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"Modularity and Monotonicity of Games"

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Modularity and Monotonicity of Games*

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Abstract

The purpose of this paper is twofold. First, we generalize Kajii et al. (2007), and provide a condition under which for a game v, its Möbius inversion is equal to zero within the framework of the k-modularity of v for $k \geq 2$. This condition is more general than that in Kajii et al. (2007). Second, we provide a condition under which for a game v for $k \geq 2$, its Möbius inversion takes non-negative values, and not just zero. This paper relates the study of totally monotone games to that of k-monotone games. Furthermore, the modularity of a game can be related to k-additive capacities proposed by Grabisch (1997). As applications of our results to economics, this paper shows that a Gini index representation of Ben-Porath and Gilboa (1994) can be characterized by using our results directly. Our results can also be applied to potential functions proposed by Hart and Mas-Colell (1989) and further analyzed by Ui et al. (2011).

JEL Classification Numbers: C71; D81; D90
Key Words: Belief Functions; Möbius Inversion; Totally Monotone Games; k-additive capacities; Gini Index; Potential Functions

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1. Introduction

In cooperative game theory and decision theory, functions on some domains play important roles. For a set $\Omega = \{1, 2, \ldots, n\}$ and the power set 2^Ω of Ω, in cooperative game theory, Ω denotes a set of players, 2^Ω is interpreted as the collection of all coalitions, and a function $v : 2^\Omega \to \mathbb{R}$ with $v(\emptyset) = 0$ is a transferable game or a game. On the other hand, in decision theory, Ω denotes a set of states of the world, 2^Ω is interpreted as the collection of all events, and $v : 2^\Omega \to \mathbb{R}$ with $v(\emptyset) = 0$ represents a decision maker's beliefs. In cooperative game theory and decision theory, it has been recognized that studies into what kinds of properties such a function v has are important. The purpose of this paper is to further investigate two properties of v, that is, v's *modularity* and *monotonicity*.

In economics and statistics, a decision maker's beliefs are usually captured by a probability measure when she is faced with "uncertain situations." However, the validity of capturing a decision maker's beliefs by a probability measure has been cast doubt on in statistics and economics.[1] Shafer (1976) defines a belief function by a totally monotone game, and shows that a game v is totally monotone if and only if its Möbius inversion is non-negative (see Section 2 for definitions).[2] Focusing on the notion of v's k-monotonicity that is a restricted notion of totally monotone games, Chateauneuf and Jaffray (1989) analyze the relation between v's k-monotonicity and its Möbius inversion, and provide some characterization of k-monotone games through the Möbius inversion. One of the purposes of this paper is to investigate properties that serve the bridge between the study of totally monotone games and that of k-monotone games.

Kajii et al. (2007) investigate the relationship between the modularity of a game and its Möbius inversion. Kajii et al. (2007) show that a game v is modular for some collection of subsets of a state space if and only if for a game v, its Möbius inversion β_T is equal to zero for all T that is not $\mathcal{E}$-complete, where $\mathcal{E}$ denotes a collection of events.[3] However, two problems remain to be solved. First, contrary to Kajii et al. (2007) in which the cases of 2-modularity are analyzed, can these results be generalized into the results of k-modularity for $k \geq 2$? Second, contrary to the case in which v's modularity is characterized by its Möbius inversion with the value being zero, can we provide a condition under which for a game v for $k \geq 2$, its Möbius inversion takes non-negative values, and not just zero?

[1] In statistics, to appropriately model uncertain situations, Dempster (1967) and Shafer (1976) propose a *belief function* to overcome shortcomings that the approach to evaluating uncertain situations by a probability measure has. To analyze uncertain situations from the point of view of economics, Schmeidler (1989) axiomatizes behaviors of a rational decision maker (the *Choquet expected utility*). See Schmeidler (1989) for details.

[2] For analyses of totally monotone games, see, for example, Chateauneuf and Rébillé (2004). Chateauneuf and Rébillé (2004) show the well-known Yosida-Hewitt (1952)'s decomposition theorem for totally monotone games on the set of all subsets of $\mathbb{N}$.

[3] The definitions of v's modularity and $\mathcal{E}$-completeness are provided in Section 3.

This task is important since such a condition enables us to characterize some class of totally monotone games by the modularity of a game v. Furthermore, the analyses of the modularity of a game are also important since they enable us to relate the Shapley value (Shapley (1953)) to potential functions (Hart and Mas-Colell (1989)). Thus, we generalize Kajii et al. (2007), and then characterize v's modularity by its Möbius inversion with the value being zero, within a more general framework than that of Kajii et al. (2007).

However, our contribution is not restricted to the generalization of Kajii et al. (2007) mentioned above. In addition to generalizing Kajii et al. (2007)'s result, by introducing the notion of *k-simpleness*, this paper characterizes some class of totally monotone games by the k-modularity and the k-monotonicity of a game v under k-simpleness through the use of Chateauneuf and Jaffray's (1989) theorem. Furthermore, our results of the modularity are closely related to the results in the literature on non-additive measure theory. Sugeno et al. (1995) propose the notion of the *inclusion-exclusion covering* characterized through the Möbius inversion of a game by Fujimoto and Murofushi (1997) (see Subsection 6.1 for details). As shown in Subsection 6.1, the modularity of a game can be related to the inclusion-exclusion covering. Furthermore, based on Miranda et al. (2005) that provide a characterization of *k-additive capacities*, the modularity of a game can be related to a k-additive capacity (see Subsection 6.2 for details).

We provide the economic interpretations of our results by applying them to existing problems. One of the applications is a Gini index representation axiomatized by Ben-Porath and Gilboa (1994). For this decision model under uncertainty, we provide an alternative characterization directly based on our results. Our results enable us to characterize the Gini index representation through Choquet integrals that cannot be characterized within the framework of Kajii et al. (2007). Furthermore, we apply our results to potential functions proposed by Hart and Mas-Colell (1989) (see Section 3 for the definition) and further analyzed by Ui et al. (2011). This application implies that our results can also be applied to cooperative game theory.

The organization of this paper is as follows. Section 2 provides the definitions and well-known results about the modularity of a game and Möbius inversions. Section 3 presents the definitions and results provided by Kajii et al. (2007). Section 4 generalizes Kajii et al. (2007), and provides a condition under which for a game v, its Möbius inversion is equal to zero within the framework of the *k-modularity* of v for $k \geq 2$. By introducing the notion of *being k-simple*, Section 5 characterizes some class of totally monotone games by the k-modularity and the k-monotonicity of a game v. That is, Section 5 provides a condition under which for a game v, its Möbius inversion takes non-negative values, and not just zero. Section 6 applies our results to previous ones. Section 7 concludes this paper.

2. Modularity and Möbius Inversion

In this section, we provide definitions and well-known results about the modularity of a game and its Möbius inversion. Let $\Omega = \{1, \dots, n\}$ be a finite set of states of the world, whose generic element is denoted by ω. A subset $E \subseteq \Omega$ is called an event. Denote by $\mathcal{F}$ the collection of all non-empty subsets of Ω, and by $\mathcal{F}_k$ the collection of subsets with k elements. For example, $\mathcal{F}_1$ denotes the set of all singleton subsets of Ω, that is, $\mathcal{F}_1 = \{\{\omega\} \,|\, \omega \in \Omega\}$.

A set function $v : 2^\Omega \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *transferable utility game*, or a *game*.[4] For a game v, we use the following definitions:

- v is *non-negative* if $v(E) \geq 0$ for all $E \in 2^\Omega$.
- v is *modular* if $v(E \cup F) = v(E) + v(F) - v(E \cap F)$ for all $E, F \in 2^\Omega$. v is *k-modular* for $k \geq 2$ if $v\left(\cup_{i=1}^k A_i\right) = \sum_{\{I : \emptyset \neq I \subset \{1,\dots,k\}\}} (-1)^{|I|+1} v\left(\cap_{i \in I} A_i\right)$ for all $A_1, \dots, A_k \in 2^\Omega$, where $|I|$ denotes the cardinality of I.
- v is *convex* (or *supermodular*) if $v(E \cup F) \geq v(E) + v(F) - v(E \cap F)$ for all $E, F \in 2^\Omega$.
- v is monotone if $E \subseteq F$ implies $v(E) \leq v(F)$ for all E, $F \in 2^\Omega$. v is *k-monotone* for $k \geq 2$ if $v\left(\cup_{i=1}^k A_i\right) \geq \sum_{\{I : \emptyset \neq I \subset \{1,\dots,k\}\}} (-1)^{|I|+1} v\left(\cap_{i \in I} A_i\right)$ for all $A_1, \dots, A_k \in 2^\Omega$.
- v is *totally monotone* if it is monotone and k-monotone for all $k \geq 2$.[5] v is called a *belief function* if it is totally monotone and $v(\Omega) = 1$.
- v is a *capacity* if $v(E) \leq v(F)$ for all $E \subseteq F$ and $v(\Omega) = 1$.

For $T \in \mathcal{F}$, let a game u_T be the *unanimity game* on T defined by the following rule: $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. Each game v is uniquely represented as a linear combination of unanimity games (Shapley (1953)): $v = \sum_{T \in \mathcal{F}} \beta_T u_T$, or equivalently $v(E) = \sum_{T \subseteq E} \beta_T$ for all $E \in \mathcal{F}$, where $\beta_T = \sum_{E \subseteq T} (-1)^{|T|-|E|} v(E)$. By convention, we omit the empty set in the summation indexed by subsets of Ω. The set of coefficients $\{\beta_T\}_{T \in \mathcal{F}}$ is referred to as the *Möbius inversion or the Harsányi dividend* (Harsányi (1959)) of v. A totally monotone game v can be characterized by the coefficients β_T for all $T \in \mathcal{F}$. The following proposition is shown by Shafer (1976).

Proposition 1 (Shafer (1976)). *For any game v, $v = \sum_{T\mathcal{F}} \beta_T u_T$ is totally monotone if and only if β_T is non-negative for all $T \in \mathcal{F}$.*

[4] In cooperative game theory, Ω and 2^Ω are interpreted as a set of players and the collection of coalitions, respectively.

[5] In the literature, the monotonicity of v is often omitted.

By Proposition 1, we obtain the following corollary.

Corollary 1. *For a non-negative game v, $\beta_{\{\omega\}} = v(\{\omega\}) \geq 0$ for all singleton sets $\{\omega\} \in \mathcal{F}_1$. Moreover, a non-negative game v is totally monotone if and only if its Möbius inversion $\beta_T \geq 0$ for all T with $|T| \geq 2$.*

Corollary 1 states that the class of totally monotone games is a finite convex cone spanned by unanimity games. Our paper characterizes games spanned by a subclass of unanimity games.[6] Chateauneuf and Jaffray (1989) study the relation between the inclusion-exclusion formula for a game v and its Möbius inversion. The following proposition shows that the inclusion-exclusion formula for v can be related to its Möbius inversion.

Proposition 2 (Chateauneuf and Jaffray (1989)). *Let $v = \sum_{T\in\mathcal{F}} \beta_T u_T$ be a game, and let k be an integer satisfying $k \geq 2$. Then,*
$v(\bigcup_{1\leq i\leq k} T_i) - \sum_{\emptyset\neq I\subseteq\{1,2,\ldots,k\}} (-1)^{|I|+1} v(\bigcap_{j\in I} T_j) = \sum_{T\subseteq\bigcup T_i, T\not\subseteq T_i (1\leq i\leq k)} \beta_T$.

Chateauneuf and Jaffray (1989) also clarify the relation between v's monotonicity and its Möbius inversion.

Proposition 3 (Chateauneuf and Jaffray (1989)). *Let $v = \sum_{T\in\mathcal{F}} \beta_T u_T$ be a game, and let k be an integer satisfying $k \geq 2$. Then, the following two statements are equivalent.*
(i) v is k-monotone.
(ii) $\sum_{A\subseteq T\subseteq B} \beta_T \geq 0$ for every $A \in 2^\Omega$ with $2 \leq |A| \leq k$ and every $B \in 2^\Omega$.

3. Complete Collection and Möbius Inversion

Before we provide generalizations of Kajii et al. (2007) in the following sections, we review Kajii et al. (2007)'s setup and results in this section. Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. By proposing the notion of *$\mathcal{E}$-completeness*, Kajii et al. (2007) analyze the modularity of a game v that is intended to clarify a condition under which its Möbius inversion takes the value $\beta_T = 0$ for all $T \notin \mathcal{E}$. Kajii et al. (2007) introduce the following definition.

Definition 1. Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. An event $T \in \mathcal{F}$ is *$\mathcal{E}$-complete* if, for any two distinct points ω_1 and ω_2 in T, there exists a set $E \in \mathcal{E}$ such that $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. The collection of all $\mathcal{E}$-complete events is called the *$\mathcal{E}$-complete collection* and denoted by $\Upsilon(\mathcal{E})$. A collection $\mathcal{E} \subseteq \mathcal{F}$ is said to be *complete* if all $\mathcal{E}$-complete subsets belong to $\mathcal{E}$, i.e., $\mathcal{E} = \Upsilon(\mathcal{E})$.

[6] Pintér (2011, Lemma 4) shows that for any game v, the class of capacities (in terms of Pintér (2011), the class of monotone games) is also a finite convex cone spanned by the generalized unanimity games that include unanimity games as a special case. For a definition of generalized unanimity games, see Pintér (2011).

Note that a singleton set is $\mathcal{E}$-complete, so is any $E \in \mathcal{E}$.[7] To relate the modularity of a game to the notion of being $\mathcal{E}$-complete, Kajii et al. (2007) introduce the following definition.

Definition 2. Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. Let $T_1, T_2 \in \mathcal{F}$. A pair of events $\{T_1, T_2\}$ with $T_1 \not\subseteq T_2$ and $T_2 \not\subseteq T_1$ are said to be an *$\mathcal{E}$-decomposition pair* for $T \in \mathcal{F}$, if $T_1 \cup T_2 = T$ and, for any $E \in \mathcal{E}$, $E \subseteq T$ implies $E \subseteq T_1$ or $E \subseteq T_2$ (or both). An event $T \in \mathcal{F}$ is *$\mathcal{E}$-decomposable* if there exists an $\mathcal{E}$-decomposition pair for T.

The following lemma in Kajii et al. (2007) clarifies the relation between the notion of being $\mathcal{E}$-decomposable and the notion of being $\mathcal{E}$-complete.

Lemma 1. *An event $T \in \mathcal{F}$ is not $\mathcal{E}$-complete if and only if T is $\mathcal{E}$-decomposable.*

If a game $v = \sum_{T\in\mathcal{F}} \beta_T u_T$ is modular, then $\beta_T = 0$ for all $T \in \mathcal{F}$ with $|T| \geq 2$. By restricting the domain on which v is modular, Kajii et al. (2007) propose the notion of being *modular for $\mathcal{E}$-decomposition pairs.*

Definition 3. Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events, and let $v = \sum_{T\in\mathcal{F}} \beta_T u_T$ be a game. A game v is said to be *modular for $\mathcal{E}$-decomposition pairs* if $v(T) = v(T_1)+v(T_2)-v(T_1\cap T_2)$ for every $\mathcal{E}$-decomposable set T and every $\mathcal{E}$-decomposition pair $\{T_1, T_2\}$ for such a T.

The modularity for $\mathcal{E}$-decomposition pairs and the coefficients of the Möbius inversion can be related by the following theorem, which is proved by Kajii et al. (2007). One of the purposes of this paper is to extend this theorem. Section 4 provides the extension.

Theorem 1 (Kajii et al. (2007)). *Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. Let $v = \sum_{T\in\mathcal{F}} \beta_T u_T$ be a game. The following statements are equivalent: (i) v is modular for $\mathcal{E}$-decomposition pairs; (ii) $\beta_T = 0$ for any T that is not $\mathcal{E}$-complete, i.e., $T \notin \Upsilon(\mathcal{E})$.*

4. A Generalization of Kajii et al. (2007)

Based on the notion of *$\mathcal{E}$-completeness*, within the framework of the 2-modularity of a game v, Kajii et al. (2007) provide a condition under which its Möbius inversion is equal to zero. In this section, we generalize Kajii et al. (2007)'s result. That is, we provide a condition under which for a game v, its Möbius inversion is equal to zero within the framework of the k-modularity of v for $k \geq 2$ that is more general than that of Kajii et al. (2007). This task is accomplished by generalizing $\mathcal{E}$-completeness into the notion of being *complete of order k* within the framework of the *k-modularity* of a game v for $k \geq 2$. For that purpose, at first, we generalize the notion of $\mathcal{E}$-completeness in Kajii et al. (2007).

[7] The term "complete" is adopted from an analogy to a complete graph. For $T \in \mathcal{F}$, let us consider an undirected graph with a vertex set T where $\{\omega, \omega'\} \subseteq T$ is an edge if there exists $E \in \mathcal{E}$ such that $\{\omega, \omega'\} \subseteq E \subseteq T$. Then, this is a complete graph if and only if T is $\mathcal{E}$-complete.

Definition 4. Let $m \geq 2$ be an integer. A set $S \in \mathcal{F}$ is said to be *$\mathcal{E}$-complete of order m* if, for every T satisfying both $T \subseteq S$ and $2 \leq |T| \leq m$, there exists an $E \in \mathcal{E}$ such that $T \subseteq E \subseteq S$. Denote by $\Upsilon^m(\mathcal{E})$ the collection of all sets that are $\mathcal{E}$-complete of order m.

Example 1. Every singleton set is $\mathcal{E}$-complete of order m for all m. That is, $\mathcal{F}_1 \subseteq \Upsilon^m(\mathcal{E})$.

Example 2. Any set $T \in \mathcal{E}$ is $\mathcal{E}$-complete of order m for all m. That is, $\mathcal{E} \subseteq \Upsilon^m(\mathcal{E})$

Example 3. Let $\Omega = \{1,2,3,4\}$ and let $\mathcal{E} = \mathcal{F}_2$. Then, $\{1,2,3\}$ is $\mathcal{E}$-complete of order 2 since $\{1,2\},\{2,3\},\{1,3\} \in \mathcal{E}$. However, $\{1,2,3\}$ is not $\mathcal{E}$-complete of order 3 since there is no $T \in \mathcal{E}$ such that $\{1,2,3\} \subseteq T$. For a finite set $\Omega = \{1,2,\dots,n\}$ and $\mathcal{E} = \mathcal{F}_2$, $\Upsilon^2(\mathcal{F}_2) = \mathcal{F}$ and $\Upsilon^3(\mathcal{F}_2) = \mathcal{F}_1 \cup \mathcal{F}_2$.

Note that for $m = 2$, the notion of being $\mathcal{E}$-complete of order m coincides with that of being $\mathcal{E}$-complete. Therefore, the idea of $\mathcal{E}$-completeness of order m is a generalization of the idea of $\mathcal{E}$-completeness in Kajii et al. (2007). Next, we introduce the notion of *k-set's $\mathcal{E}$-decomposition*.

Definition 5. Let $\mathcal{E} \subseteq \mathcal{F}$. A collection $\{T_1,\dots,T_k\}$ is said to be a *k-set's $\mathcal{E}$-decomposition* if it satisfies the following two conditions: (i) $(\cup_{i=1}^k T_i)\backslash T_j$ is a singleton $\{\omega_j\}$ for all $1 \leq j \leq k$, and (ii) for $\omega_1,\omega_2,\cdots,\omega_k$ in (i), there is no $E \in \mathcal{E}$ such that $\{\omega_1,\dots,\omega_k\} \subseteq E \subseteq \cup_{i=1}^k T_i$. Denote by $W^k(\mathcal{E})$ the collection of all k-set's $\mathcal{E}$-decompositions.

The notion of k-set's $\mathcal{E}$-decomposition is a restriction of $\mathcal{E}$-decomposition pairs for $k = 2$. If, for $\{\omega_1,\omega_2\} \subseteq T$, there is no $E \in \mathcal{E}$ such that $\{\omega_1,\omega_2\} \subseteq E \subseteq T$, then $\{\{T\backslash\{\omega_1\}\},\{T\backslash\{\omega_2\}\}\}$ are an $\mathcal{E}$-decomposition pair. This is because if $E \in \mathcal{E}$ satisfies $E \subseteq T$, then $E \subseteq T\backslash\{\omega_1\}$ or $E \subseteq T\backslash\{\omega_2\}$. However, the converse is not true. Clearly, by Definition 4, for all $k \leq m$, $\cup_{i=1}^k T_i \notin \Upsilon^m(\mathcal{E})$ if and only if $\{T_1,\dots T_k\}$ is a k-set's $\mathcal{E}$-decomposition. As such, we can also define decompositions by corresponding to each $S \notin \Upsilon^m(\mathcal{E})$ as follows.

Definition 6. Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events, and let $m \geq 2$ be an integer. A collection $\{\tilde{W}_m^k(\mathcal{E})\}_{2\leq k\leq m}$ is said to be an *$\mathcal{E}$-decomposition collection with respect to $\Upsilon^m(\mathcal{E})$* if it satisfies the following conditions: (i) $\tilde{W}_m^k(\mathcal{E}) \subseteq W^k(\mathcal{E})$ for all k with $2 \leq k \leq m$, (ii) for every $S \notin \Upsilon^m(\mathcal{E})$, there exist a unique k with $2 \leq k \leq m$ and a unique collection $\{T_1,\dots T_k\} \in \tilde{W}_m^k(\mathcal{E})$ such that $S = \cup_{i=1}^k T_i$.

Example 4. Let $\Omega = \{1,2,3\}$ and $\mathcal{E} = \{\{1,2\}\}$. Then, $\Upsilon^2(\mathcal{E}) = \{\{1\},\{2\},\{3\},\{1,2\}\}$, and $\mathcal{F}\backslash\Upsilon^2(\mathcal{E}) = \{\{1,3\},\{2,3\},\{1,2,3\}\}$,
$W^2(\mathcal{E}) = \{\{\{1\},\{3\}\},\{\{2\},\{3\}\},\{\{2,3\},\{1,2\}\},\{\{1,3\},\{1,2\}\}\}$,
$W^3(\mathcal{E}) = \{\{1,2\},\{1,3\},\{2,3\}\}$. Therefore, $\tilde{W}_2^2(\mathcal{E}) = \{\{\{1\},\{3\}\},\{\{2\},\{3\}\},\{\{2,3\},\{1,2\}\}\}$ or $\tilde{W}_2^2(\mathcal{E}) = \{\{\{1\},\{3\}\},\{\{2\},\{3\}\},\{\{1,3\},\{1,2\}\}\}$.

Note that $\sum_{2\le k\le m}|\tilde{W}^k_m(\mathcal{E})| \le |\mathcal{F}\backslash\Upsilon^m(\mathcal{E})| = 2^n-1-|\Upsilon^m(\mathcal{E})|$. We can easily show that all of our results for $\{W^k(\mathcal{E})\}_{2\le k\le m}$ hold for $\{\tilde{W}^k_m(\mathcal{E})\}_{2\le k\le m}$, which enables us to reduce the number of decompositions. Therefore, the notion of $\mathcal{E}$-decomposition collections with respect to $\Upsilon^m(\mathcal{E})$ reduces the number of equations that should be analyzed.

To relate a game v's modularity for k-set's $\mathcal{E}$-decompositions to the Möbius inversion, we define the modularity that is restricted to the collection of k-set's $\mathcal{E}$-decompositions.

Definition 7. Let $k\ge 2$ be an integer. A game v is said to be *k-modular for k-set's $\mathcal{E}$-decompositions* if $v(S) = \sum_{\emptyset\ne I\subseteq\{1,\ldots,k\}}(-1)^{|I|+1}v(\bigcap_{j\in I}T_j)$ for every $\{T_1,\ldots,T_k\}\in W^k(\mathcal{E})$, where $S=\cup_{i=1}^k T_i$.

Similarly, we define the modularity for $\mathcal{E}$-decomposition collections with respect to $\Upsilon^m(\mathcal{E})$.

Definition 8. Let $\{\tilde{W}^k_m(\mathcal{E})\}_{2\le k\le m}$ be any $\mathcal{E}$-decomposition collection with respect to $\Upsilon^m(\mathcal{E})$. Then, a game v is said to be *at most m-modular for* $\{\tilde{W}^k_m(\mathcal{E})\}_{2\le k\le m}$ if $v(S) = \sum_{\emptyset\ne I\subseteq\{1,\ldots,k\}}(-1)^{|I|+1}v(\bigcap_{j\in I}T_j)$ for every $\{T_1,\ldots,T_k\}\in\tilde{W}^k_m(\mathcal{E})$ for all k with $2\le k\le m$, where $S=\cup_{i=1}^k T_i$.

To prove Theorem 2, the following proposition is in order. The idea of the proof is based on Kojima and Ui (2007).

Proposition 4. *Fix a game v, a collection $\mathcal{E}\subseteq\mathcal{F}$, and an integer m. The following three statements about a game $w=\sum_{T\in\mathcal{F}}\gamma_T u_T$ are equivalent:*
(i) $w(S)=v(S)$ if $S\in\Upsilon^m(\mathcal{E})$, and $\gamma_T=0$ if $T\notin\Upsilon^m(\mathcal{E})$.
(ii) $\{\gamma_T\}_{T\in\mathcal{F}}$ is determined recursively by the following rule:
1. $\gamma_{\{i\}}=v(\{i\})$ for all $i\in\Omega$.
2. For $T\in\mathcal{F}$ with $|T|\ge 2$, $\gamma_T=v(T)-\sum_{S\subsetneq T}\gamma_S$ if $T\in\Upsilon^m(\mathcal{E})$, and $\gamma_T=0$ if $T\notin\Upsilon^m(\mathcal{E})$.
(iii) w satisfies the following two conditions:
(a) $w(S)=v(S)$ if $S\in\Upsilon^m(\mathcal{E})$.
(b) $w(S)=\sum_{\emptyset\ne I\subseteq\{1,\ldots,k\}}(-1)^{|I|+1}w(\bigcap_{j\in I}T_j)$ for all $\{T_1,\ldots,T_k\}\in W^k(\mathcal{E})$ satisfying both $\cup_{i=1}^m T_i=S$ and $2\le k\le m$ if $S\notin\Upsilon^m(\mathcal{E})$.

Proof. (i) $\Leftrightarrow$ (ii): The rule in (ii) is rewritten as follows: if $S\in\Upsilon^m(\mathcal{E})$, then $v(S)=\sum_{T\subseteq S}\gamma_T=w(S)$, and if $T\notin\Upsilon^m(\mathcal{E})$, then $\gamma_T=0$, which is (i).
(ii) $\Leftrightarrow$ (iii): Let w be as stated in (ii). Then, Condition (a) in (iii) is obviously satisfied. To show that Condition (b) in (iii) is satisfied, let $S\notin\Upsilon^m(\mathcal{E})$ and let $\{T_1,\ldots,T_k\}\in W^k(\mathcal{E})$ such that $\cup_{i=1}^k T_i=S$. Let $\{\omega_i\}=S\backslash T_i$ for $i=1,\ldots,k$. Since $S\notin\Upsilon^m(\mathcal{E})$, by Definition 5, there is no $E\in\mathcal{E}$ such that $\{\omega_1,\ldots,\omega_k\}\subseteq E\subseteq S$. Pick any T satisfying both $T\subseteq S$ and $T\not\subseteq T_i$ for all $1\le i\le k$. Then, it must hold that $\{\omega_1,\ldots,\omega_k\}\subseteq T\subseteq S$. Hence,

there is no $E \in \mathcal{E}$ such that $\{\omega_1, \ldots, \omega_k\} \subseteq E \subseteq T$ since there is no $E \in \mathcal{E}$ such that $\{\omega_1, \ldots, \omega_k\} \subseteq E \subseteq S$. Thus, $T \notin \Upsilon^m(\mathcal{E})$ and $\gamma_T = 0$ by (ii). Therefore, by Proposition 2, $w(S) - \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} w(\cap_{j \in I} T_j) = \sum_{T \subseteq S, T \not\subseteq T_i (1 \leq i \leq k)} \gamma_T = \sum_{\{\omega_1,\ldots,\omega_k\} \subseteq T \subseteq S} \gamma_T = 0$, which is Condition (b) in (iii). Thus, (ii) implies (iii).

Suppose that w satisfies the conditions in (iii). To prove that (iii) implies (ii), it suffices to show that w is uniquely determined because the unique game that satisfies the conditions in (ii) satisfies the conditions in (iii). To show this uniqueness, we construct w recursively such that in the h-th step, we determine the unique value of $w(S)$ with $|S| = h$ from $w(S')$ with $|S'| \leq h - 1$. Start with $w(\emptyset) = 0$. Consider the h-th step with $h \geq 1$ and pick any S with $|S| = h$. If $S \in \Upsilon^m(\mathcal{E})$, then $w(S) = v(S)$ by Condition (a) in (iii). If $S \notin \Upsilon^m(\mathcal{E})$, then there exists $\{T_1, \ldots T_k\}$, which is a k-set's $\mathcal{E}$-decomposition of S for some k with $2 \leq k \leq m$ by Definition 5. Hence, by Condition (b) in (iii), $w(S) = \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} w(\bigcap_{j \in I} T_j)$. Since the terms on the right-hand side are uniquely calculated in the earlier steps, so is $w(S)$ on the left-hand side. By this procedure, we can uniquely determine w recursively, which establishes the uniqueness. $\square$

The following result is an immediate consequence of the above proposition.

Theorem 2. *Let $m \geq 2$ be an integer and $v = \sum_{T \in \mathcal{F}} \beta_T u_T$ be a game. The following two conditions are equivalent:*
(i) v is k-modular for k-set's $\mathcal{E}$-decompositions for all $2 \leq k \leq m$.
(ii) $\beta_T = 0$ for every $T \notin \Upsilon^m(\mathcal{E})$.

This theorem states that by a game v's k-modularity, we can characterize its Möbius inversion that is equal to zero within a more general framework than that of Kajii et al. (2007). The result can be extended to any $\mathcal{E}$-decomposition collection with respect to $\Upsilon^m(\mathcal{E})$.

Corollary 2. *Let $m \geq 2$ be an integer and $v = \sum_{T \in \mathcal{F}} \beta_T u_T$ be a game. Let $\{\tilde{W}^k_m(\mathcal{E})\}_{2 \leq k \leq m}$ be any $\mathcal{E}$-decomposition collection with respect to $\Upsilon^m(\mathcal{E})$. The following two conditions are equivalent:*
(i) v is at most m-modular for $\{\tilde{W}^k_m(\mathcal{E})\}_{2 \leq k \leq m}$.
(ii) $\beta_T = 0$ for every $T \notin \Upsilon^m(\mathcal{E})$.

Proof. Clearly, Proposition 4 holds for $\{\tilde{W}^k_m(\mathcal{E})\}_{2 \leq k \leq m}$ instead of $\{W^k(\mathcal{E})\}_{2 \leq k \leq m}$. $\square$

Applying Theorem 2 and Corollary 2 to $\mathcal{E} = \cup_{i=2}^k \mathcal{F}_i$ gives the following corollary.

Corollary 3. *Let $k \geq 2$ be an integer, $|\Omega| \geq k + 1$, and $v = \sum_{T \in \mathcal{F}} \beta_T u_T$. Define a collection of decompositions, all of which consist of $(k + 1)$ sets by*

$W_{\mathcal{F}_{k+1}} \equiv \big\{\{S\backslash\{\omega_1\}\}, \{S\backslash\{\omega_2\}\}, \ldots, \{S\backslash\{\omega_{k+1}\}\} \,\big|\, |S| \geq k+1,\ \{\omega_1, \omega_2, \ldots, \omega_{k+1}\} \in \mathcal{F}_{k+1}\big\}$.
Then, the following three conditions for v are equivalent:
(1) v is $(k+1)$-modular for $(k+1)$-set's $\cup_{i=2}^{k}\mathcal{F}_i$-decompositions.
(2) v is $(k+1)$-modular for $\{\tilde{W}^j_{k+1}(\mathcal{E})\}_{2\leq j\leq k+1}$, where $\{\tilde{W}^j_{k+1}(\mathcal{E})\}_{2\leq j\leq k+1} = W_{\mathcal{F}_{k+1}}$.
(3) $\beta_T = 0$ for every T with $|T| \geq k+1$.

Proof. Let $\mathcal{E} = \cup_{i=2}^{k} \mathcal{F}_i$. Then, $\Upsilon^{k+1}(\mathcal{E}) = \mathcal{E} \cup \mathcal{F}_1$. Thus, $S \notin \Upsilon^{k+1}(\mathcal{E})$ if and only if $|S| \geq k+1$. Conditions (1) and (3) are equivalent since it follows from Theorem 2 that $W^j(\mathcal{E}) = \emptyset$ for $2 \leq j \leq k$ and $W^{k+1}(\mathcal{E}) = W_{\mathcal{F}_{k+1}}$. On the other hand, Conditions (2) and (3) are equivalent since it follows from Corollary 2 that $\{\tilde{W}^j_{k+1}(\mathcal{E})\}_{2\leq j\leq k+1} = \{\tilde{W}^{k+1}_{k+1}(\mathcal{E})\} = W_{\mathcal{F}_{k+1}}$. □

Note that in Condition (2), we use the term "$(k+1)$-modular" instead of "at most $(k+1)$-modular."

5. Characterization of Totally Monotone Games

Section 4 provided an extension of Kajii et al. (2007). However, one issue remains to be solved. That is, can we derive a condition under which for a game $v = \sum_{T\in\mathcal{F}} \beta_T u_T$, its Möbius inversion takes non-negative values, and not just zero? This issue is indeed important since it enables us to characterize totally monotone games by the k-modularity and the k-monotonicity of a game v. For that purpose, we propose the notion of being *k-simple*.

Definition 9. A collection $\mathcal{E} \subseteq \mathcal{F}$ is said to be *k-simple* if, for every S with $|S| \geq 2$, there exists T satisfying both $T \subseteq S$ and $2 \leq |T| \leq k$ such that there exists no $E \in \mathcal{E}$ with $T \subseteq E \subsetneq S$. Equivalently, for all $S \in \mathcal{F}$ with $|S| \geq 2$, if $S \in \Upsilon^k(\mathcal{E})$, then $S \notin \Upsilon^k(\mathcal{E}\backslash S)$.

Example 5. Any partition of Ω is k-simple for all $k \geq 2$, and so is $\{\{1,2\},\{2,3\},\{3,4\},\ldots,\{n-1,n\}\}$.

We need the following lemma to show Corollary 5.

Lemma 2. *The collection $\mathcal{F}_2$ is 3-simple. More generally, for all $k \geq 2$, the collection $\cup_{i=2}^{k}\mathcal{F}_i$ is $(k+1)$-simple.*

Proof. Let $\mathcal{E} = \cup_{i=2}^{k}\mathcal{F}_i$ for $k \geq 2$. Pick any S with $2 \leq |S|$. If $|S| \leq k$, then replace T in Definition 9 with S itself, which shows that there is no $E \in \mathcal{E}$ such that $T \subseteq E \subsetneq S$. When $k+1 \leq |S|$, choose any T satisfying both $T \subseteq S$ and $|T| = k+1$. Then, such a T satisfies the following: there is no $E \in \mathcal{E}$ such that $T \subseteq E \subsetneq S$ since $E \in \mathcal{E}$ implies that $|E| \leq k$. Thus, $\mathcal{E}$ must be $(k+1)$-simple. □

Note that if $\mathcal{E}$ is k-simple, then any $\mathcal{E}' \subseteq \mathcal{E}$ is k-simple. This fact considered together with Example 5 and Lemma 2 gives that the class of being k-simple is not so small.

The following lemma states that the notion of being k-simple involves the same property as completeness.

Lemma 3. *Let $\mathcal{E}$ be k-simple. Then, for S with $|S| \geq 2$, $S \in \mathcal{E}$ if and only if S is $\mathcal{E}$-complete of order k. That is, $\mathcal{E} \cup \mathcal{F}_1 = \Upsilon^k(\mathcal{E})$*

Proof. By Examples 1 and 2, $\mathcal{E} \cup \mathcal{F}_1 \subseteq \Upsilon^k(\mathcal{E})$. Then, suppose that $S \notin \mathcal{E}$ and $|S| \geq 2$. Since $\mathcal{E}$ is k-simple, there exists $T \subseteq S$ with $2 \leq |T| \leq k$ such that there exists no $E \in \mathcal{E}$ with $T \subseteq E \subsetneq S$. This means that there exists no $E \in \mathcal{E}$ with $T \subseteq E \subseteq S$ since $S \notin \mathcal{E}$, which shows that S is not $\mathcal{E}$-complete of order k. Thus, $\mathcal{E} \cup \mathcal{F}_1 = \Upsilon^k(\mathcal{E})$. $\square$

Now, we are in a position to provide our main result in this section.

Theorem 3. *Let $v = \sum_{T \in \mathcal{F}} \beta_T u_T$ be a non-negative game, and let m be an integer satisfying $m \geq 2$. Let $\mathcal{E}$ be m-simple. Let v be k-modular for k-set's $\mathcal{E}$-decompositions for all $2 \leq k \leq m$. Then, if v is m-monotone, then v is totally monotone.*

Proof. Suppose that $v = \sum_{T \in \mathcal{F}} \beta_T u_T$ is m-monotone. Since v is k-modular for k-set's $\mathcal{E}$-decompositions for all $2 \leq k \leq m$, it holds that $\beta_T = 0$ for every $T \notin \Upsilon^m(\mathcal{E})$ by Theorem 2. Thus, it holds that $\beta_T = 0$ for every $T \notin \mathcal{E} \cup \mathcal{F}_1$ by Lemma 3. It suffices to show that $\beta_S \geq 0$ for every $S \in \mathcal{E}$. Suppose that $S \in \mathcal{E}$. Since $\mathcal{E}$ is m-simple, there exists a set F with $2 \leq |F| \leq m$ such that there exists no $E \in \mathcal{E}$ with $F \subseteq E \subsetneq S$. Since v is m-monotone, Condition (ii) in Proposition 3 holds. Thus, since $2 \leq |F| \leq m$, it holds that $\sum_{F \subseteq X \subseteq S} \beta_X \geq 0$. On the other hand, every X with $F \subseteq X \subsetneq S$ satisfies $X \notin \mathcal{E}$, and thus, $\beta_X = 0$. Hence, $0 \leq \sum_{F \subseteq X \subseteq S} \beta_X = \beta_S$. By Proposition 1, it is shown that v is totally monotone. $\square$

Note that the converse is also true. Theorem 3 states that when $\mathcal{E}$ is m-simple, totally monotone games can be characterized by m-monotone games in addition to the k-modularity for k-set's $\mathcal{E}$-decompositions for all $2 \leq k \leq m$.

Corollary 4. *Let $\mathcal{E}$ be 2-simple. For a non-negative game $v \in \mathbb{R}^{\mathcal{F}}$, suppose that v is modular for $\mathcal{E}$-decomposition pairs (that is, 2-modular for 2-set's $\mathcal{E}$-decompositions). If v is convex (that is, 2-monotone), then v is totally monotone.*

Note that the converse is also true. From the viewpoint of economics, this corollary is important because two decision models under uncertainty, that is, the E-capacity expected utility model of Eichberger and Kelsey (1999) and the multiperiod decision model of Gilboa (1989), can be characterized by this corollary.[8]

Corollary 5. *If* $v : 2^{\Omega} \to \mathbb{R}$ *is non-negative, 3-modular for 3-set's* $\mathcal{F}_2$*-decompositions and 2-monotone, then* $v = \sum_{T \in \mathcal{F}_1 \cup \mathcal{F}_2} \beta_T u_T$ *where* $\beta_T \geq 0$*. More generally, for* $2 \leq k \leq n-1$*, if* $v : 2^{\Omega} \to \mathbb{R}$ *is non-negative,* $(k+1)$*-modular for* $(k+1)$*-set's* $\cup_{i=2}^{k} \mathcal{F}_i$*-decompositions and* k*-monotone, then* $v = \sum_{T \in \cup_{i=1}^{k} \mathcal{F}_i} \beta_T u_T$ *where* $\beta_T \geq 0$*.*

Proof. This corollary is proved in the same way as Theorem 3 by setting $\mathcal{E} = \cup_{i=2}^{k} \mathcal{F}_i$. By Corollary 3, it holds that $\beta_T = 0$ for every T with $|T| \geq k+1$. Thus, it suffices to show that $\beta_S \geq 0$ for every S with $|S| \leq k$. This is shown by setting $F = S$ for $\sum_{F \subseteq X \subseteq S} \beta_X \geq 0$ in the proof of Theorem 3 since v is k-monotone. $\square$

6. Applications

In this section, we compare our results with previous results, and apply our results to previously studied cases. In Subsection 6.1, we discuss the relation between Fujimoto and Murofushi (1997) and our paper. In Subsection 6.2, we discuss the relation between *k-additive measures* (or *k-additive capacities*) in Grabisch (1997) and the modularity in this paper. Furthermore, we apply our results to a Gini index representation analyzed by Ben-Porath and Gilboa (1994). In Subsection 6.3, we apply our results to potential functions proposed by Hart and Mas-Colell (1989) and further analyzed by Ui et al. (2011).

6.1. Inclusion-Exclusion Covering

One of the most relevant papers is Fujimoto and Murofushi (1997). First, we present the notion of the *inclusion-exclusion covering* proposed by Sugeno et al. (1995). Sugeno et al. (1995) provide the notion of the inclusion-exclusion covering that is further analyzed by Murofushi et al. (1998) and Fujimoto and Murofushi (1997).

Definition 10. Let $\Omega = \{1, \ldots, n\}$ be a finite set, and let v be a game. A covering $\mathcal{C} = \{C_1, \ldots, C_m\}$ of Ω, *i.e.*, $\cup_{i=1}^{m} C_i = \Omega$ and $C_i \subseteq \Omega$ for all $i = 1, \ldots, m$, is an *inclusion-exclusion covering* of Ω with respect to v if $v(A) = \sum_{I \subseteq \{1,\ldots,m\}, I \neq \emptyset} (-1)^{|I|+1} v(\cap_{i \in I} C_i \cap A)$ for every $A \subseteq \Omega$.

[8] Let $\Omega = \{1, \ldots, n\}$ be a finite set of states, and let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. For the E-capacity expected utility model (Eichberger and Kelsey (1999)), we can generalize the collection $\mathcal{E}$ into a 2-simple collection that is more general than Eichberger and Kelsey (1999) and Kajii et al. (2007). In Eichberger and Kelsey (1999) and Kajii et al. (2007), the collection $\mathcal{E} = \{E_1, \ldots, E_n\}$ is supposed to be a partition of Ω with $|E_i| \geq 2$ for each i, and to be a collection of non-empty, disjoint subsets of Ω with $|E_i| \geq 2$ for each i, respectively. For a multiperiod decision model (Gilboa (1989)), let $\mathcal{E} = \{\{i, i+1\} | 1 \leq i < n\}$. Then, $\mathcal{E}$ is the collection of adjacent time periods, and this collection $\mathcal{E}$ is 2-simple.

Fujimoto and Murofushi (1997) characterize the notion of the inclusion-exclusion covering through the Möbius inversion of a game v.

Proposition 5 (Fujimoto and Murofushi (1997)). *Let $v = \sum_{T\in\mathcal{F}} \beta_T u_T$, where the set of coefficients $\{\beta_T\}_{T\in\mathcal{F}}$ is the Möbius inversion of v. Then, the following are equivalent:*
(i) A covering $\mathcal{C} = \{C_1, \dots, C_m\}$ of Ω is an inclusion-exclusion covering of Ω with respect to v.
(ii) $\beta_T = 0$ whenever $T \nsubseteq C_i$ for every $C_i \in \mathcal{C}$.

To relate Fujimoto and Murofushi (1997) with our paper, the following lemma is in order.

Lemma 4. *Let $\mathcal{C} = \{C_1, \dots, C_m\}$ where $C_i \subseteq \Omega$, $\cup_{i=1}^m C_i = \Omega$ and $k = \max_{1\leq i\leq m} |C_i|$. Moreover, let $\mathcal{E} = \{T \,|\, |T| \geq 2, T \subseteq C_i \text{ for some } i \in \{1, \dots, m\}\}$. Then, $\Upsilon^{k+1}(\mathcal{E}) = \mathcal{E} \cup \mathcal{F}_1$.*

Proof. Assume that $|T| > 1$. Suppose that $T \notin \mathcal{E}$ and T is $\mathcal{E}$-complete of order $(k+1)$. There is at least one S such that $S \subseteq T$ and $S \in \mathcal{E}$ since T is $\mathcal{E}$-complete of order $(k+1)$. This S satisfies $S \neq T$ since $T \notin \mathcal{E}$.

Let $S^* \in \arg\max_{S\subseteq T, S\in\mathcal{E}} |S|$. There exists $\omega \in T \setminus S^*$ since $S^* \neq T$. Note that $|S^*| \leq k$ by definition of k. Hence, there must exist a set $E \in \mathcal{E}$ such that $\{\omega\} \cup S^* \subseteq E \subseteq T$ since T is $\mathcal{E}$-complete of order $k+1$. However, $E \in \mathcal{E}$ and $|S^*| < |E|$. This contradicts the assumption of maximality of S^*. $\square$

Lemma 4 together with Theorem 2 leads to the following result.

Proposition 6. *Let $\mathcal{C} = \{C_1, \dots, C_m\}$ where $C_i \subseteq \Omega$, $\cup_{i=1}^m C_i = \Omega$ and $k = \max_{1\leq i\leq m} |C_i|$. Moreover, let $\mathcal{E} = \{T \,|\, |T| \geq 2, T \subseteq C_i \text{ for some } i \in \{1, \dots, m\}\}$. The following two conditions for a game $v = \sum_{T\in\mathcal{F}} \beta_T u_T$ are equivalent:*
(i) v is p-modular for p-set's $\mathcal{E}$-decomposition for all $2 \leq p \leq k+1$.
(ii) $\beta_T = 0$ whenever $T \nsubseteq C_i$ for every $C_i \in \mathcal{C}$.

This proposition states that Theorem 2 can provide the same result in Fujimoto and Murofushi (1997) with respect to T such that $\beta_T = 0$ whenever $T \nsubseteq C_i$ for every $C_i \in \mathcal{C}$ where collection $\mathcal{C}$ satisfies the condition in Proposition 6. On the other hand, there exist many examples that are obtained from our results, but not from Fujimoto and Murofushi (1997). One of the examples is as follows.

Example 6. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$, and let $E_1 = \{\omega_1, \omega_2, \omega_3\}$, and $E_2 = \{\omega_4, \omega_5, \omega_6\}$. Moreover, let $\mathcal{E} = \{E_1, E_2\}$. Since $\Upsilon^2(\mathcal{E}) = \mathcal{E} \cup \mathcal{F}_1$, the following two conditions for a game $v = \sum_{T\in\mathcal{F}} \beta_T u_T$ are equivalent by Theorem 2: (i) v is 2-modular for 2-set's $\mathcal{E}$-decomposition. (ii) $\beta_T = 0$ whenever $T \neq E_1$, $T \neq E_2$, and $T \notin \mathcal{F}_1$. It is impossible to induce Condition (ii) by Fujimoto and Murofushi (1997).

The number of equations in (i) of Proposition 6 may be more than that in Fujimoto and Murofushi (1997) that is exactly $|\mathcal{F}| = 2^n - 1$. However, by using Corollary 2 instead of Theorem 2, we can reduce the number of equations to $2^n - 1 - |\mathcal{E} \cup \mathcal{F}_1|$.

6.2. k-Additive Capacities and the Gini Index

The notion of *k-additive measures* (or *k-additive capacities*) is introduced by Grabisch (1997). The notion of k-additive measures is proposed to decrease the complexity of capacities in applications since a capacity defined on a set Ω with n elements requires the definition of 2^n real coefficients (see also Grabisch (2000a)). Grabisch (2000b) extends Chateauneuf and Jaffray (1989) that analyze the set of probability measures dominating a given capacity, and analyzes the set of k-additive measures dominating a given capacity.[9] First, the definition of k-additive measures is provided.

Definition 11 (Grabisch (1997)). Let $\Omega = \{1, \ldots, n\}$ be a finite set. A capacity v on $(\Omega, 2^\Omega)$ is *k-order additive* or *k-additive* for some $k \in \{1, 2, \ldots, n\}$ if its Möbius inversion vanishes for any T such that $|T| > k$ and there exists at least one subset T with exactly k elements such that $\beta_T \neq 0$.

Corollary 3 in this paper provides the following characterization of k-additive capacities.[10]

Corollary 6. *Let $k \geq 2$, and let $|\Omega| \geq k+1$. Then, the following two conditions of v are equivalent:*
(i) A capacity v is at most k-additive.
(ii) For all S with $|S| \geq k+1$ and all $\{\omega_1, \omega_2, \cdots, \omega_{k+1}\} \in \mathcal{F}_{k+1}$ with $\{\omega_1, \omega_2, \cdots, \omega_{k+1}\} \subseteq S$,

$$v(S) = \sum_{\emptyset \neq I \subseteq \{1,\ldots,k+1\}} (-1)^{|I|+1} v(\cap_{j \in I}(S \backslash \{\omega_j\})).$$

The analyses of 2-additive capacities are worth mentioning. As an application of the results obtained in the previous sections, this paper generalizes Ben-Porath and Gilboa (1994)'s Gini representation. Ben-Porath and Gilboa (1994) axiomatize social welfare functions represented by a linear combination of total income and the Gini index. Gajdos (2002) provides an axiomatization that is more general than that of Ben-Porath and

[9] Miranda et al. (2006) analyze the set of k-additive belief functions dominating a given capacity, which is between the set of probability measures and the set of k-additive measures. Miranda et al. (2005) axiomatize the behaviors of a rational decision maker, and show that if she satisfies a set of axioms, then her beliefs are captured by a unique k-additive measure and her preferences are represented by the Choquet integral with respect to the k-additive measure.

[10] Miranda et al. (2005) shows that a capacity v is at most k-additive if and only if for all A such that $i_1, \ldots, i_k \in A$, $\sum_{B \subseteq \{i_1,\ldots,i_k\},\, i_1,\ldots,i_k \in A} v(A \backslash B)(-1)^{|B|} = \sum_{B \subseteq \{i_1,\ldots,i_k\}} v(\{i_1, \ldots, i_k\} \backslash B)(-1)^{|B|}$.

Gilboa (1994) and proposes a generalized Gini index.[11] In Ben-Porath and Gilboa (1994) and Gajdos (2002), policymakers' beliefs are captured by *symmetric capacities.*[12] From the viewpoint of economics, it is reasonable to assume that policymakers' beliefs are captured by symmetric capacities since the symmetricity of capacities can be interpreted as necessitating policymakers to be impartial. However, our results in this subsection do not necessarily require the symmetricity of capacities. Before we generalize Ben-Porath and Gilboa (1994)'s Gini representation, we provide the definition of *Choquet integrals* and a characterization of Choquet integrals by Möbius inversions, which is shown by Gilboa and Schmeidler (1994).

Let $\mathbb{R}^\Omega = \{x \,|\, x : \Omega \to \mathbb{R}\}$ be the set of all real valued functions on Ω. For $x \in \mathbb{R}^\Omega$ and a capacity v, the *Choquet integral* of x with respect to v is defined as $\int_\Omega x dv = \int_0^\infty v(x \geq \alpha) d\alpha + \int_{-\infty}^0 (v(x \geq \alpha) - 1)\, d\alpha$, where $v(x \geq \alpha) = v(\{\omega \in \Omega \,|\, x(\omega) \geq \alpha\})$. Gilboa and Schmeidler (1994) provide the relation between the Choquet integral and the Möbius inversion.

Theorem 4 (Gilboa and Schmeidler (1994)). *For all $x \in \mathbb{R}^\Omega$ and a capacity $v = \sum_{T\in\mathcal{F}} \beta_T u_T$, $\int x dv = \sum_{T\in\mathcal{F}} \beta_T \int x du_T = \sum_{T\in\mathcal{F}} \beta_T \min_T x$, where $\min_T x = \min_{\omega\in T} x(\omega)$.*

Let $\Omega = \{1, \dots, n\}$ be a set of individuals, and let $f = (f_1, \dots, f_n)$ be an income profile, where f_i denotes the income of individual i. Ben-Porath and Gilboa (1994) axiomatize the following operator J for an income profile $f = (f_1, \dots, f_n)$:

$$J(f) = \sum_{i=1}^n f_i - \delta \sum_{1\leq i<j\leq n} |f_i - f_j|, \tag{1}$$

where $0 < \delta < 1/(n-1)$ is constant. This representation means that a Gini preference can be represented by a linear combination of total income and the Gini index. Given $|a - b| = a + b - 2\min\{a, b\}$, (1) can be written as $J(f) = (1 - (n-1)\delta) \sum_{i=1}^n f_i + 2\delta \sum_{1\leq i<j\leq n} \min\{f_i, f_j\}$. This representation can be generalized by our results in previous sections. In this regard, let us define the following operator I for an income profile $f = (f_1, \dots, f_n)$:

$$I(f) = \sum_{i=1}^n \beta_i f_i + \sum_{1\leq i<j\leq n} \beta_{\{i,j\}} \min\{f_i, f_j\}, \tag{2}$$

where $\beta_i \geq 0$ and $\beta_{\{i,j\}} \geq 0$ are some constant numbers. We call this form the *generalized Ben-Porath-Gilboa representation.* Then, our main results in Section 4 prove the following.

[11] Gajdos (2002) calls it a *P-Gini index.* See also Section 5 in Gilboa and Schmeidler (1994).

[12] A game $v : 2^\Omega \to \mathbb{R}$ is a *symmetric capacity* if it is a capacity and $v(E) = v(F)$ for any E, $F \in 2^\Omega$ with $|E| = |F|$. Gajdos (2002) shows that if a symmetric capacity is at most k-additive, then it is expressed by a polynomial of degree at most k. Corollary 2 in this paper leads to the same result in Gajdos (2002) for symmetric capacities.

Proposition 7. *Let $f = (f_1, \dots, f_n)$ be an income profile. Then, the following two conditions for operator I are equivalent:*
(i) I is the generalized Ben-Porath-Gilboa representation defined by (2).
(ii) There exists a non-negative convex game v that is 3-modular for 3-set's $\mathcal{F}_2$-decompositions such that $I(f) = \int f dv$.

Proof. (i) $\Rightarrow$ (ii). Let I be the generalized Ben-Porath-Gilboa representation defined by $I(f) = \sum_{i=1}^{n} \beta_i f_i + \sum_{1 \le i < j \le n} \beta_{\{i,j\}} \min\{f_i, f_j\}$, where $\beta_i \ge 0$ and $\beta_{\{i,j\}} \ge 0$. Then, define a game $v = \sum_T \gamma_T u_T$ such that $\gamma_{\{i\}} = \beta_i$ for all $1 \le i \le n$, $\gamma_{\{i,j\}} = \beta_{\{i,j\}}$ for all $1 \le i < j \le n$, and $\gamma_T = 0$ for all other T. By Theorem 4, $\int f dv = \sum_{i=1}^{n} \beta_i f_i + \sum_{1 \le i < j \le n} \beta_{\{i,j\}} \min\{f_i, f_j\} = I(f)$. Since $\beta_i \ge 0$ and $\beta_{\{i,j\}} \ge 0$, v is a totally monotone game, which implies that v is a non-negative convex game. Moreover, since $\gamma_T = 0$ for all T with $|T| \ge 3$, v is 3-modular for 3-set's $\mathcal{F}_2$-decompositions by Corollary 3 .
(ii) $\Rightarrow$ (i). Let v satisfy Condition (ii) in Proposition 7. By Corollary 5, v can be written as $v = \sum_{i=1}^{n} \beta_i u_{\{i\}} + \sum_{1 \le i < j \le n} \beta_{\{i,j\}} u_{\{i,j\}}$, where $\beta_i \ge 0$ and $\beta_{\{i,j\}} \ge 0$ since v is non-negative, convex (that is, 2-monotone), and 3-modular for 3-set's $\mathcal{F}_2$-decompositions. Hence, I is the generalized Ben-Porath-Gilboa representation. $\square$

Note that v in this proposition is a 2-additive capacity. This proposition states that the generalized Ben-Porath-Gilboa representation can be characterized by 2-additive capacities. A decision maker's beliefs are not necessarily restricted to symmetric capacities, while her beliefs are captured by symmetric capacities in Ben-Porath and Gilboa (1994) and Gajdos (2002).

6.3. Potential Functions

In this subsection, we apply our results of the modularity of a game to cooperative games, and extend Ui et al. (2011) that analyze the Myerson value (Myerson (1977)).[13] First, we provide the definitions of the Shapley value. Second, we provide the result obtained by Ui et al. (2011). Finally, we provide an extension of Ui et al. (2011).

Let $\Omega = \{1, \dots, n\}$ $(n \ge 3)$ be a set of players. A subset S of Ω is a coalition, and 2^Ω denotes the set of all coalitions. Let $\mathbf{G}_\Omega$ denote the collection of all games on 2^Ω. The *Shapley value* is a function $\phi : \mathbf{G}_\Omega \to \mathbb{R}^\Omega$ satisfying the following:

$$\phi_i(v) = \sum_{S \in 2^\Omega,\, i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \left(v(S) - v(S \backslash \{i\})\right) \quad \text{for each } i \in \Omega,$$

[13] Myerson (1977) introduces the Myerson value as a solution for cooperative games under the partial cooperation structures described by networks, i.e., sets of 2-player coalitions. Myerson (1980) and van den Nouweland et al. (1992) consider the partial cooperation structures described by sets of coalitions (not necessarily 2-player coalitions), and study the Myerson value for them. The Myerson value in the partial cooperation structures coincides with the Shapley value of a specific game generated by a given game.

where $v(S) - v(S\backslash\{i\})$ denotes the marginal contribution of a player i in a coalition S.[14] It is shown that the Shapley value of $v = \sum_{T\in\mathcal{F}} \beta_T u_T$ can be represented as follows: $\phi_i(v) = \sum_{T\in\mathcal{F}} \beta_T \phi_i(u_T) = \sum_{T\in\mathcal{F},\, i\in T} \beta_T/|T|$.

Hart and Mas-Colell (1989) define a potential function for all games (transferable utility games). Let $p : \mathbf{G}_\Omega \to \mathbb{R}$. For all $i \in \Omega$, the function $Dp_i : \mathbf{G}_\Omega \to \mathbb{R}$ is defined by the marginal contribution of player i to p, that is, $Dp_i(v) = p(v) - p(v_{|\Omega\backslash\{i\}})$, where $v_{|\Omega\backslash\{i\}}$ denotes the restriction of (Ω, v) to $(\Omega\backslash\{i\}, v)$. Then, a function $p : \mathbf{G}_\Omega \to \mathbb{R}$ is a *potential function of* v if it satisfies $\sum_{i\in S} Dp_i(v) = v(S)$ for each S. Hart and Mas-Colell (1989) show that p is uniquely provided by $p(v) = \sum_{T\in\mathcal{F}} \beta_T/|T|$, and that for all $i \in \Omega$, the marginal contribution of player i coincides with the Shapley value, i.e., $Dp_i(v) = \sum_{T\in\mathcal{F}, i\in T} \beta_T/|T| = \phi_i(v)$.

Based on the extended notion of partial coalition structures, Ui et al. (2011) analyze the Myerson value. Let $\mathcal{H} \subseteq 2^\Omega\backslash\mathcal{F}_1$ be a set of coalitions.[15] We write $\mathcal{H}_S = \{H \in \mathcal{H} \,|\, H \subseteq S\}$ and $\mathcal{H}_{-i} = \mathcal{H}_{\Omega\backslash\{i\}}$ for $S \in 2^\Omega$ and $i \in \Omega$, respectively. For a game v and a set of coalitions $\mathcal{H}$, let us consider another game $v^\mathcal{H}$ as defined below.

Definition 12. A game $v^\mathcal{H} = \sum_{T\in\mathcal{F}} \beta^\mathcal{H}_T u_T$ is the $\mathcal{H}$*-projected game of* $v = \sum_{T\in\mathcal{F}} \beta_T u_T$ if $\beta^\mathcal{H}_T$ is determined recursively by the following rules:
1. $\beta^\mathcal{H}_{\{i\}} = v(\{i\})$ for all $i \in \Omega$.
2. For $T \in 2^\Omega$ with $|T| \geq 2$, $\beta^\mathcal{H}_T = v(T) - \sum_{S\subsetneq T} \beta^\mathcal{H}_S$ if $T \in \mathcal{H}$ and $\beta^\mathcal{H}_T = 0$ otherwise.

Note that the above rule is rewritten as $v^\mathcal{H}(S) = v(S)$ for each $S \in \mathcal{H} \cup \mathcal{F}_1$ and $\beta^\mathcal{H}_T = 0$ for each $T \notin \mathcal{H} \cup \mathcal{F}_1$ by Proposition 4.

Definition 13. A set of coalitions $\mathcal{H} \subseteq 2^\Omega\backslash\mathcal{F}_1$ is said to be a *complete coalition structure* if $\Upsilon^2(\mathcal{H}) = \mathcal{F}_1 \cup \mathcal{H}$.

Let **CCS** denote the set of all complete coalition structures, and let $\mathcal{H} \in \mathbf{CCS}$ be any given structure. Furthermore, let $v : 2^\Omega \to \mathbb{R}$ be a game. Ui et al. (2011) consider the following condition for a function $f^\mathcal{H} : \mathbf{G}_\Omega \to \mathbb{R}^\Omega$.[16]

(**C1**): for every v, $f^\mathcal{H}(v)$ is the vector of the marginal contributions of a game $p^\mathcal{H}$ satisfying the following two conditions:

(a) If $S \in \mathcal{H} \cup \mathcal{F}_1$, then $\sum_{i\in S} \left(p^\mathcal{H}(S) - p^\mathcal{H}(S\backslash\{i\})\right) = v(S)$.

(b) If $S \notin \mathcal{H}\cup\mathcal{F}_1$ and $\{S\backslash\{i\}, S\backslash\{j\}\} \in W^2(\mathcal{H})$,[17] then $p^\mathcal{H}(S) - p^\mathcal{H}(S\backslash\{i\}) = p^\mathcal{H}(S\backslash\{j\}) -$

[14] The Shapley value is axiomatized by four well-known axioms: efficiency, the null-player property, symmetricity, and additivity. See Shapley (1953) for details.

[15] Here, the definition of a set of coalitions $\mathcal{H}$ is slightly different from that in Ui et al. (2011).

[16] This function should be defined by $f : \mathbf{CCS} \times \mathbf{G}_\Omega \to \mathbb{R}^\Omega$. However, in this paper, any structure $\mathcal{H} \in \mathbf{CCS}$ is fixed. Therefore, f is defined by a function $f^\mathcal{H}$ that assigns a $|\Omega|$-dimensional real vector to any game on Ω.

[17] Recall that $W^2(\mathcal{H})$ denotes the collection of all 2-set's $\mathcal{H}$-decompositions. See Definition 5.

$p^{\mathcal{H}}(S\backslash\{i,j\})$. That is, $f_i^{\mathcal{H}}(v) = p^{\mathcal{H}}(\Omega) - p^{\mathcal{H}}(\Omega\backslash\{i\})$ for each $i \in \Omega$ and $\mathcal{H} \in \mathbf{CCS}$.

Condition (**C1**) can be related to the Shapley value.

Proposition 8 (Ui et al. (2011)). *A solution $f^{\mathcal{H}} : \mathbf{G}_\Omega \to \mathbb{R}^\Omega$ satisfies Condition (***C1***)*[18] *if and only if $f^{\mathcal{H}}(v) = \phi(v^{\mathcal{H}})$ for each $\mathcal{H} \in \mathbf{CCS}$ and each v where $v^{\mathcal{H}}$ is the $\mathcal{H}$-projected game of v and ϕ is the Shapley value.*

By interpreting $p^{\mathcal{H}}(S)$ given v as $p^{\mathcal{H}}(v_{|S})$, Ui et al. (2011) regard $p^{\mathcal{H}}(S)$ as a potential, and we follow this approach. Based on Theorem 2, we can extend Proposition 8 to the structures that correspond to the completeness of order m.

Definition 14. A set of coalitions $\mathcal{H} \subseteq 2^\Omega\backslash\mathcal{F}_1$ is said to be a *complete coalition structure of order* m if $\Upsilon^m(\mathcal{H}) = \mathcal{F}_1 \cup \mathcal{H}$.

Let $\mathbf{CCS}^m$ denote the set of all complete coalition structures of order m. For ease of exposition, we consider the case of $m = 3$. Let $\mathcal{H} \in \mathbf{CCS}^3$ be any given structure. We consider the following condition for a function $f^{\mathcal{H}} : \mathbf{G}_\Omega \to \mathbb{R}^\Omega$.
(**C2**): for every v, $f^{\mathcal{H}}(v)$ is the vector of the marginal contributions of a game $p^{\mathcal{H}}$ satisfying the following three conditions:
(a) If $S \in \mathcal{F}_1 \cup \mathcal{H}$, then $\sum_{i\in S}(p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\backslash\{i\})) = v(S)$.
(b) If $S \notin \mathcal{F}_1 \cup \mathcal{H}$ and $\{S\backslash\{i\}, S\backslash\{j\}, S\backslash\{k\}\} \in W^3(\mathcal{H})$, then $p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\backslash\{i\}) = (p^{\mathcal{H}}(S\backslash\{j\}) - p^{\mathcal{H}}(S\backslash\{i,j\})) + (p^{\mathcal{H}}(S\backslash\{k\}) - p^{\mathcal{H}}(S\backslash\{i,k\})) - (p^{\mathcal{H}}(S\backslash\{j,k\}) - p^{\mathcal{H}}(S\backslash\{i,j,k\}))$.
(c) If $S \notin \mathcal{F}_1 \cup \mathcal{H}$ and $\{S\backslash\{i\}, S\backslash\{j\}\} \in W^2(\mathcal{H})$, then $p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\backslash\{i\}) = p^{\mathcal{H}}(S\backslash\{j\}) - p^{\mathcal{H}}(S\backslash\{i,j\})$. That is, $f_i^{\mathcal{H}}(v) = p^{\mathcal{H}}(\Omega) - p^{\mathcal{H}}(\Omega\backslash\{i\})$ for each $i \in \Omega$ and $\mathcal{H} \in \mathbf{CCS}^3$.

Then, we can extend Proposition 8 as follows.

Proposition 9. *A solution $f^{\mathcal{H}} : \mathbf{G}_\Omega \to \mathbb{R}^\Omega$ satisfies Condition (***C2***) if and only if $f^{\mathcal{H}}(v) = \phi(v^{\mathcal{H}})$ for each $\mathcal{H} \in \mathbf{CCS}^3$ and each v where $v^{\mathcal{H}}$ is the $\mathcal{H}$-projected game of v and ϕ is the Shapley value.*

Our proof of Proposition 9 is similar to that of van den Brink (2001, Theorem 2.5). van den Brink (2001) shows that a function $f : \mathbf{G}_\Omega \to \mathbb{R}^\Omega$ is equal to the Shapley value if and only if it satisfies efficiency, the null player property, and *fairness*.[19] First, it is shown that the Shapley value satisfies the three conditions. Second, a solution f satisfying the

[18] Ui et al. (2011) call solution $f^{\mathcal{H}}$ the Myerson value for complete coalition structures.

[19] van den Brink (2001) proposes *fairness*. This property states that if to a game $v \in \mathbf{G}_\Omega$ we add a game $w \in \mathbf{G}_\Omega$ where players i and j are symmetric, then the payoffs of players i and j change by the same amount. That is, if i, $j \in \Omega$ are symmetric players in a game $w \in \mathbf{G}_\Omega$, then $f_i(v + w) - f_i(v) = f_j(v + w) - f_j(v)$ for all $v \in \mathbf{G}_\Omega$.

three conditions must be uniquely constructed from the three conditions in a recursive way, which is based on some geometric property.

Proof. Note that if $\mathcal{H} \in \mathbf{CCS}^3$, then $\Upsilon^3(\mathcal{H}) = \mathcal{F}_1 \cup \mathcal{H}$. First, we show that the potential for $v^{\mathcal{H}}$ satisfies (**C2**). Let $p^{\mathcal{H}}$ be the potential for $v^{\mathcal{H}} = \sum_{T\in\mathcal{F}} \beta_T^{\mathcal{H}} u_T$. Then, it follows from Hart and Mas-Colell (1989) that $p^{\mathcal{H}} = \sum_{T\in\mathcal{F}} (\beta_T^{\mathcal{H}}/|T|) u_T$. Therefore, if $S \in \mathcal{F}_1 \cup \mathcal{H}$, then it holds that $\sum_{i\in S} \left(p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\backslash\{i\})\right) = v^{\mathcal{H}}(S) = v(S)$, where the first equality holds because $p^{\mathcal{H}}$ is the potential for $v^{\mathcal{H}}$, and the second equality holds by the definition of the $\mathcal{H}$-projected game of v. This is Condition (a) in (**C2**). Next, since $\beta_T^{\mathcal{H}}/|T| = 0$ for all $T \notin \Upsilon^3(\mathcal{H}) = \mathcal{F}_1 \cup \mathcal{H}$, by setting $w(S) = p^{\mathcal{H}}(S)$ in Proposition 4, it holds that for $S \notin \mathcal{F}_1 \cup \mathcal{H}$ and $\{S\backslash\{i\}, S\backslash\{j\}, S\backslash\{k\}\} \in W^3(\mathcal{H})$,

$$\begin{aligned} & p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\backslash\{i\}) \\ = \; & (p^{\mathcal{H}}(S\backslash\{j\}) - p^{\mathcal{H}}(S\backslash\{i,j\})) \\ & + (p^{\mathcal{H}}(S\backslash\{k\}) - p^{\mathcal{H}}(S\backslash\{i,k\})) - (p^{\mathcal{H}}(S\backslash\{j,k\}) - p^{\mathcal{H}}(S\backslash\{i,j,k\})), \end{aligned}$$

and for $S \notin \mathcal{F}_1 \cup \mathcal{H}$ and $\{S\backslash\{i\}, S\backslash\{j\}\} \in W^2(\mathcal{H})$, $p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\backslash\{i\}) = p^{\mathcal{H}}(S\backslash\{j\}) - p^{\mathcal{H}}(S\backslash\{i,j\})$. These are Conditions (b) and (c) in (**C2**).

Second, we show that a game $p^{\mathcal{H}}$ satisfying (**C2**) must be the potential for $v^{\mathcal{H}} = \sum_{T\in\mathcal{F}} \beta_T^{\mathcal{H}} u_T$. To show this, it suffices to show that if such a $p^{\mathcal{H}}$ exists, then it exists uniquely since we have already shown that the potential for $v^{\mathcal{H}}$ satisfies (**C2**). We can show this claim by an argument similar to that in the proof of Proposition 4. We construct $p^{\mathcal{H}}$ recursively as follows. Start with $p^{\mathcal{H}}(\emptyset) = 0$ since $p^{\mathcal{H}}$ is a game. If S is a singleton, that is, $S = \{i\}$, then $p^{\mathcal{H}}(\{i\}) = v(\{i\})$ by Condition (a) in (**C2**). For $|S| \geq 2$, if $S \in \mathcal{H}$, then it follows from Condition (a) in (**C2**) that $p^{\mathcal{H}}(S) = |S|^{-1} \left(v(S) + \sum_{i\in S} p^{\mathcal{H}}(S\backslash\{i\})\right)$. If $S \notin \mathcal{H}$, $\{S\backslash\{i\}, S\backslash\{j\}, S\backslash\{k\}\} \in W^3(\mathcal{H})$, then it follows from Condition (b) in (**C2**) that

$$\begin{aligned} p^{\mathcal{H}}(S) \;=\; & p^{\mathcal{H}}(S\backslash\{i\}) + (p^{\mathcal{H}}(S\backslash\{j\}) - p^{\mathcal{H}}(S\backslash\{i,j\})) + (p^{\mathcal{H}}(S\backslash\{k\}) \\ & - p^{\mathcal{H}}(S\backslash\{i,k\})) - (p^{\mathcal{H}}(S\backslash\{j,k\}) - p^{\mathcal{H}}(S\backslash\{i,j,k\})). \end{aligned}$$

If $S \notin \mathcal{H}$, $\{S\backslash\{i\}, S\backslash\{j\}\} \in W^2(\mathcal{H})$, then it follows from Condition (c) in (**C2**) that $p^{\mathcal{H}}(S) = p^{\mathcal{H}}(S\backslash\{i\}) + p^{\mathcal{H}}(S\backslash\{j\}) - p^{\mathcal{H}}(S\backslash\{i,j\})$. □

7. Conclusion

This paper analyzes two problems that remain to be solved in Kajii et al. (2007). First, we generalize Kajii et al. (2007), and provide a condition under which for a game v, its Möbius inversion is equal to zero within the framework of the k-modularity of v for $k \geq 2$.

This condition is more general than that in Kajii et al. (2007). Second, we provide a condition under which for a game v for $k \geq 2$, its Möbius inversion takes non-negative values, and not just zero. This task is important since such a condition enables us to characterize some class of totally monotone games by the modularity of a game v.

Our results for modularity are closely related to the results in the literature on non-additive measure theory. The modularity of a game can be related to the inclusion-exclusion covering proposed by Sugeno et al. (1995). The modularity of a game can also be related to a k-additive capacity proposed by Grabisch (1997). Furthermore, we provide the economic interpretations of our results applying them to existing problems. We show that a Gini index representation axiomatized by Ben-Porath and Gilboa (1994) is characterized by the Choquet integrals satisfying some conditions. The Gini index representation cannot be obtained within the framework of Kajii et al. (2007). An application of our results to potential functions proposed by Hart and Mas-Colell (1985) and further analyzed by Ui et al. (2011) implies that our results can also be applied to cooperative game theory.

Some tasks remain for the future. This paper does not analyze axiomatizations of totally monotone games from the viewpoint of decision theory. In addition to characterizations of totally monotone games within a general framework, it is important to shed some light on totally monotone games from the normative viewpoint of economics. This is a topic for future research.

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