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“Bayesian Updating for Complementarily Additive Beliefs under Ambiguity”

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Bayesian Updating for Complementarily Additive Beliefs under Ambiguity

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Abstract

This paper proposes a formal characterization of extended Bayesian updating for complementarily additive subjective beliefs under ambiguity, which are compatible with a wide range of choice behavior toward ambiguity. The main result shows that, based on the biseparability of Ghirardato and Marinacci (2001), extended Bayesian updating characterizes the update rule which is a step-by-step composite updating for priors, where one of Dempster-Shafer rule, Bayes' update rule and Fagin-Halpern rule is applied to each step. As applications, more specific preference relations are examined, such as the maxmin expected utility, the rank-dependent expected utility, and the concave expected utility preferences by Lehrer (2009).

JEL Classification: D81

Keywords: ambiguous belief, Bayesian update rule, multiple priors, non-additive measure, subjective probability, biseparable preference

1 Introduction

In most subjectively ambiguous situations, the primitive choices are concerned with mutually complementary two events associated to a gain or loss, the more or less, the better or worse, and so forth. This paper deals with how these subjective probabilities of *winning* or *losing* events are updated after additional information arrived in a simple dynamic decision setting.

Although most experimental studies rely on choices among two-outcome alternatives, called *binary acts*, such alternatives successfully extracted a lot about various patterns of choice behavior such as common ratio/consequence effects or ambiguity aversion. However, seminal works by Machina (2009, 2014) and Baillon et. al (2011) demonstrated examples which many decision models cannot accommodate with when alternatives include *three or more outcomes*. It motivates this paper to focus on an axiomatization on preferences

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over binary acts, which encompass a wide range of choice behavior toward ambiguity on alternatives with more various outcomes.

This paper characterizes an extended Bayesian updating for complementarily additive subjective beliefs based on preferences over binary acts. As a utility representation, I adopt a class of *biseparable preferences* formulated by Ghirardato and Marinacci (2001) (hereafter GM2001), which is also a purely subjective binary version of the rank-dependent expected utility (RDU) in Nakamura (1990), imposing independence axiom only on binary comonotonic acts. The biseparability implies that the winning and losing probabilities are uniquely determined for all events and complementarily additive. It imposes less restrictions on preferences over alternatives with three or more outcomes which are vulnerable to Machina's examples, although most utility representations share this complementarily additive property.

The main result characterizes the essence of Bayesian updating consistent with a broad range of preferences, including various ambiguity averse preferences. To be concrete, Theorem 1 proves that biseparable preferences through weak Bayesian updating generate any composite update rule for priors, which surprisingly comprises only three update rules, Dempster-Shafer rule (Dempster, 1967; Shafer,1976), Bayes' update rule (Gilboa and Schmeidler, 1993; Denneberg, 1994), and Fagin-Halpern rule (Dempster, 1967; Fagin and Halpern, 1991; Walley, 1991; Jaffray, 1992).

The result remarks three features. (1) Theoretically, conventional results are based on a particular utility representation of the unconditional preferences, or intended to axiomatize model-specific updating manners. However, Theorem 1 implies that the weak Bayesian requirement on binary acts determines the *bases* for updating, which consist of the three major update rules developed in the theory of belief functions. Expanded into fairly broad range of preferences to maintain biseparability, only three rules are supported, conversely, any update rules other than these three rules are never compatible with any biseparable representations and weak Bayesian property.

(2) Not only the three rules were conventional, prominent and prevalent rules in the theory of belief functions, but also they are consistent with biseparable preferences developed in the decision theory, although the biseparability delivers less information about preferences over acts with more than three outcomes. The core nature of updating is maintained even if more axioms are imposed on the unconditional preference relation. The result is easily applied to more specific utility representations, by adding extra axioms on the unconditional preference to generate the more specific updating formula.

(3) In view of experimental design, this result gives a solid foundation to assume and concentrate on three update rules for subjects' priors from the beginning, and also provides a definite method to calculate posteriors. Moreover, in the experiments consisting of two-outcome alternatives, the result implies that conditional choices are behaviorally explained by the proper selection of a particular utility expression and the belief updating formula. It is also applied to any experiment concerning or including belief updating through a strategic interaction, such as auction, signaling, learning and so on.

This paper is organized as follows. The next section begins with illustrative examples and provides the basic definitions and a decision setting for updating under ambiguity. Section 3 begins with introducing axioms on the unconditional and conditional preferences and Bayesian updating. The main characterization result is presented in sequence. In Section 4, various utility representations, the maxmin expected utility preferences (MEU;

Gilboa and Schmeidler, 1989), the rank-dependent preferences (Nakamura , 1990), the concave expected utility preferences (LEU; Lehrer, 2009) are examined as applications of the main theorem. The last section discusses the main results, related works, applications, and further extensions.

2 The decision setting

2.1 Illustrative example

In the following epitome of a dynamic version of Ellsberg urn (Ellsberg, 1961), there are a red ball and two black or yellow balls, not informed about those composition.

	red	black	yellow
f_1	\$100	\$0	\$0
f_2	\$0	\$100	\$0
f_3	\$100	\$0	\$100
f_4	\$0	\$100	\$100

Table 1

Subjects who display $f_1 \succ f_2$ and $f_4 \succ f_3$ are supposed to be ambiguous averse. After observed the chosen ball was red or black, which color is preferred to bet 100 dollars on? The choice displays the pattern how each winning probability is updated after knowing red or black. In fact, in the pioneer experiment of Cohen et. al (2000), almost two third is compatible with the full Bayesian (FB) updating rather than the maximum likelihood updating. Dominiak et. al (2012) confirmed that more than 80 percent of ambiguity averse subjects' choices are consistent with FB rule, as well as more consequentialism rather than dynamic consistency.

To illustrate a dynamic decision setting with belief updating, consider a more elaborated example as following. There are six dice with one to six spots on every (six) face, although every number of spots is not informed. However, it is informed that, there are at least one face that has one to six each, out of 36 faces in total. For instance, there are a face with one spot, a face with two spots, ... , a face with five spots and thirty one faces with six spots, and so on. Now pick one of the dice and throw it once. According to the number of pips on the die, construct the following acts. The state space is $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $X = [0, 100]$. Consider four alternatives over six states in Table 2.

	$\{1\}$	$\{2\}$	$\{3\}$	$\{4, 5, 6\}$
f_1	\$90	\$0	\$0	\$0
f_2	\$10	\$10	\$10	\$0
f_3	\$90	\$0	\$0	\$90
f_4	\$10	\$10	\$10	\$90

Table 2

There are four alternatives, one of which f_1 gives \$90 if the die shows 1 and \$0 otherwise, and so on. The Sure-thing Principle requires that the order between f_1 and f_2 has to be the same as between f_3 and f_4.

Although there seems to be more choice patterns conformed to this situation, let us focus on a decision maker's choice: $f_1 \sim f_2$ and $f_3 \succ f_4$. The choices also imply that adding \$1 to \$10 in f_2 induces $f_1 \prec f_2$, however it cannot alter $f_3 \succ f_4$ despite adding \$1 to \$10 in f_4. This $\succsim$ violates the Sure-thing Principle.

These choices are supported to be represented by for all x, $y \in X$, $x \succsim y$

$$V\left(x_A y\right) = \nu\left(A\right) x + \left[1 - \nu\left(A\right)\right] y,$$

where the subjectively ambiguous situation is summarized in a set function (Horie, 2013)

$$\nu\left(A\right) = \frac{|A|^2}{36} \text{ for every } A \subset \Omega.$$

All alternatives f_1 to f_4 include only two outcomes, $(\$90, \$0)$, $(\$10, \$0)$, or $(\$90, \$10)$. Given such an alternative, the winning event is associated to better outcome, for example, winning event of f_1 is $\{1\}$ which gives \$90. $\nu\left(A\right)$ represents the subjective probability of winning event A.

Now suppose that event $E = \{1, 2, 3\}$ is observed. After knowing that the die showed 1, 2, or 3, which is preferred to, f_1 or f_2? For illustration, we compute the posterior winning probability for every event through three pattern of updating, Dempster-Shafer (DS) rule, Bayes' update (BU) rule, and Fagin-Halpern (FH) rule. (Detailed calculation methods of three rules are presented in Appendix A.)

	ν_E^{FH}	ν_E^{BU}	ν_E^{DS}
$\|A\| = 1$	1/21	1/9	7/27
$\|A\| = 2$	4/15	4/9	16/27
$\|A\| = 3$	1	1	1

Table 3

There are three patterns of choices between f_1 and f_2, which are represented by these winning probabilities. There are three patterns, so write $\succsim_E^{FH}$, $\succsim_E^{BU}$, and $\succsim_E^{DS}$. $V_E^{FH}\left(f_1\right) = 30/7$, $V_E^{BU}\left(f_1\right) = 10$, and $V_E^{DS}\left(f_1\right) = 70/3$, hence $f_2 \succ_E^{FH} f_1$, $f_1 \sim_E^{BU} f_2$, and $f_1 \succ_E^{DS} f_2$. Each rule ends up with the different conditional choice.

	V_E^{FH}	V_E^{BU}	V_E^{DS}
f_1	30/7	10	70/3
f_2	10	10	10

Table 4

There are three patterns of conditional choices, $f_1 \succ_E f_2$, $f_1 \sim_E f_2$ and $f_1 \prec_E f_2$, so one might think it is enough for us to capture these orders. Thus it intuitively seems sufficient to have three rules, DS, BU and FH rules in a representation. In fact this conjecture is to be proven valid in subsequent sections.

2.2 States, acts, and preferences

Let Ω be a finite *set of states* with $|\Omega| = n$ and $\Sigma = 2^{\Omega}$. A non-empty set in Σ is called an *event*. Let X be a set of *consequences*, or *outcomes* which is assumed to be a connected and separable topological space. A measurable mapping $f : \Omega \to X$ is called an *act*. Denote the set of all acts by $\mathcal{F}$. For the sake of simplicity, an element x in X also indicates a *constant act* which assigns x for all $\omega \in \Omega$. $f_A g$ denotes the act which yields $f(\omega)$ if $\omega \in A$ and $g(\omega)$ if $\omega \in A^c$. Throughout this paper $A^c \in \Sigma$ is the complement of A with respect to Ω. A two-outcome act $x_A y$ is called a *binary act* if $x, y \in X$ and $A \in \Sigma$. Let $\mathcal{F}_2$ be the set of all binary acts. Note that $X \subset \mathcal{F}_2$.

It is concerned with preference relations on $\mathcal{F}$ before and after an event $E \in \Sigma$ is realized. Given an event $E \in \Sigma$, a binary relation $\succsim_E$ is called a *conditional preference relation given* E. For every event $E \in \Sigma$, as usual, $\succ_E$ and $\sim_E$ refer to asymmetric and symmetric parts of $\succsim_E$ respectively. When $E = \Omega$, $\succsim_\Omega$ is interpreted as the *unconditional preference relation* and we simply express it as $\succsim$. The preference relation on X is invoked from $\succsim_E$ on constant acts.

Let μ express a finite monotone set function $\mu : \Sigma \to [0,1]$ satisfying (i) $\mu(\varnothing) = 0$ and $\mu(\Omega) = 1$, and (ii) for all A, $B \in \Sigma$ with $A \subset B$, we have $\mu(A) \leqq \mu(B)$. μ is also called a capacity. Let Δ be the set of all finite monotone set function μ on Ω. μ is *superadditive* if for all A, $B \in \Sigma$ with $A \cap B = \varnothing$, $\mu(A \cup B) \geqq \mu(A) + \mu(B)$. μ is said to be *convex* if for all A, $B \in \Sigma$, $\mu(A \cup B) + \mu(A \cap B) \geqq \mu(A) + \mu(B)$. If the inequality holds in equality for all A and B in Σ, μ is called *additive*. Let Δ^0 be the set of additive set functions on Ω.

Given an event E, μ_E is the *conditional*, or *updated set function of* μ *given* E, i.e. for all $A \in \Sigma$ with $A \cap E = E$, $\mu_E(A \cap E) = 1$. Note that for any event $E \in \Sigma$, μ_E has domain Σ. When $E = \Omega$, μ_Ω is interpreted as *unconditional* and we simply write it μ. As above, when μ_E is additive, write it as p_E, which is also belong to Δ^0.

We are concerned with an update rule of μ which is a transformation of μ to generate the conditional set function of μ given E as a posterior. Formally, a mapping $\Phi : \Delta \times \Sigma \to \Delta$ is called an *update rule of* μ if it maps μ and a conditioning event E to μ_E. Note that $\Phi(\mu|E)$ is the updated set function of μ given E, which can be also written as μ_E in terms of μ.

It is possible to construct an update of μ compositely event by event through different manners. Fix a μ, a conditioning event E and a partition of E^c, $(T_k, T_{k-1}, \ldots, T_1)$. Let us consider the step-by-step updating which applies an update rule Φ^i to every conditioning event $E^i = \Omega \backslash (T_k \cup \cdots \cup T_i)$. Denote $\Phi^i(\mu)$ in place of $\Phi^i(\mu|E^i)$ if the conditioning event E^i is explicitly given.

Definition 1 *A conditional monotone set function given E, μ_E is a composite update of μ if, there exists a partition of E^c, $(T_k, T_{k-1}, \ldots, T_1)$, $k \leqq |E^c|$,*

$$\mu_E(A) = \Phi^k \circ \Phi^{k-1} \circ \cdots \circ \Phi^1(\mu)(A) \text{ for all } A \subset E,$$

where every Φ^i is an update rule given $E^i = \Omega \backslash (T_k \cup \cdots \cup T_i)$, $i = 1, \ldots, k$.

Subjective beliefs are sometimes represented in the form of a *set of priors*, a nonempty closed and convex set $C \subset \Delta^0$, thus we deal with an update rule for a prior set. Fix a C, a conditioning event E, and a partition of E^c, $(T_k, T_{k-1}, \ldots, T_1)$. The step-by-step updating which applies an update rule Φ^i to every conditioning event $E^i = \Omega \backslash (T_k \cup \cdots \cup T_i)$. Denote $\Phi^i(p)$ in place of $\Phi^i(p|E^i)$ if the conditioning event E^i is explicitly given.

Definition 2 *A $C_E \subset \Delta^0$ is a composite update of C if, there exists a partition of E^c, $(T_k, T_{k-1}, \ldots, T_1)$, $k \leqq |E^c|$, for every $i = 1, \ldots, k$, $C_i = \{p_i \in \Delta^0 \mid p \in C\}$ such that*

$$p_i(A) = \Phi^i \circ \Phi^{i-1} \circ \cdots \circ \Phi^1(p)(A) \text{ for all } A \subset E^i.$$

3 Characterization

3.1 Axioms

3.1.1 Axioms for unconditional preference

This section presents axioms for unconditional preferences, which are mainly standard necessary and sufficient conditions for a certain class of utility representations. Therefore, most axioms are indebted to preceding representation results to be clarified here. Notice that all axioms are subjectively and behaviorally described to conform to the Savage-style framework.

First, throughout this paper, it is assumed that every conditional preference exists as a weak order on $\mathcal{F}$.

A0 (Weak order) A preference relation $\succsim_A$ given $A \in \Sigma$ is a weak order on $\mathcal{F}$: (i) for all f, $g \in \mathcal{F}$, $f \succsim_A g$ or $g \succsim_A f$, (ii) for all f, g, $h \in \mathcal{F}$, if $f \succsim_A g$ and $g \succsim_A h$, then $f \succsim_A g$.

The existence of conditional preference relations as a weak order is thoroughly examined in more general decision circumstances in Siniscalchi (2011).

For future references, the axiom which is defined only by binary acts is put an asterisk against the number.

For the purpose of our analyses here, the next axiom is necessary to describe the conditional preference as a non-trivial binary relation.

A1* (Essentiality) There exist an event $A \in \Sigma$ and consequences x, $y \in X$ such that $x \succ x_A y \succ y$.

An event $E \in \Sigma$ is *essential* if $x \succ x_E y \succ y$ for some x, $y \in X$. Event $E \in \Sigma$ is *null* if $x_E y \sim y$ for all x, $y \in X$. Let N be a set of all null events and $\Sigma^\circ = \Sigma \backslash N$ be the set of all non-null events with respect to $\succsim$. An event E^c is *universal* if $E \in N$. An essential event is not null, nor universal. A1* also implies there are at least one (two, in practice) conditioning event to be effective.

A2* (Boundedness) There exist consequences x^*, $x_* \in X$ such that $x^* \succ x_*$ and for all $x \in X$, $x^* \succsim x \succsim x_*$.

A2* states that there are the maximum and minimum consequences in X in terms of $\succsim$. Since this axiom may not be crucial for deriving update rule, we may drop this axiom. However, this paper still deals with degree of tractability as priority.

A3 (Monotonicity) For all acts f, $g \in \mathcal{F}$, if $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$, then $f \succsim g$.

A3 is standard monotonicity for general acts.

A4* (Eventwise Monotonicity) For any event $A \in \Sigma$, if for some $x \succ y$, $x_A y \succ y$ (resp. $x \succ x_A y$), then for all $a \succ b \succsim c$, $a_A c \succ b_A c$ (resp. for all $c \succsim a \succ b$, $c_A a \succ c_A b$).

A4* is originated from A3* in Alon and Schmeidler (2014), and it is equivalent to B3 in GM2001. If-part implies that event $A \in \Sigma$ is essential or universal, hence non-null. As noted, the axiom implies that on a non-null event, choice is responsive eventwisely to any strictly better outcome. This axiom is also crucial to characterize conditional choices given the same subact (here, same outcomes), as well as it is critical to obtain biseparable expression.

The next is the standard axiom for continuity and boundedness. The definitions below follow Wakker (1991) and Nakamura (1990).

Given an essential event $A \in \Sigma$ and $\succsim$, we call $\alpha^1, \alpha^2, \ldots$ a *standard sequence* if there exist x, $y \in X$ such that $x \nsim y$ and either $\{x, y\} \succsim \alpha^k$ and $x_A \alpha^k \sim y_A \alpha^{k+1}$ for all $k = 1, 2, \ldots$, or $\{x, y\} \precsim \alpha^k$ and ${\alpha^k}_A x \sim {\alpha^{k+1}}_A y$ for all $k = 1, 2, \ldots$.[1] A sequence $\alpha^1, \alpha^2, \ldots$ is *bounded* if there exist $\alpha^{\sup}$ and $\alpha^{\inf}$ such that $\alpha^{\sup} \succsim \alpha^k \succsim \alpha^{\inf}$ for all k with $x = \alpha^{\sup}$ and $y = \alpha^{\inf}$ for some x, $y \in X$. A sequence is *strictly bounded* when $\succsim$ in the above expression is replaced with $\succ$.

A5*a (Archimedean) Every strictly bounded standard sequence is finite.

Given an essential event $A \in \Sigma$ and $\succsim$, we call $\alpha^1, \alpha^2, \ldots$ a *second-order standard sequence* if for every k there exists a standard sequence $\beta^1, \beta^2, \ldots$ such that for some m, $l \in \mathbb{N}$ $\alpha^k \sim \beta^m$, $\alpha^{k+1} \sim \beta^{m+l}$, and $\alpha^{k+2} \sim \beta^{m+2l}$.

A5*b (Second-order Archimedean) Every bounded second-order standard sequence is finite.

A5*b (Second-order Archimedian) is necessary and sufficient for finite utility values on maximal and minimal outcomes in X, thoroughly examined in Wakker (1991, 1993) and Chateauneuf & Wakker (1993).

Let us introduce a mixture of two acts, whose definition is from GM2001.

Definition 3 (B-mixture of acts) *Given f, $g \in \mathcal{F}$ and $B \in \Sigma$, the B-mixture of f and g is the act $h \in \mathcal{F}$ such that $h(\omega) \sim f(\omega)_B\, g(\omega)$ for every $\omega \in \Omega$.*

Write the B-mixture of f and g as $c(fBg)$. Note that for some event $B \in \Sigma$, such $c(fBg)$ may not exist.

A6* (Binary Comonotonic Act Independence) For every essential $A \in \Sigma$, every $B \in \Sigma$, and for all f, g, $h \in \mathcal{F}_2$ such that $f = x_A y$, $g = {x'}_A y'$ and $h = {x''}_A y''$, if f, g, h are pairwise binary monotonic, and $\{x, x'\} \succsim x''$ and $\{y, y'\} \succsim y''$ (or $\{x, x'\} \precsim x''$ and $\{y, y'\} \precsim y''$), then $f \succsim g \Longrightarrow c(fBh) \succsim c(gBh)$.

[1] $\{x, y\} \succsim z$ denotes $x \succsim z$ and $y \succsim z$ for short.

A6* is one of the necessary and sufficient conditions for a biseparable expression. Compared to comonotonic act independence for the rank-dependent axiom, A6* is quite weaker in the sense that, for any $A \subset B$, $x_A y$ and $x_B y$ is comonotonic, although they are not binary comonotonic if $A \neq B$. In this sense, rank-dependent preferences are biseparable, however, biseparability still suggests more flexibility. If anything, any two acts which have the same winning event, hence a losing event, are evaluated the common winning or losing probability, which are summing up to one for any event.

Proposition 1 (GM2001) *The following statements are equivalent:*

(i) $\succsim$ satisfies A0 (Weak Order), A1 (Essentiality), A3 (Monotonicity), A4* (Eventwise Monotonicity), A5*a (Archimedean), A6* (Binary Comonotonic Act Independence).

(ii) There exist a continuous nontrivial monotonic representation $V : \mathcal{F} \to \mathbb{R}$ of $\succsim$ and a monotone set function $\mu : \Sigma \to [0,1]$ such that for all $f \in \mathcal{F}$, all $x \succsim y$ in X, all $A \in \Sigma$, letting $u(x) \equiv V(x)$ for all $x \in X$,

$$V(x_A y) = \mu(A) u(x) + (1 - \mu(A)) u(y) . \tag{1}$$

The representation V is unique up to positive affine transformations and the set function μ is unique.

In addition, A2 and A5*b are added instead of A5*a to the statement (i) if and only if V in (ii) is bounded.

We call the representation (1) *biseparable representation*, which is a complementarily additive preference relation. It is also a purely subjective binary version of the rank-dependent utility in Nakamura (1990).

As noted above, A5*b (Second-order Archimedian) is necessary and sufficient to obtain finite utility values on extreme alternatives, thoroughly examined in Wakker (1991, 1993) and Chateauneuf & Wakker (1993). It is shown in the proof of Theorem 3.3.(c) in Wakker (1993). This paper deals with finite utility-valued acts especially on counterfactual events, which compensates for some technical complexity of the axiom.

We know many biseparable preferences such as RDU, MEU, Hurwicz criterion model (Hurwicz, 1951; Ghirardato et. al, 2004), the disappointment aversion utility model by Gul (1991) and so on. Here three variations in such basic models are introduced as illustrations. Example 1 originates from Gul and Pesendorfer (2015), and Example 2 and 3 are spacial cases in Chew and Epstein (1989), Chew et. al (1993), Grant et. al (2000).

Example 1 (Hurwicz rank-dependent utility)

$$\begin{aligned} V^1(x_A y) &= \alpha \min_{p \in C} \left[\pi(p_A) u(x) + (1 - \pi(p_A)) u(y)\right] \\ &\qquad + (1-\alpha) \max_{p \in C} \left[\pi(p_A) u(x) + (1 - \pi(p_A)) u(y)\right] \\ &= \left[\alpha \pi \left(\min_{p \in C} p_A\right) + (1-\alpha) \left(1 - \pi\left(\min_{p \in C} p_A\right)\right)\right] u(x) \\ &\qquad + \left[1 - \left\{\pi\left(\min_{p \in C} p_A\right) u(x) + (1-\alpha)\left(1 - \pi\left(\min_{p \in C} p_A\right)\right)\right\}\right] u(y), \end{aligned}$$

where $\alpha \in [0,1]$ and a weight function $\pi : [0,1] \to [0,1]$ is increasing continuous and onto.

Example 2 (max-min disappointment aversion utility)

$$V^2(x_A y) = \min_{p \in C} p_A \frac{u(x) + \beta v}{1+\beta} + \left(1 - \min_{p \in C} p_A\right) u(y),$$

where $\beta \in (-1, \infty)$ is an index of disappointment aversion. It can be written in the explicit biseparable formula

$$V^2(x_A y) = \frac{\min_{p \in C} p_A}{1 + (1 - \min_{p \in C} p_A)\beta} u(x) + \left[1 - \frac{\min_{p \in C} p_A}{1 + (1 - \min_{p \in C} p_A)\beta}\right] u(y).$$

Example 3 (rank-dependent weighted utility)

$$\pi(A)\phi(x)[u(x) - v] + (1 - \pi(A))\phi(y)[u(y) - v] = 0,$$

where $\pi : \Sigma \to [0,1]$ is a monotone set function, and $\phi : X \to \mathbb{R}$ is a weight function for consequences. It can be also written in the explicit biseparable formula

$$\begin{aligned} V^3(x_A y) &= \frac{\pi(A)\phi(x)}{\pi(A)\phi(x) + (1 - \pi(A))\phi(y)} u(x) \\ &\quad + \frac{(1 - \pi(A))\phi(y)}{\pi(A)\phi(x) + (1 - \pi(A))\phi(y)} u(y). \end{aligned}$$

3.1.2 Axioms for Conditional Preferences: Bayesian updating

This subsection introduces two categorized axioms on conditional preference relations. The first one is for the axioms concerning an extension of Bayesian updating, relevant to the way to revise preferences in the course of updating. The other is for the axioms prescribing properties that conditional preferences succeed from the unconditional preference, concerning actual utility representations. While these two might be logically integrated into a consistency property, separating two properties is a crucial factor for further generalizations.

As in Savage's interpretation of conditional preferences (Savage, 1954; p. 22), a modification for two acts *so as to agree with one another outside of* B leads to define conditional preference relations. Formally, the *Bayesian updating* is described as follows.

Bayesian updating There exists an act $a \in \mathcal{F}$ such that for all $E \in \Sigma^\circ$, all acts $f, g \in \mathcal{F}$, $f \succsim_E g$ if and only if $f_E a \succsim g_E a$.

Bayesian updating implies that the modification is made by a *common* subact a so as to agree with each other. In fact, the conditional preference relation given E is equivalent to the unconditional preference relation after modification. Note that Bayesian updating requests that a particular $a \in \mathcal{F}$ have to be commonly used for modifying any pair of acts. In this regard, Bayesian updating is equivalent to the *f-Bayesian update rule* in Gilboa and Schmeidler (1993), $\overline{h}$-Bayesian updating in Ghirardato (2002) and Siniscalchi (2011). However, "a common subact for any act" is fairly strong requirement to be relaxed below.

B1 (Weak Bayesian updating) For every $x \in X$ there exists an act $a \in \mathcal{F}$ such that for all $E \in \Sigma^\circ$, all $f \in \mathcal{F}$, $f \sim_E x$ if and only if $f_E a \sim x_E a$.

B1 (Weak Bayesian Updating) is the extended version of Bayesian Updating, accommodating the common subact $a \in \mathcal{F}$ to vary in the set of acts which are not indifferent. In other words, the subact $a \in \mathcal{F}$ is common to indifferent acts, more concretely, varying dependent on the certainty equivalent of indifferent acts. After careful consideration of this dependency, we may rewrite $f \sim_E g$ if and only if $f_E a \sim g_E a$.

B1 is to restrict its application only to binary acts for binary characterizations.

B2* (Weak Bayesian updating for Binary Acts) For every $x \in X$ there exists an act $a \in \mathcal{F}$ such that for all $E \in \Sigma^{\circ}$, all $f \in \mathcal{F}_2$, $f \sim_E x$ if and only if $f_E a \sim x_E a$.

While B2* is imposed only on binary acts, since as a representation, biseparable preferences are examined, therefore it is why axioms are imposed only on binary acts.

Let us turn to the second category of axioms, prescribing the properties conditional preferences preserve.

B3 (Independence Preserved)

If a set of acts f, g, $h \in \mathcal{F}$ satisfies independence under $\succsim$: for every $B \in \Sigma$

$$\text{if } f \succsim g, \text{ then } c(fBh) \succsim c(gBh),$$

then the set of acts f, g, $h \in \mathcal{F}$ satisfies independence under $\succsim_E$: for every $B \in \Sigma$

$$\text{if } f \succsim_E g, \text{ then } c(fBh) \succsim_E c(gBh).$$

The next axiom is the binary act version of B3.

B4* (Binary Independence Preserved)

If a set of acts f, g, $h \in \mathcal{F}$ satisfies independence under $\succsim$: for every $B \in \Sigma$

$$\text{if } f \succsim g, \text{ then } c(fBh) \succsim c(gBh),$$

then the set of acts f, g, $h \in \mathcal{F}$ satisfies independence under $\succsim_E$: for every $B \in \Sigma$

$$\text{if } f \succsim_E g, \text{ then } c(fBh) \succsim_E c(gBh).$$

B3 and B4* assert that "the set of acts satisfying independence axiom" never shrink. More specifically, ex ante MEU preferences satisfy binary comonotonic independence and constant act independence, hence B2 requires that ex post preferences $\succsim_E$ also satisfy these two types of independence. As for RDU preferences, $\succsim$ satisfies comonotonic independence, hence B3 asserts that $\succsim_E$ also satisfies comonotonic independence. It also suggests that unconditional SEU never generates conditional MEU.

3.2 Elicitation of the update rule

The subsection presents the main result of this paper. Theorem 1 provides the formal characterization of the weak Bayesian updating under biseparable preferences.

Theorem 1 *Suppose that every* $\succsim_E \in \{\succsim_A\}_{A\in\Sigma}$ *satisfies A0. The following statements are equivalent.*

(i) $\succsim$ *satisfies*

A1 Essentiality*

A2 Boundedness*

A3 Monotonicity

A4 Eventwise Monotonicity*

*A5*b Second-order Archimedian*

A6 Binary Comonotonic Act Independence*

and every $\succsim_E \in \{\succsim_A\}_{A\in\Sigma}$ *satisfies*

B2 Weak Bayesian Updating for Binary Acts*

B4 Binary Independence Preserved.*

(ii) For all $E \in \Sigma^\circ$*, there exist a unique monotone set function* $\mu_E : \Sigma \to [0,1]$ *and a non-constant continuous monotonic affine representation* $V_E : \mathcal{F}_2 \to \mathbb{R}$ *such that for all* x*,* $y \in X$ *with* $x \succsim y$*, all* $A \in \Sigma$*,*

$$V_E(x_A y) = \mu_E(A)\, u(x) + [1 - \mu_E(A)]\, u(y),$$

where $u(x) \equiv V_E(x)$ *for all* $x \in X$*.*

Moreover, μ_E *is a composite update of* μ *by BU, DS, and FH rules: there exists a partition of* E^c*,* $(T_k, T_{k-1}, \ldots, T_1)$*,* $k \leqq |E^c|$*,*

$$\mu_E = \Phi^k \circ \Phi^{k-1} \circ \cdots \circ \Phi^1(\mu)$$

where given $E^i = \Omega \backslash (T_k \cup \cdots \cup T_i)$*, for all* $A \subset E$*,*

$$\Phi^i(\mu)(A) \in \left\{ \frac{\mu((A\cap E^i)\cup T_i) - \mu(T_i)}{1-\mu(T_i)},\ \frac{\mu(A\cap E^i)}{\mu(E^i)},\ \frac{\mu(A\cap E^i)}{\mu(A\cap E^i) + 1 - \mu(A\cup T_i)} \right\}.$$

Theorem 1 consists of two arguments. The first part demonstrates that if the ex ante and ex post preferences satisfy the set of axioms there exists a collection of complementarily additive conditional probabilities such that the probabilities represent the conditional preference over binary acts. The latter part characterizes the update rule for calculating such conditional probabilities constitutes a composite update of priors. In the course of compositely updating, every update step is conducted only by one of DS, BU, and FH update rule.

The result is most remarkable in two aspects. (i) Expanded into fairly broad range of preferences, only three update rules are supported, conversely, any update rules other than three rules are never compatible with any biseparable preferences with weak Bayesian updating. (ii) Not only the *big three* rules were conventional, prominent, and prevalent rules in the theory of belief functions, but also they are consistent with biseparable preferences developed in the decision theory, although the biseparability delivers few information about preferences over acts with more than three outcomes.

The three rules are considered to share the common property, Weak Bayesian property. As seen in the proof of Theorem 1, these rules are characterized through the conditioning act on the counterfactual event E^c, x^* for DS, x for FH, and x_* for BU rule. When $\succsim$ has a biseparable representation, a *subjective mixture of f and g* is well-defined as in Ghirardato, et. al (2003). For any acts $f, g \in \mathcal{F}$ and any $\alpha \in [0,1]$, $h = \alpha f + (1-\alpha) g$ is a *subjective mixture of f and g* if $h(\omega) \sim \alpha f(\omega) + (1-\alpha) g(\omega)$ for all $\omega \in \Omega$.

For every $b \succsim w$ and some $\alpha \in [0,1]$

$$\begin{aligned} \alpha x_E x^* + (1-\alpha) x_E x_* &\sim \alpha b_A w_{E\backslash A} x^* + (1-\alpha) b_A w_{E\backslash A} x_* \\ &\Leftrightarrow x_E (\alpha x^* + (1-\alpha) x_*) \sim b_A w_{E\backslash A} (\alpha x^* + (1-\alpha) x_*) . \end{aligned}$$

By Theorem 1, the above relationship is equivalent to $x \sim_E b_A w$ if α satisfies the following three pattern:

$\alpha = 1 \Leftrightarrow$ DS, $\mu_E(A) = \frac{\mu((A\cap E)\cup E^c) - \mu(E^c)}{1-\mu(E^c)}$

$\alpha = 0 \Leftrightarrow$ BU, $\mu_E(A) = \frac{\mu(A\cap E)}{\mu(E)}$

α is such that $x \sim \alpha x^* + (1-\alpha) x_* \ \Leftrightarrow$ FH, $\mu_E(A) = \frac{\mu(A\cap E)}{\mu(A\cap E) + 1 - \mu(A\cup E^c)}$,

where $a \in \{x^*, x_*, x\}$ generates μ_E obtained by DH, BU, FH rule, respectively.

To see more detailed updating formula, let us calculate the exact updated probabilities in Example 1 to 3.

Example 1′ (Hurwicz rank-dependent utility updated by FH rule) Suppose that $\nu(A) = \pi(\min_{p\in C} p_A)$ is updated by FH rule, which is correspond to $a(x) = x$. Then $\nu_E(A) = \frac{\nu(A\cap E)}{\nu(A\cap E) + \overline{\nu}(A^c\cap E)}$, and $\overline{\nu_E}(A) = \frac{\overline{\nu}(A^c\cap E)}{\overline{\nu}(A\cap E) + \nu(A^c\cap E)}$.[2] On the other hand, letting $\rho(A) = \alpha\nu(A) + (1-\alpha)\overline{\nu}(A)$,

$$\rho_E(A) = \frac{\alpha\nu(A\cap E) + (1-\alpha)\overline{\nu}(A\cap E)}{[\alpha\nu(A\cap E) + (1-\alpha)\overline{\nu}(A\cap E)] + 1 - [\alpha\nu(A^c\cap E) + (1-\alpha)\overline{\nu}(A^c\cap E)]}.$$

However,

$$\begin{aligned} &\alpha\nu_E(A) + (1-\alpha)\overline{\nu_E}(A) \\ = \ &\alpha\frac{\nu(A\cap E)}{\nu(A\cap E) + \overline{\nu}(A^c\cap E)} + (1-\alpha)\frac{\overline{\nu}(A\cap E)}{\overline{\nu}(A\cap E) + \nu(A^c\cap E)}, \end{aligned}$$

which is not equal to $\rho_E(A)$ unless ν is additive or $\alpha \in \{0,1\}$. Although the unconditional and conditional preferences are biseparable, they does not satisfy weak Bayesian property (B2*) if ν is not additive and $\alpha \in (0,1)$.

Example 2′ (max-min disappointment aversion utility updated by BU rule) Suppose that $\rho(A) = \min_{p\in C} p_A$ is updated by BU rule, which is correspond to $a = x_*$. Then $\rho_E(A) = \frac{\rho(A\cap E)}{\rho(E)}$. Letting $\rho(A) = \frac{\min_{p\in C} p_A}{1 + \left(1-\min_{p\in C} p_A\right)\beta}$,

$$\rho_E(A) = \frac{\rho(A\cap E)}{\rho(E)} = \frac{1 + (1-\min_{p\in C} p_E)\beta}{1 + (1-\min_{p\in C} p_{A\cap E})\beta}\frac{\min_{p\in C} p_{A\cap E}}{\min_{p\in C} p_E}.$$

[2] $\overline{\nu}$ is the *conjugate of* ν, $\overline{\nu}(A) = 1 - \nu(A^c)$ for any $A \in \Sigma$.

However, applying Theorem 1 to implicit formula for every $A \subset E$

$$V_E^2(x_A y) = \frac{V^2(x_A y)}{\rho(E)} = \rho_E(A)\, u(x) + (1 - \rho_E(A))\, u(y),$$

which implies B2*.

Example 3′ (updating rank-dependent weighted utility) Suppose π in implicit expression

$$\pi(A)\,\phi(x)\,[u(x) - v] + (1 - \pi(A))\,\phi(y)\,[u(y) - v] = 0,$$

is updated by DS rule, to obtain $\pi_E(A) = \frac{\pi((A \cap E) \cup E^{c \cup}) - \pi(E^c)}{1 - \pi(E^c)}$. The conditional preference is represented by

$$V_E^3(x_A y) = \frac{\pi_E(A)\phi(x)}{\pi_E(A)\phi(x) + (1-\pi_E(A))\phi(y)} u(x) + \frac{(1-\pi_E(A))\phi(y)}{\pi_E(A)\phi(x) + (1-\pi_E(A))\phi(y)} u(y).$$

It is also verified to satisfy B2*.

4 Application

4.1 Maxmin expected utility preferences

The preferences represented by the maxmin expected utility proposed by Gilboa and Schmeidler (1989) and Casadesus-Masanell et. al (2000), and the purely subjective version is constructed by Alon and Schmeidler (2014), which is explicitly characterized by using biseparable framework. As the update rule for its prior set in the MEU model, most decision settings adopt the full Bayesian update rule axiomatized by Pires (2001). The main axiom is called *conditional certainty equivalent consistency*: for all non-null event $E \in \Sigma^{\circ}$, all $f \in \mathcal{F}$ and all $x \in X$, if $f \sim_E x$ then $f_E x \sim x$. It is somewhat opposite direction in this paper, in the sense that $\succsim_E$ inherits certain properties from $\succsim$. In fact, the opposite direction is true as long as $\succsim$ and $\succsim_E$ are represented by MEU, as shown in Hanany and Klibanoff (2007). However, it is also verified through our axiomatization in this paper, MEU preferences are biseparable, thus all lower probabilities are compositely updated by BU, DS, or FH rule. Pires's certainty equivalent consistency correspond to $a = x$ in B2*. It is due to, MEU preferences are axiomatized by binary comonotonic acts independence plus other axioms.

To see this, it is necessary to introduce more definitions and axioms to obtain the MEU representation. Let us introduce the preference average of constant acts. The definition is from Ghirardato et. al (2003) and Alon & Schmeidler (2014).

Definition 4 (Preference average) *Given $x, y \in X$ and an essential event $A \in \Sigma$, $z \in X$ is called a preference average of x and y given A if $x \succsim z \succsim y$ and $x_A y \sim c(xAz)_A\, c(zAy)$.*

If $y \succsim x$, z is also called a preference average of x and y if it is a preference average of y and x. Let us denote the preference average of x and y by $m(xAy)$. Write $m(fAg)$ for the act $h \in \mathcal{F}$ such that $h(\omega)$ is a preference average of $f(\omega)$ and $g(\omega)$ given A for every $\omega \in \Omega$.

A7 (Uncertainty aversion) For all $f, g \in \mathcal{F}$, if $f \sim g$, then $m(fAg) \succsim f$.

A8 (Certainty Independence) For all $f, g \in \mathcal{F}$ and all $x \in X$, if $f \succsim g$, then $m(fAx) \succsim m(gAx)$.

A9 (Certainty Covariance) For all $f, g \in \mathcal{F}$ and all $x, y \in X$ such that $m(fAy) \sim m(gAx)$, $f \sim x$ if and only if $g \sim y$.

Theorem 2 *Suppose that every $\succsim_E \in \{\succsim_A\}_{A\in\Sigma}$ satisfies A0. The following statements are equivalent.*

(i) $\succsim$ satisfies A1, A2*, A3, A4*, A5*b, A6**

A7 Uncertainty aversion

A8 Certainty Independence

A9 Certainty Covariance,

and every $\succsim_E \in \{\succsim_A\}_{A\in\Sigma}$ satisfies

B1 Weak Bayesian Updating

B3 Independence Preserved.

(ii) For every $E \in \Sigma^\circ$, there exist a unique nonempty closed convex set $C_E \subset \Delta^0$ and a non-constant continuous increasing affine function $u : X \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$

$$f \succsim_E g \Longleftrightarrow \min_{p\in C_E} \int u \circ f dp \geqq \min_{p\in C_E} \int u \circ g dp,$$

where C_E is a composite update of C: there exists $T = (T_3, T_2, T_1)$ such that $\mu_E : \Sigma \to [0,1]$ defined as $\mu_E(A) \equiv \min_{p\in C_E} p(A)$ for every $A \in \Sigma$ is a composite update of μ, $\mu_E = \Phi^3 \circ \Phi^2 \circ \Phi^1(\mu)$, where $\Phi^3(\mu) = \mu^{FH}_{E^3}$, $\Phi^2(\mu) = \mu^{BU}_{E^2}$ and $\Phi^1(\mu) = \mu^{DS}_{E^1}$,

$$\mu_E(A) = \frac{\mu(A \cup T_1) - \mu(T_1)}{[\mu(A \cup T_1) - \mu(T_1)] + [\mu(E \cup T_1 \cup T_3) - \mu(A \cup T_1 \cup T_3)]}$$

for every $A \subset E$.

B1-1 (Certainty Equivalent Consistency) For all $E \in \Sigma^\circ$ and all $f \in \mathcal{F}$, $f \sim_E x$ if and only if $f_E x \sim x$.

If we assume B1-1 in place of B1, we obtain the characterization of FB rule.

Corollary 1 *Suppose that every $\succsim_E \in \{\succsim_A\}_{A\in\Sigma}$ satisfies A0. The following statements are equivalent.*

(i) $\succsim$ satisfies A1, A2*, A3, A4*, A5*b, A6*, A7, A8, A9*

and every $\succsim_E \in \{\succsim_A\}_{A\in\Sigma}$ satisfies

B1-1 Certainty Equivalent Consistency

B3 Independence Preserved.

(ii) For every $E \in \Sigma^\circ$, there exist a unique nonempty closed convex set $C_E \subset \Delta^0$ and a non-constant continuous increasing affine function $u : X \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$

$$f \succsim_E g \Longleftrightarrow \min_{p\in C_E} \int u \circ f dp \geqq \min_{p\in C_E} \int u \circ g dp,$$

where C_E is updated by

$$C_E = \left\{p_E \in \Delta^0 \mid p \in C\right\} \text{ where } p_E(A) = \frac{p(A \cap E)}{p(E)} \text{ for every } A \in \Sigma.$$

4.2 Rank-dependent expected utility preferences

In this subsection, more generalized characterization for updating monotone set functions, as seen in Horie (2006) is investigated. RDU preferences are also biseparable, since the main axiom is called *comonotonic independence*, the very full version of A6*. Therefore, the formulation can be made only to plus an additional axiom A10 (comonotonic act independence) in place of A6* in (i) of Theorem 1. It enables us to give the full characterization of BU, DS, FH update rule for monotone set functions.

Let $\int_{\mathcal{C}} f d\mu$ express the Choquet integral of f with respect μ. Two acts f, $g \in \mathcal{F}$ are *comonotonic*, if there are no ω, $\omega' \in \Omega$ such that $f(\omega) \succ f(\omega')$ and $g(\omega) \succ g(\omega')$.

A10 (Comonotonic Independence) For every essential $A \in \Sigma$, every $B \in \Sigma$, and for all comonotonic f, g, $h \in \mathcal{F}$ such that h weakly dominates f and g, or h is weakly dominated by f and g, $f \succsim g \Longrightarrow c(fBh) \succsim c(gBh)$.

Theorem 3 *Suppose that every $\succsim_E \in \{\succsim_A\}_{A\in\Sigma}$ satisfies A0. The following statements are equivalent.*

(i) $\succsim$ satisfies A1, A2*, A3, A4*, A5*b,*

A10 Comonotonic Independence

and every $\succsim_E \in \{\succsim_A\}_{A\in\Sigma}$ satisfies

B2 Weak Bayesian Updating for Binary Acts*

B3 Independence Preserved.

(ii) For every $E \in \Sigma^\circ$, there exist a unique monotone set function on E $\mu_E : \Sigma \to [0,1]$ and a non-constant increasing affine function $u : X \to \mathbb{R}$ such that for all f, $g \in \mathcal{F}$,

$$f \succsim_E g \Longleftrightarrow \int_{\mathcal{C}} u \circ f d\mu_E \geqq \int_{\mathcal{C}} u \circ f d\mu_E,$$

where μ_E is a composite update of μ: there exists $T = (T_3, T_2, T_1)$ such that $\mu_E = \Phi^3 \circ \Phi^2 \circ \Phi^1(\mu)$, where $\Phi^3(\mu) = \mu^{FH}_{E^3}$, $\Phi^2(\mu) = \mu^{BU}_{E^2}$ and $\Phi^1(\mu) = \mu^{DS}_{E^1}$,

$$\mu_E(A) = \frac{\mu(A \cup T_1) - \mu(T_1)}{[\mu(A \cup T_1) - \mu(T_1)] + [\mu(E \cup T_1 \cup T_3) - \mu(A \cup T_1 \cup T_3)]}$$

for every $A \subset E$.

4.3 Concave expected utility preferences

The concave integral proposed by Lehrer (2009) and Even & Lehrer (2014) is an extended version of Choquet integral. In this section, consider the update rule tie up with preferences represented by *Lehrer expected utility* (LEU).

To define the concave integral, more notations are incorporated in addition to the setting in the previous section.

- $X = \mathbb{R}_+$
- $u : X \to \mathbb{R}_+$: a non-constant continuous increasing utility function
- $\mathbb{I}_A$ expresses the act $1_A 0 \in \mathcal{F}$ called the *indicator of* A
- $\sum_{i=1}^{k} \alpha_i \mathbb{I}_{A_i}$ is 2^N-*decomposition of* f if (i) $\sum_{i=1}^{k} \alpha_i \mathbb{I}_{A_i} = u \circ f$, (ii)$\alpha_i \geqq 0$ and $A_i \in \Sigma$ for every $k = 1, \ldots, n$.
- Concave integral of $u \circ f$:

$$\int_{\mathcal{L}} u \circ f d\mu = \max \left\{ \sum_{i=1}^{k} \alpha_i \mu(A_i) \mid \sum_{i=1}^{k} \alpha_i \mathbb{I}_{A_i} \text{ is } 2^N\text{-decomposition of } f \right\}$$

- μ is superadditive.

If monotone set function μ is convex, LEU and CEU coincide, hence it is extension of Choquet integral.

The concave integral with respect to μ requires quite complicated calculations in general, in this section, μ is assumed to be superadditive. The symmetric consequences does not hold under μ is subadditive, since it does not imply biseparability.

The following proposition shows that, under μ's superadditivity, LEU is biseparable.

Proposition 2 *Suppose that μ is superadditive. Then, $\int_{\mathcal{L}} u \circ f d\mu$ is biseparable.*

From this proposition, $\succsim$ represented by LEU with superadditive μ satisfies binary comonotonic independence. It implies that on binary acts, $\succsim$ behaves as preferences represented by CEU. However, $\succsim$'s behavior on trinary acts is quite different as in the following example.

Example 1 *Consider the four states $\{a, b, c, d\}$ and ν as following.*

$$\begin{aligned}
\nu(a) &= \nu(b) = \nu(c) = \nu(d) = \frac{1}{8} \\
\nu(ab) &= \nu(bc) = \frac{2.4}{8}, \ \nu(ac) = \frac{4.8}{8}, \ \nu(ad) = \nu(bd) = \nu(cd) = \frac{2.1}{8} \\
\nu(abc) &= \nu(bcd) = \nu(abd) = \nu(acd) = \frac{6}{8}, \ \nu(abcd) = 1
\end{aligned}$$

Assume a linear utility $u(x) = x$ for all $x \in X$. Consider the act $g = (3, 2, 1, 0)$. The optimal decomposition of CEU is $[\{a, b, c, d\}, \{a, b, c\}, \{ab\}, \{a\}]$ and the CEU value of g is

$$\int_{\mathcal{C}} g d\nu = \frac{1}{8} \times 3 + \left(\frac{2.4}{8} - \frac{1}{8} \right) \times 2 + \left(\frac{6}{8} - \frac{2.4}{8} \right) \times 1 = \frac{9.4}{8}.$$

However the optimal decomposition of LEU is $[\{a, b\}, \{a, c\}]$, and the LEU value of g is

$$\int_{\mathcal{L}} g d\nu = \frac{2.4}{8} \times 2 + \frac{4.8}{8} = \frac{9.6}{8}.$$

It is caused by that $\{ab\}$ and $\{ac\}$ are pairwise concave, i.e. $\nu\{abc\} - \nu\{ab\} < \nu\{ac\} - \nu\{a\}$.

Theorem 4 *Suppose that every $\succsim_E \in \{\succsim_A\}_{A\in\Sigma}$ satisfies A0. The following statements are equivalent.*

(i) $\succsim$ is represented by the concave expected utility with a superadditive monotone set function μ and u,

and $\succsim_E \in \{\succsim_A\}_{A\in\Sigma}$ satisfies

B2 Weak Bayesian Updating for Binary Acts*

B3 Independence Preserved.

(ii) For every $E \in \Sigma^\circ$, there exist a unique monotone set function on E, $\mu_E : \Sigma \to [0,1]$ such that for all acts f and g,

$$f \succsim_E g \Longleftrightarrow \int_{\mathcal{L}} u \circ f d\mu_E \geqq \int_{\mathcal{L}} u \circ f d\mu_E,$$

where μ_E is updating of μ by

$$\mu_E(A) = \frac{\mu(A\cap E)}{\mu(E)}$$

for every $A \in \Sigma$.

5 Discussion

Gilboa and Schmeidler (1993) formulated Savage's Bayesian paradigm as the *f-Bayesian update rule*, which is formally applied to the CEU preferences with convex capacities. Their significant contribution is to establish a formal framework where the unconditional and conditional preferences are linked through a subact f which is assumed on the unobserved event, and to characterize the formation of posteriors through an update rule for prior capacities. Pires (2001) and Hanany and Klibanoff (2007) also adopted this approach to axiomatize the full Bayesian updating for a given set of priors under MEU preferences. Eichberger et. al (2007) and Horie (2013) characterized the full Bayesian update rule, FH rule in this paper, for RDU preferences.

In the above literature, a set of consistent counterfactual subacts characterizes an update rule, and also gives an introspectional interpretation to a decision maker's pattern of conditioning. For example, Gilboa and Schmeidler (1993) showed that DS rule (Shafer, 1976) (resp. BU rule) is pessimistic (resp. optimistic). As for FH update rule, the consistent counterfactual act is described as the certainty equivalence, hence which exhibits a fair degree of neutrality.

We have investigated an extension of the Bayesian updating under biseparable preferences. Only three update rules are compatible with this fairly broad range of biseparability. However, it might also suggest the limitation in the method of unconditional-conditional preferences approach, since further widening the sets of counterfactual subacts may not generate any consistent preferences, and become disconnected with reality. Although it is quite difficult to test and measure out how to revise subjective uncertainty, it is the matter for future investigation anticipated eagerly in a behavioral sense.

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Appendix A

Update rules for ambiguous beliefs

Several update rules for non-additive measures are distinguished as an extension of classical Bayes' rule for conditional probabilities. Most update rules are initially proposed and investigated in the theory of belief functions. In our decision theoretic framework, they are mostly applied to monotone set functions.

Formally, an *update rule* of μ is a transformation of μ to generate the conditional monotone set function as a posterior.

Dempster-Shafer update rule (DS rule; Dempster,1967; Shafer, 1976) is defined through

$$\mu_E^{DS}(A) = \frac{\mu((A \cap E) \cup E^c) - \mu(E^c)}{1 - \mu(E^c)} \quad \text{for every } A \in \Sigma,$$

which is well-defined if $1 - \mu(E^c) > 0$.

The rule updating μ like in the additive case, is called *Bayes' update rule* (BU rule): Given an event $E \in \Sigma$ such that $\mu(E) > 0$

$$\mu_E^{BU}(A) = \frac{\mu(A \cap E)}{\mu(E)} \quad \text{for every } A \in \Sigma.$$

The generalized Bayes update rule, called *Fagin-Halpern update rule* (FH rule; Dempster, Fagin and Halpern, 1991; Jaffray, 1992) is defined through

$$\mu_E^{FH}(A) = \frac{\mu(A \cap E)}{\mu(A \cap E) + 1 - \mu((A \cap E) \cup E^c)} \quad \text{for every } A \in \Sigma,$$

if it is well-defined.

As for the update rule for a nonempty, closed and convex prior set $C \subset \Delta^0$, the *full Bayesian update rule* (FB rule) updates all priors in C through Bayes' rule:

$$C_E = \left\{p_E \in \Delta^0 \mid p \in C\right\} \text{ where } p_E(A) = \frac{p(A \cap E)}{p(E)} \text{ for every } A \in \Sigma \text{ and } p(E) > 0.$$

Appendix B

B1. Proof of Theorem 1

(i)⇒(ii)

Throughout the proof, it is assumed that every preference relation in $\{\succsim_A\}_{A \in \Sigma}$ satisfying A0 and a non-null event $E \in \Sigma^\circ$ are all fixed.

A0-A6* is summarized into the following observations:

- $\succsim$ has a biseparable representation. $\cdots$ Proposition 1
- An event A is essential if and only if $\mu(A) \in (0,1)$, A is null if and only if $\mu(A) = 0$, and A is universal if and only if $\mu(A) = 1$ $\cdots$ Proposition 3(i) in GM2001

- $V(X) \subset \mathbb{R}^n$ and $V(\mathcal{F}_2) \subset \mathbb{R}^{2n}$ are bounded. $\cdots$ Wakker (1991, 1993)
- Since $\succsim$ has a biseparable representation, a subjective mixture of f and g is well-defined. $\cdots$ Ghirardato, et. al (2003)

If there is no essential event $A \subsetneq E$ with respect to $\succsim_E$, $\mu_E(A) = 0$ for all $A \subsetneq E$ in (2) satisfies (ii), therefore we consider the case where there exists at least one essential event $A \subsetneq E$ for the rest of the proof.

At first, Lemma 1 is to specify the set of possible conditioning acts a to satisfy A6*.

Lemma 1 *A constant act $a(\cdot)$ satisfies $x \succsim y \Leftrightarrow x_E a(x) \succsim y_E a(y)$ for all x, $y \in X$ if and only if $a(x) \in \{x^*, x, x_*\}$.*

Proof. At first, assume $a(x) \in \{x^*, x, x_*\}$. Pick arbitrary x, y such that $x \succsim y$. If $a(x) = x$, then $x_E a(x) = x$ and $y_E a(y) = y$, the argument holds immediately. When $a = x^*$ or x_*, for any pair of $x \succsim y$, we have $x^* \succsim x \succsim y \succsim x_*$ by definition. If $x \sim y$, then the statement holds immediately. Assume $x \succ y$. Then, by monotonicity, $x_E a \succsim y_E a$, $a \in \{x^*, x_*\}$. In any case, $x_E a$ and $y_E a$ are pairwise binary comonotonic, hence the statement holds.

For sufficiency, to lead to a contradiction, assume $a \notin \{x^*, x, x_*\}$ satisfies B2*. If $a(\cdot)$ is such that for some $x \succ y$, $a(x) \succ x$ and $a(y) \prec y$, $x_E a(x)$ and $y_E a(y)$ are not pairwise binary comonotonic, thus such $a(\cdot)$ cannot satisfy B2*.

Assume then $a(x)$ is such that $a(x) \succ x$ for any $x \in X$. However, there is no outcome $a(x^*) \succ x^*$ in X by A2*. On the contrary, assume $a(x)$ is such that $a(x) \prec x$ for any $x \in X$. However, there is no outcome $a(x_*) \prec x_*$ in X, again contradiction.

Suppose $a(x)$ is such that $x \prec a(x) \prec x^*$ for all $x \prec x^*$. Take an essential event E, $x \in X$ and choose $\alpha \in (0,1)$ satisfying $\alpha x^* + (1-\alpha) x \sim a(x)$. By assumption, we have $a(a(x)) \succ a(x)$, then by A4* $a(x)_E a(a(x)) \succ \alpha x^*_E a(x^*) + (1-\alpha) x_E a(x)$, contradicting A6*. The case where $a(x)$ is such that $x \succ a(x) \succ x_*$ for all $x \succ x_*$ can be shown in the same way. It follows that $a(x) \in \{x^*, x, x_*\}$ if it is constant. ■

Write $a \in \{x^*, x, x_*\}$ instead of $a(x)$ hereafter.

Lemma 2 *$\succsim_E$ satisfies A1*-A6* on $\mathcal{F}_2$.*

Proof. $\succsim_E$ on $\mathcal{F}_2$ agrees with $\succsim$ through B2*, it is straightforward to satisfy A1* (essentiality), A2* (Boundedness), A3 (monotonicity), A4*(Eventwise Monotonicity), A5*b (2nd-order Archimedian) on $\mathcal{F}_2$. Note that the axiom with * hold on $\mathcal{F}_2$. Since $\succsim$ satisfies A6* (binary comonotonic acts independence), B2* is applied to any set of binary comonotonic acts, say $x_A y$, $x'_A y'$ and $x''_A y''$ (w.l.o.g $x \succsim y$, $x' \succsim y'$ and $x'' \succsim y''$) satisfying independence, so that the set of $x_A y_{E \backslash A} a$, $x'_A y'_{E \backslash A} a$ and $x''_A y''_{E \backslash A} a$ also satisfies independence. Through B4*, the same set of $x_A y$, $x'_A y'$ and $x''_A y''$ satisfies binary comonotonic independence in terms of $\succsim_E$. ■

It implies that, by Proposition 1, $\succsim_E$ also has a biseparable representation V_E such that $V_E(x) = V(x) = u(x)$ and $V_E(x_A y) = \mu_E(A) u(x) + [1 - \mu_E(A)] u(y)$, where $u(x) = V(x)$ since $\succsim_E$ on constant acts agrees with $\succsim$.

Lemma 3 *If $\succsim_E$ satisfies B2* with $a \in \{x^*, x_*, x\}$, then $\succsim_E$ is represented by V_E such that for all $A \in \Sigma$*

$$\mu_E(A) = \begin{cases} \dfrac{\mu((A \cap E) \cup E^c) - \mu(E^c)}{1 - \mu(E^c)} & \text{if } a = x^* \\ \dfrac{\mu(A \cap E)}{\mu(E)} & \text{if } a = x_* \\ \dfrac{\mu(A \cap E)}{\mu(A \cap E) + 1 - \mu(A \cup E^c)} & \text{if } a = x \end{cases}.$$

Proof. By Proposition 1, $\succsim_E$ has a representation V_E such that for any $A \subset E$ and any binary act $x_A y \in \mathcal{F}_2$ with $x \succsim y$, $V_E(x_A y) = \mu_E(A)\, u(x) + [1 - \mu_E(A)]\, u(y)$.

(1) $a = x^*$

Pick an arbitrary essential event $A \subset E$ and consider x, $w \in X$ satisfying $x_E x^* \sim x^*_A w_{E \setminus A} x^*$. Then

$$\begin{aligned} x_E x^* &\sim x^*_A w_{E\setminus A} x^* \\ &\Leftrightarrow V(x_E x^*) - V\left(x^*_A w_{E\setminus A} x^*\right) = 0 \\ &\Leftrightarrow \{\mu(E^c)\, u(x^*) + [1 - \mu(E^c)]\, u(x)\} - \{\mu(A \cup E^c)\, u(x^*) + [1 - \mu(A \cup E^c)]\, u(w)\} = 0 \\ &\Leftrightarrow u(x) - \left\{\tfrac{\mu(A\cup E^c) - \mu(E^c)}{1 - \mu(E^c)} u(x^*) + \left[1 - \tfrac{\mu(A\cup E^c) - \mu(E^c)}{1-\mu(E^c)}\right] u(w)\right\} = 0 \\ &\Leftrightarrow u(x) - \{\mu_E(A)\, u(x^*) + [1 - \mu_E(A)]\, u(w)\} = 0 \\ &\Leftrightarrow x \sim_E x^*_A w \end{aligned}$$

It generates Dempster-Shafer update rule of μ: for all $A \in \Sigma$

$$\mu_E(A) = \frac{\mu((A \cap E) \cup E^c) - \mu(E^c)}{1 - \mu(E^c)}.$$

For this x, $w \in X$, consider b', $w' \in X$ such that $b' \sim \alpha x^* + (1-\alpha)\, x$ and $w' \sim \alpha x + (1-\alpha)\, w$ for some $\alpha \in [0,1]$. Due to A3, $b' \succsim w'$, $x \precsim b' \precsim x^*$, and $x \succsim w' \succsim w$. A6* tells that $x_E x^* \sim b'_A w'_{E\setminus A} x^* \sim x^*_A w_{E\setminus A} x^*$. Then

$$\begin{aligned} x_E x^* &\sim b'_A w'_{E\setminus A} x^* \sim \alpha(x_E x^*) + (1-\alpha)\left(x^*_A w_{E\setminus A} x^*\right) \\ &\Leftrightarrow V(x_E x^*) = \alpha V(x_E x^*) + (1-\alpha)\, V\left(x^*_A w_{E\setminus A} x^*\right) \\ &\Leftrightarrow u(x) = \tfrac{\mu(A\cup E^c) - \mu(E^c)}{1-\mu(E^c)} [\alpha u(x^*) + (1-\alpha)\, u(x)] \\ &\qquad\qquad + \left[1 - \tfrac{\mu(A\cup E^c)-\mu(E^c)}{1-\mu(E^c)}\right] [\alpha u(x) + (1-\alpha)\, u(w)] \\ &\Leftrightarrow u(x, v) = \mu_E(A)\, u(b') + [1 - \mu_E(A)]\, u(w') \\ &\Leftrightarrow x \sim_E b'_A w'. \end{aligned}$$

It implies that, for any binary act $b_A w$ and $b'_A w'$, $b_A w \succsim_E b'_A w'$ if and only if $b_A w_{E\setminus A} x^* \succsim_E b'_A w'_{E\setminus A} x^*$. Therefore, if $a = x^*$, μ is updated by DS rule.

(ii) $a = x_*$

Pick an arbitrary essential event $A \in \Sigma$ and consider x, $b \in X$ satisfying $x_E x_* \sim b_A x_*$.

Then

$$\begin{aligned}
x_E x_* &\sim b_A x_* \\
&\Leftrightarrow V(x_E x_*) = V(b_A x_*) \\
&\Leftrightarrow u(x) = \{\mu(E)\,u(x) + [1-\mu(E)]\,u(x_*)\} - \{\mu(A)\,u(b) + [1-\mu(A)]\,u(x_*)\} \\
&\Leftrightarrow u(x) = \tfrac{\mu(A)}{\mu(E)} u(b) + \left[1 - \tfrac{\mu(A)}{\mu(E)}\right] u(x_*) \\
&\Leftrightarrow u(x) = \mu_E(A)\,u(b) + [1-\mu_E(A)]\,u(x_*) \\
&\Leftrightarrow x \sim_E b_A x_*.
\end{aligned}$$

It generates the Bayesian update rule of μ: for all $A \in \Sigma$

$$\mu_E(A) = \frac{\mu(A \cap E)}{\mu(E)}.$$

For this x, $b \in X$, consider b'', $w'' \in X$ such that $b'' \sim \alpha x + (1-\alpha) b$ and $w'' \sim \alpha x + (1-\alpha) x_*$ for some $\alpha \in [0,1]$. Due to A3, $b'' \succsim w''$, $x \precsim b'' \precsim b$, and $x \succsim w'' \succsim x_*$. A6* tells that $x_E x_* \sim b''_A w''_{E\setminus A} x_* \sim b_A x_*$. Then

$$\begin{aligned}
x_E x_* &\sim b''_A w''_{E\setminus A} x_* \sim \alpha(x_E x_*) + (1-\alpha) b_A x_* \\
&\Leftrightarrow V(x_E x_*) - \{\alpha V(x_E x_*) + (1-\alpha) V(b_A x_*)\} = 0 \\
&\Leftrightarrow u(x) - \left\{\tfrac{\mu(A)}{\mu(E)}[\alpha u(x) + (1-\alpha) u(b)] + \left[1 - \tfrac{\mu(A)}{\mu(E)}\right][\alpha u(x) + (1-\alpha) u(x_*)]\right\} = 0 \\
&\Leftrightarrow u(x) - \{\mu_E(A)\,u(b'') + [1-\mu_E(A)]\,u(w'')\} = 0 \\
&\Leftrightarrow x \sim_E b''_A w''
\end{aligned}$$

It implies that, for any binary act $b_A w$ and $b''_A w''$, $b_A w \succsim_E b''_A w''$ if and only if $b_A w_{E\setminus A} x_* \succsim b''_A w''_{E\setminus A} x_*$. Therefore, if $a = x_*$, μ is updated by BU rule.

(iii) $a = x$

Now pick arbitrary b, $w \in X$ such that $b \succ w$, and consider x to satisfy $x \sim b_A w_{E\setminus A} x$. When $x \sim \alpha b + (1-\alpha) w$, x is indifferent to $\alpha b_A w_{E\setminus A} b + (1-\alpha) b_A w_{E\setminus A} w$ by construction. Then

$$\begin{aligned}
x &\sim \alpha b_A w_{E\setminus A} b + (1-\alpha) b_A w_{E\setminus A} w \\
&\Leftrightarrow u(x) - \left\{\alpha V\left(b_A w_{E\setminus A} b\right) + (1-\alpha) V\left(b_A w_{E\setminus A} w\right)\right\} = 0 \\
&\Leftrightarrow \begin{array}{r} u(x) - \alpha\{\mu(A \cup E^c)\,u(b) + [1-\mu(A \cup E^c)]\,u(w)\} \\ -(1-\alpha)\{\mu(A)\,u(b) + [1-\mu(A)]\,u(w)\} = 0 \end{array} \\
&\Leftrightarrow \begin{array}{r} u(x) + \mu(A)\{\alpha u(b) + (1-\alpha) u(w)\} - \mu(A \cup E^c)\{\alpha u(b) + (1-\alpha) u(w)\} \\ -\mu(A)\,u(b) - [1-\mu(A \cup E^c)]\,u(w) = 0 \end{array} \\
&\Leftrightarrow u(x) - \left\{\tfrac{\mu(A)}{\mu(A)+1-\mu(A\cup E^c)} u(b) + \tfrac{1-\mu(A\cup E^c)}{\mu(A)+1-\mu(A\cup E^c)} u(w)\right\} = 0 \\
&\Leftrightarrow u(x) - \{\mu_E(A)\,u(b) + [1-\mu_E(A)]\,u(w)\} = 0 \\
&\Leftrightarrow x \sim_E b_A w
\end{aligned}$$

It generates FH update rule of μ

$$\mu_E(A) = \frac{\mu(A \cap E)}{\mu(A \cap E) + 1 - \mu(A \cup E^c)}.$$

Since the choice of b and w are arbitrary as long as $x \sim \alpha b + (1-\alpha) w$, it concludes that, for any b, $w \in X$ with $b \succsim w$, $V_E(b_A w) = \mu_E(A) u(b) + [1 - \mu_E(A)] u(w)$. Hence if $a = x$, the statement holds and μ_E is transformed by FH rule. ■

Lemma 4 *Suppose that (i) is satisfied. Then, there exists a partition of E^c, $(T_k, T_{k-1}, \ldots, T_1)$, $k \leqq |E^c|$, $\succsim_E$ on $\mathcal{F}_2$ is represented by u and μ_E, which is a composite update of μ: for all $A \in \Sigma$*

$$\mu_E(A) = \Phi^k \circ \Phi^{k-1} \circ \cdots \circ \Phi^1 \circ \mu(A)$$

where

$$\Phi^i \circ \mu\left(A|E^i\right) \in \left\{ \frac{\mu((A \cap E_i) \cup T_i) - \mu(T_i)}{1 - \mu(T_i)}, \ \frac{\mu(A \cap E^i)}{\mu(E^i)}, \ \frac{\mu(A \cap E^i)}{\mu(A \cap E^i) + 1 - \mu(A \cup T_i)} \right\}.$$

Proof. If E^c is compositely updated, E^c is partitioned according to the order of updating and $a \in \{x^*, x_*, x\}$. Let $\mathbf{S} = \{S_m, S_{m-1}, \ldots, S_1\}$ be a partition of Ω. Such $\mathbf{S}$ is determined by $\succsim_{\Omega \setminus S_i}$ and $a^i \in \{x^*, x_*, x\}$, $i = 1, \ldots, m$. Let $T_i = S_i \cap E^c$, and $\mathbf{T} = (T_k, T_{k-1}, \ldots, T_1)$ is renumbered according to the same order in $\mathbf{S}$ if $T_i \neq \varnothing$. Let $a^i \in \{x^*, x_*, x\}$ express the conditioning act on T_i. Write $a_{\mathbf{T}} = a^k \cdots a^1$.

From lemma above, Φ^i updating every T_i by using $a^i \in \{x^*, x_*, x\}$ is consistent with B2* and B4*. It remains to prove that the sequence of updating $(T_k, T_{k-1}, \ldots, T_1)$ is also consistent with B2* and B4*. Assume $x \in X$ is such that $x_E a_{\mathbf{T}} \sim x^*_A x_{*E \setminus A} a_{\mathbf{T}}$. By construction, we have $x \sim_E x^*_A x_*$. Let $\alpha \in [0,1]$ satisfy $x \sim \alpha x^* + (1-\alpha) x_*$. Note that, since every a^i is x^*, x_*, or $\alpha x^* + (1-\alpha) x_*$, $a_{\mathbf{S}}$ can be expressed by an α-subjective mixture of two binary acts $x^*_B x_*$ and $x^*_{B'} x_*$ for some B, $B' \in \Sigma$. Furthermore, $x_E a_{\mathbf{T}}$ and $x^*_A x_{*E \setminus A} a_{\mathbf{T}}$ are also represented by a subjective mixture of two binary acts. It implies that, if they are indifferent comonotonic acts and a^i is common, then they are also indifferent in terms of $\succsim_{E^{i-1}}$.

Starting with $\succsim_E$

$$\begin{array}{lll}
x & \sim & {}_E\, x^*_A x_* \quad (\succsim_E \text{ is represented by } u \text{ and } \Phi^k \circ \Phi^{k-1} \circ \cdots \circ \Phi^1 \circ \mu) \\
 & \Leftrightarrow & x_E a^k (a^{k-1} \cdots a^1) \sim_{E^{k-1}} x^*_A x_{*E \setminus A} a^k (a^{k-1} \cdots a^1) \\
(& \succsim & {}_{E^{k-1}} \text{ is represented by } u \text{ and } \Phi^{k-1} \circ \cdots \circ \Phi^1 \circ \mu) \\
 & \Leftrightarrow & x_E a^k a^{k-1} (a^{k-2} \cdots a^1) \sim_{E^{k-2}} x^*_A x_{*E \setminus A} a^k a^{k-1} (a^{k-2} \cdots a^1) \\
(& \succsim & {}_{E^{k-2}} \text{ is represented by } u \text{ and } \Phi^{k-2} \circ \cdots \circ \Phi^1 \circ \mu) \\
 & & \vdots \\
 & \Leftrightarrow & x_E a^k a^{k-1} \cdots (a^1) \sim_{E^1} x^*_A x_{*E \setminus A} a^k a^{k-1} \cdots (a^1) \\
(& \succsim & {}_{E^{k-2}} \text{ is represented by } u \text{ and } \Phi^1 \circ \mu) \\
 & \Leftrightarrow & x_E a^k a^{k-1} \cdots a^1 \sim x^*_A x_{*E \setminus A} a^k a^{k-1} \cdots a^1 \\
(& \succsim & \text{ is represented by } u \text{ and } \mu).
\end{array}$$

By Lemma 3 and A6*, in every step, x^* and x_* are replaced by b and w with $b \succsim w$ since they produce only binary comonotonic acts. Every a^i implies Φ^i to be one of BU, DH, and FH rule, thus the indifference relation

$$x \sim_E b_A w \Leftrightarrow x_E(a^k \cdots a^1) \sim b_A w_{E\setminus A}(a^k \cdots a^1)$$

generate the composite update, which completes the proof of lemma. ■

(ii)⇒(i)

Proof. It is proper to assume the following:

- $\succsim$ has a continuous nontrivial monotonic biseparable representation $V : \mathcal{F} \to \mathbb{R}$ and a unique monotone set function $\mu : \Sigma \to [0,1]$ with $\mu(E) \in (0,1)$ as a conditioning event.
- $u(X)$ is bounded.
- $\succsim_E$ has a continuous nontrivial monotonic biseparable representation $V_E : \mathcal{F}_2 \to \mathbb{R}$ and a unique monotone set function $\mu_E : \Sigma \to [0,1]$.
- μ_E is a well-defined compositely update of μ by $\left(a^k a^{k-1} \cdots a^1\right)$, $a^i \in \{x^*, x_*, x\}$.

On $\mathcal{F}_2$, $\succsim_E$ is represented by V_E with μ_E in (ii). The reverse sequence generates μ. With this μ, define $V : \mathcal{F}_2 \to \mathbb{R}$ by $V(b_A w) = \mu(A) u(b) + [1 - \mu(A)] u(w)$, where $u(x) \equiv V_E(x)$. By Proposition 1, A3, A4*, A5*b, A6* are satisfied immediately under biseparable V.

(A1*) The event E such that $\mu(E) \in (0,1)$ is essential.

(A2*) $u(X)$ is bounded, thus there exist consequences x^*, $x_* \in X$, $x^* \succ x_*$ such that for all event $A \in \Sigma$ and all x, $y \in X$, $x^* \succsim x_A y \succsim x_*$.

(B2*) μ_E is composite update of μ by $\left(a^k a^{k-1} \cdots a^1\right)$, $a^i \in \{x^*, x_*, x\}$. For any f, $g \in \mathcal{F}_2$

$$f \sim_E g \Leftrightarrow f_E a^k a^{k-1} \cdots a^1 \sim f_E a^k a^{k-1} \cdots a^1.$$

(B4*) Since $\succsim_E$ on $\mathcal{F}_2$ has a biseparable representation, for any b, w, b', $w' \in X$, $b \succsim w$, $b' \succsim w'$,

$$V_E\left(\alpha b_A w + (1-\alpha) b'_A w'\right) = \alpha V_E\left(b_A w\right) + (1-\alpha) V_E\left(b'_A w'\right),$$

combined with monotonicity of V, $\succsim_E$ satisfies A6*. Since $\succsim$ on $\mathcal{F}_2$ also has biseparable representation V, B4* holds. ■

B2. Proofs in Applications

Proof of Theorem 2 : MEU case

Proof. (i)⇒(ii) Theorem 2 in Alon & Schmeidler (2014) proved that A0-A9 in (i) are sufficient to obtain MEU representation in a purely subjective setting. Proposition 1 assures finite utility values on extreme alternatives x^*, x_*.

Since the statement (i) is more restrictive than (i) in Theorem 1, it may begin with the result of Theorem 1. That is, there exists a partition of E^c, $(T_k, T_{k-1}, \ldots, T_1)$, $k \leqq |E^c|$ and an act on E^c, $\left(a^k a^{k-1}\cdots a^1\right)$, $a^i\in\{x^*,x_*,x\}$, $i=1,\ldots,k$ such that C_E is a composite update of C. Let $T_1=\{\omega\in E^c\mid a^i(\omega)=x^* \text{ for some } i\}$, $T_2=\{\omega\in E^c\mid a^i(\omega)=x_* \text{ for some } i\}$, and $T_3=\{\omega\in E^c\mid a^i(\omega)=x \text{ for some } i\}$. Rewrite $\left(a^k a^{k-1}\cdots a^1\right)$ as $a_{\mathbf{T}}=x_{T_3}x_{*T_2}x^*_{T_1}$. Let μ define the corresponding biseparable expression, which constitutes lower probabilities of C, $\mu(A):=\min_{p\in C}p(A)$ for every $A\in\Sigma$.

(a) Suppose $T_3=T_2=\varnothing$. Then

$$\begin{aligned}
& x_E x^* \sim f_E x^* \\
\Leftrightarrow\quad & \min_{p\in C}\int u\circ x_E x^*\, dp-\min_{p\in C}\int u\circ f_E x^*\, dp=0 \\
\Leftrightarrow\quad & u(x)-\min_{p\in C}\sum_{\omega\in E}\tfrac{p((\omega\cap E)\cup E^c)-\mu(E^c)}{1-\mu(E^c)}u\circ f=0 \\
\Leftrightarrow\quad & u(x)-\min_{p_1\in C_E}\sum_{\omega\in E}p_1(\omega)\,u\circ f=0 \\
\Leftrightarrow\quad & x\sim_E f,
\end{aligned}$$

where $C_1=\{p_1\in\Delta^0\mid p\in C\}$ such that $p_1(A)=\frac{p((A\cap E)\cup E^c)-\mu(E^c)}{1-\mu(E^c)}$ for every $A\in\Sigma$. The lower probabilities of C_1 is updated by DS rule, $\mu_1(A)=\frac{\mu((A\cap E)\cup E^c)-\mu(E^c)}{1-\mu(E^c)}$ for every $A\in\Sigma$.

(b) Suppose $T_3=T_1=\varnothing$. Then

$$\begin{aligned}
& x_E x_* \sim f_E x_* \\
\Leftrightarrow\quad & \min_{p\in C}\int u\circ x_E x_*\, dp-\min_{p\in C}\int u\circ f_E x_*\, dp=0 \\
\Leftrightarrow\quad & u(x)-\min_{p\in C}\sum_{\omega\in E}\tfrac{p(\omega\cap E)}{\mu(E)}u\circ f=0 \\
\Leftrightarrow\quad & u(x)-\min_{p_2\in C_E}\sum_{\omega\in E}p_2(\omega)\,u\circ f=0 \\
\Leftrightarrow\quad & x\sim_E f,
\end{aligned}$$

where $C_2=\{p_2\in\Delta^0\mid p\in C\}$ such that $p_2(A)=\frac{p(A\cap E)}{\mu(E)}$ for every $A\in\Sigma$. The lower probabilities of C_2 is updated by BU rule, $\mu_1(A)=\frac{\mu(A\cap E)}{\mu(E)}$.

(c) Suppose $T_2=T_1=\varnothing$. Then C_3 is updated via FB rule, $C_3=\{p_3\in\Delta^0\mid p\in C\}$ such that $p_3(A)=\frac{p(A\cap E)}{p(E)}$ for every $A\in\Sigma$. The lower probabilities of C_3 is updated by FH rule, $\mu_3(A)=\frac{\mu(A\cap E)}{\mu(A\cap E)+1-\mu((A\cap E)\cup E^c)}$.

(d) When T_i, $i=1,2,3$ includes at most one nonempty set, C is updated compositely.

$$\begin{aligned}
& x_E a_{\mathbf{T}} \sim f_E a_{\mathbf{T}} \\
\Leftrightarrow\quad & \min_{p\in C}\int u\circ x_E a_{\mathbf{T}}\, dp-\min_{p\in C}\int u\circ f_E a_{\mathbf{T}}\, d\mu=0 \\
\Leftrightarrow\quad & u(x)-\min_{p\in C}\sum_{\omega\in E^2}\frac{p\left(\left(\omega\cap E^2\right)\cup T_1\right)-\mu(T_1)}{1-\mu(T_1)}u\circ f=0 \\
\Leftrightarrow\quad & u(x)-\min_{p\in C_E}\int u\circ f\, dp=0 \\
\Leftrightarrow\quad & x\sim_E f,
\end{aligned}$$

where C_E is compositely updated by (T_3, T_2, T_1) and

$$\mu_E(A) = \frac{\mu(A \cup T_1) - \mu(T_1)}{[\mu(A \cup T_1) - \mu(T_1)] + [\mu(E \cup T_1 \cup T_3) - \mu(A \cup T_1 \cup T_3)]}$$

for every $A \subset E$.

The proof of the direction **(ii)⇒(i)** is straightforward. ■

Proof of Theorem 3 : RDU case

Proof. (i)⇒(ii) Theorem 1 in Nakamura(1990) and GM2001 proved that A0-A5b plus A10 in (i) is sufficient to obtain RDU representation. Proposition 1 assures finite utility values on extreme alternatives x^*, x_*.

Since the statement (i) is more restrictive than (i) in Theorem 1, it may begin with the result of Theorem 1. That is, there exists a partition of E^c, (T_3, T_2, T_1) such that μ_E is compositely update of μ via conditioning act $a_{\mathbf{T}} = x_{T_3} x_{*T_2} x^*_{T_1}$ defined in the proof of Theorem 2.

Choose an arbitrary non-null event $E \in \Sigma^\circ$. From the above argument, $\succsim$ and $\succsim_E$ are represented by RDU with (u, μ) and (u, μ_E) respectively.

By assumption, $\succsim_E$ satisfy B2* with $a_{\mathbf{T}}$. For any $x \in X$, any b, $w \in X$ with $b \succsim w$, and any event $A \subset E$,

$$\begin{aligned}
& x_E a_{\mathbf{T}} \sim b_A w_{E \setminus A} a_{\mathbf{T}} \\
\Leftrightarrow\ & \int u \circ x_E a_{\mathbf{T}}\, d\mu - \int u \circ b_A w_{E \setminus A} a_{\mathbf{T}}\, d\mu = 0 \\
\Leftrightarrow\ & \{[\mu(A \cup T_1) - \mu(T_1)] + [\mu(E \cup T_1 \cup T_3) - \mu(A \cup T_1 \cup T_3)]\} \times \\
& \Big\{ u(x) - \Big[\tfrac{\mu(A \cup T_1) - \mu(T_1)}{[\mu(A \cup T_1) - \mu(T_1)] + [\mu(E \cup T_1 \cup T_3) - \mu(A \cup T_1 \cup T_3)]} u(b) \\
& \qquad + \tfrac{\mu(E \cup T_3 \cup T_3) - \mu(A \cup T_1 \cup T_3)}{[\mu(A \cup T_1) - \mu(T_1)] + [\mu(E \cup T_1 \cup T_3) - \mu(A \cup T_1 \cup T_3)]} u(w) \Big] \Big\} = 0 \\
\Leftrightarrow\ & u(x) - \{\mu_E(A) u(b) + [1 - \mu_E(A)] u(w)\} = 0 \\
\Leftrightarrow\ & \int u \circ x\, d\mu_E - \int u \circ b_A w\, d\mu_E = 0 \\
\Leftrightarrow\ & x \sim_E b_A w.
\end{aligned}$$

Therefore, any binary comonotonic act f and g in $\mathcal{F}_2$, we have $f_E a_{\mathbf{T}} \sim g_E a_{\mathbf{T}} \Leftrightarrow f \sim_E g$. It holds for any $A \in \Sigma$, hence $\mu_E(A)$ is calculated as in the statement.

(ii)⇒(i) To show this part, assume the following:

- $\succsim$ is represented by $\int u \circ f\, d\mu$ and $\succsim_E$ is represented by $\int u \circ f\, d\mu_E$ as stated in (ii).
- $u(X)$ is bounded.
- μ_E is a well-defined compositely update of μ by $a_{\mathbf{T}} = x_{T_3} x_{*T_2} x^*_{T_1}$.

From CEU representations for $\succsim$, $\succsim_E$ and above arguments, A2*, A3, A4, A5*b, A10 are immediately satisfied. B2* is also true since the aforementioned equations are effective retrospectively. ■

Proof of Theorem 4 : LEU case

Proof of Proposition 2. Consider $b_A w$, b, $w \in X$ with $b \succsim w$, $A \in \Sigma^\circ$. For any $B \subset A$, $\mu(B) + \mu(A \backslash B) \leqq \mu(A)$, and for any $B' \subset A^c$, $\mu(B') + \mu(A^c \backslash B') \leqq \mu(A^c)$. If outcomes in event A or A^c are divided between any two events, it cause to smaller values. If w were evaluated by $\mu(A^c)$, then b was also evaluated by $\mu(A)$, however by superadditivity we have $\mu(A^c)\,u(w) \leqq [1 - \mu(A)]\,u(w)$, which leads a contradiction. Therefore, the maximum is $\int_L u \circ f d\mu = \mu(A)\,u(b) + [1 - \mu(A)]\,u(w)$. ■

Proof of Theorem 4. When μ is convex, the concave integral is equivalent to Choquet integral, hence it is sufficient to begin with examining $a_{\mathbf{T}} = x_{T_3} x_{*T_2} x^*_{T_1}$.

Example 4 provides an example of $\succsim$ and $\succsim_E$ that violates B2* when S_1 or S_2 is non-empty. Therefore it remains to show that $a = x_*$ meets B2* and B3.

Now, consider a binary act $b_A w$ with $b \succsim w$. As seen in the proof of Theorem 1, B2* and B3 asserts that

$$x \sim_E b_A w \Longleftrightarrow x_E x_* \sim b_A w_{E \backslash A} x_*,$$

which leads to BU rule. The optimal decomposition is $\alpha(A) = \mu(A)$, $\alpha(E \backslash A) = \mu(E) - \mu(A)$. To see this suppose that $\beta \neq \alpha$ is also optimal decomposition of Ω. If so, $E \backslash A$ is included in the optimal decomposition β by Co-decomposition additivity in Proposition 5 (Even and Lehrer, 2014), that is, $\mu(E) - \mu(A) < \mu(E \backslash A)$, contradicting μ's superadditivity. ■

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