(\tau_v) = \sum_{(x,y)\in\tau_v} c_i(x,y), \qquad\qquad c_i^*(v) = \min_{\tau_v\in\Upsilon_i(v)} c_i(\tau_v).$$

Define

$$M(V^i) = \left\{ v \in V^i : c_i^*(v) = \min_{w\in V^i} c_i^*(w) \right\}$$

Note that Theorem 1 shows that the set of stochastically stable alternatives is given by $M(V^0)$.

We will prove that

$$M(V^i) = \Lambda\left(M(V^{i+1})\right).$$

It suffices to show that there exists a constant β^i such that $c_i^*(v) = c_{i+1}^*(\mathbf{v}) + \beta^i$ for all $v \in \mathbf{v} \in V^{i+1}$. Then we have that $v \in \mathbf{v}$ minimizes $c_i^*(\cdot)$ if and only if $\mathbf{v}$ minimizes $c_{i+1}^*(\cdot)$.

An iterative use of this argument will imply that $v \in M(V^0)$ if and only if $v \in \Lambda^{\bar{i}}(V^{\bar{i}})$.[27]

We will show that $\beta^i = (|V^{i+1}| - 1)\,\tilde{c}_i^*$. First, we show that $c_i^*(v) \leq c_{i+1}^*(\mathbf{v}) + \tilde{c}_i^*(|V^{i+1}| - 1)$. Let $\tau_{\mathbf{v}}$ be a minimum cost tree for $\mathbf{v}$ on V^{i+1}. Suppose a v-tree constructed by the steps below.

(i) For $(\hat{\mathbf{v}}, \hat{\mathbf{w}}) \in \tau_{\mathbf{v}}$, choose $\hat{v} \in \hat{\mathbf{v}}$ and $\hat{w} \in \hat{\mathbf{w}}$ such that there exists a path $d(\hat{v}, \hat{w})$ on V^i satisfying the two conditions:

$$\begin{aligned}
&\sum_{(x,y)\in d(\hat{v},\hat{w})} c_i(x,y) = \tilde{c}_i(\hat{v},\hat{w}), \\
&x \notin \hat{\mathbf{v}} \setminus \{\hat{v}\},\ y \notin \hat{\mathbf{w}} \setminus \{\hat{w}\} \qquad\qquad \forall (x,y) \in d(\hat{v},\hat{w}).
\end{aligned}$$

Add edges of $d(\hat{v}, \hat{w})$. For $z \in \hat{\mathbf{v}} \setminus \hat{v}$, find a path $d(z, \hat{v})$ such that $\sum_{(x,y)\in d(z,\hat{v})} c_i(x,y) = 0$.[28] Add its edges. Similarly, for $z' \in \hat{\mathbf{w}} \setminus \hat{w}$, add edges of a path $d(z', \hat{w})$ such that $\sum_{(x,y)\in d(z',\hat{w})} c_i(x,y) = 0$. Apply this step for all $(\hat{\mathbf{v}}, \hat{\mathbf{w}}) \in \tau_{\mathbf{v}}$.

(ii) Let $z \in V^i$ be a node which is not contained in any V^i-recurrent set, i.e., $z \notin \hat{\mathbf{v}}$ for all $\hat{\mathbf{v}} \in V^{i+1}$. There must exist $\hat{v} \in \hat{\mathbf{v}} \in V^{i+1}$ such that there is a path $d(z, \hat{v})$ with $\sum_{(x,y)\in d(z,\hat{v})} c_i(x,y) = 0$.[29] Add edges of the path $d(z, \hat{v})$. Apply this step for all such z.

The resulting set of edges must be an v-tree, say τ_v, such that

$$\begin{aligned}
c_i(\tau_v) &= \sum_{(x,y)\in\tau_v} c_i(x,y) = \sum_{(\hat{\mathbf{v}},\hat{\mathbf{w}})\in\tau_{\mathbf{v}}} \tilde{c}_i(\hat{v},\hat{w}) \\
&= \sum_{(\hat{\mathbf{v}},\hat{\mathbf{w}})\in\tau_{\mathbf{v}}} \left[c_{i+1}(\hat{\mathbf{v}},\hat{\mathbf{w}}) + \tilde{c}_i^*\right] = \sum_{(\hat{\mathbf{v}},\hat{\mathbf{w}})\in\tau_{\mathbf{v}}} \left[c_{i+1}(\hat{\mathbf{v}},\hat{\mathbf{w}})\right] + \tilde{c}_i^*(|V^{i+1}| - 1) \\
&= c_{i+1}(\tau_{\mathbf{v}}) + \tilde{c}_i^*(|V^{i+1}| - 1).
\end{aligned}$$

In the last term in the first line of the equations, $\hat{v} \in \hat{\mathbf{v}}$ and $\hat{w} \in \hat{\mathbf{w}}$ are the ones chosen in step (i) above. The above equation implies that

$$c_i^*(v) \leq c_i(\tau_v) = c_{i+1}(\tau_{\mathbf{v}}) + \tilde{c}_i^*(|V^{i+1}| - 1) = c_{i+1}^*(\mathbf{v}) + \tilde{c}_i^*(|V^{i+1}| - 1).$$

Next, we show that $c_i^*(v) \geq c_{i+1}^*(\mathbf{v}) + \tilde{c}_i^*(|V^{i+1}| - 1)$. We first prove the following lemma.

[27] $M(V^{\bar{i}}) = V^{\bar{i}}$ since $V^{\bar{i}}$ is singleton.

[28] Such a path exists due to the definition of V^i-recurrent.

[29] If such a zero-cost path does not exist, then z must be contained in some V^i-recurrent set. This contradicts.

Lemma 8. *For every v-tree for $v \in \mathbf{v} \in V^{i+1}$, say τ, there exists a graph $T(\tau)$ on V^{i+1} such that (a) $T(\tau)$ is a $\mathbf{v}$-tree on V^{i+1},*
(b) $\sum_{(\hat{\mathbf{v}},\hat{\mathbf{w}})\in T(\tau)} \tilde{c}_i(\hat{v},\hat{w}) \le \sum_{(x,y)\in\tau} c_i(x,y)$, where $\hat{v} \in \hat{\mathbf{v}}$ and $\hat{w} \in \hat{\mathbf{w}}$.[30]

Proof. For $\hat{v},\hat{w} \in V^i$, let $d(\hat{v},\hat{w}) \subseteq \tau$ denote a path on τ from $\hat{v}$ to $\hat{w}$. Suppose a graph $T(\tau)$ constructed the following steps.

(I) For all $\hat{\mathbf{v}} \in V^{i+1}$, find $\hat{\mathbf{w}} \neq \hat{\mathbf{v}}$ such that there exists a path $d(\hat{v},\hat{w}) \subset \tau$ for some $\hat{v} \in \hat{\mathbf{v}}$, $\hat{w} \in \hat{\mathbf{w}}$ satisfying the condition below:

$$y \notin \mathbf{z} \qquad\qquad \forall (x,y) \in d(\hat{v},\hat{w}), \quad \forall \mathbf{z} \in V^{i+1} \setminus \{\hat{\mathbf{v}},\hat{\mathbf{w}}\}.$$

Add $(\hat{\mathbf{v}},\hat{\mathbf{w}})$ to $T(\tau)$. Note that the resulting graph is a $\mathbf{v}$-tree on V^{i+1}.

(II) Make the following modification on $T(\tau)$ for satisfying property (b). For the remainder, let $\hat{v}_j$, $\hat{w}$ and z be an arbitrary choice of nodes from $\hat{\mathbf{v}}_j$, $\hat{\mathbf{w}}$, $\mathbf{z} \in V^{i+1}$, respectively.

If there exist $(\hat{\mathbf{v}}_1,\hat{\mathbf{w}}),\ldots,(\hat{\mathbf{v}}_h,\hat{\mathbf{w}}) \in T(\tau)$ such that $d(\hat{v}_1,\hat{w}) \cap \ldots \cap d(\hat{v}_h,\hat{w}) \neq \varnothing$, then do the following things.

(II a) If $c_i(x,y) = 0$ for all $(x,y) \in d(\hat{v}_1,\hat{w}) \cap \ldots \cap d(\hat{v}_h,\hat{w})$, then do nothing.[31]

(II b) If $c_i(x,y) > 0$ for some $(x,y) \in d(\hat{v}_1,\hat{w}) \cap \ldots \cap d(\hat{v}_h,\hat{w})$, find $\mathbf{z} \in V^{i+1} \setminus \hat{\mathbf{w}}$ such that $\tilde{c}_i(x,z) = 0$.[32] If $\mathbf{z} = \hat{\mathbf{v}}_{j^*}$ for some $j^* \in \{1,\ldots,h\}$, then replace edge $(\hat{\mathbf{v}}_j,\hat{\mathbf{w}})$ with $(\hat{\mathbf{v}}_j,\hat{\mathbf{v}}_{j^*})$ for all $j \neq j^*$.[33] This will reduce the cost of $T(\tau)$ by at least $(h-1)c_i(x,y)$.

If $\mathbf{z} \neq \hat{\mathbf{v}}_j$ for all $j \in \{1,\ldots,h\}$, then do the following things.

(II b 1) If $y \notin \hat{\mathbf{v}}_1 \cup \ldots \cup \hat{\mathbf{v}}_h$ for all $(x,y) \in d(z,v)$, then replace edge $(\hat{\mathbf{v}}_j,\hat{\mathbf{w}})$ with $(\hat{\mathbf{v}}_j,\mathbf{z})$ for all $j \in \{1,\ldots,h\}$.[34] This will reduce the cost of $T(\tau)$ by at least $hc_i(x,y)$.

[30]The choice of $\hat{v}$ and $\hat{w}$ is arbitrary since $\tilde{c}_i(\hat{v},\hat{w})$ has the same value for all $\hat{v} \in \hat{\mathbf{v}}$ and $\hat{w} \in \hat{\mathbf{w}}$.

[31]For this case, observe that

$$\sum_{j=1}^{h} \tilde{c}_i(\hat{v}_j,\hat{w}) \le \sum_{(x,y)\in d(\hat{v}_1,\hat{w})\cup\ldots\cup d(\hat{v}_h,\hat{w})} c_i(x,y).$$

[32]Such $\mathbf{z}$ must exits. Otherwise, x must be V^i-recurrent, which contradicts the construction of $T(\tau)$.

[33]Edge $(\hat{\mathbf{v}}_{j^*},\hat{w})$ will remain in $T(\tau)$, and added edges are directed to $\hat{\mathbf{v}}_{j^*}$. The resulting graph is still an $\mathbf{v}$-tree.

[34]If $y \notin \hat{\mathbf{v}}_1 \cup \ldots \cup \hat{\mathbf{v}}_h$ for all $(x,y) \in d(z,v)$, then there exists a path $\{(\mathbf{z},\mathbf{x}_1),(\mathbf{x}_1,\mathbf{x}_2),\ldots,(\mathbf{x}_H,\mathbf{v})\}$ on $T(\tau)$ such that $\mathbf{x}_J \neq \hat{\mathbf{v}}_j$ for all $J \in \{1,\ldots,H\}$ and $j \in \{1,\ldots,h\}$. The resulting graph is still a $\mathbf{v}$-tree.

(II b 2) If $y \in \hat{\mathbf{v}}_{j^{**}}$ for some j^{**} and $(x,y) \in d(z,v)$, then replace edge $(\hat{\mathbf{v}}_j, \hat{\mathbf{w}})$ with $(\hat{\mathbf{v}}_j, \mathbf{z})$ for all $j \neq j^{**}$.[35] This will reduce the cost of $T(\tau)$ by at least $(h-1)c_i(x,y)$.

Note that the following inequality will hold if $d(\hat{v}_1, \hat{w}) \cap \ldots \cap d(\hat{v}_h, \hat{w}) \neq \varnothing$:

$$\sum_{(x,y)\in d(\hat{v}_1,\hat{w})\cup \ldots \cup d(\hat{v}_h,\hat{w})} c_i(x,y) \geq \sum_{j=1}^{h} \tilde{c}_i(\hat{v}_j,\hat{w}) - \sum_{(x,y)\in d(\hat{v}_1,\hat{w})\cap \ldots \cap d(\hat{v}_h,\hat{w})} (h-1)c_i(x,y).$$

The operation (II) will keep $T(\tau)$ being a $\mathbf{v}$-tree and make it satisfy the property (b). □

Let τ_v^* be a minimum cost tree for v on V^i. Let $\tau_{\mathbf{v}}$ denote $T(\tau_v^*)$ in Lemma 8. Observe that

$$\begin{aligned} c_i^*(v) = c_i(\tau_v^*) &= \sum_{(x,y)\in\tau_v^*} c_i(x,y) \geq \sum_{(\hat{\mathbf{v}},\hat{\mathbf{w}})\in\tau_{\mathbf{v}}} \tilde{c}_i(\hat{v},\hat{w}) \quad \text{ for } \hat{v}\in\hat{\mathbf{v}},\ \hat{w}\in\hat{\mathbf{w}} \\ &= \sum_{(\hat{\mathbf{v}},\hat{\mathbf{w}})\in\tau_{\mathbf{v}}} \left[c_{i+1}(\hat{\mathbf{v}},\hat{\mathbf{w}}) + \tilde{c}_i^*\right] = c_{i+1}(\tau_{\mathbf{v}}) + \tilde{c}_i^*(|V^{i+1}|-1) \\ &\geq c_{i+1}^*(\mathbf{v}) + \tilde{c}_i^*(|V^{i+1}|-1). \end{aligned}$$

The first inequality comes from the property (b) of $T(\tau_v^*)$. This together with the first observation implies that $c_i^*(v) = c_{i+1}^*(\mathbf{v}) + \tilde{c}_i^*(|V^{i+1}|-1)$. □

Section 4

We will prove Lemma 4 and Theorem 5. As for Lemma 4, we first prove Lemmas 9 and 10. The former shows that if an alternative a is stochastically stable for quota $q > \underline{q}$, a must be in the top cycle under $\underline{q}$-quota rule, i.e. $a \in \mathcal{A}_{\underline{q}}$. The latter shows that any minimum cost tree of stochastically stable alternatives has no edge emanating from some alternative in $\mathcal{A}_{\underline{q}}$ to one not in $\mathcal{A}_{\underline{q}}$. It implies that if a is stochastically stable with $q > \underline{q}$, then τ_a minimizing $c_q(\cdot)$ must have an a-subtree over $\mathcal{A}_{\underline{q}}$.

Lemma 9. $\lim_{\varepsilon\to 0} \pi_\varepsilon^q(a) > 0$ *for* $q > \underline{q}$ *only if* $a \in \mathcal{A}_{\underline{q}} \in \Theta(\underline{q})$.

Proof. By a way of contradiction, suppose that $\pi_\varepsilon^q(a) > 0$ for $a \notin \mathcal{A}_{\underline{q}}$. Note that $n(a,a') = n - n(a',a) \geq n - \underline{q} + 1$ for all $a' \in \mathcal{A}_{\underline{q}}$ because $a \notin \mathcal{A}_{\underline{q}}$ implies that $n(a',a) < \underline{q}$ for $a' \in \mathcal{A}_{\underline{q}}$.

Let τ_a be an a-tree minimizing its cost, i.e. $c_q(\tau_a) = c_q^*$. Choose $(a',a'') \in \tau_a$ such that $a' \in \mathcal{A}_{\underline{q}}$ and $a'' \notin \mathcal{A}_{\underline{q}}$. Such an edge must exist since the root of τ_a is not in $\mathcal{A}_{\underline{q}}$.

[35] If $y \in \hat{\mathbf{v}}_{j^{**}}$ for j^{**} and $(x,y) \in d(z,v)$, then there exists a path $\{(\mathbf{z},\mathbf{y}_1),\ldots,(\mathbf{y}_H,\hat{\mathbf{v}}_{j^{**}})\}$ on $T(\tau)$. Edge $(\hat{\mathbf{v}}_{j^*},\hat{\mathbf{w}})$ will remain in $T(\tau)$, and added edges are directed to $\mathbf{z}$. The resulting graph is still a $\mathbf{v}$-tree.

Remove (a', a'') from τ_a. This will reduce the cost of τ_a by at least $q - (\underline{q} - 1)$ because $n(a', a'') \leq \underline{q} - 1$ for $a' \in \mathcal{A}_{\underline{q}}$ and $a'' \notin \mathcal{A}_{\underline{q}}$. Then, add an edge (a, a') to τ_a. This will increase the cost by $q - n(a, a')$, which is at most $q - (n - \underline{q} + 1)$. It is easy to see that $\underline{q} - 1 < n/2 \leq n - \underline{q} + 1$. The resulting tree is an a'-tree with the cost strictly less than $c_q(\tau_a)$. This contradicts that $c_q(\tau_a) = c_q^*$. $\square$

Lemma 10. *Suppose that $a \in \mathcal{A}$ is stochastically stable with $q > \underline{q}$. Let τ_a be such that $c_q(\tau_a) = c_q^*$. If $a' \in \mathcal{A}_{\underline{q}}$ and $(a', a'') \in \tau_a$, then $a'' \in \mathcal{A}_{\underline{q}}$.*

Proof. Observe that Lemma 9 implies that $a \in \mathcal{A}_{\underline{q}}$. By a way of contradiction, suppose that there exists $(a', a'') \in \tau_a$ such that $a' \in \mathcal{A}_{\underline{q}}$ and $a'' \notin \mathcal{A}_{\underline{q}}$. Since $a'' \notin \mathcal{A}_{\underline{q}}$, $n(a', a'') < \underline{q}$. Remove edge (a', a'') from τ_a. This will reduce the cost of τ_a by at least $q - \underline{q} + 1$. Let τ_a^1 denote the resulting set of edges.

If $n(a', a) \geq \underline{q}$, add edge (a', a) to τ_a^1. This will increase the cost by at most $q - \underline{q}$. The resulting set is an a-tree, say τ_a^2. Observe that $c_q(\tau_a^2) \leq c_q(\tau_a) - (q - \underline{q} + 1) + q - \underline{q} = c_q(\tau_a) - 1$.

If $n(a', a) < \underline{q}$, add edge (a, a') to τ_a^1. This will increase the cost by at most $q - \underline{q}$.[36] The resulting set is an a'-tree, say $\tau_{a'}$. Observe that $c_q(\tau_{a'}) \leq c_q(\tau_a) - (q - \underline{q} + 1) + q - \underline{q} = c_q(\tau_a) - 1$. Those observations contradict that $c_q(\tau_a) = c_q^*$. $\square$

Now, we are ready to prove Lemma 4.

Proof of Lemma 4. The proof for each part is conducted by a way of contradiction.
'only if' part: Suppose that $a \in \mathcal{A}_{\underline{q}} \setminus M_q(\mathcal{A}_{\underline{q}})$ is stochastically stable.[37] Let τ_a be the minimum cost tree for a. Lemma 10 implies that τ_a has an a-subtree over $\mathcal{A}_{\underline{q}}$, say $\tau_a^\sharp$.

Let $b \in M_q(\mathcal{A}_{\underline{q}})$ with $\tau_b^\sharp$ being a b-subtree over $\mathcal{A}_{\underline{q}}$ such that $c_q(\tau_b^\sharp) = c^*_{q, \mathcal{A}_{\underline{q}}}$. Replace $\tau_a^\sharp$ with $\tau_b^\sharp$ in τ_a. The resulting set of edges, say τ_b^*, must be a b-tree. Observe that

$$c_q(\tau_b^*) = c_q(\tau_a) - c_q(\tau_a^\sharp) + c_q(\tau_b^\sharp) < c_q(\tau_a).$$

The inequality comes from the fact that $a \notin M_q(\mathcal{A}_{\underline{q}})$. This contradicts that $c_q(\tau_a) = c_q^*$.
'if' part: Suppose that $a \in M_q(\mathcal{A}_{\underline{q}})$ is not stochastically stable. Let $\tau_a^\sharp$ be an a-subtree over $\mathcal{A}_{\underline{q}}$ such that $c_q(\tau_a^\sharp) = c^*_{q, \mathcal{A}_{\underline{q}}}$. Let a' be some stochastically stable alternative with a minimum cost tree $\tau_{a'}$. Lemmas 9 and 10 imply that $\tau_{a'}$ has an a'-subtree over $\mathcal{A}_{\underline{q}}$, say $\tau_{a'}^\sharp$.

[36] Since $n(a', a)$ is an integer, it must be that $n(a', a) \leq \underline{q} - 1$. Then, $n(a, a') \geq \underline{q}$.

[37] Lemma 9 allows us to restrict our attention to $\mathcal{A}_{\underline{q}}$.

Replace $\tau^\sharp_{a'}$ with $\tau^\sharp_a$ in $\tau_{a'}$. The resulting set of edges is an a-tree, say τ^*_a. Observe that

$$c_q(\tau^*_a) = c_q(\tau_{a'}) - c_q(\tau^\sharp_{a'}) + c_q(\tau^\sharp_a) \leq c_q(\tau_{a'}) = c^*_q.$$

Then, Theorem 1 suggests that a is stochastically stable. A contradiction. □

Proof of Theorem 5

We first prove Lemma 11. It shows that, when identifying stochastically stable alternatives, we can ignore edges which have positive costs under $\underline{q}$-quota rule. This is because the cost-minimizing trees for $q > \underline{q}$ will not contain such edges.

Lemma 11. *Suppose that $a \in \mathcal{A}$ is stochastically stable with $q > \underline{q}$. Let τ_a be such that $c_q(\tau_a) = c^*_q$. Then, $P^{0,\underline{q}}_{x,y} = 0$ for all $(x, y) \in \tau_a$.*

Proof. The proof is the way of contradiction. Suppose that $P^{0,\underline{q}}_{x,y} > 0$ for some $(x, y) \in \tau_a$. This implies that $n(x, y) < n/2 \leq \underline{q}$ and that $c^q_{xy} > 0$ for $q > \underline{q}$. Recall that $n(x, a) + n(a, x) = n$. Since $n(x, y) < n/2$, either $n(x, a) > n(x, y)$ or $n(a, x) > n(x, y)$ must hold.

First, suppose that $n(x, a) > n(x, y)$. Observe that $c^q_{xa} < c^q_{xy}$ for $q > \underline{q}$.[38] Replace (x, y) with (x, a) in tree τ_a. The resulting set of edges must be another a-tree, say τ'_a. Then,

$$c_q(\tau'_a) = c_q(\tau_a) - c^q_{xy} + c^q_{xa} < c_q(\tau_a).$$

This contradicts that $c_q(\tau_a) = c^*_q$.

Next, suppose that $n(a, x) > n(x, y)$. Replace (x, y) with (a, x) in tree τ_a. The resulting set of edges must be an x-tree, say τ_x. Observe that

$$c_q(\tau_x) = c_q(\tau_a) - c^q_{xy} + c^q_{ax} < c_q(\tau_a).$$

This contradicts that a is stochastically stable. Thus, it holds that $P^{0,\underline{q}}_{x,y} = 0$ for all $(x, y) \in \tau_a$. □

Proof of Theorem 5. Lemma 4 allows us to restrict our attention to $\mathcal{A}_{\underline{q}}$. That is, $a \in \mathcal{A}$ is stochastically stable if and only if $a \in \mathcal{A}_{\underline{q}}$ and it has an a-subtree on $\mathcal{A}_{\underline{q}}$ that minimizes the cost over all subtrees on $\mathcal{A}_{\underline{q}}$. Lemma 11 guarantees that we can ignore the set of edges having positive costs under q-quota rule, i.e. $\hat{E}$. Thus, we can let $V^0 = \mathcal{A}_{\underline{q}}$, $E^0 = (\mathcal{A}_{\underline{q}} \times \mathcal{A}_{\underline{q}}) \setminus \hat{E}$. The rest of the proof is the same as in that of Theorem 3. □

[38] The strict inequality holds since $q - n(x, y) > 0$ for $q > \underline{q}$. Then, $c^q_{xa} = \max\{q - n(x, a), 0\} < q - n(x, y) = c^q_{xy}$.

Section 6
Recall that $d(a, A') = \inf_{a' \in A'} d(a, a')$ for $a \in \mathcal{A}^0$ and $A' \subset \mathcal{A}^0$, i.e. the distance between a point and a set.

Proof of Lemma 5. (i): By definition of n^*, it holds that $n(a, a') \leq n^*$ for $a \in \mathcal{A}^\delta \cap \mathfrak{C}^*$ and all $a' \in \mathcal{A}^\delta \setminus \{a\}$. This implies that $\overline{n}^\delta \leq n(a) \leq n^*$. Suppose that $\lim_{\delta \to 0} \overline{n}^\delta < n^*$. Choose $a \in \mathcal{A}^\delta$ such that $n(a) = \overline{n}$. By the definition of n^*, there exists some $a' \in \mathcal{A}^0$ such that $|i \in N : d(s_i, a') < d(s_i, a)| \geq n^*$. Pick $a'' \in \mathcal{A}^\delta$ with $d(a', a'') < \delta$. By the continuity of d, we must have that $|i \in N : u_i(a'') > u_i(a)| \geq n^*$ for all sufficiently small δ. This contradicts that $a \in \mathfrak{C}^*$.

(ii): Since $\overline{n}^\delta$ takes only finite integers, (i) implies that $n(a^\delta) = \overline{n}^\delta = n^*$ for all $a^\delta \in \mathcal{A}^{*,\delta}$ and all sufficiently small δ. Suppose that δ is small enough that $\overline{n}^\delta = n^*$.

It suffices to show that $\mathcal{A}^{*,\delta} \cap (\mathcal{A}^0 \setminus \mathfrak{C}^{*,\sigma}) = \varnothing$. Choose $r \in \mathcal{A}^0 \setminus \mathfrak{C}^{*,\sigma}$. Note that $d(a, \mathfrak{C}^*) \geq \sigma$. Since $r \notin \mathfrak{C}^*$, there exists some $r^* \in \mathcal{A}^0$ such that $|i \in N : u_i(r^*) > u_i(r)| \geq n^* + 1$. By the continuity of d, there exists $a^\delta \in \mathcal{A}^\delta$ with $d(r^*, a^\delta) < \delta$ such that $|i \in N : u_i(a^\delta) > u_i(r)| \geq n^* + 1$ for all sufficiently small δ. Since the choice of r is arbitrary, this implies that, for all $r \in \mathcal{A}^0 \setminus \mathfrak{C}^{*,\sigma}$, $r \notin \mathcal{A}^{*,\delta}$ for all sufficiently small δ. $\square$

Proof of Lemma 6. The proof follows the technique used in McKelvey (1976). Our algorithm generalizes his to any $q \leq \overline{n}$.

Fix $q \leq \overline{n}$. For $y \in \mathbb{R}^h$ and $c \in \mathbb{R}$, let $H_c(y) = \{r \in \mathbb{R}^h : r' \cdot y = c\}$. $H_c(y)$ is a hyperplane associated with y and c. Let $H_c^+(y) = \{r \in \mathbb{R}^h : r' \cdot y \geq c\}$ be a half space separated by $H_c(y)$. We say $H_c^+(y)$ is a *q-winning space* if $|\{i \in N : s_i \in H_c^+(y)\}| = q$.[39] Note that if a is not in a q-winning space, then there exists another alternative in that space which wins at least q votes against a.

For each $y \in \mathbb{R}^h$, with $\|y\| = 1$, define C_y to be the set of c satisfying $H_c^+(y)$ being a q-winning space. Note that C_y is a bounded interval which is closed above. So we can set $c_y = \max C_y$, and we define $H_y^+ = H_{c_y}^+(y)$. Since $q \leq \overline{n}$, for all $a \in \mathbb{R}^h$, there exists a q-winning space which does not include a. We will prove that

$$\bigcap_{\|y\|=1} H_y^+ = \varnothing.$$

By way of contradiction, suppose that there exists $a \in \bigcap_{\|y\|=1} H_y^+$. Choose $b \neq a$. Let $y = (b-a)/\|b-a\|$ and $c = \frac{1}{2}(b+a)'y$. Observe that $a'y < c$ and $b'y > c$. Let $H_c^+(y)$ be a half space separated by the hyperplane $H_c(y)$. $a \notin H_c^+(y)$ implies that $H_c^+(y)$ is not

[39] A q-winning space is a version of the median hyperplane and its upper space for super-majority rules.

a q-winning space. Thus, $\{i \in N : s_i \in H_c^+(y)\} < q$.[40] This implies that $n(a,b) \leq q-1$. Since the choice of b is arbitrary, it must be that $n(a) \leq q-1 < \overline{n}$. This contradicts that $\overline{n}$ is the min-max quota.

By Helly's theorem, we can find a set of $h+1$ vectors, $y_1 \dots y_{h+1}$, such that[41]

$$\bigcap_{y \in \{y_1, \dots, y_{h+1}\}} H_y^+ = \varnothing.$$

Choose a minimal collection of points $\{y_1, \dots, y_p\}$ which satisfies the condition above. It holds that $\bigcap_{y \in \{y_1,\dots,y_p\}} H_y^+ = \varnothing$, and moreover that, for all $j \in \{1, \dots, p\}$,[42]

$$\exists r \in \mathbb{R}^h \quad \text{such that} \quad r' \cdot y_i \geq c_i \quad \forall 1 \leq i \leq p \text{ with } i \neq j,$$

where $c_i = \max C_{y_i}$. That is, $\{y_1, \dots, y_p\}$ is one of the minimum subsets of vectors which have no solution to $r' \cdot y_i \geq c_i$ for all $1 \leq i \leq p$.

For each $1 \leq j \leq p$, let z_j be a solution to[43]

$$z_j' \cdot y_i = c_i \quad \forall 1 \leq i \leq p \text{ with } i \neq j.$$

Note that for any $r \in \mathbb{R}^h$, there exists some y_i such that $r' \cdot y_i \leq 0$. Otherwise, $\alpha r' \cdot y_i \geq c_i$ holds for all $1 \leq i \leq p$ for some large α. It contradicts that $\bigcap_{y \in \{y_1,\dots,y_p\}} H_y = \varnothing$. We assume that $\sum_{1 \leq j \leq p} z_j = 0$.[44] Then, $0 = \sum_{1 \leq j \leq p} z_j' \cdot y_i = (p-1)c_i + z_i' \cdot y_i < pc_i$. This ensures that $c_i > 0$.

Now, for a_k, we construct $a_{k+1} \in \mathcal{A}^\delta$ as follows. Pick y_i such that $a_k' \cdot y_i \leq 0$. We define $r_{k+1} \in \mathcal{A}^0$ as below.[45] Then, pick $a_{k+1} \in \mathcal{A}^\delta$ such that $d(a_{k+1}, r_{k+1}) < \delta$.

$$r_{k+1} = a_k + [c_i - 2y_i' \cdot a_k] y_i \quad \text{if} \quad c_i > 0.$$

Observe that

$$\|a_k\|^2 = (y_i' \cdot a_k)^2 + \|a_k - (y_i' \cdot a_k) y_i)\|^2,$$

[40] If $\{i \in N : s_i \in H_c^+(y)\} \geq q$, then it must be that $c \leq c_y$ by the definition of c_y. However, $c \leq c_y$ implies that $a \in H_c^+(y)$. It contradicts.

[41] For Helly's theorem, see Danzer et al. (1963) for example.

[42] Without loss of generality, let the first p vectors of the $h+1$ vectors, $y_1, \dots, y_p$ (for $p \leq h+1$), satisfy this property.

[43] Note that $z_i' \cdot y_i < c_i$. This is because $\bigcap_{y \in \{y_1,\dots,y_p\}} H_y = \varnothing$.

[44] That is, we set the origin of the vector space to the point where $\sum_{1 \leq j \leq p} z_j = 0$. There is no loss of generality since the distance between every pair of points will be preserved.

[45] We will later show that all r_{k+1} we consider satisfy $r_{k+1} \in B(5\rho)$ which ensures that $r_{k+1} \in \mathcal{A}^0$.

$$\|r_{k+1}\|^2 = (y_i' \cdot r_{k+1})^2 + \|r_{k+1} - (y_i' \cdot r_{k+1})y_i)\|^2.$$

Observe that for $c_i > 0$,

$$\begin{aligned}
\|r_{k+1}\|^2 - \|a_k\|^2 &= (y_i' \cdot r_{k+1})^2 - (y_i' \cdot a_k)^2 = [y_i' \cdot (a_k + [c_i - 2y_i' \cdot a_k]y_i)]^2 - (y_i' \cdot a_k)^2 \\
&= [y_i' \cdot a_k + c_i - 2y_i' \cdot a_k]^2 - (y_i' \cdot a_k)^2 = (c_i - y_i' \cdot a_k)^2 - (y_i' \cdot a_k)^2 \\
&= c_i^2 - 2c_i\, y_i' \cdot a_k \geq c_i^2.
\end{aligned}$$

The third equality comes from that $y_i' \cdot y_i = \|y_i\| = 1$. The last inequality is from that $y_i' \cdot a_k \leq 0$. Let $\mathbb{I}_r = r_{k+1}/\|r_{k+1}\|$. For a_{k+1}, we have that

$$\begin{aligned}
\|a_{k+1}\|^2 - \|a_k\|^2 \geq \|r_{k+1} - \delta \cdot \mathbb{I}_r\|^2 - \|a_k\|^2 &= \|r_{k+1}(1 - \delta/\|r_{k+1}\|)\|^2 - \|a_k\|^2 \\
&= c_i^2 - 2c_i\, y_i' \cdot a_k + \delta^2 - 2\delta\|r_{k+1}\|.
\end{aligned}$$

The right hand side of the last expression is positive for all sufficiently small δ. Note that a_{k+1} is more distant away from the origin than a_k is. A successive application of the algorithm will get a_k as far from the origin as we want.

We show that $|\{j \in N : u_j(a_{k+1}) > u_j(a_k)\}| \geq q$. Observe that

$$\begin{aligned}
u_j(r_{k+1}) > u_j(a_k) &\Leftrightarrow \|s_j - r_{k+1}\| < \|s_j - a_k\| \Leftrightarrow \|s_j - r_{k+1}\|^2 < \|s_j - a_k\|^2 \\
&\Leftrightarrow \|s_j\|^2 + \|r_{k+1}\|^2 - 2s_j' \cdot r_{k+1} < \|s_j\|^2 + \|a_k\|^2 - 2s_j' \cdot a_k \\
&\Leftrightarrow 2s_j' \cdot (r_{k+1} - a_k) > r_{k+1}' \cdot r_{k+1} - a_k' \cdot a_k \\
&\Leftrightarrow 2s_j' \cdot (r_{k+1} - a_k) > (r_{k+1} + a_k)'(r_{k+1} - a_k) \\
&\Leftrightarrow 2s_j' \cdot (c_i - 2y_i' \cdot a_k)y_i > (r_{k+1} + a_k)' \cdot (c_i - 2y_i' \cdot a_k)y_i \\
&\Leftrightarrow 2s_j' \cdot y_i > (r_{k+1} + a_k)' \cdot y_i \\
&\Leftrightarrow s_j' \cdot y_i > c_i/2.
\end{aligned}$$

For the last claim, observe that $(r_{k+1} + a_k)' \cdot y_i = (2a_k + [c_i - 2y_i' \cdot a_k]y_i)' \cdot y_i = c_i$. Recall that we choose y_i such that $|s_i \in H_{y_i}^+| = q$ where $H_{y_i}^+ = \{r \in \mathbb{R}^h : r' \cdot y \geq c_i\}$. That $s_j' \cdot y_i > c_i/2$ implies that there are at least q players who prefer r_{k+1} to a_k. Observe that a similar computation will obtain that

$$\begin{aligned}
u_j(a_{k+1}) > u_j(a_k) &\Leftrightarrow \|s_j - a_{k+1}\| < \|s_j - a_k\| \\
&\Leftarrow \|s_j - r_{k+1}\| + \delta < \|s_j - a_k\| \Leftrightarrow s_j' \cdot y_i > c_i/2 + \delta\, \frac{\|s_j - r_{k+1}\| + 1}{c_i - 2y_i' \cdot a_k}
\end{aligned}$$

If δ is sufficiently small, the right hand side must be smaller than c_i. By the definition of c_i, it must be that for sufficiently small δ,

$$\begin{aligned}|\{j \in N : u_j(a_{k+1}) > u_j(a_k)\}| &\geq \left| \left\{ j \in N : s'_j \cdot y_i > c_i/2 + \delta \ \frac{\|s_j - r_{k+1}\| + 1}{c_i - 2y'_i \cdot a_k} \right\} \right| \\ &\geq |\{j \in N : s'_j \cdot y_i > c_i\}| \geq q.\end{aligned}$$

We show the upper bound of $\|r_{k+1}\|$ for given a_k. Recall that r_{k+1} is preferred to a_k by at least q players. This implies that $r_{k+1} \in B(3\rho)$ if $a_k \in B(\rho)$, and that $r_{k+1} \in B(5\rho)$ if $a_k \in B(3\rho)$. For the first case, observe that r_{k+1} will be at least 2ρ distant away from s_i for all $i \in N$, if $r_{k+1} \notin B(3\rho)$. While $a_k \in B(\rho)$ implies that $\|s_i - a_k\| < 2\rho$ for all $i \in N$. Then, for at least q players to prefer r_{k+1}, it must be that $r_{k+1} \in B(3\rho)$. Similarly, r_{k+1} will be at least 4ρ distant away from all ideal points if $r_{k+1} \notin B(5\rho)$, while $a_k \in B(3\rho)$ implies that $\|s_i - a_k\| < 4\rho$ for all $i \in N$.

Finally, with the above discussion in hand, we show that the process can reach a_L. Let $B^* = B(5\rho) \setminus B(3\rho)$, i.e., the distance from any point in B^* to one in $B(\rho)$ is at least 2ρ. Since the algorithm will get a_k as far as we want, we can pick a sequence $\{a_1, \ldots, a_{L-1}\}$ such that $a_{L-2} \in B(3\rho)$ and $a_{L-1} \in B^*$. Then, the proof is complete by observing that $|\{i \in N : u_i(a_L) > u_i(a_{L-1})\}| \geq q$. □

To prove Lemma 7, we first prove our version of Kramer (1977)'s Lemma 3 below.

Lemma 12. *Fix small $\delta > 0$ such that $\overline{n}^\delta = n^*$. $d(a, \mathfrak{C}^*) > d(a', \mathfrak{C}^*)$ for $a \notin \mathcal{A}^{*,\delta}$ and $a' \in \overline{Q}(a)$.*

Proof of Lemma 12. Observe that $\mathfrak{C}(J) = hull(J)$, where $hull(J)$ is the convex hull of ideal points $\{s_i : i \in J\}$. $\mathfrak{C}(J) \supseteq hull(J)$ is obvious. To see $\mathfrak{C}(J) \subseteq hull(J)$, let $r^* \in \text{argmin}_{r' \in hull(J)} d(r, r')$ for $r \notin hull(J)$. Then, r^* dominates r via J. Note that $\mathfrak{C}(J) = hull(J)$ implies that $\mathfrak{C}(J) \subseteq \mathfrak{C}(J')$ for $J \subseteq J'$.

Suppose that $a \notin \mathcal{A}^{*,\delta}$ and $a' \in \overline{Q}(a)$. Let $N(a, a') = J$. The definition of $\mathcal{A}^{*,\delta}$ implies that $|J| \geq \overline{n} + 1$. Define an open half space $V_{a'} = \{x \in \mathbb{R}^h : d(x, a) > d(x, a')\}$. It must hold that $\mathfrak{C}(J) = hull(J) \subset V_{a'}$. Otherwise, some player of $N(a, a')$ must prefer a to a', which contradicts the definition of $N(a, a')$. Then,

$$\mathfrak{C}^* = \bigcap_{J' \in N_{\overline{n}+1}} \mathfrak{C}(J') \subseteq \bigcap_{J'' \in N_{|J|}} \mathfrak{C}(J'') \subseteq \mathfrak{C}(J) \subset V_{a'}.$$

This proves the claim that $d(a, \mathfrak{C}^*) > d(a', \mathfrak{C}^*)$. □

Proof of Lemma 7. Note that sequentially choosing $a_{i+1} \in \overline{Q}(a_i)$ must result in a cycle due to the finiteness of $\mathcal{A}^{\delta}$. Let $\{a_1, a_2, \ldots, a_L\}$ denote such a cyclic sequence of alternatives, that is, $a_{i+1} \in \overline{Q}(a_i)$ for all $i \in \{1, \ldots, L\}$ with a convention that $a_{L+1} = a_1$. We show that such a cycle must include $a_i \in \mathcal{A}^{*,\delta}$ for some i.

By way of contradiction, suppose that there exists a sequence $\{a_1, a_2, \ldots, a_L\}$ such that $a_i \notin \mathcal{A}^{*,\delta}$ for all i. Then, Lemma 12 implies that $d(a_i, \mathfrak{C}^*) > d(a_{i+1}, \mathfrak{C}^*)$ for all $i \in \{1, \ldots, L\}$, i.e., the distance between a_i and $\mathfrak{C}^*$ is strictly decreasing as the sequence $\{a_1, a_2, \ldots\}$ progresses. Since $d(a_1, \mathfrak{C}^*) > \ldots > d(a_{L+1}, \mathfrak{C}^*)$ implies that $a_1 \neq a_{L+1}$, this contradicts that the sequence is cyclic.

Note that the distance increases, $d(a_i, \mathfrak{C}^*) \leq d(a_{i+1}, \mathfrak{C}^*)$, only if $a_i \in \mathcal{A}^{*,\delta}$. By sequentially choosing $a_{i+1} \in \overline{Q}(a_i)$, the process must reach some $a_i \in \mathcal{A}^{*,\delta}$. □

We first prove Lemmas 13 and 14 below in order to prove Proposition 5.

Lemma 13. *Let a_1 and a_2 be an arbitrary pair of alternatives with $a_1 < a_2$. Let $a_0 = a_1 - \delta$ and $a_3 = a_2 + \delta$. Then*

$$c^q_{a_1 a_2} > c^q_{a_1 a_0} \quad \text{if } a^{**} < a_1 < a_2, \tag{21}$$

$$c^q_{a_2 a_1} > c^q_{a_2 a_3} \quad \text{if } a_1 < a_2 < a^*. \tag{22}$$

Proof. Let $a_{jk} = (a_j + a_k)/2$. For the inequality (21), let

$Y_1 = \{i \in N : s_{\underline{q}} < s_i \leq a_{01}\}$, $Y_2 = \{i \in N : a_{01} < s_i \leq a_1\}$, $Y_3 = \{i \in N : a_1 < s_i \leq a_{12}\}$, $Y_4 = \{i \in N : a_{12} < s_i\}$.

Note that some of sets above may be empty. Let $y_i = |Y_i|$ and $Y_i' = \{i \in Y_i : i \leq q\}$. Observe that

$$c^q_{a_1 a_2} = \left[q - \underline{q} + 1 + \sum_{1 \leq j \leq 2} y_j \right] (a_2 - a_1) + \sum_{i \in Y_3} (a_1 + a_2 - 2s_i),$$
$$c^q_{a_1 a_0} = \sum_{i \in Y_2'} [2s_i - (a_0 + a_1)] + \sum_{3 \leq j \leq 4} y_j' \delta \leq \sum_{2 \leq j \leq 4} y_j' \delta.$$

The cost of voting for a_2 is $a_2 - a_1$ for any player i with $s_i \leq a_1$, and $a_1 + a_2 - 2s_i$ for $i \in Y_3$. For a_2 to obtain q votes, it needs votes of $(q - \underline{q} + 1)$ players with $s_i \leq s_{\underline{q}}$ and votes of players in $Y_1 \cup \ldots \cup Y_4$.[46] This observation gives $c^q_{a_1 a_2}$ above. For $c^q_{a_1 a_0}$, the cost of voting for a_0 is zero for i with $s_i \leq a_{01}$, $2s_i - (a_0 + a_1)$ for $i \in Y_2$, and $(a_1 - a_0) = \delta$ for $i \in Y_3 \cup Y_4$.

[46] There are other cases of obtaining q votes. The described one is a minimum-cost one.

Observe that

$$c^q_{a_1a_2} \geq (q - \underline{q} + 1 + \sum_{1\leq j\leq 3} y_j)(a_2 - a_1) \geq (q - \underline{q} + 1 + \sum_{1\leq j\leq 3} y_j)\delta.$$

Note that $s_{\underline{q}} < s_i \leq q$ for all $i \in Y'_2 \cup Y'_3 \cup Y'_4$. This implies that $y'_2 + y'_3 + y'_4 \leq q - \underline{q}$. Thus, the inequality (21) holds. We can prove (22) similarly. □

Lemma 14. *Let $\{a_0, a_3\}$ be an arbitrary pair of alternatives with $a_0 < a_3$. Let $a_1 = a_0 + \delta$ and $a_2 = a_3 - \delta$. Then,*

$$c^q_{a_3a_0} \geq c^q_{a_3a_2}, \tag{23}$$
$$c^q_{a_0a_3} \geq c^q_{a_0a_1}. \tag{24}$$

Proof. We prove the inequality (23). If $a_0 = a_2$, then it obviously holds. So suppose that $a_0 < a_2$. Let $a_{jk} = (a_j + a_k)/2$. Also let

$$Y_1 = \{i \in N : a_{03} < s_i \leq a_{23}\}, \quad Y_2 = \{i \in N : a_{23} < s_i \leq a_3\}, \quad Y_3 = \{i \in N : a_3 < s_i\}.$$

Let $y_i = |Y_i|$, $Y'_i = \{i \in Y_i : i \leq q\}$. Observe that

$$c^q_{a_3a_0} = \sum_{i\in Y'_1\cup Y'_2} [2s_i - (a_0 + a_3)] + y'_3(a_3 - a_0), \qquad c^q_{a_3a_2} = \sum_{i\in Y'_2} [2s_i - (a_2 + a_3)] + y'_3\delta.$$

As for $c^q_{a_3a_0}$, the cost of voting for a_0 is zero for any player i with $s_i \leq a_{03}$, that is $2s_i - (a_0 + a_3)$ for $i \in Y_1 \cup Y_2$, and that is $a_3 - a_0$ for $i \in Y_3$. For a_0 to obtain q votes, it needs votes of $Y'_1 \cup \ldots \cup Y'_3$.[47] This gives $c^q_{a_3a_0}$ above. For $c^q_{a_3a_2}$, the cost of voting for a_2 is zero for i with $s_i \leq a_{23}$, $2s_i - (a_2 + a_3)$ for $i \in Y_2$, and $(a_3 - a_2) = \delta$ for $i \in Y_3$. Since $a_0 < a_2$ and $a_3 - a_0 > \delta$, it holds that $c^q_{a_3a_0} \geq c^q_{a_3a_2}$. We can prove (24) similarly. □

Proof of Proposition 5. Suppose the contrary, that is, there exists $\hat{a} \notin \{a^*, a^{**}\}$ which is stochastically stable. Suppose the case that $\hat{a} < a^*$. Let $\tau_{\hat{a}}$ denote an $\hat{a}$-tree which minimizes the cost of $\hat{a}$-trees. Let a_1 and a_2 be such that (a^*, a_1), $(a^{**}, a_2) \in \tau_{\hat{a}}$.

For all $a < a^*$ with $a \neq \hat{a}$, replace the edge $(a, \cdot) \in \tau_a$ with $(a, a + \delta)$. Let τ_1 denote the resulting set of edges. τ_1 may not be a tree, but observe that, for any $a \notin \{\hat{a}, a^*\}$ τ_1 has a path from a to either $\hat{a}$ or a^*. Lemmas 13 and 14 imply that $c_q(\tau_1) = \sum_{(v,w)\in\tau_1} c^q_{vw} \leq c_q(\tau_{\hat{a}})$. If $a_1 = a^{**}$ with $s_{\underline{q}} - a^* > \delta/2$, then replace (a^{**}, a_2) with $(\hat{a}, \hat{a} + \delta)$ in τ_1. Let τ_2

[47] This is a minimum-cost case since the cost of voting for a_0 is smaller for players with smaller s_i than that for those with greater s_i.

denote the resulting set. Otherwise, replace (a^*, a_1) with $(\hat{a}, \hat{a}+\delta)$ in τ_1. Let τ_3 denote the resulting set of edges. τ_2 must be an a^{**}-tree, while τ_3 must be an a^*-tree.

We will show a contradiction, that is, τ_2 or τ_3 will have a strictly smaller cost than $\tau_{\hat{a}}$. First, we consider τ_2. Either $a_2 < a^*$ or $a_2 > a^{**}$ must hold since $a_1 = a^{**}$. Observe that

$$c^q_{a^{**}a_2} > (q - \underline{q} + 1)\delta, \qquad\qquad c^q_{\hat{a}\hat{a}+\delta} \leq (q - \underline{q})\delta.$$

If $a_2 < a^*$, then $s_{\underline{q}} - a_2 > \frac{3}{2}\delta$. Recall that $a^{**} - s_{\underline{q}} < \delta/2$. For any player i with $i \geq \underline{q}$, her cost of transition (a^{**}, a_2) is given by $(a_2 - s_i) - |a^{**} - s_i| > \delta$. If $a_2 > a^{**}$, then $a_2 - a^{**} \geq \delta$. For any player i with $i \leq \underline{q}$, her cost of the transition will be greater than δ. Those observations imply the first inequality. The second inequality comes from that ideal points of at least $\underline{q}$ players are greater than $\hat{a}+\delta$. Then, we have that $c^*_q(a^{**}) \leq c_q(\tau_2) < c_q(\tau_{\hat{a}})$. A contradiction.

Next, we consider τ_3. If $a_1 < a^*$, then

$$c^q_{a^*a_1} \geq (q - \underline{q} + 1)\delta, \qquad\qquad c^q_{\hat{a}\hat{a}+\delta} \leq (q - \underline{q})\delta.$$

The first inequality comes from that ideal points of at least $\underline{q}$ players are greater than a^* since $a^* \leq s_{\underline{q}}$. Similarly, the second one comes from that ideal points of at least $\underline{q}$ players are greater than $\hat{a}+\delta$ since $\hat{a} < \hat{a}+\delta \leq a^* \leq s_{\underline{q}}$. This implies that $c^*_q(a^*) \leq c_q(\tau_3) < c_q(\tau_1) \leq c_q(\tau_{\hat{a}})$. A contradiction.

For $a_1 > a^*$, note that the $\underline{q}$-th player prefers a^* to a_1. If $a_1 \neq a^{**}$, then $a_1 - s_{\underline{q}} > \delta$. If $a_1 = a^{**}$ with $s_{\underline{q}} - a^* \leq \delta/2$, then $a_1 - s_{\underline{q}} > \delta/2$. Thus, player $\underline{q}$ prefers a^*. Let

$$Y_1 = \{i \in N : s_i \leq a^*\}, \quad Y_2 = \{i \in N : a^* < s_i \leq a_{1*}\}, \quad Y_3 = \{i \in N : a_{1*} < s_i\},$$

where $a_{1*} = (a_1 + a^*)/2$. Let $y_i = |Y_i|$ and $Y'_i = \{i \in Y_i : i \geq n+1-q\}$. Then, observe that

$$c^q_{a^*a_1} = y'_1\delta + \sum_{i \in Y'_2}[a_1 + a^* - 2s_i], \qquad\qquad c^q_{\hat{a}\hat{a}+\delta} \leq y'_1\delta.$$

Note that $c^q_{\hat{a}\hat{a}+\delta}$ has no summand over Y'_2 since players in Y_2 prefer $\hat{a}+\delta$ to $\hat{a}$. Also observe that the $\underline{q}$-th player is always in Y'_2, i.e., $y'_2 \geq 1$. This is because $a^* < s_{\underline{q}} \leq a^* + \delta/2$ implies that $a^* < s_{\underline{q}} \leq a_{1*}$. Thus $c^q_{a^*a_1} > c^q_{\hat{a}\hat{a}+\delta}$. Then, $c^*_q(a^*) < c_q(\tau_{\hat{a}})$. A contradiction.

We omit the proof for the case that $\hat{a} > a^{**}$ since it is very similar. □

Section 7

Proof of Theorem 8. (A) and (N) are obvious. (P) and (SS) are implied by Corollary 1 and Theorem 1, respectively.
proof for (M):
The monotonicity is implied by Theorem 5. Suppose that the profile changes in the way described in Definition 14. Let c^n_{xy} and $\mathbf{c}^n_{xy}$ denote the cost of (x, y) for the original profile and the changed one, respectively. Also $\tilde{c}_i(x, y)$ and $\tilde{\mathbf{c}}_i(x, y)$ denote the cost defined in Theorem 5 for the two profiles, respectively. Observe that under the changed profile, it is hard to move away from a and it is easier to get into a, that is, $c^n_{ab} \leq \mathbf{c}^n_{ab}$ and $c^n_{ba} \geq \mathbf{c}^n_{ba}$ for all $b \in \mathcal{A}$. Then, this implies that

$$\tilde{c}_0(a, b) \leq \tilde{\mathbf{c}}_0(a, b), \quad \tilde{c}_0(b, a) \geq \tilde{\mathbf{c}}_0(b, a) \qquad\qquad \forall b \in \mathcal{A}_{\underline{q}}.$$

With the costs above in hand, the algorithm in Theorem 5 implies that if a is V^i-recurrent under the original profile, it must be so under the changed one. And if a is stochastically stable, then it must be so under the changed one.

proof for (S):
Recall that $\underline{q} = \lfloor (n+1)/2 \rfloor$ and that $\mathcal{A}_{\underline{q}}$ is the unique recurrent class under $\underline{q}$-quota rule. We show that $\mathcal{A}_{\underline{q}} = B_1$, where B_1 is defined in Definition 15. Then, the claim will be implied by Lemma 4, which shows that stochastically stable alternatives for $q \geq \underline{q}$ must be in $\mathcal{A}_{\underline{q}}$.

For odd n, it is clear that $\mathcal{A}_{\underline{q}} = B_1$ since the strict majority is $\underline{q}$. For even n, the strict majority is $q = \underline{q} + 1$. The definition of the recurrent class implies that, if $n(a, a') \geq n/2$ for some $a \in \mathcal{A}_{\underline{q}}$, then such a' must be $\mathcal{A}_{\underline{q}}$. This further implies that, for all $a \in \mathcal{A}_{\underline{q}}$ and $a' \in \mathcal{A} \setminus \mathcal{A}_{\underline{q}}$, $n(a, a') < n/2$ and $n(a', a) = n - n(a, a') \geq n/2 + 1$. Thus, $\mathcal{A}_{\underline{q}} = B_1$.

proof for (IC):
For $\hat{c}_0 \in \mathcal{A}$, let $\hat{A} = \{\hat{c}_1, \ldots, \hat{c}_K\}$ be a set of clones of $\hat{c}_0$. Let $\hat{\mathcal{A}} = \mathcal{A} \cup \hat{A}$, i.e. a set generated by adding clones to $\mathcal{A}$. $c_n(\cdot)$ and $\hat{c}_n(\cdot)$ denote costs on $\mathcal{A}$ and on $\hat{\mathcal{A}}$, respectively. Let τ^*_a and $\hat{\tau}^*_a$ be the minimum cost a-trees on $\mathcal{A}$ and on $\hat{\mathcal{A}}$, respectively. Let $\hat{c}_h \in \hat{A} \cup \{\hat{c}_0\}$ be such that the path $d(\hat{c}_h, a) \subset \hat{\tau}^*_a$ does not include any edge having other clones, i.e., $x \notin \hat{A} \cup \{\hat{c}_0\}$ for all $(\cdot, x) \in d(\hat{c}_h, a)$. Let a_0 be such that $(\hat{c}_0, a_0) \in \tau^*_a$, and $a_h \in \mathcal{A}$ such that $(\hat{c}_h, a_h) \in \hat{\tau}^*_a$. In the proof, we write "$k \neq h$" for "$k \in \{0, \ldots, h-1, h+1, \ldots, K\}$".

We will show that $\hat{c}_n(\hat{\tau}^*_a) - c_n(\tau^*_a)$ is constant for all $a \in \mathcal{A} \setminus \{\hat{c}_0\}$ and that a similar property holds for clones. Then, it will imply that a minimizes the cost on $\mathcal{A}$ if and only if a or its clone minimizes the cost on $\hat{\mathcal{A}}$.

Firstly, we show that by a way of contradiction,

$$\hat{c}_n(\hat{\tau}_a^*) - c_n(\tau_a^*) = \sum_{(\hat{c}_k,\cdot)\in\hat{\tau}_a^*:k\neq h} c^n_{\hat{c}_k,\cdot} \qquad \forall a \in \mathcal{A}\setminus\{\hat{c}_0\}. \tag{25}$$

Suppose that $\hat{c}_n(\hat{\tau}_a^*) - c_n(\tau_a^*) > \sum_{(\hat{c}_k,\cdot)\in\hat{\tau}_a^*:k\neq h} c^n_{\hat{c}_k,\cdot}$. Add to τ_a^* edge $(\hat{c}_k,\cdot) \in \hat{\tau}_a^*$ for $k \neq h$. Replace $(\hat{c}_0, a_0) \in \tau_a^*$ with $(\hat{c}_h, a_0)$. This replacement will not increase the cost since $c^n_{\hat{c}_0,a_0} = c^n_{\hat{c}_h,a_0}$. The resulting set must be an a-tree on $\hat{\mathcal{A}}$, say $\hat{\tau}_a$. Observe that

$$\hat{c}_n(\hat{\tau}_a) = c_n(\tau_a^*) - c^n_{\hat{c}_0,a_0} + c^n_{\hat{c}_h,a_0} + \sum_{(\hat{c}_k,\cdot)\in\hat{\tau}_a^*:k\neq h} c^n_{\hat{c}_k,\cdot} = c_n(\tau_a^*) + \sum_{(\hat{c}_k,\cdot)\in\hat{\tau}_a^*:k\neq h} c^n_{\hat{c}_k,\cdot} < \hat{c}_n(\hat{\tau}_a^*).$$

This contradicts that $\hat{\tau}_a^*$ minimizes the cost of a on $\hat{\mathcal{A}}$. Next suppose that $\hat{c}_n(\hat{\tau}_a^*) - c_n(\tau_a^*) < \sum_{(\hat{c}_k,\cdot)\in\hat{\tau}_a^*:k\neq h} c^n_{\hat{c}_k,\cdot}$. Remove from $\hat{\tau}_a^*$ edges $(\hat{c}_k,\cdot) \in \hat{\tau}_a^*$ for $k \neq h$. Replace $(\hat{c}_h, a_h) \in \hat{\tau}_a^*$ with $(\hat{c}_0, a_h)$. Also replace edges $(y, \hat{c}_k) \in \hat{\tau}_a^*$, if any, with $(y, \hat{c}_0)$. The resulting set of edges must be an a-tree on $\mathcal{A}$, say τ_a. Observe that

$$c_n(\tau_a) = \hat{c}_n(\hat{\tau}_a^*) - \sum_{(\hat{c}_k,\cdot)\in\hat{\tau}_a^*:k\neq h} c^n_{\hat{c}_k,\cdot} < c_n(\tau_a^*).$$

This contradicts that τ_a^* minimizes the cost of a on $\mathcal{A}$.

Secondly, we show that $\hat{c}_n(\hat{\tau}_a^*) - c_n(\tau_a^*) = \hat{c}_n(\hat{\tau}_b^*) - c_n(\tau_b^*)$ for all $a, b \in \mathcal{A}\setminus\{\hat{c}_0\}$. Suppose that $\hat{c}_n(\hat{\tau}_a^*) - c_n(\tau_a^*) < \hat{c}_n(\hat{\tau}_b^*) - c_n(\tau_b^*)$. Let b_0 be such that $(\hat{c}_0, b_0) \in \tau_b^*$. Add to τ_b^* edges $(\hat{c}_k,\cdot) \in \hat{\tau}_a^*$ for $k \neq h$.[48] Replace $(\hat{c}_0, b_0) \in \tau_b^*$ with $(\hat{c}_h, b_0)$. The resulting set of edges must be a b-tree on $\hat{\mathcal{A}}$.[49] Let $\hat{\tau}_b$ denote it. Observe that

$$\hat{c}_n(\hat{\tau}_b) = c_n(\tau_b^*) + \sum_{(\hat{c}_k,\cdot)\in\hat{\tau}_a^*:k\neq h} c^n_{\hat{c}_k,\cdot} < c_n(\tau_b^*) + \sum_{(\hat{c}_k,\cdot)\in\hat{\tau}_b^*:k\neq h} c^n_{\hat{c}_k,\cdot} = \hat{c}_n(\hat{\tau}_b^*).$$

The last equality comes from (25). This contradicts that $\hat{\tau}_b^*$ minimizes the cost of b on $\hat{\mathcal{A}}$. Since the choice of a and b is arbitrary, this implies that $\hat{c}_n(\hat{\tau}_a^*) - c_n(\tau_a^*) = \hat{c}_n(\hat{\tau}_b^*) - c_n(\tau_b^*)$ for all $a, b \in \mathcal{A}\setminus\{\hat{c}_0\}$.

Finally, we consider trees of clones. Let $\hat{\tau}_k^*$ denote the minimum cost tree of $\hat{c}_k \in \hat{A}\cup\{\hat{c}_0\}$ on $\hat{\mathcal{A}}$. Let τ_0^* denote that of $\hat{c}_0$ on $\mathcal{A}$. Then, we have that $\hat{c}_n(\hat{\tau}_h^*) - c_n(\tau_0^*) =$

[48] Recall that h is such that the path $d(\hat{c}_h, a) \subset \hat{\tau}_a^*$ has no clone other than $\hat{c}_h$.

[49] For $\hat{c}_k$ with $k \neq h$, the set of edges must have a path from $\hat{c}_k$ to $\hat{c}_h$ or a path from $\hat{c}_k$ to some $a' \in \mathcal{A}$. Since τ_b is a tree, there must be paths from b_0 to b and from a' to b. Thus, $\hat{\tau}_b$ is a b-tree on $\hat{\mathcal{A}}$.

$\hat{c}_n(\hat{\tau}_a^*) - c_n(\tau_a^*)$ for h and all $a \in \mathcal{A} \setminus \{\hat{c}_0\}$.[50] To see this, observe that the arguments above still apply by removing the two operations: "replace $(\hat{c}_h, a_h) \in \hat{\tau}_a^*$ with $(\hat{c}_0, a_h)$" and "replace $(\hat{c}_0, b_0) \in \tau_b^*$ with $(\hat{c}_h, b_0)$". Since $\hat{c}_h$ is the root of the tree $\hat{\tau}_h^*$. The two operations in the previous arguments are not necessary. For $k \neq h$, we can similarly show that $\hat{c}_n(\hat{\tau}_k^*) - c_n(\tau_0^*) \geq \hat{c}_n(\hat{\tau}_a^*) - c_n(\tau_a^*)$.

That $\hat{c}_n(\hat{\tau}_a^*) - c_n(\tau_a^*) = \hat{c}_n(\hat{\tau}_b^*) - c_n(\tau_b^*)$ for all $a, b \in \mathcal{A} \setminus \{\hat{c}_0\}$ and that a similar equality holds for clone alternatives imply that an alternative minimizes the cost on $\mathcal{A}$ if and only if the alternative or its clone minimizes the cost on $\hat{\mathcal{A}}$. □

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[50] h is such that $d(\hat{c}_h, a) \subset \hat{\tau}_a^*$ has no clone other than $\hat{c}_h$. The equality holds for all such h.

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