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Time Preference and Income Convergence in a Dynamic Heckscher-Ohlin Model*

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Abstract

This paper shows that income convergence in an open-economy setting hinges upon how the time discount rate of the households is determined. As opposed to the case of constant time discount rate where cross-country income divergence may emerge, the small open economy may catch up with the rest of the world if the time discount rate increases with consumption. In contrast, either if the time discount rate decreases with consumption or if future-oriented investment of the household lowers the time discount rate, then the small open economy fails to catch up with the rest of the world under free trade of commodities.

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1 Introduction

According to the neoclassical growth theory, closed economies with identical technologies and preferences converge to the same steady state. A large number of studies on cross-country income convergence assume that each country behaves as a closed economy and, hence, they attribute international income disparity to the differences in fundamentals of each country such as technology, preferences, policy as well as institution.[1] This line of research, however, ignores the unambiguous fact that every country is interdependent: as Matsuyama (2009) emphasizes, there is no virtually closed economy in our real world and the only closed economy we know is the global economy itself. Therefore, the issue of cross-country income comparison should be discussed in an open-economy setting.

As for this issue, Atkeson and Kehoe (2000) present an insightful example of income divergence. Utilizing the standard Heckscher-Ohlin model, they focus on the behavior of a small country in the global economy where the rest of the world has already reached the steady-state equilibrium. The small country has the same technology and preferences as those of the rest of the world and the only difference is that her capital stock is smaller than the steady-state level of other countries. Now suppose that the small country opens up international trade at some specific date. If the consumption goods sector uses less capital intensive technology than the investment goods sector and if the initial capital stock of the small country is low enough, then the small country initially specializes in consumption goods production. Atkeson and Kehoe (2000) confirm that in this case the small country converges to the lower boundary of the diversification cone, so that her steady-state income is less than that of the rest of the world whose steady state is inside the diversification cone. In addition, if the initial capital of the small country is large enough to be in the diversification cone (but it is smaller than the steady-state capital of the rest of the world), then capital accumulation stops when the small country starts international trade. Therefore, the 'late-bloomers' never catch up with the 'early-bloomers' unless they refuse to open up international transactions.[2]

[1] See, for example, Barro (1997) and Barro and Sala-i-Martin (2004, Chapter 12).

[2] Chen (1992) constructs a two-country, dynamic Heckscher-Ohlin model of the world economy and analyzes the model behavior inside the diversification cone. He demonstrates that the steady-state distribution of capital between the two countries is path dependent: it is determined by a specification of the initial distribution of capital in the world economy. Restricting their attention to the small country model, Atkeson and Kehoe (2000) consider the dynamics of the small country inside as well as outside the diversification cone. Recently, Caliendo (2010) examines a two-country Heckscher-Ohlin model with Cobb-Douglas production functions and completely characterizes the global behavior of the world economy.

Atkeson and Kehoe's finding demonstrates how growth pattern of an open economy may dramatically differ from growth process of the closed economy counterpart. In this paper we reconsider the issue of income divergence in an open economy under alternative specifications of time preference of the households. While Atkeson and Kehoe (2000) fix the rate of time preference of the representative household, we assume that the rate of time preference is endogenously determined. We first assume that the time discount rate increases with consumption (the case of increasing marginal impatience). This formulation is initiated by Uzawa (1968) and the majority of literature assuming endogenous time preference has employed this hypothesis. Given this assumption, we show that the steady state of the small country coincides with the steady state of the rest of the world: there is no other stationary state for the small country. Moreover, the steady state is saddle stable under the standard conditions on the utility and time discount functions. Therefore, in this case the small country ultimately catches up with the rest of the world regardless of her initial level of capital stock.

We then examine the opposite case where the time discount rate decreases with consumption (the case of decreasing marginal impatience). By contrast to the model with increasing marginal impatience, in this case the small country may have a steady state out of the diversification cone. Moreover, while the steady state inside the diversification cone is the same as that of the rest of the world, it is generally unstable. As a result, if the time discount rate is a decreasing function of consumption, the small country starting with a lower capital cannot catch up with the rest of the world.

We also examine the model behavior under the Becker and Mulligan's (1997) hypothesis. Here, we assume that the time discount rate is a decreasing function of the future-oriented investment of the household. In this formulation, patience of the household can be strengthened by the purposeful investment. We show that the Becker-Mulligan modeling yields the same outcome as that of the model with decreasing marginal impatience. Namely, the small country cannot converge to the steady state of the rest of the world either.

The next section sets up the analytical framework. Section 3 discusses the model with increasing marginal impatience, while Section 4 considers the case of decreasing marginal

See also Bajona and Kehoe (2010), Cunat and Maffezzoli (2001) and Gaitan and Roe (2007) for further studies on the two-country, dynamic Hechscher-Ohlin models. All of the contributions mentioned above assume that the time discount rate of households stays constant over time.

impatience. Section 5 uses the Becker-Mulligan hypothesis to show that the basic results are close to the case of decreasing marginal impatience. Concluding remarks are given in Section 6.

2 The Model

Consider a small country whose production technology and preference structure are the same as those of the rest of the world. Following Atkeson and Kehoe (2000), it is assumed that the rest of the world consists of identical countries engaging in free trade of commodities. Hence, under the standard conditions of Heckscher-Ohlin model, the rest of the world behaves as a closed economy. We assume that the rest of the world has already reached the steady-state equilibrium and the world relative price is set by the steady-state conditions in the rest of the world.

2.1 Production

There are two production sectors: the first sector produces investment goods and the second sector produces consumption goods. The production function of each sector satisfies the neoclassical properties and it is given by

$$Y_i = L_i f_i\left(k_i\right), \quad k_i = K_i/L_i, \quad i = 1, 2,$$

where Y_i, K_i and L_i are output, capital and labor of sector i. It is assumed that the productivity function, $f_i\left(k_i\right)$, is strictly increasing and concave in k_i, and it satisfies the Inada conditions. In the competitive factor and product markets, the real rent r and the real wage w satisfy:

$$r = p f_1'\left(k_1\right) = f_2'\left(k_2\right),$$

$$w = p[f_1\left(k_1\right) - k_1 f_1'\left(k_1\right)] = f_2\left(k_2\right) - k_2 f_2'\left(k_2\right),$$

where p denotes the price of investment goods in terms of consumption goods. As is well known, these conditions relate k_i to p in the following manner:[3]

$$k_i = k_i(p), \quad i = 1, 2, \tag{1}$$

$$\text{sign}[k_i'(p)] = \text{sign}\left[k_2(p) - k_1(p)\right]. \tag{2}$$

We assume that the consumption goods sector uses less capital intensive technology than the investment goods sector:

$$k_1(p) > k_2(p) \quad \text{for all feasible } p.$$

In view of (1), the factor prices are expressed as

$$r(p) = f_2'(k_2(p)), \tag{3}$$

$$w(p) = f_2(k_2(p)) - k_2(p) f_2'(k_2(p)). \tag{4}$$

Production factors cannot cross the borders, so that their full employment conditions are

$$K_1 + K_2 = K, \quad L_1 + L_2 = L = 1.$$

Here, we assume that labor supply is constant and normalized to unity. The full employment conditions yield

$$L_1 k_1(p) + (1 - L_1) k_2(p) = k \,(= K/L = K).$$

As a result, the small country specializes in consumption (investment) goods production for $k \in (0, k_2(p)]$ ($k \in [k_1(p), \infty)$), whereas she produces both consumption and investment

[3] The factor price equations yield

$$\frac{f_i(k_i) - k_i f_i'(k_i)}{f_i'(k_i)} = \frac{w}{r}, \quad i = 1, 2.$$

This shows that k_i is positively related to w/r. Thus, the relative price satisfies

$$p = \frac{f_2'(k_2(\omega))}{f_1'(k_1(\omega))},$$

where $\omega = w/r$ and $k_i'(\omega) > 0$. Using the above, we can derive (2).

goods for $k \in [k_2(p), k_1(p)]$.

2.2 Households

The objective function of the representative household is

$$U = \int_0^\infty u(c) e^{-z} dt, \tag{5}$$

where the instantaneous utility function $u(c)$ is assumed to be strictly negative, monotonically increasing and strictly concave in c. Here, z represents the endogenous time-preference rate and is formed as follows:

$$\dot{z} = \rho(c). \tag{6}$$

In this formulation of endogenous time preference, the literature has employed two alternative assumptions. The majority of the studies follow Uzawa's (1968) formulation in which the instantaneous time discount rate, $\rho(c)$, monotonically increases with c. This assumption is called *increasing marginal impatience*: a higher level of current consumption makes an individual less patient. Since this assumption does not fit well the conventional wisdom, some authors such as Das (2003), Chang (2009) and Hirose and Ikeda (2012a) assume the opposite formulation in which a higher consumption lowers the time discount rate (the case of *decreasing marginal impatience*). Since the distinction between increasing and decreasing marginal impatience will be critical for our discussion, at this stage we do not specify the sign of $\rho'(c)$. We also specify the sign of $\rho''(c)$ later.

We first assume that the small country taking as given the world price $\bar{p}$ opens up international trade at $t = 0$, and that her initial capital stock satisfies $k_0 < k_2(\bar{p})$. Then the small country produces consumption goods alone, so that the representative household maximizes U subject to (6) and

$$\dot{k} = \frac{1}{\bar{p}}(f_2(k) - c) - \delta k, \quad 0 < \delta < 1, \tag{7}$$

where δ is the depreciation rate of capital. We set up the Hamiltonian function in such a way that

$$H = u(c) e^{-z} + \hat{q}\left[\frac{1}{\bar{p}}(f_2(k) - c) - \delta k\right] - \hat{\mu}\rho(c),$$

where $\hat{q}$ and $\hat{\mu}$ respectively denote the costate variables of k and z. Note that $\hat{\mu}$ has a negative value. Letting $q = e^z \hat{q}$ and $\mu = e^z \hat{\mu}$, we find that the optimization conditions include the following:

$$q = \bar{p}u'(c) - \mu \bar{p}\rho'(c), \tag{8}$$

$$\dot{q} = q \left[\rho(c) + \delta - \frac{f_2'(k)}{\bar{p}} \right], \tag{9}$$

$$\dot{\mu} = -u(c) + \mu\rho(c). \tag{10}$$

Note that the second-order condition for maximizing the Hamiltonian function with respect to c requires that

$$u''(c) - \mu\rho''(c) < 0. \tag{11}$$

Condition (8) gives

$$c = c(q, \mu),$$

where

$$c_q = \frac{1}{\bar{p}(u'' - \mu\rho'')}, \qquad c_\mu = \frac{\rho'}{u'' - \mu\rho''}. \tag{12}$$

From the second-order condition (11), we find that

$$c_q(c, \mu) < 0, \qquad \text{sign}[c_\mu(q, \mu)] = -\ \text{sign}[\rho'(c)]. \tag{13}$$

When the small country is in the diversification cone where she produces both goods, the factor price equalization holds. Hence, the household maximizes U subject to

$$\dot{k} = \frac{r(\bar{p})k + w(\bar{p}) - c}{\bar{p}} - \delta k \tag{14}$$

and (6). In this case, the behavior of the utility price of capital is given by

$$\dot{q} = q \left[\rho(c) + \delta - \frac{r(\bar{p})}{\bar{p}} \right]. \tag{15}$$

2.3 Trade Patterns and Dynamic Systems

To sum up, the small country specializes in consumption goods production when her per capita capital is less than $k_2(\bar{p})$, while she specializes in investment goods production when her per capita capital is higher than $k_1(\bar{p})$. Inside the diversification cone where

$k_2(\bar{p}) < k < k_1(\bar{p})$, the small country produces both consumption and investment goods. Therefore, the dynamic behavior of the small country in each trade regime is respectively given by the following:

$$\left.\begin{aligned} \dot{q} &= q\left[\rho\left(c\left(q,\mu\right)\right) + \delta - \frac{f_2'\left(k\right)}{\bar{p}}\right], \\ \dot{k} &= \frac{1}{\bar{p}}[f_2\left(k\right) - c\left(q,\mu\right)] - \delta k, \\ \dot{\mu} &= -u\left(c\left(q,\mu\right)\right) + \mu\rho\left(c\left(q,\mu\right)\right), \end{aligned}\right\} \text{ for } k \in (0, k_2(\bar{p})], \tag{S1}$$

$$\left.\begin{aligned} \dot{q} &= q\left[\rho\left(c\left(q,\mu\right)\right) + \delta - \frac{r\left(\bar{p}\right)}{\bar{p}}\right], \\ \dot{k} &= \frac{r\left(\bar{p}\right)k + w\left(\bar{p}\right) - c\left(q,\mu\right)}{\bar{p}} - \delta k, \\ \dot{\mu} &= -u\left(c\left(q,\mu\right)\right) + \mu\rho\left(c\left(q,\mu\right)\right), \end{aligned}\right\} \text{ for } k \in [k_2(\bar{p}), k_1(\bar{p})], \tag{S2}$$

$$\left.\begin{aligned} \dot{q} &= q[\rho\left(c\left(q,\mu\right)\right) + \delta - f_1'\left(k\right)], \\ \dot{k} &= f_1\left(k\right) - \frac{c\left(q,\mu\right)}{\bar{p}} - \delta k, \\ \dot{\mu} &= -u\left(c\left(q,\mu\right)\right) + \mu\rho\left(c\left(q,\mu\right)\right), \end{aligned}\right\} \text{ for } k \in [k_1(\bar{p}), +\infty). \tag{S3}$$

Since we intend to explore whether the small country can catch up with the rest of the world, in what follows, we restrict our attention to systems (S1) and (S2).

3 The Case of Increasing Marginal Impatience

In this section, we assume that $\rho(c)$ monotonically increases with c. This assumption has been most frequently employed in the literature.[4]

3.1 Steady State

In the case of increasing marginal impatience, conditions in (13) show $c_q(q,\mu) < 0$ and $c_\mu(q,\mu) < 0$. First, consider (S1). If system (S1) has a steady state, it should satisfy the following conditions:

$$\rho(c^*) + \delta = \frac{f_2'(k^*)}{\bar{p}}, \tag{16}$$

[4] Implication and analytical properties of Uzawa's (1968) formulation were discussed in detail by Epstein (1987) and Obstfeld (1990). See also Chang (1994) for further investigation.

$$c^* = f_2(k^*) - \delta \bar{p} k^*, \tag{17}$$

where k^* and c^* denote the steady-state values of k and c in the small country. Since the rest of the world stays inside the diversification cone, their steady state is described by the steady-state equilibrium of (S2). Hence, the steady-state levels of k and c in the rest of the world (denoted by $\bar{k}$ and $\bar{c}$) are determined by

$$\rho(\bar{c}) + \delta = \frac{1}{\bar{p}} f_2'(k_2(\bar{p})) = f_1'(k_1(\bar{p})), \tag{18}$$

$$\begin{aligned} \bar{c} &= r(\bar{p})\,\bar{k} + w(\bar{p}) - \delta \bar{p} \bar{k} \\ &= f_2'(k_2(\bar{p}))\,\bar{k} + f_2(k_2(\bar{p})) - k_2(\bar{p})\, f_2'(k_2(\bar{p})) - \delta \bar{p} \bar{k}. \end{aligned} \tag{19}$$

Inspecting the steady-state conditions given above reveals the following:

Lemma 1. *If the time discount rate increases with consumption, then (i) system (S1) has no feasible steady state; and (ii) system (S2) has a unique steady state that is identical to the steady state of the rest of the world.*

Proof. (i) Suppose that there is a steady-state level of $k^* \in (0, k_2(\bar{p})]$. Since $f_2(k)$ is strictly concave in k and $k^* \leq k_2(\bar{p})$, it holds that

$$\frac{f_2(k^*) - f_2(k_2(\bar{p}))}{k^* - k_2(\bar{p})} \geq f_2'(k_2(\bar{p})),$$

or

$$f_2(k^*) - f_2(k_2(\bar{p})) \leq f_2'(k_2(\bar{p}))\,(k^* - k_2(\bar{p})).$$

Therefore, from (17), (18) and (19) the following relations hold:

$$\begin{aligned} c^* - \bar{c} &= f_2'(k_2(\bar{p}))\left(k_2(\bar{p}) - \bar{k}\right) + f_2(k^*) - f_2(k_2(\bar{p})) + \delta \bar{p}\left(\bar{k} - k^*\right) \\ &\leq f_2'(k_2(\bar{p}))\left(k_2(\bar{p}) - \bar{k}\right) + f_2'(k_2(\bar{p}))\,(k^* - k_2(\bar{p})) + \delta \bar{p}\left(\bar{k} - k^*\right) \\ &= \left\{f_2'(k_2(\bar{p})) - \delta \bar{p}\right\}\left(k^* - \bar{k}\right) \\ &= \bar{p}\rho(\bar{c})\left(k^* - \bar{k}\right) < 0. \end{aligned}$$

However, from (16) and (18) we obtain

$$\rho(c^*) + \delta = \frac{f_2'(k^*)}{\bar{p}} \geq \frac{f_2'(k_2(\bar{p}))}{\bar{p}} = \rho(\bar{c}) + \delta.$$

Since $\rho(c)$ is assumed to be strictly increasing in c, the above shows $c^* \geq \bar{c}$, implying that there is no steady state of the small country for $k \in [0, k_2(\bar{p})]$. Hence, system (S1) has no stationary-state solution.[5]

(ii) The steady-state values of k and c in system (S2) satisfy the following:

$$\rho(c^*) + \delta = f_1'(k_1(\bar{p})), \tag{20}$$

$$c^* = r(\bar{p})k^* + w(\bar{p}) - \bar{p}\delta k^*. \tag{21}$$

Since $\rho(c)$ monotonically increases with c, (18) and (20) mean that $c^* = \bar{c}$. Then in view of (19) and (21), we find that $k^* = \bar{k}$. ■

Notice that in the steady state of (S2) q and μ are given by the following conditions:

$$\bar{q} = \bar{p}u'(\bar{c}) - \bar{\mu}\bar{p}\rho'(\bar{c}),$$

$$-u(\bar{c}) + \bar{\mu}\rho(\bar{c}) = 0.$$

The above equations show that the steady-state levels of $\bar{q}$ and $\bar{\mu}$ are also uniquely determined.

In view of the above lemma, we obtain the following proposition:

Proposition 1 *If the time discount rate monotonically increases with consumption, the small country has a unique steady state that is identical to the steady state of the rest of the world.*

This result is in stark contrast to the model with a fixed time discount rate. As proved by Atkeson and Kehoe (2000), if the initial level of capital of the small country is $k_0 \in (0, k_2(\bar{p})]$, then the steady-state conditions of the small country outside the diversification cone include $f_2'(k^*)/\bar{p} = \rho + \delta$. Since $f_2'(k_2(\bar{p}))/\bar{p} = \rho + \delta$ holds in the rest of the world, the steady-state capital of the small country is $k^* = k_2(\bar{p}) < \bar{k}$. If $k_0 \in [k_2(\bar{p}), k_1(\bar{p})]$, the shadow value of capital follows

$$\dot{q} = q\left[\rho + \delta - \frac{r(\bar{p})}{\bar{p}}\right] = 0 \quad \text{for all } t \geq 0,$$

[5] In the similar manner, we can show that there is no steady state for $k \in [k_1(\bar{p}), \infty)$. Therefore, the small country has a stationary equilibrium only within the diversification cone.

because $\rho + \delta = r(\bar{p})/\bar{p}$ is established in the rest of the world. Thus, q stays constant over time and, hence, $c = u'^{-1}(q)$ is fixed as well. As a result, the optimal level of c is determined to satisfy

$$\frac{1}{\bar{p}}\left[r(\bar{p})k + w(\bar{p}) - c\right] - \delta k = \dot{k} = 0 \text{ for all } t \geq 0.$$

Therefore, the steady-state level of capital of the small country is given by $k^* = k_0$.

In contrast to the model with a fixed time discount rate, in our setting the household is more patient when her consumption level is low. Therefore, a late-bloomer who starts with lower level of capital stock than the rest of the world will attain a higher rate of savings, which promotes capital accumulation.

3.2 Stability

As for the stability of the steady-state equilibrium, the following result holds:

Proposition 2 *If the time discount rate is a strictly increasing function of consumption, the steady-state equilibrium of the small country satisfies local saddle stability.*

Proof. We consider local stability of dynamic system (S2). The coefficient matrix of system linearized at $(\bar{q}, \bar{k}, \bar{\mu})$ is given by

$$J_1 = \begin{bmatrix} q\rho' c_q & 0 & q\rho' c_\mu \\ -c_q/\bar{p} & \rho & -c_\mu/\bar{p} \\ (-u' + \mu\rho')c_q & 0 & (-u' + \mu\rho')c_\mu + \rho \end{bmatrix},$$

where each element of J_1 is evaluated at $(\bar{q}, \bar{k}, \bar{\mu})$. Denoting the eigenvalue of J_1 by x, the eigen equation of J_1 is as follows:

$$(x - \rho)(x^2 - \rho x + q\rho\rho' c_q) = 0.$$

Since $q\rho\rho' c_q < 0$ holds under $\rho' > 0$ and $c_q < 0$, the eigen equation of J_1 has one stable and two unstable roots. Noting that system (S2) has two jumpable variables (q and μ) and one non-jumpable variable (k), we confirm that the steady state is saddle stable. ■

If the stable saddle path spans over the entire domain of $[k_2(\bar{p}), k_1(\bar{p})]$, then we can find a unique path converging to $(\bar{k}, \bar{c})$ starting from $k_0 = k_2(\bar{p})$. Denote the initial value of optimal consumption by $\hat{c}_0$ corresponding to $k_0 = k_2(\bar{p})$. Then if the small country starts with the initial capital $k_0 \in (0, k_2(\bar{p})]$, we may find a unique path governed by (S1) which passed point $(k_2(\bar{p}), \hat{c}_0)$. If this is the case, regardless of her initial level of capital, the small country ultimately converges to the same steady state the rest of the world has realized. In other words, if the preference structure exhibits increasing marginal impatience, the late-bloomer can ultimately catch up with the early-bloomers even in the presence of free trade of commodities.

4 The Case of Decreasing Marginal Impatience

We now assume that the time discount rate monotonically decreases with consumption. As oppose to the Uzawa formulation, this assumption has been often supported by the conventional wisdom claiming that the rich must be patient to accumulate a large amount of wealth.[6]

4.1 Steady State

First, suppose that the small country specializes in consumption goods production. The steady-state characterization of (S1) is identical to the case of increasing marginal impatience, so that the steady-state conditions are given by (16) and (17). In contrast to the case where $\rho'(c) > 0$, if $\rho'(c) < 0$, the small country specializing in consumption goods production may have a feasible steady state.

Lemma 2. *If the time discount rate is a strictly decreasing function of consumption, then system (S1) may have an interior steady state.*

Proof. As in the first half of the proof of Proposition 1, we can show that the steady-state level of consumption of the small country, c^*, satisfies $c^* < \bar{c}$, if $k^* \leq k_2(\bar{p}) < \bar{k}$. In addition, conditions (16) and (18) present

$$\rho(c^*) + \delta = \frac{f_2'(k^*)}{\bar{p}} \geq \frac{f_2'(k_2(\bar{p}))}{\bar{p}} = \rho(\bar{c}) + \delta.$$

Since $\rho'(c) < 0$, the above conditions may hold when $c^* < \bar{c}$. ■

[6] Das (2003) and Chang (2009) inspect the analytical property of a one-sector optimal growth model with decreasing marginal impatience. Using two-country models where labor is the only factor of production, Hirose and Ikeda (2012a and 2012b) discuss consequences of the assumption of decreasing marginal impatience in open economies.

Note that if $k \in (k_2(\bar{p}), k_1(\bar{p}))$ and thus the small country produces both investment and consumption goods, then monotonicity of $\rho(c)$ ensures that in the steady-state equilibrium the small country attains the same levels of capital and consumption as those of the rest of the world.

4.2 Stability

Now examine stability of the steady-state equilibrium of the small country. The main findings are as follows:

Proposition 3 *Suppose that the time discount rate strictly decreases with consumption. Then (i) the steady state of (S1) is saddle stable if it holds that* $\rho(c^*)\rho'(c^*)\bar{p}^2 > f_2''(k^*)$*; and (ii) the steady state of* $(S2)$ *is totally unstable.*

Proof. (i) The coefficient matrix of $(S1)$ linearized at the steady state is given by

$$J_2 = \begin{bmatrix} q\rho' c_q & -qf_2''/\bar{p} & q\rho' c_\mu \\ -c_q/\bar{p} & \rho & -c_\mu/\bar{p} \\ (-u' + \mu\rho')\, c_q & 0 & (-u' + \mu\rho')\, c_\mu + \rho \end{bmatrix},$$

where each element is evaluated at the steady state. In view of (8) and (12), the trace of J_2 is written as

$$\begin{aligned} \mathrm{tr}J_2 &= \left(-u' + \mu\rho'\right) c_\mu + q\rho' c_q + 2\rho \\ &= -c_\mu q/\bar{p} + q\rho' c_q + 2\rho = 2\rho > 0. \end{aligned}$$

Similarly, the determinant of J_2 is

$$\det J_2 = q\rho c_q \left(\rho\rho' - \frac{f_2''}{\bar{p}^2}\right).$$

Since $c_q < 0$, $\det J_2 < 0$ if and only if $\rho\rho'\bar{p}^2 > f_2''$. If this condition holds, the eigen equation of J_2 has one stable and two unstable roots, so that local saddle stability is established.

(ii) As shown in the proof of Proposition 2, the eigen equation of matrix J_1 in the proof of Proposition 2 is

$$(x - \rho)\left(x^2 - \rho x + q\rho\rho' c_q\right) = 0.$$

When $\rho' < 0$, equation $x^2 - \rho x + q\rho\rho' c_q = 0$ has two roots with positive real parts, so the system has three unstable roots and thus the steady state is unstable around the steady state. ■

Consequently, if the initial capital of the small country satisfies $k_0 \in (k_2(\bar{p}), k_1(\bar{p}))$, there is no converging path leading to the steady state of the rest of the world. On the other hand, saddle stability of the steady state outside the diversification cone means that there may exist a unique path that starts from $k_0 = k_2(\bar{p})$ and converges to (k^*, c^*). Denoting the corresponding initial value of c by $\hat{c}_0$, we may find that the path starting with $k_0 \in \left(k_2(\bar{p}), \bar{k}\right)$ passes point $(k_2(\bar{p}), \hat{c}_0)$ and converges to (k^*, c^*). In this case, the small country whose initial capital is less than $\bar{k}$ initially produces both consumption and investment goods, and then she will end up with specialization in consumption goods. Namely, free trade of commodities prevents the small country from catching up with the rest of the world.

When the time discount rate decreases with consumption, a late-bloomer with low levels of income and consumption is less patient than the households in the rest of the world. As a result, the small country fails to accumulate enough capital to attain the steady-state equilibrium which the rest of the world has already reached.

5 Investment for Patience

So far, we have assumed that the time discount rate is endogenously determined by consumption activities of the household. In this section we examine an alternative hypothesis of endogenous time preferences based on Becker and Mulligan (1997). Those authors assume that the time discount rate may be reduced by the future-oriented investment spending of the household. In their formulation, patience is a kind of capital that can be accumulated by purposeful investment such as spending for education, information and health. The Becker-Mulligan hypothesis has attracted considerable attention in various fields in economics and its macroeconomic implication has been investigated by several authors.[7]

[7] For example, Stern (2006) explores a one-sector optimal growth model with the Becker-Mulligan type time preference. See also Nakamoto (2009) and Kawagishi (2012) for applications of the Becker-Mulligan approach.

5.1 Model with the Becker-Mulligan Hypothesis

Under the Becker-Mulligan hypothesis, the time discount rate is a decreasing function of individual investment in future-oriented resources denoted by s:

$$\rho = \rho(s), \qquad \rho'(s) < 0.$$

The future-oriented investment is assumed to use investment goods.[8] In addition, along the lines of the existing studies on the Becker-Mulligan hypothesis, we assume that $\rho''(s) > 0$. We also assume that $u(c) > 0$ to ensure that an increase in investment in future-oriented resources has a positive effect on utility. Given these conditions, the representative household of the small country maximizes U in (5) subject to

$$\dot{k} = \frac{1}{\bar{p}}\big(rk + w - c\big) - s - \delta k, \tag{22}$$

$$\dot{z} = \rho(s), \tag{23}$$

together with the initial conditions on k_0 and $z_0(= 0)$.

The Hamiltonian function for this problem is given by

$$H = u\left(c\right) e^{-z} + \hat{q}\left[\psi\left(k, c, s; \bar{p}\right)\right] - \hat{\mu}\rho\left(s\right),$$

where

$$\psi\left(k, c, s; \bar{p}\right) = \begin{cases} \dfrac{1}{\bar{p}}\big(f_2(k) - c\big) - s - \delta k \quad \text{for} \;\; k \in (0, k_2(\bar{p})), \\ \dfrac{1}{\bar{p}}\big(r(\bar{p})k + w(\bar{p}) - c\big) - s - \delta k \quad \text{for } k \in [k_2(\bar{p}), k_1(\bar{p})]. \end{cases}$$

Letting $q = e^z \hat{q}$ and $\mu = e^z \hat{\mu}$, we see that the optimal choice of consumption and that of investment for patience respectively satisfy

$$u'\left(c\right) = \frac{q}{\bar{p}} \;\; \text{and} \;\; \rho'\left(s\right) = -\frac{q}{\mu}.$$

Using these first-order conditions as well as the canonical equations of q and μ, we can derive the following dynamic systems, each of which respectively corresponds to (S1) and

[8] In an open economy with free trade, the main results do not hinge upon whether the future-oriented investment needs consumption goods or investment goods.

(S2) used in the previous sections:

$$\left.\begin{aligned}
\dot{c} &= \{\sigma(c)\}^{-1}\left[\frac{f_2'(k)}{\bar{p}} - \delta - \rho(s)\right], \\
\dot{s} &= \{\gamma(s)\}^{-1}\left[\frac{f_2'(k)}{\bar{p}} - \delta + \frac{u(c)}{\bar{p}u'(c)}\rho'(s)\right], \\
\dot{k} &= \frac{1}{\bar{p}}\big(f_2(k) - c\big) - s - \delta k,
\end{aligned}\right\} \text{for} \quad k \in (0, k_2(\bar{p})), \tag{S1'}$$

$$\left.\begin{aligned}
\dot{c} &= \{\sigma(c)\}^{-1}\left[\frac{r(\bar{p})}{\bar{p}} - \delta - \rho(s)\right], \\
\dot{s} &= \{\gamma(s)\}^{-1}\left[\frac{r(\bar{p})}{\bar{p}} - \delta + \frac{u(c)}{\bar{p}u'(c)}\rho'(s)\right], \\
\dot{k} &= \frac{1}{\bar{p}}\big(r(\bar{p})k + w(\bar{p}) - c\big) - s - \delta k,
\end{aligned}\right\} \text{for} \quad k \in [k_2(\bar{p}), k_1(\bar{p})], \tag{S2'}$$

where

$$\sigma(c) \equiv -\frac{u''(c)}{u'(c)}, \quad \gamma(s) \equiv -\frac{\rho''(s)}{\rho'(s)}.$$

Again, since we are concerned with the small country whose initial capital is less than that of the rest of the world, we do not consider the case where $k \in [k_1(\bar{p}), \infty)$.

5.2 Steady State

We first consider dynamic system (S1'). If system (S1') has an interior steady state, the following expressions hold:

$$\frac{f_2'(k^*)}{\bar{p}} = \delta + \rho(s^*), \tag{24a}$$

$$\frac{f_2'(k^*)}{\bar{p}} = \delta - \frac{u(c^*)}{\bar{p}u'(c^*)}\rho'(s^*), \tag{24b}$$

$$c^* = f_2(k^*) - \bar{p}\big(s^* + \delta k^*\big), \tag{24c}$$

where k^*, c^* and s^* respectively denote the steady-state values of k, c and s of the small country. Since the rest of the world stays inside the diversification cone, their steady-state levels of k, c and s (denoted by $\bar{k}$, $\bar{c}$ and $\bar{s}$) satisfy the following conditions:

$$\frac{f_2'(k_2(\bar{p}))}{\bar{p}} = f_1'(k_1(\bar{p})) = \delta + \rho(\bar{s}), \tag{25a}$$

$$\frac{f_2'(k_2(\bar{p}))}{\bar{p}} = f_1'(k_1(\bar{p})) = \delta - \frac{u(\bar{c})}{\bar{p}u'(\bar{c})}\rho'(\bar{s}), \tag{25b}$$

$$\begin{aligned} \bar{c} &= r(\bar{p})\bar{k} + w(\bar{p}) - \bar{p}\big(\bar{s} + \delta\bar{k}\big) \\ &= f_2'(k_2(\bar{p}))\bar{k} + f_2(k_2(\bar{p})) - k_2(\bar{p})f_2'(k_2(\bar{p})) - \bar{p}\big(\bar{s} + \delta\bar{k}\big). \end{aligned} \tag{25c}$$

Using the steady-state conditions displayed above, we obtain the following:

Lemma 3. *Under the Becker-Mulligan hypothesis, (i) system (S1') may involve an interior steady state; and (ii) the steady state of system (S2') is the same as the steady state of the rest of the world.*

Proof. See Appendix. ■

The above lemma demonstrates that the steady-state characterization under the Becker-Mulligan hypothesis is similar to the case of decreasing marginal impatience.

5.3 Stability

We find that stability properties of systems (S1') and (S2') are also close to the dynamic property of the model with decreasing marginal impatience.

Proposition 4 *Under the Becker-Mulligan hypothesis, (i) the steady state in (S1') is saddle stable if* $-\frac{(\rho')^2}{f_2''} < \frac{u'}{u} < -\frac{1}{\bar{p}}\left(\frac{\rho''}{\rho'}\right)$*; and (ii) the steady state of (S2') is totally unstable.*

Proof. (i) The coefficient matrix of (S1') linearized at the steady state is given by

$$J_3 = \begin{bmatrix} 0 & -\sigma^{-1}\rho' & \sigma^{-1}\left(\frac{f_2''}{\bar{p}}\right) \\ \gamma^{-1}\left\{1 - \frac{uu''}{(u')^2}\right\}\frac{\rho'}{\bar{p}} & \gamma^{-1}\left(\frac{u}{\bar{p}u'}\right)\rho'' & \gamma^{-1}\left(\frac{f_2''}{\bar{p}}\right) \\ -\frac{1}{\bar{p}} & -1 & \frac{f_2'}{\bar{p}} - \delta \end{bmatrix}.$$

Calculating $\mathrm{tr}J_3$ and $\det J_3$, we obtain

$$\mathrm{tr}J_3 = 2\rho > 0,$$
$$\det J_3 = \sigma^{-1}\gamma^{-1}\rho'\left(\frac{u}{u'}\right)\left(\frac{f_2''}{\bar{p}^2}\right)\Omega,$$

where

$$\Omega \equiv \frac{1}{\bar{p}}\left(\frac{\rho''}{\rho'}\right) - \frac{(\rho')^2}{f_2''} + \frac{u''}{u'}\left[1 + \left(\frac{u}{u'}\right)\left\{\frac{(\rho')^2}{f_2''}\right\}\right].$$

Now assume that

$$-\frac{(\rho')^2}{f_2''} < \frac{u'}{u} < -\frac{1}{\bar{p}}\left(\frac{\rho''}{\rho'}\right). \tag{26}$$

Under (26), it is ensured that there exists a unique steady state. In addition, it follows from (26) that $\Omega < 0$, implying that $\mathrm{tr}J_3 > 0$ and $\det J_3 < 0$, so that the eigen equation of J_3 has one stable and two unstable roots. Since (S1') involves two jumpable variables, c and s, the steady state is saddle stable.

(ii) The coefficient matrix of (S2') linearized at the steady state is

$$J_4 = \begin{bmatrix} 0 & -\dfrac{\rho'}{\sigma} & 0 \\ \gamma^{-1}\left\{1 - \dfrac{uu''}{(u')^2}\right\}\dfrac{\rho'}{\bar{p}} & \gamma^{-1}\left(\dfrac{u}{\bar{p}u'}\right)\rho'' & 0 \\ -\dfrac{1}{\bar{p}} & -1 & \dfrac{r}{\bar{p}} - \delta \end{bmatrix}.$$

The eigen equation of J_4 is written as

$$(x - \rho)\left[x^2 - \rho x + \frac{(\rho')^2}{\bar{p}\sigma\gamma}\left\{1 - \frac{uu''}{(u')^2}\right\}\right] = 0,$$

where x denotes eigenvalue of J_4. The above shows that the eigen equation of J_4 has three unstable roots. Since the dynamic system (S2') has one non-jumpable variable, k, the steady state is locally unstable. ■

The above proposition means that if $k_0 \in (0, k_2(\bar{p}))$, the small country may converge to the interior steady state of the regime where she specializes in consumption goods production. By contrast, the small country cannot reach the interior steady state inside the diversification cone if $k_0 \in [k_2(\bar{p}), k_1(\bar{p})]$. Consequently, the dynamic behavior of the small country exhibits essentially the same patterns as those obtained in the model discussed in

Section 4.

6 Conclusion

We have reconsidered the Atkeson-Kehoe proposition of income divergence in the Heckscher-Ohlin setting. We have confirmed that the income divergence under free trade critically hinges upon the assumption that the time discount rate of the representative household is fixed. If the time discount rate increases with consumption, the poor tend to be more patient than the rich. This promotes savings and capital accumulation of the households in the small country who open up international trade with lower levels of capital and income than the foreign households. As a result, unlike the case with a constant time discount rate, free trade of commodities will not prevent the small country from catching up with the rest of the world. By contrast, either if the time discount rate decreases with consumption or if it may be lowered by the purposeful investment of the household, then the poor who can conduct a smaller level of consumption or investment for patience have a higher rate of time discount than the rich. Hence, the small country starting with low levels of capital and income would converge to a steady state with a strictly lower level of income than that attained in the rest of the world. Furthermore, the steady state realized by the rest of the world tends to be unattainable for the small country, because it is totally unstable from the view point of the small country. Consequently, the income-divergence result obtained in the case of a fixed time discount rate takes a more prominent form under the assumption of decreasing marginal impatience or under the Becker-Mulligan hypothesis.

In this paper, using the traditional Heckscher-Ohlin framework, we have focused on the role of variable rate of time preference in a small country model where the rest of the world has already reached its steady-state equilibrium. Since our discussion treats a restrictive environment, future studies on the relation between time preference and international income convergence should consider more general models. In the existing literature on trade and growth, many authors have extended the baseline, dynamic Heckscher-Ohlin model.[9] Since most of those studies assume that the time discount rate is fixed, it deserves

[9] For example, Nishimura and Shimomura (2002) and Hu and Mino (2013) introduce production externalities into two-country, dynamic Heckscher-Ohlin models. Hu and Mino (2013) also consider international lending and borrowing. Chen et al. (2008) consider consumption externalities in the Heckscher-Ohlin setting. Acemoglu and Ventura (2002) and Ventura (1997) explore growth models of the world economy with

further studies to explore the role of variable time preference in those generalized models.

Appendix: Proof of Lemma 3

(i) If the dynamic system (S1') has an interior steady state, it follows that $k^* < k_2(\bar{p})$, so that $f_2'(k^*) > f_2'(k_2(\bar{p}))$. Thus, from (24a) and (25a), we find that $s^* < \bar{s}$ holds because $\rho(\cdot)$ is a decreasing function of s.

From (24b), (25b) and $f_2'(k^*) > f_2'(k_2(\bar{p}))$, we obtain the following inequality:

$$-\frac{u(c^*)}{\bar{p}u'(c^*)}\rho'(s^*) > -\frac{u(\bar{c})}{\bar{p}u'(\bar{c})}\rho'(\bar{s}). \tag{A1}$$

Noting that $s^* < \bar{s}$ holds if there exists an interior steady state and that $\rho(\cdot)$ is a convex function, we have

$$-\rho'(s^*) > -\rho'(\bar{s}). \tag{A2}$$

Taking (A2) into consideration, we see that the sufficient condition for (A1) to hold is given as follows:

$$\frac{u(c^*)}{u'(c^*)} > \frac{u(\bar{c})}{u'(\bar{c})}. \tag{A3}$$

Since $u(\cdot)/u'(\cdot)$ is an increasing function of c, it follows that $c^* > \bar{c}$ under (A3).

From (24c) and (25c), we find that

$$\begin{aligned}
c^* - \bar{c} &= f_2'(k_2(\bar{p}))(k_2(\bar{p}) - \bar{k}) + f_2(k^*) - f_2(k_2(\bar{p})) + \bar{p}\big\{(\bar{s} - s^*) + \delta(\bar{k} - k^*)\big\} \\
&< f_2'(k_2(\bar{p}))(k_2(\bar{p}) - \bar{k}) + f_2'(k_2(\bar{p}))(k^* - k_2(\bar{p})) + \bar{p}\big\{(\bar{s} - s^*) + \delta(\bar{k} - k^*)\big\} \\
&= \big\{f_2'(k_2(\bar{p})) - \bar{p}\delta\big\}(k^* - \bar{k}) + \bar{p}(\bar{s} - s^*) \\
&= \bar{p}\rho(\bar{s})(k^* - \bar{k}) + \bar{p}(\bar{s} - s^*).
\end{aligned}$$

In the second line of the above calculation, we employ the concavity of the production function. In addition, we use (25a) in the fourth line. Note that there is the possibility that $\bar{p}\rho(\bar{s})(k^* - \bar{k}) + \bar{p}(\bar{s} - s^*) > 0$. In this case, the small country may have an interior steady state outside the diversification cone.

(ii) In the case where $k \in [k_2(\bar{p}), k_1(\bar{p})]$, if the small country has a steady state in the

intermediate goods trade. See Acemoglu (Chapter 19, 2009) for further discussion on trade and growth.

diversification cone, it follows that

$$\frac{f_2'(k_2(\bar{p}))}{\bar{p}} = f_1'(k_1(\bar{p})) = \delta + \rho(s^*), \tag{A4}$$

$$\frac{f_2'(k_2(\bar{p}))}{\bar{p}} = f_1'(k_1(\bar{p})) = \delta - \frac{u(c^*)}{\bar{p}u'(c^*)}\rho'(s^*), \tag{A5}$$

$$c^* = r(\bar{p})k^* + w(\bar{p}) - \bar{p}\big(s^* + \delta k^*\big). \tag{A6}$$

From (25a) and (A4), it follows that $s^* = \bar{s}$ because $\rho(\cdot)$ is a monotonic function. In addition, (25b), (A5) and $s^* = \bar{s}$ yield $c^* = \bar{c}$ because $u(\cdot)/u'(\cdot)$ is a monotonic function. Since $c^* = \bar{c}$ and $s^* = \bar{s}$, we obtain $k^* = \bar{k}$ from (25c) and (A6).

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