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"An Extension of the Chaos Expansion Approximation for the Pricing of Exotic Basket Options"

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AN EXTENSION OF THE CHAOS EXPANSION APPROXIMATION FOR THE PRICING OF EXOTIC BASKET OPTIONS

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ABSTRACT. Funahashi and Kijima (2013) have proposed an approximation method based on the Wiener–Ito chaos expansion for the pricing of European-style contingent claims. In this paper, we extend the method to the multi-asset case with general local volatility structure for the pricing of exotic basket options such as Asian basket options. Through ample numerical experiments, we show that the accuracy of our approximation remains quite high even for a complex basket option with long maturity and high volatility.

Keywords: Wiener–Ito chaos expansion, local volatility, average option, basket option, spread option, Asian basket option

1. INTRODUCTION

The aim of this paper is to provide an approximation method for the pricing of European-style *Asian basket* options and their variants.

Asian options belong to the class of path-dependent options whose payoff functions are determined by the average of underlying asset price process over some pre-determined period of time.[1] Asian options are popular in the foreign exchange and commodities markets, since they can help corporate firms hedge risks arising from their businesses. In addition, Asian options are cheaper than the corresponding vanilla options, and hence they are preferred by practitioners. On the other hand, basket options are also exotic options, whose payoff functions are based on more than one underlying assets. Examples of basket options include index options, spread options, rainbow options and options on a portfolio. This type of options is also popular in the foreign exchange market, because financial corporations with multiple currency exposures can hedge their exposures less expensively by purchasing a basket option than by purchasing vanilla options on each currency individually. Asian basket options are the combination of these popular options. In general, both Asian and basket options are known to be difficult to price analytically and numerically.

A large number of numerical methods have been proposed in the literature for the pricing of Asian and/or basket options. For Asian options, numerical methods based on the partial differential equation (PDE for short) or Monte Carlo methods are proposed by Kemna and Vorst (1990), Dewynne and Wilmott (1993), Lapeyre and Temam (2001), Vecer (2001) and many others, whereas the literature for basket options includes Rubinstein (1991), Pellizzari (2001) and

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[1] Unless stated otherwise, 'average' always means *arithmetic* average in this paper.

Hager et al. (2010). Unfortunately, these methods are in general computationally too burdensome for practical use. In particular, for general basket options, any standard method such as finite difference method and tree method is subject to the 'curse of dimensionality' and cannot solve the problem for the high-dimensional case within a reasonable time for practice. While Monte Carlo simulation seems to work for the pricing problem even for the high-dimensional case, this method is in general too computationally expensive to be used for calibration purpose. Therefore, closed-form approximation formulas might be the only feasible solution to practitioners for the pricing of Asian basket options.

When the underlying asset model is restricted to the geometric Brownian motion (GBM for short) case, several effective approximations have been proposed in the literature. Tumbull and Wakeman (1991) and Ritchken et al. (1993) apply a fourth-order Edgeworth series approximation to the log-normal distribution and obtain an analytical approximation for Asian option prices. Levy (1992) modifies the Tumbull–Wakeman approximation to derive another analytic approximation method which is considered to give more accurate results. Milevsky and Posner (1998) propose closed-form approximation formulas for Asian options based on reciprocal Gamma distributions. Posner and Milevsky (1998) use Johnson functions to approximate the state price density by matching the first four moments and use them in integral formulas for pricing Asian options. Ju (2002) approximates the ratio of the characteristic function of the arithmetic average to that of the approximating log-normal random variable by using the sixth-order Taylor expansion around zero and derives a very accurate approximated closed-form solution. It should be noted that these methods can be directly applied for the pricing of basket options (see Ju (2002) and Krekel et al. (2004) for detailed discussions). On the other hand, Zhang (2001) proposes a semi-analytical approach that uses a singularity-removing technique to derive an analytical approximation formula of Asian options and derives the PDE for the correction term between the exact price and the analytical approximation.

It is well known that the Black and Scholes model (1973) cannot consistently price European options in the market, since implied volatility surfaces are usually skew- or smile-shaped. This tendency holds for the case of Asian basket options as well, and the above mentioned approximations are not suitable for practical use because they are based on the GBM assumption. Hence, it is required to develop some approximation method for more general underlying processes. In this regard, Takahashi (1999) applies the Malliavin–Watanabe theory to derive second-order asymptotic approximation formulas for both Asian and basket options under a general class of diffusion processes. Also, Fouque and Han (2003) use perturbation techniques to approximate Asian option prices under a stochastic volatility environment.

In this paper, we propose an approximation method based on the chaos expansion approach, recently proposed by Funahashi and Kijima (2013), for the pricing of Asian basket options. Through ample numerical examples, we show that the accuracy of our approximation remains quite high even for a complex basket option with long maturity and high volatility under various diffusion models. By the comparison with the previous works, we show that our approximation provides highly accurate results over a wide range of data sets even for the GBM case. Moreover, our approximation formulas can capture the skew and smile effects. Also, our approach can not only save computational time without sacrificing much accuracy, but also lead to the effective and/or stable calculation of Greeks.

This paper is organized as follows. After explaining our problem concisely in the next section, we extend the chaos expansion approach of Funahashi and Kijima (2013) to the multi-asset case in Section 3. Each asset price is approximated by a truncated sum of iterated Ito stochastic integrals, and the Asian basket variable is also described, after the change of order of integration, as a sum of iterated Ito stochastic integrals. An approximated formula of Asian basket options can then be derived in closed form. In Section 4, we consider a special case where each local volatility function depends only on its price, not on the other asset prices. Approximation for the ordinary Black–Scholes setting (1973) is also considered and compared with the previous approximation results mentioned above. Section 5 is devoted to numerical examples. Comparing the closed formulas with Monte Carlo simulation results, it is observed that our approximation remains quite accurate even for a complex basket option with long maturity and high volatility. Finally, Section 6 concludes this paper.

Throughout this paper, $(\Omega, \mathcal{F}, \mathbb{Q}, \{\mathcal{F}_t\}_{t\geq 0})$ will be a filtered probability space where the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions. The probability measure $\mathbb{Q}$ is a risk-neutral measure and the expectation operator under $\mathbb{Q}$ is denoted by $\mathbb{E}$.

2. THE SETUP

In this paper, we consider a financial market with N risky assets $\{S_{i,t}\}_{0\leq t\leq T}$, $i = 1, 2, \ldots, N$, and one risk-free asset $\{S_{0,t}\}_{0\leq t\leq T}$. The risk-free asset is a money-market account with spot interest rate $r(t)$, that is a deterministic function of time t. On the other hand, the risky assets are modeled by the following stochastic differential equation (SDE for short):

$$\frac{\mathrm{d}S_{i,t}}{S_{i,t}} = r(t)\mathrm{d}t + \sigma_i(\mathbf{S}_t, t)\mathrm{d}W_{i,t}, \qquad 0 \leq t \leq T, \tag{2.1}$$

under the risk-neutral measure $\mathbb{Q}$, where $\mathbf{S}_t = (S_{1,t}, \ldots, S_{N,t})$ and the volatilities $\sigma_i(\mathbf{s}, t)$ are deterministic functions of both asset prices $\mathbf{s} = (s_1, \ldots, s_N)$ and time t, and where $\{W_{i,t}\}_{t\geq 0}$ are standard Brownian motions under $\mathbb{Q}$ with correlation $\mathrm{d}W_{i,t}\mathrm{d}W_{j,t} = \rho_{i,j}\mathrm{d}t$. It is assumed throughout that each volatility $\sigma_i(\mathbf{s}, t)$ is an analytic function of $(\mathbf{s}, t)$.

For the multivariate *local volatility* model (2.1), we consider the following exotic basket options. Namely, for weighting (deterministic) functions $w_{i,t}$, define the random variable

$$V_T := \sum_{i=1}^{N} \int_0^T w_{i,t} S_{i,t} \mathrm{d}t, \qquad V_0 = V, \tag{2.2}$$

and consider a call option written on V_T with exercise price K, i.e.,

$$C(V, K, T) = \mathbb{E}\left[\mathrm{e}^{-\int_0^T r(u)\mathrm{d}u} (V_T - K)^+\right]. \tag{2.3}$$

The aim of this paper is to derive the option value $C(V, K, T)$.

To this end, there may be several approaches that are applicable for the pricing problem. For example, in principle, any standard method such as finite difference method and tree method can be applied to solve the problem. However, these methods are subject to the curse of dimensionality and cannot solve the problem within a reasonable time for practice. On the other hand, Monte Carlo simulation seems to work for the pricing problem even for the high-dimensional case. However, when calibration is required to the market, this method is in general too computationally expensive to be used in practice, because the entire optimization procedure is extremely time-consuming. Therefore, closed-form approximation formulas might be the only feasible solution for practitioners.

In this paper, we apply the chaos expansion approach recently developed by Funahashi and Kijima (2013) to approximate the random variable V_T by a truncated sum of iterated Ito stochastic integrals. We then derive the probability density function (PDF for short) of the approximated random variable. The value of the call option can be derived in closed form by the approximated PDF.

Before proceeding, we emphasize that the above formulation is a generalization of the following well-known options:

European Option: Let $N = 1$ and $w_{1,t} = \delta(T-t)$, where $\delta(u)$ represents the Dirac delta function. Then, $V_T = S_{1,T}$ so that the option (2.3) reduces to the plain-vanilla European option.

Asian Option: Let $N = 1$ and $w_{1,t} = 1/T$ for all $0 \le t \le T$. Then, $V_T = \frac{1}{T}\int_0^T S_{1,t}\mathrm{d}t$. This option is called an average or (arithmetic) Asian option.

Partial Average Option: Let $N = 1$ and $w_{1,t} = 1/(T_1 - T_2)$ for all $T_1 \le t \le T_2$ and $w_{1,t} = 0$ otherwise, where $0 \le T_1 < T_2 \le T$. In this case, we have $V_T = \frac{1}{(T_2-T_1)}\int_{T_1}^{T_2} S_{1,t}\mathrm{d}t$.

Basket Option: Let $w_{i,t} = w_i\delta(T-t)$, where w_i are some constants for $i = 1, 2, \dots, N$. In this case, we have $V_T = \sum_{i=1}^N w_i S_{i,T}$, the so-called basket option.

Spread Option: As a special case of basket options, if $N = 2$, $w_1 = 1$ and $w_2 = -1$, then we have $V_T = S_{1,T} - S_{2,T}$. Hence, the option is reduced to a spread option.

Asian Basket Option: Let $w_{i,t} = 1/T$, $i = 1, 2, \dots, N$, for all $0 \le t \le T$. Then, $V_T = \sum_{i=1}^N \frac{1}{T}\int_0^T S_{i,t}\mathrm{d}t$, and the option is called an Asian basket option.

3. THE CHAOS EXPANSION APPROACH

The chaos expansion approach is composed by the following four steps:

(1) Represent the underlying asset by Hermite polynomials,
(2) Expand the underlying dynamics by means of successive substitution, and approximate it by a truncated sum of iterated Ito stochastic integrals using the Wiener–Ito chaos expansion,
(3) Derive the PDF of the approximated underlying variable, and
(4) Compute the value of a European contingent claim in closed form by using the PDF.

In the following, we extend the results of Funahashi and Kijima (2013) to the multi-dimensional case. Some of the extensions are straightforward; in that case, we omit our exposition largely.

3.1. Approximation of the underlying assets.

First, by applying Ito's formula to the SDE (2.1), we obtain

$$S_{i,t} = F_i(0,t)\exp\left[J_{i,t}(\sigma_i) - \frac{1}{2}\|\sigma_i\|_t^2\right], \tag{3.1}$$

where $F_i(0,t) = S_{i,0}\mathrm{e}^{\int_0^t r(u)\mathrm{d}u}$ is the forward price of the underlying asset with delivery date t, $J_{i,t}(g) = \int_0^t g(u)\mathrm{d}W_{i,u}$ and $\|g\|_t^2 = \int_0^t g^2(u)\mathrm{d}u$. It is well known (see, e.g., Chapter 1 of Nualart (2006)) that the above expression can be written as

$$S_{i,t} = F_i(0,t)\sum_{n=0}^{\infty}\frac{\|\sigma_i\|_t^n}{n!}h_n\left(\frac{J_{i,t}(\sigma_i)}{\|\sigma_i\|_t}\right) \tag{3.2}$$

for any $\sigma_i \in L^2([0,T])$, where $h_n(x)$ denotes the Hermite polynomial of order n.

Second, let $S_{i,t}^{(0)} = F_i(0,t)$ and define $S_{i,t}^{(m)}$ successively by

$$S_{i,t}^{(m+1)} = F_i(0,t)\exp\left[J_t(\sigma_i^{(m)}) - \frac{1}{2}\|\sigma_i^{(m)}\|_t^2\right], \tag{3.3}$$

where $\sigma_i^{(m)}(t) = \sigma_i(\mathbf{S}_t^{(m)}, t)$ with $\mathbf{S}_t^{(m)} = (S_{1,t}^{(m)}, \ldots, S_{N,t}^{(m)})$. As in Funahashi and Kijima (2013), we assume that $\mathbf{S}_t^{(m)}$ converges to $\mathbf{S}_t$ componentwise almost surely as $m \to \infty$. Then, we have

$$S_{i,t} = S_{i,t}^{(1)} + \sum_{m=1}^{\infty} \left\{ S_{i,t}^{(m+1)} - S_{i,t}^{(m)} \right\}.$$

Also, from (3.2) and (3.3), we obtain

$$\frac{S_{i,t}^{(m+1)}}{F_i(0,t)} = \sum_{n=0}^{\infty} \frac{\|\sigma_i^{(m)}\|_t^n}{n!} h_n \left(\frac{J_{i,t}(\sigma_i^{(m)})}{\|\sigma_i^{(m)}\|_t} \right).$$

It follows that

$$S_{i,t} = S_{i,t}^{(1)} + F_i(0,t) \sum_{m,n=1}^{\infty} I_{i:m,n}(t),$$

where

$$I_{i:m,n}(t) = \frac{1}{n!} \left\{ \|\sigma_i^{(m)}\|_t^n h_n \left(\frac{J_{i,t}(\sigma_i^{(m)})}{\|\sigma_i^{(m)}\|_t} \right) - \|\sigma_i^{(m-1)}\|_t^n h_n \left(\frac{J_{i,t}(\sigma_i^{(m-1)})}{\|\sigma_i^{(m-1)}\|_t} \right) \right\}. \tag{3.4}$$

As in Funahashi and Kijima (2013), we approximate $S_{i,t}$ by a truncated sum at $m + n \leq 3$. Namely, our approximation is given by

$$S_{i,t} \;\approx\; S_{i,t}^{(1)} + F_i(0,t) \sum_{m+n \leq 3} I_{i:m,n}(t). \tag{3.5}$$

This approximation is justified by Proposition 2.2 of Funahashi and Kijima (2013), when the volatility term is small in the L_2 sense.

It remains to approximate the remaining terms in (3.5). To this end, we invoke Proposition 1.14 of Nualart (2006) to derive

$$\frac{S_{i,t}^{(1)}}{F_i(0,t)} = 1 + \sum_{n=1}^{\infty} \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \sigma_i^{(0)}(t_1)\sigma_i^{(0)}(t_2) \cdots \sigma_i^{(0)}(t_n) \mathrm{d}W_{i,t_1} \cdots \mathrm{d}W_{i,t_n},$$

where $\sigma_i^{(0)}(t) = \sigma_i(\mathbf{S}_t^{(0)}, t)$, $\mathbf{S}_t^{(0)} = (F_1(0,t), \ldots, F_N(0,t))$, is a deterministic function. According to our strategy, we approximate it as

$$\begin{aligned} S_{i,t}^{(1)} \;\approx\; F_i(0,t) \Bigg[1 &+ \int_0^t \sigma_i^{(0)}(t_1) \mathrm{d}W_{t_1} + \int_0^t \int_0^{t_2} \sigma_i^{(0)}(t_1)\sigma_i^{(0)}(t_2) \mathrm{d}W_{i,t_1} \mathrm{d}W_{i,t_2} \\ &+ \int_0^t \int_0^{t_3} \int_0^{t_2} \sigma_i^{(0)}(t_1)\sigma_i^{(0)}(t_2)\sigma_i^{(0)}(t_3) \mathrm{d}W_{i,t_1} \mathrm{d}W_{i,t_2} \mathrm{d}W_{i,t_3} \Bigg], \end{aligned} \tag{3.6}$$

the third-order approximation.

In order to approximate $I_{i:m,n}(t)$, $m + n \leq 3$, we employ Taylor's expansion around $\mathbf{S}_t^{(m-1)}$, as in Funahashi and Kijima (2013). Namely, since $J_{i,t}(\sigma_i^{(m)}) = \int_0^t \sigma_i(\mathbf{S}_u^{(m)}, u) \mathrm{d}W_{i,u}$, it follows that

$$\begin{aligned} J_{i,t}(\sigma_i^{(m)}) \;\approx\; & J_{i,t}(\sigma_i^{(m-1)}) + \sum_{p=1}^{N} \int_0^t \partial_p \sigma_i^{(m-1)}(u) \{S_{p,u}^{(m)} - S_{p,u}^{(m-1)}\} \mathrm{d}W_{i,u} \\ & + \frac{1}{2} \sum_{p,q=1}^{N} \int_0^t \partial_{pq} \sigma_i^{(m-1)}(u) \{S_{p,u}^{(m)} - S_{p,u}^{(m-1)}\}\{S_{q,u}^{(m)} - S_{q,u}^{(m-1)}\} \mathrm{d}W_{i,u}, \end{aligned} \tag{3.7}$$

where $\partial_p \sigma_i^{(m)}(u)$ denotes the partial derivative of $\sigma_i^{(m)}(u)$ with respect to the pth variable and $\partial_{pq}\sigma^{(m)}(u)$ represents the second-order partial derivative with respect to the pth and qth variables. Furthermore, we use the approximation

$$J_{i,t}^2(\sigma_i^{(m)}) \;\approx\; J_{i,t}^2(\sigma_i^{(m-1)}) + 2J_{i,t}(\sigma_i^{(m-1)}) \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(m-1)}(u)\{S_{p,u}^{(m)} - S_{p,u}^{(m-1)}\}\mathrm{d}W_{i,u}. \tag{3.8}$$

Repeated application of the expansion results (3.7) and (3.8) leads to the following. Recall that our strategy is to neglect those terms that produce fourth- or higher-order iterated Ito stochastic integrals. The proof is given in Appendix A.

Lemma 3.1. *Each term $I_{i:m,n}(t)$ defined by (3.4) is approximated as follows:*

$$\begin{aligned}
I_{i:1,1}(t) \;\approx\; & \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0,s) \left(\int_0^s \sigma_p^{(0)}(u)\mathrm{d}W_{p,u} \right) \mathrm{d}W_{i,s} \\
& + \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0,s) \left(\int_0^s \sigma_p^{(0)}(u) \left(\int_0^u \sigma_p^{(0)}(r)\mathrm{d}W_{p.r} \right) \mathrm{d}W_{p,u} \right) \mathrm{d}W_{i,s} \\
& + \sum_{p,q=1}^N \int_0^t \partial_{pq} \sigma_i^{(0)}(s) F_p(0,s) F_q(0,s) \left(\int_0^s \sigma_p^{(0)}(u) \left(\int_0^u \sigma_q^{(0)}(r)\mathrm{d}W_{q.r} \right) \mathrm{d}W_{p,u} \right) \mathrm{d}W_{i,s} \\
& + \frac{1}{2} \sum_{p,q=1}^N \int_0^t \partial_{pq} \sigma_i^{(0)}(s) F_p(0,s) F_q(0,s) \left(\int_0^s \sigma_p^{(0)}(u)\sigma_q^{(0)}(u)\mathrm{d}u \right) \mathrm{d}W_{i,s},
\end{aligned}$$

$$\begin{aligned}
I_{i:1,2}(t) \;\approx\; & \sum_{p=1}^N \int_0^t \sigma_i^{(0)}(s) \left(\int_0^s \partial_p \sigma_i^{(0)}(u) F_p(0,u) \left(\int_0^u \sigma_p^{(0)}(r)\mathrm{d}W_{p,r} \right) \mathrm{d}W_{p,u} \right) \mathrm{d}W_{i,s} \\
& + \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0,s) \left(\int_0^s \sigma_i^{(0)}(u) \left(\int_0^u \sigma_p^{(0)}(r)\mathrm{d}W_{p,r} \right) \mathrm{d}W_{i,u} \right) \mathrm{d}W_{i,s} \\
& + \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0,s) \left(\int_0^s \sigma_p^{(0)}(u) \left(\int_0^u \sigma_i^{(0)}(r)\mathrm{d}W_{i,r} \right) \mathrm{d}W_{p,u} \right) \mathrm{d}W_{i,s} \\
& + \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0,s) \left(\int_0^s \sigma_i^{(0)}(u)\sigma_p^{(0)}(u)\mathrm{d}u \right) \mathrm{d}W_{i,s},
\end{aligned}$$

and

$$I_{i:2,1}(t) \approx \sum_{p,q=1}^N \int_0^t \partial_p \sigma_i^{(0)} F_p(0,s) \left(\int_0^s \partial_q \sigma_p^{(0)} F_q(0,s) \left(\int_0^u \sigma_q^{(0)}(r)\mathrm{d}W_{q,r} \right) \mathrm{d}W_{p,u} \right) \mathrm{d}W_{i,s}.$$

Note that all the integrands in Lemma 3.1 are deterministic functions, since $\sigma_i^{(0)}(t) = \sigma_i(\mathbf{S}_t^{(0)}, t)$ with $S_{i,t}^{(0)} = F_i(0,t)$ being the forward price of asset $S_{i,t}$ that is observed in the market.

We are now in a position to state our approximation result. The next result is obtained by putting above approximation results all together.

Theorem 3.1. *Each asset price $S_{i,t}$ is approximated as*

$$S_{i,t} \;\approx\; F_i(0,t)\left[1 + A_{i,t}^1 + A_{i,t}^2 + A_{i,t}^3\right], \qquad i = 1, 2, \ldots, N, \tag{3.9}$$

where

$$\begin{aligned} A^1_{i,t} &= \int_0^t P^1_i(s)\mathrm{d}W_{i,s}, \\ A^2_{i,t} &= \int_0^t \sigma^{(0)}_i(s)\left(\int_0^s \sigma^{(0)}_i(u)\mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s} \\ &\quad + \sum_{p=1}^N \int_0^t P^2_{i:p}(s)\left(\int_0^s \sigma^{(0)}_p(u)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s}, \end{aligned}$$

and $A^3_{i,t} = \sum_{k=1}^7 A^3_{i,t}(k)$ *with* $A^3_{i,t}(k)$ *being defined by*

$$\begin{aligned} A^3_{i,t}(1) &= \int_0^t \sigma^{(0)}_i(s)\left(\int_0^s \sigma^{(0)}_i(u)\left(\int_0^u \sigma^{(0)}_i(r)\mathrm{d}W_{i,r}\right)\mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s}, \\ A^3_{i,t}(2) &= \sum_{p=1}^N \int_0^t P^2_{i:p}(s)\left(\int_0^s \sigma^{(0)}_p(u)\left(\int_0^u \sigma^{(0)}_p(r)\mathrm{d}W_{p.r}\right)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s}, \\ A^3_{i,t}(3) &= \sum_{p=1}^N \int_0^t \sigma^{(0)}_i(s)\left(\int_0^s P^2_{i:p}(u)\left(\int_0^u \sigma^{(0)}_p(r)\mathrm{d}W_{p,r}\right)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s}, \\ A^3_{i,t}(4) &= \sum_{p=1}^N \int_0^t P^2_{i:p}(s)\left(\int_0^s \sigma^{(0)}_i(u)\left(\int_0^u \sigma^{(0)}_p(r)\mathrm{d}W_{p,r}\right)\mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s}, \\ A^3_{i,t}(5) &= \sum_{p=1}^N \int_0^t P^2_{i:p}(s)\left(\int_0^s \sigma^{(0)}_p(u)\left(\int_0^u \sigma^{(0)}_i(r)\mathrm{d}W_{i,r}\right)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s}, \\ A^3_{i,t}(6) &= \sum_{p,q=1}^N \int_0^t P^2_{i:p}(s)\left(\int_0^s P^2_{p:q}(s)\left(\int_0^u \sigma^{(0)}_q(r)\mathrm{d}W_{q,r}\right)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s}, \\ A^3_{i,t}(7) &= \sum_{p,q=1}^N \int_0^t P^3_{i:p,q}(s)\left(\int_0^s \sigma^{(0)}_p(u)\left(\int_0^u \sigma^{(0)}_q(r)\mathrm{d}W_{q.r}\right)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s}. \end{aligned}$$

Note that $A^k_{i,t}$*,* $k = 1, 2, 3$*, corresponds to the* k*th-order iterated Ito stochastic integrals. In particular,* $A^1_{i,t}$ *follows a normal distribution with zero mean. Here, we define*

$$\begin{aligned} P^1_i(s) &:= \sigma^{(0)}_i(s) + \sum_{p=1}^N \partial_p\sigma^{(0)}_i(s)F_p(0,s)\left(\int_0^s \sigma^{(0)}_i(u)\sigma^{(0)}_p(u)\mathrm{d}u\right) \\ &\quad + \frac{1}{2}\sum_{p,q=1}^N \partial_{pq}\sigma^{(0)}_i(s)F_p(0,s)F_q(0,s)\left(\int_0^s \sigma^{(0)}_p(u)\sigma^{(0)}_q(u)\mathrm{d}u\right), \\ P^2_{i:p}(s) &:= \partial_p\sigma^{(0)}_i(s)F_p(0,s), \qquad p = 1, 2, \ldots, N, \\ P^3_{i:p,q}(s) &:= \partial_{pq}\sigma^{(0)}_i(s)F_p(0,s)F_q(0,s), \qquad p, q = 1, 2, \ldots, N. \end{aligned}$$

Note that $P^k_i(t)$*'s are all deterministic functions.*

Now, from (3.9), we have

$$\int_0^T w_{i,t}S_{i,t}\mathrm{d}t = \int_0^T w_{i,t}F_i(0,t)\mathrm{d}t + a_{i,1}(T) + a_{i,2}(T) + a_{i,3}(T), \tag{3.10}$$

where

$$a_{i,k}(T) = \int_0^T w_{i,t}F_i(0,t)A^k_{i,t}\mathrm{d}t, \qquad k = 1,2,3.$$

By changing the order of integration, we obtain

$$a_{i,1}(T) = \int_0^T \bar{p}^1_i(t,T)\mathrm{d}W_{i,t}, \qquad \bar{p}^1_i(t,T) := P^1_i(t)\int_t^T w_{i,s}F_i(0,s)\mathrm{d}s,$$

and

$$\begin{aligned} a_{i,2}(T) &= \int_0^T \bar{s}_i(t,T)\left(\int_0^t \sigma^{(0)}_p(s)\mathrm{d}W_{p,s}\right)\mathrm{d}W_{i,t} \\ &\quad + \sum_{p=1}^N \int_0^T \bar{p}^2_{i:p}(t,T)\left(\int_0^t \sigma^{(0)}_p(s)\mathrm{d}W_{p,s}\right)\mathrm{d}W_{i,t}, \end{aligned}$$

where

$$\bar{s}_i(t,T) := \sigma^{(0)}_i(t)\int_t^T w_{i,s}F_i(0,s)\mathrm{d}s, \qquad \bar{p}^2_{i:p}(t,T) := P^2_{i:p}(t)\int_t^T w_{i,s}F_i(0,s)\mathrm{d}s.$$

Similar representation holds for $a_{i,3}(T)$. Namely, the first term of $a_{i,3}(T)$ is given by

$$\begin{aligned} a^1_{i,3}(T) &:= \int_0^T w_{i,t}F_i(0,t)A^3_{i,t}(1)\mathrm{d}t \\ &= \int_0^T \bar{s}_i(t,T)\left(\int_0^s \sigma^{(0)}_i(u)\left(\int_0^u \sigma^{(0)}_i(r)\mathrm{d}W_{i,r}\right)\mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s}. \end{aligned}$$

Other terms $a^k_{i,3}(T)$, $k = 2,\ldots,7$, are provided in Appendix B. The random variable V_T in (2.2) is now approximated as

$$V_T \approx \sum_{i=1}^N \int_0^T w_{i,t}F_i(0,t)\mathrm{d}t + a_1(T) + a_2(T) + a_3(T), \tag{3.11}$$

where $a_k(T) = \sum_{i=1}^N a_{i,k}(T)$.

3.2. **Option pricing formula.** Let $X_T := V_T - \sum_{i=1}^N \int_0^T w_{i,t}F_i(0,t)\mathrm{d}t$ so that

$$X_t \approx a_1(t) + a_2(t) + a_3(t).$$

Since $a_1(T)$ is a mixture of normal random variables, $a_1(t)$ follows a normal distribution with zero mean and variance

$$\Sigma_t = \sum_{i,j=1}^N \int_0^t \rho_{i,j}\bar{p}_{i,1}(s)\bar{p}_{j,1}(s)\mathrm{d}s.$$

It follows that $a_1(t)$ can be rewritten as

$$a_1(t) = \int_0^t \sqrt{\Lambda_s}\mathrm{d}\hat{W}_s, \tag{3.12}$$

where $\Lambda_t = \sum_{i,j=1}^N \rho_{i,j}\bar{p}_{i,1}(t)\bar{p}_{j,1}(t)$ and $\mathrm{d}\hat{W}_t = \sum_{i=1}^N \left(\bar{p}_{i,1}(t)/\sqrt{\Lambda_t}\right)\mathrm{d}W_{i,t}$. Note that $\hat{W}_t$ is considered to be a standard Brownian motion under $\mathbb{Q}$.

By applying the following result, an approximation of the density function of X_t can be obtained. The proof is found in Funahashi and Kijima (2013). The density function of X_t is denoted by $f_{X_t}(x)$.

Lemma 3.2. *The density function of X_t is approximated by*

$$\begin{aligned} f_{X_t}(x) &\approx n\left(x;0,\Sigma_t\right) - \frac{\partial}{\partial x}\left\{\mathbb{E}[a_2(t)|a_1(t)=x]n\left(x;0,\Sigma_t\right)\right\} \\ &\quad - \frac{\partial}{\partial x}\left\{\mathbb{E}[a_3(t)|a_1(t)=x]n\left(x;0,\Sigma_t\right)\right\} \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{\mathbb{E}[a_2(t)^2|a_1(t)=x]n\left(x;0,\Sigma_t\right)\right\}, \end{aligned}$$

where $n(x;a,b)$ denotes the normal density function with mean a and variance b.

The conditional expectations in Lemma 3.2 can be evaluated explicitly. Some key results are provided in Appendix C. Using the approximated density function, say $\tilde{f}_{X_T}(x)$, the call option price (2.3) can now be approximated by

$$\begin{aligned} C(V,K,T) &= \mathbb{E}\left[\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\left(V(T)-K\right)^+\right] \\ &= \mathbb{E}\left[\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\left(X_T+\bar{K}\right)^+\right] \\ &\approx \mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\int_{-\bar{K}}^{\infty}(x+\bar{K})\tilde{f}_{X_T}(x)\mathrm{d}x, \end{aligned}$$

where $\bar{K} := \sum_{i=1}^N \int_0^T w_{i,t}F_i(0,t)\mathrm{d}t - K$. The derivation of the approximated option price is tedious but straightforward. The resulting formula is complicated and omitted. The complete derivation and formulas are available from the authors upon request.

4. Some Special Cases

So far, we have considered the general local volatility model (2.1) whose volatility value depends not only on its price but also the other asset prices. However, it is usually very difficult to specify the volatility functions in its full generality for practical uses. In this section, as a special case of (2.1), we assume that

$$\frac{\mathrm{d}S_{i,t}}{S_{i,t}} = r(t)\mathrm{d}t + \sigma_i(S_{i,t},t)\mathrm{d}W_{i,t}, \qquad 0\le t\le T, \tag{4.1}$$

under the risk-neutral measure $\mathbb{Q}$. That is, we assume that the local volatility depends only on the price of its own (not the others), while assuming that the Brownian motions are correlated as $\mathrm{d}W_{i,t}\mathrm{d}W_{i,t} = \rho_{i,j}\mathrm{d}t$. Then, all the cross partial-derivatives in Theorem 3.1 disappear. Namely, in this case, $P_i^k(s)$ in Theorem 3.1 are reduced to

$$\begin{aligned} P_i^1(s) &:= \sigma_i^{(0)}(s) + \partial_i\sigma_i^{(0)}(s)F_i(0,s)\left(\int_0^s\left\{\sigma_i^{(0)}(u)\right\}^2\mathrm{d}u\right) \\ &\quad + \frac{1}{2}\partial_{ii}\sigma_i^{(0)}(s)F_i^2(0,s)\left(\int_0^s\left\{\sigma_i^{(0)}(u)\right\}^2\mathrm{d}u\right), \\ P_{i:i}^2(s) &:= \partial_i\sigma_i^{(0)}(s)F_i(0,s), \\ P_{i:i,i}^3(s) &:= \partial_{ii}\sigma_i^{(0)}(s)F_i^2(0,s). \end{aligned} \tag{4.2}$$

Hence, each asset price is approximated as

$$
\begin{aligned}
S_{i,t} \approx\; & F_i(0,t)\left[1+\int_0^t r_{i,1}(s)\mathrm{d}W_{i,s}+\int_0^t r_{i,2}(s)\left(\int_0^s \sigma_i^{(0)}(u)\mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s}\right. \\
& +\int_0^t r_{i,3}(s)\left(\int_0^s \sigma_i^{(0)}(u)\left(\int_0^u \sigma_i^{(0)}(r)\mathrm{d}W_{i,r}\right)\mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s} \\
& \left.+\int_0^t r_{i,4}(s)\left(\int_0^s r_{i,5}(u)\left(\int_0^u \sigma_i^{(0)}(r)\mathrm{d}W_{i,r}\right)\mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s}\right],
\end{aligned} \tag{4.3}
$$

where $\sigma_i^{(0)}(s) := \sigma_i(F_i(0,s),s)$ and

$$
\begin{aligned}
r_{i,1}(s) &:= P_i^1(s), \\
r_{i,2}(s) &:= \sigma_i^{(0)}(s)+P_{i:i}^2(s), \\
r_{i,3}(s) &:= \sigma_i^{(0)}(s)+3P_{i:i}^2(s)+P_{i:i,i}^3(s), \\
r_{i,4}(s) &:= \sigma_i^{(0)}(s)+P_{i:i}^2(s), \\
r_{i,5}(s) &:= P_{i:i}^2(s).
\end{aligned}
$$

The iterative integrals $a_{i,k}(T)$, $k=1,2,3$, in (3.10) can also be simplified as

$$
\begin{aligned}
a_{i,1}(T) &= \int_0^T \bar{r}_{i,1}(t)\mathrm{d}W_{i,t}, \\
a_{i,2}(T) &= \int_0^T \bar{r}_{i,2}(t)\left(\int_0^t \sigma_i^{(0)}(s)\mathrm{d}W_{i,s}\right)\mathrm{d}W_{i,t}, \\
a_{i,3}(T) &= \int_0^T \bar{r}_{i,3}(t)\left(\int_0^t \sigma_i^{(0)}(s)\left(\int_0^s \sigma_i^{(0)}(u)\mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s}\right)\mathrm{d}W_{i,t} \\
&\quad +\int_0^T \bar{r}_{i,4}(t)\left(\int_0^s r_{i,5}(u)\left(\int_0^u \sigma_i^{(0)}(r)\mathrm{d}W_{i,r}\right)\mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s},
\end{aligned} \tag{4.4}
$$

where $\bar{r}_{i,k}(t)=r_{i,k}(t)\int_t^T w_{i,s}F(0,s)\mathrm{d}s$ for $k=1,2,3,4$.

Moreover, the conditional expectations in Lemma 3.2 are derived explicitly as

$$
\begin{aligned}
\mathbb{E}[a_2(t)|a_1(t)=x] &= q_1(t)\left(\frac{x^2}{\Sigma_t^2}-\frac{1}{\Sigma_t}\right), \\
\mathbb{E}[a_3(t)|a_1(t)=x] &= q_2(t)\left(\frac{x^3}{\Sigma_t^3}-\frac{3x}{\Sigma_t^2}\right), \\
\mathbb{E}[a_2^2(t)|a_1(t)=x] &= q_3(t)\left(\frac{x^4}{\Sigma_t^4}-\frac{6x^2}{\Sigma_t^3}+\frac{3}{\Sigma_t^2}\right)+q_4(t)\left(\frac{x^2}{\Sigma_t^2}-\frac{1}{\Sigma_t}\right)+q_5(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma_t &= \sum_{i,j=1}^{N}\int_0^t \rho_{i,j}\bar{r}_{i,1}(s)\bar{r}_{j,1}(s)\mathrm{d}s,\\
q_1(t) &= \sum_{i,j,k=1}^{N}\int_0^t \rho_{i,j}\bar{r}_{j,1}(s)\bar{r}_{i,2}(s)\left(\int_0^s \rho_{i,k}\sigma_i^{(0)}(u)\bar{r}_{k,1}(u)\mathrm{d}u\right)\mathrm{d}s,\\
q_2(t) &= \sum_{i,j,k,l=1}^{N}\int_0^t \rho_{i,j}\bar{r}_{j,1}(s)\bar{r}_{i,3}(s)\left(\int_0^s \rho_{i,k}\sigma_i^{(0)}(u)\bar{r}_{k,1}(u)\left(\int_0^u \rho_{i,l}\sigma_i^{(0)}(r)\bar{r}_{l,1}(r)\mathrm{d}r\right)\mathrm{d}u\right)\mathrm{d}s\\
&+ \sum_{i,j,k,l=1}^{N}\int_0^t \rho_{i,j}\bar{r}_{j,1}(s)\bar{r}_{i,4}(s)\left(\int_0^s \rho_{i,k}\bar{r}_{k,1}(u)r_{i,5}(u)\left(\int_0^u \rho_{i,l}\sigma_i^{(0)}(r)\bar{r}_{l,1}(r)\mathrm{d}r\right)\mathrm{d}u\right)\mathrm{d}s,\\
q_3(t) &= q_1^2(t),\\
q_4(t) &= 2\sum_{i,j,k,l=1}^{N}\int_0^t \rho_{i,k}\bar{r}_{k,1}(s)\bar{r}_{i,2}(s)\left(\int_0^s \rho_{j,l}\bar{r}_{l,1}(u)\bar{r}_{j,2}(u)\left(\int_0^u \rho_{i,j}\sigma_i^{(0)}(r)\sigma_{j,0}(r)\mathrm{d}r\right)\mathrm{d}u\right)\mathrm{d}s\\
&+ 2\sum_{i,j,k,l=1}^{N}\int_0^t \rho_{i,k}\bar{r}_{k,1}(s)\bar{r}_{i,2}(s)\left(\int_0^s \rho_{i,j}\sigma_i^{(0)}(u)\bar{r}_{j,2}(u)\left(\int_0^u \rho_{j,l}\sigma_{j,0}(r)\bar{r}_{l,1}(r)\mathrm{d}r\right)\mathrm{d}u\right)\mathrm{d}s\\
&+ \sum_{i,j,k,l=1}^{N}\int_0^t \rho_{i,j}\bar{r}_{i,2}(s)\bar{r}_{j,2}(s)\left(\int_0^s \rho_{i,l}\sigma_i^{(0)}(u)\bar{r}_{l,1}(u)\mathrm{d}u\right)\left(\int_0^s \rho_{j,m}\sigma_{j,0}(u)\bar{r}_{m,1}(u)\mathrm{d}u\right)\mathrm{d}s,\\
q_5(t) &= \sum_{i,j=1}^{N}\int_0^t \rho_{i,j}\bar{r}_{i,2}(s)\bar{r}_{j,2}(s)\left(\int_0^s \rho_{i,j}\sigma_i^{(0)}(u)\sigma_{j,0}(u)\mathrm{d}u\right)\mathrm{d}s.
\end{aligned}
$$

By substituting these results into Lemma 3.2, we can obtain the following results.

Theorem 4.1. *The probability density function of X_t is approximated as*

$$
\begin{aligned}
f_{X_t}(x) \approx\ & \frac{1}{2}n\left(x;0,\Sigma_t\right)\left[\frac{q_3(t)}{\Sigma_t^3}h_6\left(\frac{x}{\sqrt{\Sigma_t}}\right)+\frac{(2q_2(t)+q_4(t))}{\Sigma_t^2}h_4\left(\frac{x}{\sqrt{\Sigma_t}}\right)\right.\\
&\left.+\frac{2q_1(t)}{\left(\sqrt{\Sigma_t}\right)^3}h_3\left(\frac{x}{\sqrt{\Sigma_t}}\right)+\frac{q_5(t)}{\Sigma_t}h_2\left(\frac{x}{\sqrt{\Sigma_t}}\right)+2\right],
\end{aligned}
$$

where Σ_t and $q_i(t)$ are defined above and $n(x;a,b)$ denotes the normal density function with mean a and variance b.

Theorem 4.2. *The value of a European call option with maturity T and strike K is approximated as*

$$
\begin{aligned}
C(T) \approx\ & \frac{\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}n(\bar{K};0,\Sigma_t)}{2\Sigma_t^4}\Big[q_3(T)(\bar{K}^4-6\bar{K}^2\Sigma_t+3\Sigma_t^2)\\
&+\Sigma_t^2\left(q_4(T)+2q_2(T)\right)\left(\bar{K}^2-\Sigma_t\right)\\
&+\Sigma_t^3\left\{-2q_1(T)\bar{K}+q_5(T)\Sigma_t+2\Sigma_t^2\right\}\Big]\\
&+\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\bar{K}\left(1-\Phi(-\bar{K}/\sqrt{\Sigma_t})\right),
\end{aligned}
$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution.

4.1. **The GBM case.** As in the previous papers, we now assume that the underlying assets follow ordinary geometric Brownian motions, i.e.,

$$\frac{\mathrm{d}S_{i,t}}{S_{i,t}} = r\mathrm{d}t + \sigma_i \mathrm{d}W_{i,t}, \qquad 0 \le t \le T, \tag{4.5}$$

where r and σ_i are some constants with $\mathrm{d}W_{i,t}\mathrm{d}W_{i,t} = \rho_{i,j}\mathrm{d}t$. Then, by definition, we have $\sigma_i^{(0)}(s) = \sigma_i$ so that $P_i^k(s)$ in (4.2) are reduced to

$$P_i^1(s) = \sigma_i, \qquad P_{i:i}^2(s) = P_{i:i,i}^3(s) = 0.$$

It follows from (4.3) that each asset price is approximated as

$$\begin{aligned} S_{i,t} \approx S_{i,0}\mathrm{e}^{rt}\Bigg[1 &+ \int_0^t \sigma_i \mathrm{d}W_{i,s} + \int_0^t \sigma_i \left(\int_0^s \sigma_i \mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s} \\ &+ \int_0^t \sigma_i \left(\int_0^s \sigma_i \left(\int_0^u \sigma_i \mathrm{d}W_{i,r}\right)\mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s}\Bigg], \end{aligned} \tag{4.6}$$

since $\sigma_i^{(0)}(s) = r_{i,k}(s) = \sigma_i$, $k = 1, \ldots, 4$, and $r_{i,5}(s) = 0$ in this case. The iterative integrals $a_{i,k}(T)$ in (4.4) are further simplified as

$$\begin{aligned} a_{i,1}(T) &= \int_0^T \bar{r}_i(t)\mathrm{d}W_{i,t}, \\ a_{i,2}(T) &= \int_0^T \bar{r}_i(t)\left(\int_0^t \sigma_i \mathrm{d}W_{i,s}\right)\mathrm{d}W_{i,t}, \\ a_{i,3}(T) &= \int_0^T \bar{r}_i(t)\left(\int_0^t \sigma_i \left(\int_0^s \sigma_i \mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s}\right)\mathrm{d}W_{i,t}, \end{aligned}$$

where $\bar{r}_i(t) = S_{i,0}\sigma_i \int_t^T w_{i,s}\mathrm{e}^{rs}\mathrm{d}s$. The option price formula in Theorem 4.2 is simplified accordingly.

4.2. **Numerical comparison for the GBM case.** In this subsection, we calculate ordinary Asian option and Basket option prices by using our method and compare them with those obtained by several approximation methods previously proposed in the literature under the Black–Scholes setting (1973). In order to avoid intentional choice of parameter values, we adopt the same parameter setting as those used in Table 2 of Ju (2002).

Table 1 shows the Asian option prices calculated by the semi-analytic method (SA (Exact)) of Zhang (2001), the log-normal approximation (LN) of Levy (1992), the Edgeworth expansion method (EW) of Tumbull and Wakeman (1991) and Ritchken et al. (1993), the reciprocal gamma approximation (RG) of Milevsky and Posner (1998), the 4th-order moment approximation (FM4) of Posner and Milevsky (1998), the 6th-order Taylor expansion approximation (TE6) of Ju (2002) and our approximation (WIC). We use SA (Exact) to be the benchmark values for our comparison purposes. In the table, volatility σ varies from 5% to 50% and strike K from 95 to 105. The other parameters are set as $S_0 = 100$, $r = 0.09$, and $T = 3$. In order to check the accuracy, the residual mean squared error (labeled by RMSE) as well as the maximum absolute error (labeled by MAE) from the SA (Exact) is considered. The results of SA, LN, EW, RG, FM4 and TE6 are quoted from Table 2 of Ju (2002). The ranking based on the

σ	K	SA (Exact)	TE6	LN	EW	RG	FM	WIC
0.05	95	15.11626	15.11626	15.1163	15.11628	15.11624	15.11626	15.1162
	100	11.30361	11.3036	11.30422	11.30368	11.30318	11.30361	11.30353
	105	7.55332	7.55335	7.5567	7.55335	7.55075	7.55333	7.55312
0.1	95	15.2138	15.21396	15.22546	15.21443	15.20538	15.21383	15.21282
	100	11.63766	11.63798	11.65759	11.6351	11.62237	11.63764	11.63681
	105	8.39122	8.3914	8.41475	8.3863	8.37232	8.39115	8.39084
0.2	95	16.63723	16.63942	16.74023	16.52766	16.55504	16.63634	16.6326
	100	13.76693	13.7677	13.86951	13.6658	13.68133	13.76559	13.76443
	105	11.21985	11.21879	11.31054	11.14619	11.14037	11.21835	11.21842
0.3	95	19.0232	19.02652	19.2791	18.36063	18.80529	19.0162	19.01279
	100	16.58613	16.58509	16.82823	16.10382	16.37079	16.57768	16.57895
	105	14.39295	14.38751	14.6101	14.0994	14.19042	14.38394	14.38808
0.4	95	21.74097	21.74461	22.2318	19.63683	21.30176	21.71507	21.71776
	100	19.5883	19.58355	20.05569	18.26416	19.15285	19.5579	19.57066
	105	17.62548	17.61269	18.05875	16.98036	17.20469	17.59266	17.61289
0.5	95	24.5719	24.5774	25.40607	20.2883	23.79644	24.50412	24.52514
	100	22.63085	22.62276	23.43633	20.82553	21.85456	22.55035	22.59402
	105	20.84322	20.82213	21.60941	21.00517	20.07802	20.7542	20.81615
		RMSE	0.00662	0.39303	1.26967	0.37305	0.03486	0.01748
		MAE	0.02108	0.83417	4.2836	0.77629	0.08902	0.04676
		Rank	No1				No3	No2

Table 1: Asian option prices for the GBM case. The results of SA, LN, EW, RG, FM and TE6 are quoted from Table 2 of Ju (2002), where SA (Exact) stands for the semi-analytic method of Zhang (2001), LN the log-normal approximation of Levy (1992), EW the Edgeworth expansion method of Tumbull and Wakeman (1991) and Ritchken et al. (1993), RG the reciprocal gamma approximation of Milevsky and Posner (1998), FM4 the 4th-order moment approximation of Posner and Milevsky (1998), TE6 the 6th-order Taylor expansion approximation of Ju (2002), and WIC the approximation method proposed in this paper. The other parameters are set as $S = 100$, $r = 0.09$, and $T = 3$. The CPU time to calculate our results (WIC) was 0.001 second for each case in average.

RMSE (also MAE) is also appended in the table. The CPU time to calculate our results (WIC) was 0.001 second for each case in average.[2]

On the other hand, Table 2 shows the basket option prices calculated by the Monte Carlo simulation (MC), LN, RG, FM4 TE6, the geometric conditioning method (GC) of Curran (1994) and our approximation method WIC, where each basket consists of five *homogeneous* stocks. In the table, strike K varies from 90 to 110, short rate r from 0.05 to 0.1, volatility σ from 20$\%$ to 50$\%$, and correlation ρ from 0 to 0.5. The weights are $w_1 = 0.05$, $w_2 = 0.15$, $w_3 = 0.2$, $w_4 = 0.25$ and $w_5 = 0.35$. The other parameters are set as $S_{i,0} = 100$ and $T = 3$. The results of MC, LN, RG, FM4, GC and TE6 are quoted from Table 6 of Ju (2002). As in Table 1, the residual mean squared error (RMSE) as well as the maximum absolute error (MAE) from Monte Carlo simulation (MC) and the ranking are also appended. The CPU time to calculate our results (WIS) was 0.826 second in average.

From these tables, it is explicitly observed that our approximation (WIC) is quite comparable to the existing accurate methods even for the GBM case. In particular, even in the case of high volatility, we can see that the accuracy of our approximation remains quite high. Note that we have made no intention for the parameter choice. Also, our approximation is not restricted to

[2]The CPU time is measured by the time function equipped by C++. Note that $q_i(T)$ in the option pricing formula (Theorem 4.2) are evaluated by numerical integration.

K	r	σ	ρ	MC	TE6	LN	RG	FM	GC	WIC
90	0.05	0.2	0	23.0121	23.0148	23.0561	22.9501	23.0169	22.9777	23.03326
100	0.1	0.2	0	26.1671	26.1706	26.2005	26.1232	26.1738	26.1463	26.18798
110	0.05	0.5	0	21.0184	21.0437	21.8495	20.4446	20.5303	19.3543	21.39616
90	0.1	0.5	0	37.2287	37.1973	37.969	36.5845	37.3486	36.3446	37.86944
100	0.05	0.2	0.5	18.5798	18.5812	18.5875	18.193	18.5809	18.5653	18.58114
110	0.1	0.2	0.5	21.7596	21.76	21.7664	21.3672	21.7598	21.7465	21.75889
90	0.05	0.5	0.5	36.828	36.8255	36.9131	33.4933	36.8083	36.5786	36.83792
100	0.1	0.5	0.5	38.5906	38.5874	38.6742	35.3043	38.5789	38.3534	38.59878
110	0.05	0.2	0	9.8016	9.8013	9.8546	9.7366	9.796	9.6887	9.81099
90	0.1	0.2	0	33.3711	33.3707	33.381	33.3551	33.3735	33.3639	33.38699
100	0.05	0.5	0	25.161	25.1394	26.0042	24.4746	24.7427	23.6466	25.60056
110	0.1	0.5	0	27.6233	27.619	28.4929	26.9366	27.3201	26.2559	28.13687
90	0.05	0.2	0.5	24.8104	24.8111	24.8172	24.4386	24.811	24.7999	24.80759
100	0.1	0.2	0.5	27.5462	27.5463	27.5519	27.2072	27.5463	27.537	27.53928
110	0.05	0.5	0.5	29.1034	29.1026	29.1871	25.8326	29.0556	28.8042	29.12288
90	0.1	0.5	0.5	42.7673	42.7625	42.8455	39.678	42.776	42.5596	42.767
100	0.05	0.2	0	15.678	15.6802	15.7425	15.5903	15.6775	15.6027	15.69741
110	0.1	0.2	0	19.4368	19.4357	19.4894	19.3541	19.4358	19.3805	19.45461
90	0.05	0.5	0	29.9998	29.9817	30.8485	29.2973	29.7887	28.7457	30.54791
100	0.1	0.5	0	32.1145	32.1032	32.9523	31.4284	32.0113	30.9819	32.70757
110	0.05	0.2	0.5	13.4902	13.4905	13.4954	13.1811	13.4901	13.4713	13.49058
90	0.1	0.2	0.5	34.0088	34.0101	34.014	33.7796	34.0102	34.0047	33.9919
100	0.05	0.5	0.5	32.7054	32.7176	32.8051	29.3599	32.6827	32.4421	32.73262
110	0.1	0.5	0.5	34.8364	34.8388	34.9267	31.4782	34.8126	34.5777	34.85218
				RMSE	0.01078	0.41767	1.68341	0.1553	0.67642	0.26348
				MAE	0.0314	0.8696	3.3582	0.4881	1.6641	0.64074
				Rank	No1			No2		No3

Table 2: Basket option prices for the GBM case. The results of MC, LN, RG, FM4, GC and TE6 are quoted from the Table 6 of Ju (2002), where MC stands for the Monte Carlo simulation, LN the log-normal approximation of Levy (1992), RG the reciprocal gamma approximation of Milevsky and Posner (1998), FM4 the 4th-order moment approximation of Posner and Milevsky (1998), TE6 the 6th-order Taylor expansion approximation of Ju (2002), GC the geometric conditioning method of Curran (1994) and WIC the proposed method in this paper. Here, each basket consists of five homogeneous stocks with initial price $S_{i,0} = 100$. The option maturity is $T = 3$ years, and the weights are $w_1 = 0.05$, $w_2 = 0.15$, $w_3 = 0.2$, $w_4 = 0.25$ and $w_5 = 0.35$. The CPU time to calculate our results (WIC) was 0.826 second in average.

the GBM case. In the next section, we shall show the applicability of our approximation method for more complex models.

5. NUMERICAL EXAMPLES

In this section, we examine the accuracy of our approximation method under more general settings. Since there are no closed form solutions for Asian basket options, we compare our approximation results with Monte Carlo simulation.

In the following, we investigate three settings. First, we test the so-called constant elasticity of variance (CEV) model. Second, we examine a complex nonlinear volatility model and test the performance of our approximation formulas by calibrating it to the real market. Finally, we consider a general Asian basket option.

5.1. **The CEV Model.** In this subsection, we suppose that the underlying asset price follows the CEV model. To be more specific, we assume that the volatility in the SDE (4.1) is specified

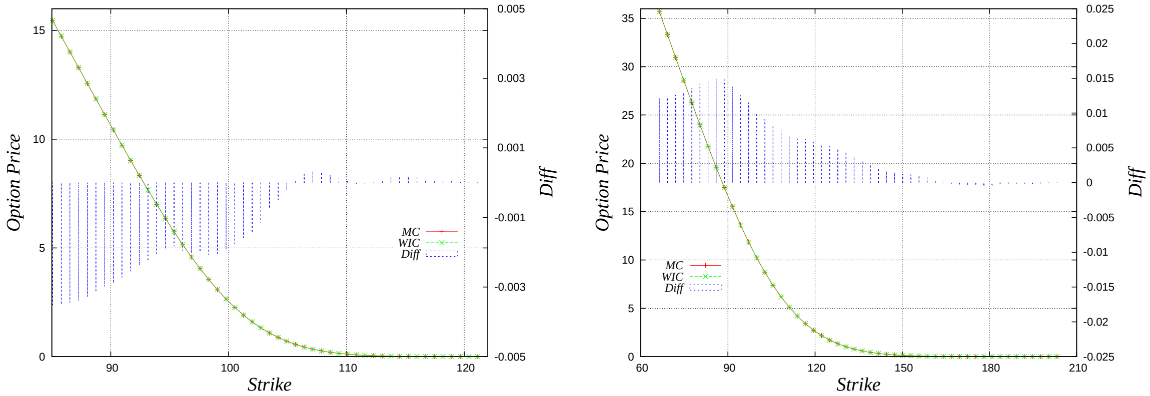

Figure 1: Asian option prices for the square-root model. The left-hand side panel corresponds to the short maturity ($T = 6$ months) case and the right-hand side panel corresponds to the long maturity ($T = 5$ years) case. MC means the option prices calculated by Monte Carlo simulation, while WIC indicates the approximate prices calculated by the formula given in Theorem 4.2. The parameters are set as $S_0 = 100$, $r = 0.03$, and $\sigma = 1.33$.

for each i as

$$\sigma_i(S_{i,t}, t) := \sigma_i S_{i,t}^{\beta_i - 1}, \qquad t \geq 0,$$

where σ_i and β_i are some constants. In particular, if $\beta_i = 1$ then the model becomes the Black-Scholes setting, whereas it is called the square-root model if $\beta_i = 0.5$.

We show in Figures 1 and 2 the differences between the option prices calculated by our approximation formula in Theorem 4.2 and the corresponding Monte Carlo simulation results for short maturity (6 months, left-hand-side panel) and long maturity (5 years, right-hand-side panel) cases.

Figure 1 reports the Asian option prices under the square-root model ($\beta = 0.5$). The parameters are set as $S_0 = 100$, $r = 0.03$, and $\sigma = 1.33$.

On the other hand, in Figure 2, we consider a European basket option that consists of 2 asymmetric stocks. The parameters are set as $r = 0.03$, $S_{1,0} = S_{2,0} = 100$, $\beta_1 = 1$, $\beta_2 = 0.5$, $\sigma_1 = 0.15$ and $\sigma_2 = 1.33$. Thus, while asset 1 follows the square-root process, asset 2 follows the GBM, in order to make the underlying assets asymmetric. Figure 2 reports the results for the cases of $\rho = 0.75$, $\rho = 0$ and $\rho = -0.75$ from the top two panels to the bottom two panels, respectively. The left-hand-side panel corresponds to the short maturity (6 months) case, whereas the right-hand-side panel corresponds to the long maturity (5 years) case.

From these figures, for both Asian and Basket options, we observe that the error becomes slightly large for long maturity and/or far in-the- and out-of-the money strikes. But, the errors of our approximation are small enough for practical uses.

5.2. **Calibration to the market.** We examine the performance of our approximation formula in Theorem 4.2 by testing it on real market data. For this purpose, we consider two examples of volatility surface observed in the currency options market of 5 year JPY/USD and JPY/AUD options.

To this end, we need to slightly modify the SDE (4.1) in order to apply our results to the FX options market. Namely, we assume that the i-th spot exchange rate $S_{i,t}$ follows the SDE

$$\frac{\mathrm{d}S_{i,t}}{S_{i,t}} = (r_d(t) - r_{i,f}(t))\mathrm{d}t + \sigma_i(S_{i,t}, t)\mathrm{d}W_{i,t}, \qquad i = 1, 2, \tag{5.1}$$

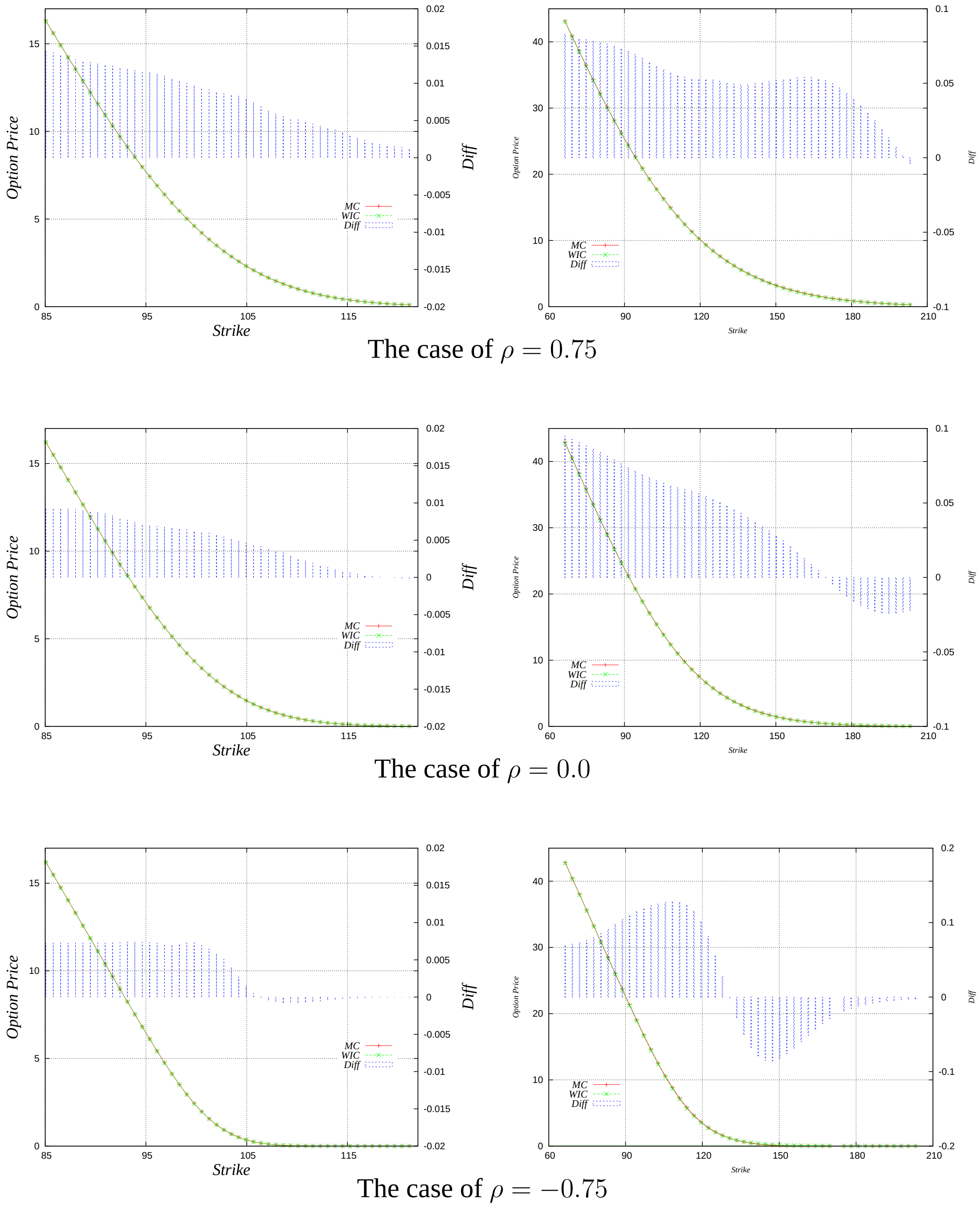

Figure 2: Basket option prices for the CEV model. The left-hand side panel corresponds to the short maturity ($T = 6$ months) case and the right-hand side panel corresponds to the long maturity ($T = 5$ years) case. The top two panels report the results for the case of $\rho = 0.75$, the middle panels $\rho = 0$ and the bottom panels $\rho = -0.75$, respectively. MC means the option prices calculated by Monte Carlo simulation, while WIC indicates the approximate prices calculated by the formula given in Theorem 4.2. The parameters are set as $S_{1,0} = S_{2,0} = 100$, $r = 0.03$, $\beta_1 = 1$, $\beta_2 = 0.5$, $\sigma_1 = 0.15$ and $\sigma_2 = 1.33$.

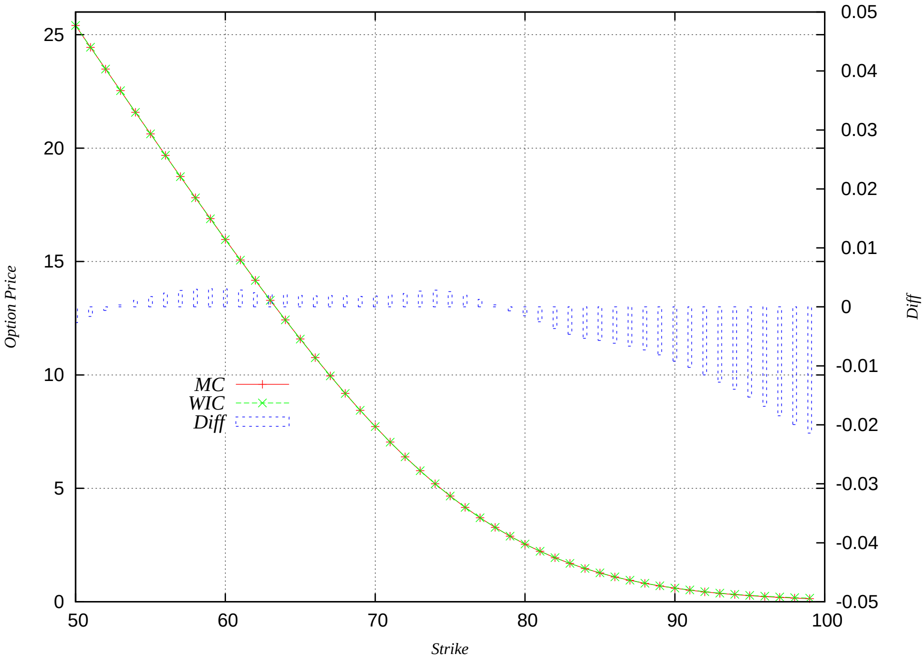

Figure 3: Asian option prices for the calibrated parameters. MC indicates the results calculated by Monte Carlo simulation, while WIC means that the results are obtained our approximation method. The other parameters are set as $r_d(t) = 0.00977$, $r_{1,f}(t) = 0.0222$, $S_{1,0} = 77.54$, $T = 3$.

where $r_d(t)$ and $r_{i,f}(t)$ represent the domestic and ith foreign short rates, respectively. In this setting, the forward price in Theorem 4.2 is modified as $F_i(0,t) = \exp\{\int_0^t (r_d(s) - r_{i,f}(s))\mathrm{d}s\}$.

In order to capture the volatility smile in the FX options market, we assume that the volatility in (5.1) is given by

$$\sigma_i(S_{i,t}, t) := \left(\alpha_i + \beta_i x + \frac{\gamma_i}{x} + \delta_i x^2\right) \mathrm{e}^{-\epsilon_i x}, \qquad t \geq 0, \tag{5.2}$$

where $x = S_{i,t}/F_i(0,t)$. The idea behind this specification is found in Funahashi and Kijima (2013).

First, we examine Asian options under the above setting. We calibrate our model to the 5 year JPY/USD option market dated on November 11 2011. The calibrated parameters are given as

α_1	β_1	γ_1	δ_1	ϵ_1
1.29531	-1.99897	-0.00018	1.014743	0.809651

Figure 3 reports the prices of 3-year Asian options computed from these calibrated parameters. In the figure, MC indicates the prices calculated by Monte Carlo simulation and WIC means the approximate prices calculated by the formula given in Theorem 4.2. The other parameters are set as $r_d(t) = 0.00977$, $r_{1,f}(t) = 0.0222$, and $S_{1,0} = 77.54$.

Second, we examine the basket options on two currencies, JPY/USD and JPY/AUD. In addition to the JPY/USD case, we calibrate our model to the 5 year JPY/AUD option market on the same date (November 11, 2011). The calibration results are given as

α_2	β_2	γ_2	δ_2	ϵ_2
0.853855	-0.699724	-0.001258	0.221596	0.627338

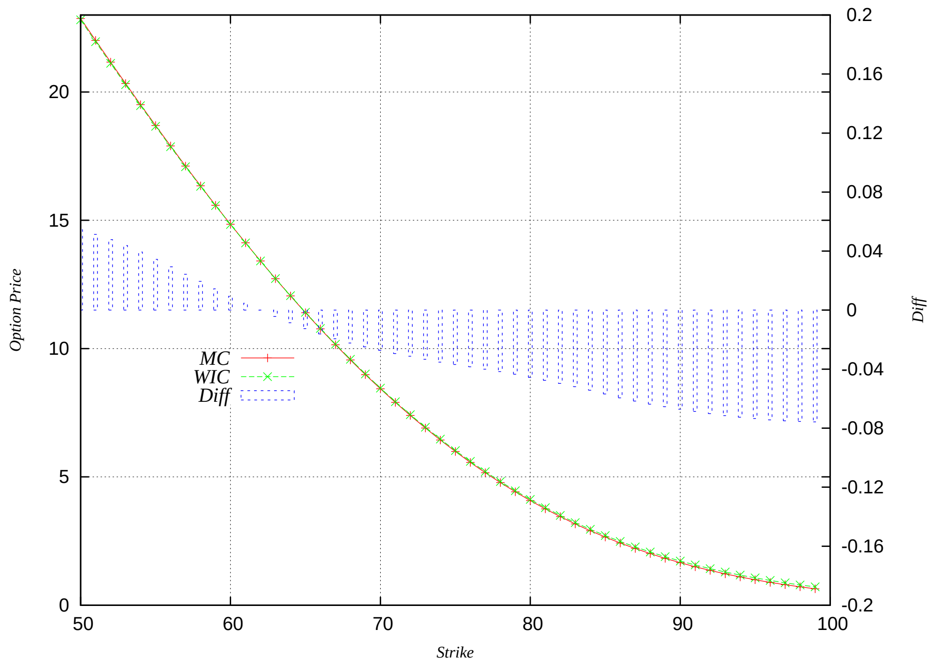

Figure 4: Basket option prices for the calibrated parameters. MC and WIC indicate Monte Carlo simulation and our approximation results, respectively. The other parameters are set as $r_d(t) = 0.00977$, $r_{1,f}(t) = 0.0222$, $r_{2,f}(t) = 0.04848$, $S_{1,0} = 77.54$, $S_{2,0} = 78.60$, $T = 3$ and $w_1 = w_2 = 0.5$. The correlation between them is set as $\rho = 0.5$.

Figure 4 reports the prices of 3-year basket options computed from these parameters. In the figure, MC and WIC indicate Monte Carlo simulation and our approximation results, respectively. The other parameters are set as $r_d(t) = 0.00977$, $r_{1,f}(t) = 0.0222$, $r_{2,f}(t) = 0.04848$, $S_{1,0} = 77.54$, $S_{2,0} = 78.60$, and $w_1 = w_2 = 0.5$. The correlation between them is set as $\rho = 0.5$.

From these figures, we observe that the accuracy of our approximation method is satisfactory for the practical cases in the wide ranges of strikes and maturities.

5.3. **Valuation of complex basket options.** In this last subsection, we consider the valuation of a fictitious, complex basket option that consists of 6 underlying assets. By complex, we mean that the basket consists of *inhomogeneous* underlyings with different asset dynamics, different weighting functions, and different correlation structures.

The asset dynamics are modeled by the SDE (4.1) with volatility functions $\sigma_i(x,t)$, $i = 1, 2$, given in (5.2) and

$$\sigma_i(S_i, t) = \begin{cases} \sigma_i, & i = 3, 4, \\ \sigma_i/\sqrt{S_i}, & i = 5, 6, \end{cases}$$

with $\sigma_3 = 0.1$, $\sigma_4 = 0.2$, $\sigma_5 = 1$ and $\sigma_6 = 2$.[3] The initial prices are assumed to be $S_{i,0} = 100$ for all i. The spot interest rate $r(t)$ is assumed to be a constant, say $r = 0.02$.

[3]In order to keep the asset volatilities similar levels, we set $\sigma_3 = 0.1$ and $\sigma_5 = 1$, because $S_{3,0} = S_{5,0} = 100$ so that $\sigma_3(S_{3,0}, 0) = \sigma_5(S_{5,0}, 0) = 0.1$. The same reasoning applies for assets 4 and 6.

The weighting functions are assumed to be given by

$$w_{i,t} = \begin{cases} 0.1\delta(T-t) & \text{(maturity price)}, & i=1,4, \\ 0.2\dfrac{1}{T_2-T_1}1_{\{T_1\le t\le T_2\}} & \text{(partial average)}, & i=2,5, \\ 0.2\dfrac{\eta}{1-\mathrm{e}^{-\eta T}}\mathrm{e}^{-\eta(T-t)} & \text{(exponential weighting)}, & i=3,6, \end{cases}$$

where T denotes the maturity of the complex basket option and $0 < T_1 < T_2 \le T$. Then, $\bar{r}_{i,k}(t)$ defined in (4.4) become

$$\bar{r}_{i,k}(t) = \begin{cases} 0.1 r_{i,k}(t) S_{i,0}\mathrm{e}^{rT}, & i=1,4, \\ 0.2 r_{i,k}(t) S_{i,0}\left(\dfrac{\mathrm{e}^{rT_2}-\mathrm{e}^{rT_1}}{r(T_2-T_1)}1_{\{t<T_1\}} + \dfrac{\mathrm{e}^{rT_2}-\mathrm{e}^{rt}}{r(T_2-T_1)}1_{\{T_1\le t<T_2\}}\right), & i=2,5, \\ 0.2 r_{i,k}(t) S_{i,0}\dfrac{\eta\left(\mathrm{e}^{rT}-\mathrm{e}^{(r+\eta)t-\eta T}\right)}{(1-\mathrm{e}^{-\eta T})(r+\eta)}, & i=3,6. \end{cases}$$

In this numerical example, we assume that $T=3$, $T_1=1$, $T_2=2$ and $\eta=0.95$.

Finally, we assume that the correlations between the assets are given by

$$(\rho_{jk}) = \begin{pmatrix} 1 & 0.7 & 0.5 & 0.3 & 0.2 & 0.1 \\ 0.7 & 1 & 0.6 & 0.4 & 0.3 & 0.2 \\ 0.5 & 0.6 & 1 & 0.5 & 0.4 & 0.3 \\ 0.3 & 0.4 & 0.5 & 1 & 0.5 & 0.4 \\ 0.2 & 0.3 & 0.4 & 0.5 & 1 & 0.5 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 1 \end{pmatrix}.$$

Note that the correlation matrix needs to be positive definite.

The option prices calculated by our formula are compared with those calculated by Monte Carlo simulation. The results are depicted in Figure 5. The CPU time to calculate our results (WIC) was 2.612 seconds for each case in average. Even such a complex basket product can be evaluated reasonably fast.

We can see from Figure 5 that the errors of our approximation are very small even in this complex setting. Hence, we conclude that our approximation formula is flexible enough for the pricing of exotic basket derivatives.

6. CONCLUSION

In this paper, we propose an approximation method based on the chaos expansion approach proposed by Funahashi and Kijima (2013) for the pricing of complex basket options such as Asian basket options, which are known to be difficult to price both analytically and numerically. Our approximation is not restricted to the case of geometric Brownian motions, and can be applied to the multi-dimensional local volatility model. Also, it is possible to extend our approach to include stochastic volatility models by following the idea of Funahashi (2012). Through ample numerical examples, we show that the accuracy of our approximation remains quite high even for the long maturity and/or the high volatility cases under various diffusion models.

We note that differentiation calculated from the closed-form approximation provides a good approximation for the option delta. Higher-order differentiation (maybe numerical) provides an approximated gamma, and the other Greeks are evaluated similarly.

As future works, we will extend our approach to include stochastic volatility models that are modeled by fractional Brownian motions as studied by Comte and Renault (1998). Applications

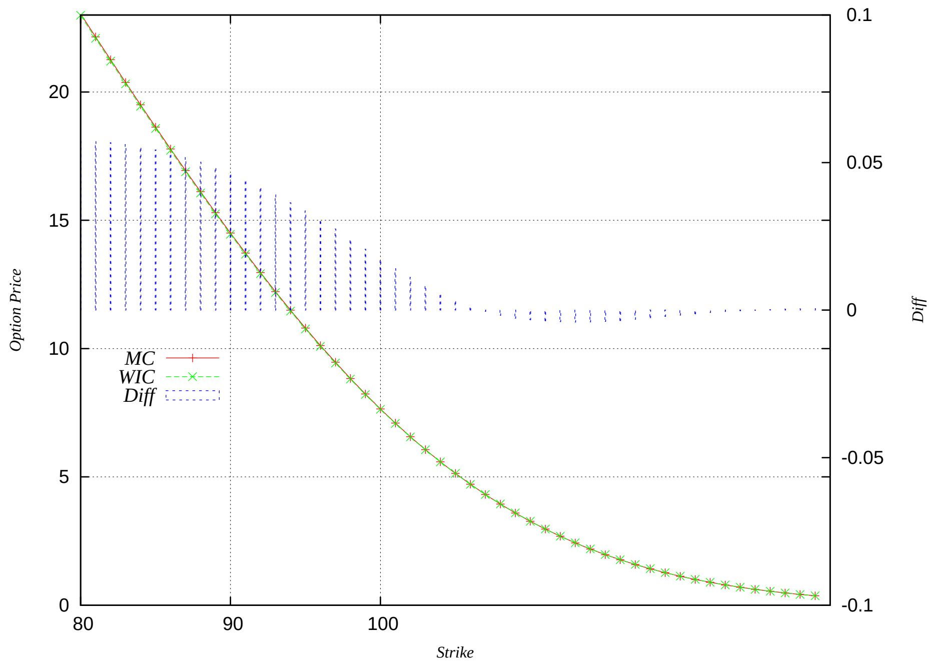

Figure 5: Complex basket option prices that contains of 6 underlying assets. MC means the option prices calculated by Monte Carlo simulation, while WIC indicates the approximate prices calculated by the formula given in Theorem 4.2. The CPU time to calculate our results (WIC) was 2.612 seconds for each case in average.

for the valuation problems of other financial contingent claims such as barrier options are also considered.

Appendix A. Proof of Lemma 3.1

In this appendix, we approximate each $I_{i:m,n}(t)$ by using the approximations (3.7) and (3.8). The strategy is to neglect the terms of iterative integrals of higher than the third order. We call such terms 'higher terms". The proof is similar to the one given by Funahashi and Kijima (2013). However, we provide a concise proof for the reader's convenience.

A.1. **Approximation of $I_{i:1,1}(t)$.** By definition, $I_{i:1,1}(t) = J_{i,t}(\sigma_i^{(1)}) - J_{i,t}(\sigma_i^{(0)})$ and so, from (3.7), we have

$$I_{i:1,1}(t) \approx \sum_{p=1}^{N} \int_0^t \partial_p \sigma_i^{(0)}(u)\{S_{p,u}^{(1)} - S_{p,u}^{(0)}\}\mathrm{d}W_{i,u} + \frac{1}{2}\sum_{p,q=1}^{N} \int_0^t \partial_{pq}\sigma_i^{(0)}(u)\{S_{p,u}^{(1)} - S_{p,u}^{(0)}\}\{S_{q,u}^{(1)} - S_{q,u}^{(0)}\}\mathrm{d}W_{i,u}.$$

Since $S_{i,t}^{(0)} = F_i(0,t)$ and $S_{i,t}^{(1)}$ is approximated by (3.6), by ignoring the higher terms, we obtain

$$
\begin{aligned}
\text{(A.1)}\quad & I_{i:1,1}(t) \\
\approx\ & \sum_{p=1}^{N}\int_0^t \partial_p\sigma_i^{(0)}(s)F_p(0,s)\left(\int_0^s \sigma_p^{(0)}(u)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s} \\
+\ & \sum_{p=1}^{N}\int_0^t \partial_p\sigma_i^{(0)}(s)F_p(0,s)\left(\int_0^s \sigma_p^{(0)}(u)\left(\int_0^u \sigma_p^{(0)}(r)\mathrm{d}W_{p,r}\right)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s} \\
+\ & \frac{1}{2}\sum_{p,q=1}^{N}\int_0^t \partial_{pq}\sigma_i^{(0)}(u)F_p(0,s)F_q(0,s)\left(\int_0^s \sigma_p^{(0)}(u)\mathrm{d}W_{p,u}\right)\left(\int_0^s \sigma_q^{(0)}(u)\mathrm{d}W_{q,u}\right)\mathrm{d}W_{i,u}.
\end{aligned}
$$

Further, by Ito's formula, we get

$$
\begin{aligned}
\left(\int_0^t \sigma_p^{(0)}(s)\mathrm{d}W_{p,s}\right)\left(\int_0^t \sigma_q^{(0)}(s)\mathrm{d}W_{q,s}\right) &= \int_0^t \sigma_q^{(0)}(s)\left(\int_0^t \sigma_p^{(0)}(s)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{q,s} \\
&\quad + \int_0^t \sigma_p^{(0)}(s)\left(\int_0^t \sigma_q^{(0)}(s)\mathrm{d}W_{q,u}\right)\mathrm{d}W_{p,s} \\
\text{(A.2)}\qquad &\quad + \int_0^t \sigma_p^{(0)}(s)\sigma_q^{(0)}(s)\mathrm{d}s.
\end{aligned}
$$

Finally, substitution of (A.2) into (A.1) yields the result.

A.2. **Approximation of $I_{i:1,2}(t)$.** By the definition of Hermite polynomials, we have

$$
\begin{aligned}
I_{i:1,2}(t) &= \frac{1}{2}\left\{\left(J_{i,t}^2(\sigma_i^{(1)}) - J_{i,t}^2(\sigma_i^{(0)})\right) - \left(\|\sigma_i^{(1)}\|_t^2 - \|\sigma_i^{(0)}\|_t^2\right)\right\} \\
&\approx J_{i,t}(\sigma_i^{(0)})\left(\sum_{i=1}^{N}\int_0^t \partial_p\sigma_i^{(0)}(u)\{S_{p,u}^{(1)} - S_{p,u}^{(0)}\}\mathrm{d}W_{p,u}\right) - \frac{1}{2}\left(\|\sigma_i^{(1)}\|_t^2 - \|\sigma_i^{(0)}\|_t^2\right),
\end{aligned}
$$

where we have used (3.7) for the approximation. Hence, since $J_{i,t}(\sigma_i^{(0)}) = \int_0^t \sigma_i^{(0)}(u)\mathrm{d}W_u$ and $\sigma_i^{(0)}(t) = \sigma_i(\mathbf{S}_t^{(0)}, t)$, $\mathbf{S}_t^{(0)} = (F_1(0,t), \ldots, F_N(0,t))$, by ignoring the higher terms, we obtain

$$
\begin{aligned}
I_{i:1,2}(t) &\approx \left(\int_0^t \sigma_i^{(0)}(s)\mathrm{d}W_{i,s}\right)\left(\sum_{p=1}^{N}\int_0^t \partial_p\sigma_i^{(0)}(s)F_p(0,s)\left(\int_0^s \sigma_p^{(0)}(u)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s}\right) \\
\text{(A.3)}\qquad &\quad - \frac{1}{2}\left(\|\sigma_i^{(1)}\|_t^2 - \|\sigma_i^{(0)}\|_t^2\right),
\end{aligned}
$$

where we have applied (3.6) for the further approximation.

Now, by Ito's formula, the first term in (A.3) is rewritten as

$$
\begin{aligned}
&\left(\int_0^t \sigma_i^{(0)}(s)\mathrm{d}W_{i,s}\right)\left(\sum_{p=1}^N \int_0^t \partial_p\sigma_i^{(0)}(s)F_p(0,s)\left(\int_0^s \sigma_p^{(0)}(u)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s}\right)\\
= &\sum_{p=1}^N \int_0^t \sigma_i^{(0)}(s)\left(\int_0^s \partial_p\sigma_p^{(0)}(u)F_p(0,u)\left(\int_0^u \sigma_p^{(0)}(r)\mathrm{d}W_{p,r}\right)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s}\\
&+\sum_{p=1}^N \int_0^t \partial_p\sigma_i^{(0)}(s)F_p(0,s)\left(\int_0^s \sigma_i^{(0)}(u)\mathrm{d}W_{i,u}\right)\left(\int_0^s \sigma_p^{(0)}(u)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s}\\
&+\sum_{p=1}^N \int_0^t \sigma_i^{(0)}(s)\partial_p\sigma_i^{(0)}(s)F_p(0,s)\left(\int_0^s \sigma_p^{(0)}(u)\mathrm{d}W_{p,u}\right)\mathrm{d}s.
\end{aligned}
$$

On the other hand, the second term in (A.3) is approximated as

$$
\begin{aligned}
\|\sigma_i^{(1)}\|_t^2 - \|\sigma_i^{(0)}\|_t^2 &= \int_0^t \left\{(\sigma_i^{(1)}(s))^2 - (\sigma_i^{(0)}(s))^2\right\}\mathrm{d}s\\
&\approx 2\sum_{p=1}^N \int_0^t \sigma_i^{(0)}(u)\partial_p\sigma_i^{(0)}(u)\{S_{p,u}^{(1)} - S_{p,u}^{(0)}\}\mathrm{d}s,
\end{aligned}
$$

by Taylor's expansion around $S_t^{(0)}$.

Similarly, we get

$$
\begin{aligned}
&\sum_{p=1}^N \int_0^t \partial_p\sigma_i^{(0)}(s)F_p(0,s)\left(\int_0^s \sigma_p^{(0)}(u)\mathrm{d}W_{p,u}\right)^2\mathrm{d}W_{p,s}\\
= &\sum_{p=1}^N \int_0^t \partial_p\sigma_i^{(0)}(s)F_p(0,s)\left(\int_0^s \sigma_p^{(0)}(u)\left(\int_0^u \sigma_i^{(0)}(r)\mathrm{d}W_{i,r}\right)\mathrm{d}W_{p,u}\right)\mathrm{d}W_{i,s}\\
&+\sum_{p=1}^N \int_0^t \partial_p\sigma_i^{(0)}(s)F_p(0,s)\left(\int_0^s \sigma_i^{(0)}(u)\left(\int_0^u \sigma_p^{(0)}(r)\mathrm{d}W_{p,r}\right)\mathrm{d}W_{i,u}\right)\mathrm{d}W_{i,s}\\
&+\sum_{p=1}^N \int_0^t \partial_p\sigma_i^{(0)}(s)F_p(0,s)\left(\int_0^s \sigma_p^{(0)}(u)\sigma_i^{(0)}(u)\mathrm{d}u\right)\mathrm{d}W_{i,s}.
\end{aligned}
$$

Finally, we put these results together to obtain the approximation result for $I_{i:1,2}(t)$.

A.3. **Approximation of $I_{i:2,1}(t)$.** By definition, $I_{i:2,1}(t) = J_{i,t}(\sigma_i^{(2)}) - J_{i,t}(\sigma_i^{(1)})$ and so, from (3.7), we have

$$
I_{i:2,1}(t) \approx \sum_{p=1}^N \int_0^t \partial_p\sigma_i^{(1)}(u)(S_{p,u}^{(2)} - S_{p,u}^{(1)})\mathrm{d}W_{i,s}.
$$

Since $S_{i,u}^{(2)} - S_{i,u}^{(1)} = F_i(0,t)I_{i:1,1}(t)$, and since $I_{i:1,1}(t) = J_{i,t}(\sigma_i^{(1)}) - J_{i,t}(\sigma_i^{(0)})$, we have

$$
I_{i:2,1}(t) \approx \sum_{p=1}^N \int_0^t \partial_p\sigma_i^{(1)}(u)F_p(0,t)I_{i:1,1}(t)\mathrm{d}W_{i,s}.
$$

Hence, from (A.1), by ignoring the higher terms, we obtain

$$I_{i:2,1}(t) \approx \sum_{p,q=1}^{N} \int_0^t \partial_p \sigma_i^{(1)}(u) F_p(0,t) \Big(\int_0^t \partial_q \sigma_p^{(0)}(s) F_q(0,s) \left(\int_0^s \sigma_q^{(0)}(u) \mathrm{d}W_{q,u} \right) \mathrm{d}W_{p,s} \Big) W_{i,s}.$$

Now, we apply Taylor's expansion to $\partial_p \sigma_i^{(1)}(t)$ around $S_t^{(0)}$. It follows by ignoring the higher terms again that

$$\begin{aligned} I_{i:2,1}(t) \quad \approx \quad & \sum_{p,q,r=1}^{N} \int_0^t \left\{ \partial_p \sigma_i^{(0)}(u) + \partial_{pp} \sigma_i^{(0)}(u) \{ S_{r,u}^{(1)} - S_{r,u}^{(0)} \} \right\} \\ & \times F_p(0,t) \Big(\int_0^t \partial_q \sigma_p^{(0)}(s) F_q(0,s) \big(\int_0^s \sigma_q^{(0)}(u) \mathrm{d}W_{q,u} \big) \mathrm{d}W_{p,s} \Big) W_{i,s}. \end{aligned}$$

By applying (3.6) and ignoring the higher terms again, we finally get the result.

Appendix B. Explicit Formulas of $a_{i,3}^k(T)$

From Theorem 3.1, the second term of $a_{i,3}(T)$ is given by

$$a_{i,3}^2(T) = \sum_{p=1}^{N} \int_0^T w_{i,t} F_i(0,t) \int_0^t P_{i:p}^2(s) \left(\int_0^s \sigma_p^{(0)}(u) \left(\int_0^u \sigma_p^{(0)}(r) \mathrm{d}W_{p.r} \right) \mathrm{d}W_{p,u} \right) \mathrm{d}W_{i,s} \mathrm{d}t.$$

By changing the order of integration, we get

$$a_{i,3}^2(T) = \sum_{p=1}^{N} \int_0^T \bar{p}_{i:p}^3(t,T) \left(\int_0^s \sigma_p^{(0)}(u) \left(\int_0^u \sigma_p^{(0)}(r) \mathrm{d}W_{p.r} \right) \mathrm{d}W_{p,u} \right) \mathrm{d}W_{i,s},$$

where

$$\bar{p}_{i:p}^3(t,T) := P_{i:p}^2(t) \int_t^T w_{i,s} F_i(0,s) \mathrm{d}s.$$

Similarly, we obtain

$$\begin{aligned} a_{i,3}^3(T) &= \sum_{p=1}^{N} \int_0^T \bar{s}_i(t,T) \left(\int_0^s P_{i:p}^2(u) \left(\int_0^u \sigma_p^{(0)}(r) \mathrm{d}W_{p,r} \right) \mathrm{d}W_{p,u} \right) \mathrm{d}W_{i,s}, \\ a_{i,3}^4(T) &= \sum_{p=1}^{N} \int_0^T \bar{p}_{i:p}^2(t,T) \left(\int_0^s \sigma_i^{(0)}(u) \left(\int_0^u \sigma_p^{(0)}(r) \mathrm{d}W_{p,r} \right) \mathrm{d}W_{i,u} \right) \mathrm{d}W_{i,s}, \\ a_{i,3}^5(T) &= \sum_{p=1}^{N} \int_0^T \bar{p}_{i:p}^2(t,T) \left(\int_0^s \sigma_p^{(0)}(u) \left(\int_0^u \sigma_i^{(0)}(r) \mathrm{d}W_{i,r} \right) \mathrm{d}W_{p,u} \right) \mathrm{d}W_{i,s}, \\ a_{i,3}^6(T) &= \sum_{p,q=1}^{N} \int_0^T \bar{p}_{i:p}^2(t,T) \left(\int_0^s P_{p:q}^2(s) \left(\int_0^u \sigma_q^{(0)}(r) \mathrm{d}W_{q,r} \right) \mathrm{d}W_{p,u} \right) \mathrm{d}W_{i,s} \end{aligned}$$

and

$$a_{i,3}^7(T) = \sum_{p,q=1}^{N} \int_0^T \bar{p}_{i:p,q}^4(t,T) \left(\int_0^s \sigma_p^{(0)}(u) \left(\int_0^u \sigma_q^{(0)}(r) \mathrm{d}W_{q.r} \right) \mathrm{d}W_{p,u} \right) \mathrm{d}W_{i,s}.$$

Here, we define

$$\bar{p}^4_{i:p,q}(t,T) := P^3_{i:p,q}(t)\int_t^T w_{i,s}F_i(0,s)\mathrm{d}s.$$

Appendix C. Formulas for Conditional Expectations

Let W^i_t, $i = 1,\ldots,5$, be standard Brownian motions with correlation $\mathrm{d}W^i_t\mathrm{d}W^j_t = \eta_{i,j}\mathrm{d}t$, and let $y_i(x)$, $i = 1,\ldots,5$, be some deterministic functions. Moreover, let $\Sigma := \int_0^T y_1^2(t)\mathrm{d}t$, and denote $J_T(y_1) = \int_0^T y_1(t)\mathrm{d}W^1_t$. In order to derive the conditional expectations of Lemma 3.2, the following well-known results[4] are sufficient:

$$\mathbb{E}\left[\int_0^T y_3(t)\left(\int_0^t y_2(s)\mathrm{d}W^2_s\right)\mathrm{d}W^3_t \middle| J_T(y_1)=x\right] = v_1\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right), \tag{C.1}$$

where

$$v_1 = \int_0^T \eta_{1,3}y_3(t)y_1(t)\left(\int_0^t \eta_{1,2}y_2(s)y_1(s)\mathrm{d}s\right)\mathrm{d}t,$$

$$\begin{aligned}\mathbb{E}\left[\int_0^T y_4(t)\left(\int_0^t y_3(s)\left(\int_0^s y_2(u)\mathrm{d}W^2_u\right)\mathrm{d}W^3_s\right)\mathrm{d}W^4_t \middle| J_T(y_1)=x\right]&\\ = v_2\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)&,\end{aligned} \tag{C.2}$$

where

$$v_2 = \int_0^T \eta_{1,4}y_4(t)y_1(t)\left(\int_0^t \eta_{1,3}y_3(s)y_1(s)\left(\int_0^s \eta_{1,2}y_2(u)y_1(u)\mathrm{d}u\right)\mathrm{d}s\right)\mathrm{d}t,$$

and

$$\begin{aligned}\mathbb{E}\Bigg[\left(\int_0^T y_3(t)\left(\int_0^t y_2(s)\mathrm{d}W^2_s\right)\mathrm{d}W^3_t\right)\left(\int_0^T y_5(t)\left(\int_0^t y_4(s)\mathrm{d}W^2_s\right)\mathrm{d}W^3_t\right)\Bigg| J_T(y_1)=x\Bigg]&\\ = v_3\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}-\frac{3}{\Sigma^2}\right)+v_4\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)+v_5&,\end{aligned} \tag{C.3}$$

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[4]Formulas (C.1), (C.2) and (C.3) are one-dimensional versions of Lemma 2.1 in Takahashi (1999). See also Yoshida (1992) for detailed discussions on the conditional expectations.

where

$$\begin{aligned}
v_3 &= \left(\int_0^T \eta_{1,3} y_3(t) y_1(t) \left(\int_0^t \eta_{1,2} y_2(t) y_1(t) \mathrm{d}s\right) \mathrm{d}t\right) \\
&\qquad \times \left(\int_0^T \eta_{1,5} y_5(t) y_1(t) \left(\int_0^t \eta_{1,4} y_4(t) y_1(t) \mathrm{d}s\right) \mathrm{d}t\right), \\
v_4 &= \int_0^T \eta_{1,3} y_3(t) y_1(t) \left(\int_0^t \eta_{1,5} y_5(s) y_1(s) \left(\int_0^s \eta_{2,4} y_4(u) y_2(u) \mathrm{d}u\right) \mathrm{d}s\right) \mathrm{d}t \\
&\quad + \int_0^T \eta_{1,5} y_5(t) y_1(t) \left(\int_0^t \eta_{1,3} y_1(s) y_3(s) \left(\int_0^s \eta_{2,4} y_4(u) y_2(u) \mathrm{d}u\right) \mathrm{d}s\right) \mathrm{d}t \\
&\quad + \int_0^T \eta_{1,3} y_3(t) y_1(t) \left(\int_0^t \eta_{2,5} y_2(s) y_5(s) \left(\int_0^s \eta_{1,4} y_4(u) y_1(u) \mathrm{d}u\right) \mathrm{d}s\right) \mathrm{d}t \\
&\quad + \int_0^T \eta_{1,5} y_5(t) y_1(t) \left(\int_0^t \eta_{3,4} y_3(s) y_4(s) \left(\int_0^s \eta_{1,2} y_2(u) y_1(u) \mathrm{d}u\right) \mathrm{d}s\right) \mathrm{d}t \\
&\quad + \left\{ \int_0^T \eta_{3,5} y_5(t) y_3(t) \left(\int_0^t \eta_{1,2} y_2(s) y_1(s) \mathrm{d}s\right) \left(\int_0^t \eta_{1,4} y_4(s) y_1(s) \mathrm{d}s\right) \mathrm{d}t \right\},
\end{aligned}$$

and

$$v_5 = \int_0^T \eta_{3,5} y_5(t) y_3(t) \left(\int_0^t \eta_{2,4} y_4(u) y_2(u) \mathrm{d}u\right) \mathrm{d}t.$$

Namely, for the conditional expectation $\mathbb{E}[a_2(t)|a_1(t)=x]$, since $a_2(t) = \sum_{i=1}^N a_{i,2}(T)$ and

$$\begin{aligned}
a_{i,2}(T) &= \int_0^T \bar{s}_i(t,T) \left(\int_0^t \sigma_i^{(0)}(s) \mathrm{d}W_{i,s}\right) \mathrm{d}W_{i,t} \\
&\qquad + \sum_{p=1}^N \int_0^T \bar{p}^2_{i:p}(t,T) \left(\int_0^t \sigma_p^{(0)}(s) \mathrm{d}W_{p,s}\right) \mathrm{d}W_{i,t},
\end{aligned}$$

we can apply (C.1) to calculate $\mathbb{E}\left[\int_0^T \bar{s}_i(t,T) \left(\int_0^t \sigma_p^{(0)}(s) \mathrm{d}W_{p,s}\right) \mathrm{d}W_{i,t} \middle| a_1(t) = x\right]$. To this end, we set $y_1(x) = \sqrt{\Lambda_x}$, $y_2(x) = \sigma_p^{(0)}(x)$, $y_3(x) = \bar{s}_i(x,T)$, and note that

$$\begin{aligned}
\eta_{1,2} &= \mathrm{d}W_{p,t} \mathrm{d}\hat{W}_t = \sum_{k=1}^N \rho_{kp} \left(\bar{p}_{k,1}(t)/\sqrt{\Lambda_t}\right) \mathrm{d}t, \\
\eta_{1,3} &= \mathrm{d}W_{i,t} \mathrm{d}\hat{W}_t = \sum_{k=1}^N \rho_{ik} \left(\bar{p}_{k,1}(t)/\sqrt{\Lambda_t}\right) \mathrm{d}t.
\end{aligned}$$

It follows that

$$\mathbb{E}\left[\int_0^T \bar{s}_i(t,T) \left(\int_0^t \sigma_p^{(0)}(s) \mathrm{d}W_{p,s}\right) \mathrm{d}W_{i,t} \middle| a_1(t) = x\right] = v_1 \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right),$$

where

$$\begin{aligned}
v_1 &= \int_0^T \sum_{k=1}^N \rho_{ik} \left(\bar{p}_{k,1}(t)/\sqrt{\Lambda_t}\right) \bar{s}_i(t,T)\sqrt{\Lambda_t} \left(\int_0^t \sum_{l=1}^N \rho_{lp} \left(\bar{p}_{k,1}(s)/\sqrt{\Lambda_s}\right) \sigma_p^{(0)}(s)\sqrt{\Lambda_s}\mathrm{d}s\right)\mathrm{d}t \\
&= \sum_{k,l=1}^N \int_0^T \rho_{ik}\bar{p}_{k,1}(t)\bar{s}_i(t,T)\left(\int_0^t \rho_{lp}\bar{p}_{k,1}(s)\sigma_p^{(0)}(s)\mathrm{d}s\right)\mathrm{d}t.
\end{aligned}$$

Other terms are similarly calculated.

The calculation of the conditional expectations $\mathbb{E}[a_3(t)|a_1(t)=x]$ and $\mathbb{E}[a_2(t)^2|a_1(t)=x]$ are rather complicated and tedious, but straightforward. We omit the details. The detailed results are available from the authors upon request.

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