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"Two-Dimensional Constrained Chaos and Industrial Revolution Cycles with Mathemetical Appendices"

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Two-Dimensional Constrained Chaos and Industrial Revolution Cycles with Mathemetical Appendices.[1]

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Abstract

Between the 1760s and 1980s, we have experienced at least three industrial revolutions. We explain such cycles as ergodic chaos and relate it to the average long-run interest rate and intellectual property protection. Because innovation dynamics is intrinsically multi-dimensional, we need newly to develop a structural characterization of multi-dimensional ergodic chaos suitable for an economic analysis. Introducing such a characterization for the two-dimensional case, we show that if the monopolistic use of a new invention lasts eight years, an industrial-revolution-like burst of new technologies recurs about every one hundred years, given empirically reasonable values of the determinants of a long-run interest rate.

Keywords: industrial revolutions, chaotic cycles, intellectual properties, market quality dynamics.

JEL Classification Codes: C62; E32; O41

1 Introduction

It is often said that we are currently in the middle of the third industrial revolution. This observation leads to a question as to why an industrial revolution, or a period of very fast and fundamental technological progress, has emerged cyclically just about every one hundred years. On the one hand, the first industrial revolution is often attributed to various institutional factors (North, 1981). On the other hand, however, the underlying mechanism of industrial revolution cycles has not yet been studied in the existing literature.

Kondratieff (1925, 1935) discovers fifty to sixty year cycles of innovation, which he regards as a deterministic phenomenon. Calling them long waves, he explains,

> "In asserting the existence of long waves and in denying that they arise out of random causes, we are also of the opinion that the long waves arise out of causes which are inherent in the essence of the capitalistic economy" (see Kondratieff (1935, p. 115)).

About 35 years after the start of the third industrial evolution, we now face its second wave. This is consistent with Kondratieff's observation.[1]

This study intends to explain the coexistence, and in particular frequencies and magnitudes, of industrial-revolution-like cycles and Kondratieff-like waves. It has been known that ergodic chaos is a perfect analytical tool to explain the average frequency (or time average) of waves by a probability distribution (or space average); see Birkhoff (1931) and von Neumann (1932). A difficulty is that while innovation dynamics intrinsically involves two state variables, no characterization of ergodic chaos has been known by which an economic structure behind two-dimensional ergodic chaos can be explained.

We develop a new characterization for ergodic chaos that makes it possible to relate two-dimensional innovation cycles to the determinants of an average long-run interest rate as well as intellectual property protection. Judd (1985) and Deneckere and Judd (1992) build models in which innovation is completed within a single period and in which innovation dynamics may follow single-dimensional ergodic chaos of Lasota and Yorke (1973). Because of the atemporality of innovation, however, innovation dynamics becomes independent of the long-run interest rate in their models, which makes it difficult to characterize the frequency of innovation cycles in an economically meaningful manner. Because innovation is an intrinsically intertemporal activity,

[1] Many people, including policymakers, view the current technogical progress based on the new use of data as the fourth industrial revolution; see, for examaple, Peccarelli (2020).

it is naturally guided by interest rates (or rates of return to investment in innovation), which this study highlights.

Our two-dimensional chaos is an extension of what may be called constrained chaos, which emerges from the interaction between unstable interior dynamics in the feasible set of an economy and the boundary of the feasibility constraint; see Nishimura and Yano (1994, 1995a, 1995b).[2] While the existing studies on constrained chaos have been based on a single-dimensional system, the present study extends it in a two-dimensional system. In this system, what may be called a "core" of the feasible set exists in which force is to push the state variable vector out of it; once the state variable vector leaves the core, it is pushed to the boundary of the feasible set from which dynamics in the core resumes. Since, on the boundary, the value of one state variable determines that of the other, double-period dynamics in the constrained system may follow a single-dimensional system. We prove this double-period system is expansive and unimodal system, which Lasota and Yorke (1973) show ergodic chaos.

Our model captures a phase shift from a period of no innovation to that of positive innovation, which we call an innovation take-off. Along an equilibrium path, three patterns of innovation take-offs will alternate irregularly. The frequency of take-offs in each pattern depends on four factors, the discount factor of future utilities, the growth rate of total factor productivity, that of labor productivity, and the length of time for which a new technology can be used monopolistically. These parameters jointly determine the steady state interest rate.

A take-off in the first pattern is larger than any take-off in the second and third patterns. Our simulation shows that, along an equilibrium path, the first pattern take-off occurs about every one hundred years if parameter values are consistent with historical values or, for example, if we set the annual TFP growth rate around 1.7 percent, the annual per-capita productivity growth rate around 1.6 percent, the steady state annual interest rate between 1.9 percent to 2.6 percent, and the length of a period in which a new technology is monopolistically used about 8 years. With a probability about one quarter to one third, it is followed by an even larger second wave

[2]Those studies are concerned with the existencee of an economic stucture in which an optimal path obeys chaotic dynamics; see Mitra and Khan (2005) for a similar structural approach. In contrast, Boldirn and Montrucchio (1986) and Deneckere and Polikan (1986) show the possibility of chaotic optimal dynamics by demonstrating the exististence of an optimal growth model for a given chaotic system. For early studies in a broader context, see Benhabib and Dei (1980) and Grandmont (1886), who are concerned with chaotic economic dynamics, and Scheinkman and LeBaron (1989), who study chaos in financial markets.

about 30 years later.

Our result is related to Matsuyama (1999, 2001), who incorporate the accumulation of physical capital into Judd's model. He demonstrates that innovation takes place along with capital accumulation; also see Matsuyama, Sushko, and Gardini (2014), who study the international interaction of chaotic business cycles in a similar basic setting. The possibility with which single-dimensional chaos emerges in his model has been studied extensively.[3]

In what follows, we will specify our model by introducing time in innovation into the model of Judd (1985) and characterize the two-dimensional constrained chaos. In Section 3, we will apply this result to obtain a sufficient condition under which our model of innovation dynamics is in fact ergodically chaotic. In Section 4, we relate the length of a period in which inventions are protected to the frequency of innovation take-offs in a model with parameter values consistent with real-world data.

2 Time in Innovation Dynamics

In Judd (1985), innovation and differentiated goods production are assumed to take place within a single period. This study introduces time in innovation by assuming new technologies will become available one period after labor input is made for invention.

Let $N_t \in \mathbb{R}_+$ be the number of technologies existing in period t. Assume that λv_{tj} and κZ_t are labor inputs in period t, respectively, to supply $v_{tj} \in \mathbb{R}_+$ units of product j in period t and to invent $Z_t \in \mathbb{R}_+$ technologies by which new differentiated products can be produced from period $t+1$. Then, the labor market clearing condition is

$$\int_0^{N_t} \lambda v_{tj} dj + \kappa Z_t = E_t, \tag{1}$$

where E_t is the amount of effective labor in period t; the number of technologies available for production in period $t+1$ is

$$N_{t+1} = N_t + Z_t. \tag{2}$$

The rest of the model is much the same as in Judd (1985). That is, the amount effective labor grows at a constant rate, $E_t = \alpha^t \bar{E}$. The perfectly

[3]See Mitra (2001), Mukherji (2005), Gardinia, Iryna, and Naimzada (2008), Yano, Sato, and Furukawa (2011), and Deng and Kahn (2018).

competitive retail sector produces the final consumption good, X, from the existing differentiated products by

$$X_t = \left(\int_0^{N_t} {v_{tj}}^{1-\theta} dj \right)^{\frac{1}{1-\theta}}, \quad 0 < \theta < 1. \tag{3}$$

The representative consumer chooses consumption stream X_t by maximizing

$$U = \sum_{t=1}^{\infty} \beta^{t-1} \ln X_t, \; 0 < \beta < 1, \tag{4}$$

subject to the standard budget constraint. The first order condition of optimization is

$$X_{t+1}/X_t = \beta(1 + r_t), \tag{5}$$

where r_t is the interest rate in period t. Given (4), the inverse demand for the product may be expressed as

$$p_{tj} = p(v_{tj}; X_t) = (X_t/v_{tj})^{\theta}. \tag{6}$$

The license for a new invention is traded at P_t; its purchaser can use the the invention monopolistically only for one period. The licence market is perfectly competitive. Given that j is a new product, the price of a license is equal to, or above, the monopolistic profit, i.e,

$$(p(v_{tj}; X_t) - w_t\lambda)v_{tj} \leq P_{tj}, \tag{7}$$

where w_t is the wage rate. If j is a product invented in the past, it is traded perfectly competitively at

$$p_{tj} = \lambda w_t. \tag{8}$$

Let

$$x_t = \frac{N_t}{\theta \alpha^t} \tag{9}$$

and

$$y_t = \frac{1}{\alpha} \left(\frac{N_t}{\theta \alpha^t} + \frac{Z_t}{\theta \alpha^t} \right). \tag{10}$$

Our model boils down to the following system with two state variables, x_t and y_t :We may prove that our model boils down to

$$\begin{cases} x_{t+1} &= y_t \\ y_{t+1} &= \max \left\{ \frac{1}{\alpha} y_t, \;\; \frac{1}{\alpha} y_t + \frac{1}{\alpha} \dfrac{\frac{\bar{E}\beta(1-\theta)^{1/\theta-1}}{\kappa} \frac{(1-\theta)^{1/\theta-1}(\alpha y_t - x_t)+x_t}{(1-\theta)^{1/\theta}(\alpha y_t - x_t)+x_t} - y_t}{\left(\beta\theta \frac{(1-\theta)^{1/\theta-1}(\alpha y_t - x_t)+x_t}{(1-\theta)^{1/\theta}(\alpha y_t - x_t)+x_t} + 1\right)(1-\theta)^{1/\theta-1}} \right\} \end{cases}; \tag{11}$$

see Appendix A for a derivation.

3 Two-Dimensional Constrained Chaos

Given a solution to a two-dimensional system, $(x_{t+1}, y_{t+1}) = f(x_t, y_t)$, we call $(y_{2\tau}, y_{2(\tau+1)})$, $\tau = 0, 1, 2, ...$, a double-period solution. In Figure 1A, a double-period solution to (11) is plotted for

$$(\alpha, \theta, \beta, \bar{E}/\kappa) = (1.12, 0.05, 0.94, 10). \tag{12}$$

It appears to follow a unimodal and expansive system.[4] Figure 1B shows the probability with which $y_{2\tau}$, $\tau = 0, 1, 2, ...$, falls in each interval; this distribution is independent of initial states.

Lasota and Yorke (1973) show that if a single-period system is unimodal and expansive, it is ergodic chaos. That is, the ergodic theorem of Birkhoff (1931) and von-Neumann (1932) holds, or that the relative frequency with which $y_{2\tau}$ falls in each Borel set can be described by a probability distribution independent of initial states.

3.1 Analytical Characterization

Figure 1 may be explained by the fact that the domain of system (11) is constrained; $y_t \geq \frac{1}{\alpha} x_t$. Think of a two-dimensional unconstrained system, $F : \mathbb{R}_2 \to \mathbb{R}_2$,

$$(x_{t+1}, y_{t+1}) = F(x_t, y_t) = (y_t, Y(x_t, y_t)), \tag{13}$$

and the associated constrained system

$$(x_{t+1}, y_{t+1}) = (y_t, \max\{A(y_t), Y(x_t, y_t)\}) = f(x_t, y_t). \tag{14}$$

Assume:

Assumption 1 For almost every (x, y), $||F^t(x, y)|| \to \infty$ as $t \to \infty$.

Assumption 2 Function A satisfies $y - \varepsilon > A(y)$ with some $\varepsilon > 0$ and $A(0) = 0$.

Assumption 3 Equation $Y(x, y) = A(y)$ can be solved for $y = B(x)$, which is continuous and monotone decreasing. Moreover, $y \leq B(x)$ if and only if $Y(x, y) \geq A(y)$.

[4] A system is unimodal and expansive if its graph consists of two parts with one of the slopes uniformly larger than 1 and the other uniformly smaller than -1.

Let

$$C = \{(x, y) : y \geq A(x) \text{ and } Y(x, y) \geq A(y)\}. \tag{15}$$

The following two conditions are crucial:

$$1.\ y_{t+1} = A(y_t) \text{ implies } Y(y_t, y_{t+1}) \geq A(y_{t+1}); \tag{16}$$

$$2.\ y_{t+1} = A(y_t) \text{ and } Y(y_{t+1}, y_{t+2}) \geq A(y_{t+2}) \text{ imply } Y(y_{t+2}, y_{t+3}) \geq A(y_{t+3}). \tag{17}$$

Under these conditions, a solution goes in and out of subset C, which may be called the "core" of nonlinear dynamics. If $y_{t+1} = A(y_t)$, by (16), $(y_t, y_{t+1}) \in C$, which implies $y_{t+2} = r(y_{t+1}) = R(y_t)$, where

$$r(y) = Y(A^{-1}(y), y) \tag{18}$$

and

$$R(y) = Y(y, A(y)). \tag{19}$$

If $(y_{t+1}, y_{t+2}) \notin C$, $Y(y_{t+1}, y_{t+2}) < A(y_{t+2})$. Thus, by (14), $y_{t+3} = A(y_{t+2})$, which implies, by (16), $y_{t+4} = R(y_{t+2})$. If, instead, $(y_{t+1}, y_{t+2}) \in C$, $Y(y_{t+1}, y_{t+2}) \geq A(y_{t+2})$; thus, by (14) and $y_{t+1} = A(R^{-1}(y_{t+2}))$, $y_{t+3} = G(y_{t+2})$ with

$$G(y) = Y(A(R^{-1}(y)), y). \tag{20}$$

If, moreover, $(y_{t+2}, y_{t+3}) \in C$, by (17), $Y(y_{t+2}, y_{t+3}) \geq A(y_{t+3})$; thus, $y_{t+4} = L(y_{t+2}) = l(y_{t+3})$, where

$$L(y) = Y(y, G(y)) \tag{21}$$

and

$$l(y) = Y(G^{-1}(y), y). \tag{22}$$

By characterizing (16) and (17), our main theorem provides a sufficient condition under which double-period solutions to (14) follow $y_{2(\tau+1)} = T(y_{2\tau})$ with

$$T(y) = \min_y\{L(y), R(y)\}, \tag{23}$$

which is expansive and unimodal. Let

$$S(y) = \begin{cases} G(y) & \text{if } L(y) \leq R(y) \\ A(y) & \text{if } L(y) \geq R(y) \end{cases}. \tag{24}$$

Theorem 1 *Let y_H satisfy $A(y_H) = B(y_H)$. Moreover, let $y_C = \arg\max_x T(x)$, $y_{\max} = \max_x T(x)$, and $y_L = R(y_{\max})$. Suppose that the following three conditions are met.*

Condition 1: $L'(x) > 1$ and $R'(x) < -1$,

Condition 2: $y_C < y_{\max} < y_H$ and $y_L < L(y_L)$,

Condition 3: If $B(x) \leq y < y_H$ and $R(y) < B(A(y))$, it holds that

$$G(R(y)) < B(R(y)) \tag{25}$$

and

$$L(R(y)) \geq B(G(R(y))). \tag{26}$$

Then, the solution, (x_t, y_t), $t = 0, 1, ...$, to the original dynamical system, (14), from (x_0, y_0), $B(x_0) < y_0 < y_H$, follows an ergodically chaotic system on $[y_L, y_{\max}]^2$

$$\left(x_{2(\tau+1)}, y_{2(\tau+1)}\right) = \left(S(y_{2\tau}), T(y_{2\tau})\right). \tag{27}$$

In order to prove the theorem, we first obtain a sufficient condition under which $B(x_t) < y_t < y_H$ implies $y_{t+2} = R(y_t)$ and $y_{t+4} = L(y_{t+2})$.

Lemma 1 *Suppose that an equilibrium path (x_t, y_t), solving system (11), satisfies the following conditions:*

$$B(x_t) \leq y_t < y_H; \tag{28}$$

$$y_{t+2} < B(x_{t+2}) < y_H; \tag{29}$$

$$y_{t+3} < B(x_{t+3}) < y_H. \tag{30}$$

If that R^{-1} exists, the equilibrium path satisfies $(x_{t+2}, y_{t+2}) = (A(y_t), R(y_t))$ and $(x_{t+4}, y_{t+4}) = (G(y_{t+2}), L(y_{t+2}))$.

Proof. Let y_t satisfy (28). Then, $B(x_t, y_t) \leq 0$. Thus, by (11),

$$y_{t+1} = A(y_t) \text{ and } x_{t+1} = y_t. \tag{31}$$

Since B and A are, respectively, decreasing and increasing and intersect each other at y_t, $y_t < y_H$ implies $y_{t+1} < B(x_{t+1})$. Since this implies $Y(x_{t+1}, y_{t+1}) > A(y_{t+1})$, by $y_{t+1} = A(y_t)$ and (11), we have

$$y_{t+2} = Y(y_t, A(y_t)) \text{ and } x_{t+2} = A(y_t), \tag{32}$$

which implies $y_{t+2} = R(y_t)$. Since $y_{t+2} < B(x_{t+2}) < y_H$ by (29), similarly, we have

$$y_{t+3} = Y(y_{t+1}, y_{t+2}) \text{ and } x_{t+3} = y_{t+2}, \tag{33}$$

which implies $x_{t+4} = y_{t+3} = G(y_{t+2})$. Moreover, since $y_{t+3} < B(x_{t+3}) < y_H$ by (30),

$$y_{t+4} = Y(y_{t+2}, y_{t+3}) \text{ and } x_{t+4} = y_{t+3}, \tag{34}$$

which implies $y_{t+4} = L(y_{t+2})$ by $y_{t+3} = G(y_{t+2})$. ■

We are now ready to prove Theorem 1.

Proof of Theorem 1. By Condition 1, $y_{\max} = R(y_C) = L(y_C)$. Since $R' < -1$, $y_{\max} = R(y_C)$ implies $y_C > R(y_{\max}) = y_L$. Since $y_L < L(y_L)$ by Condition 2, $T(y) = L(y)$ on $[y_L, y_C]$ and $T(y) = R(y)$ on $[y_C, y_{\max}]$. Moreover, $R([y_C, y_{\max}]) = [y_L, y_{\max}]$ and $L([y_L, y_C]) \subset [y_L, y_{\max}]$ by $y_L < L(y_L)$. This implies that $T(x)$ is a function on $[y_L, y_{\max}]$ onto itself. Thus, by Lasota and Yorke's theorem, T is an ergodically chaotic dynamical system on $[y_L, y_{\max}]$.

Suppose $y_C \leq B(x_t) \leq y_t < y_H$ for an arbitrary t. As is shown in the proof of Lemma 1, this implies $(x_{t+2}, y_{t+2}) = (A(y_t), T(y_t))$. Since $y_{\max}$ is achieved at y_C, by Condition 2,

$$y_{t+2} = R(y_t) = T(y_t) \leq y_{\max} < y_H.$$

Either $y_{t+2} \geq B(x_{t+2})$ of $y_{t+2} < B(x_{t+2})$. If $y_{t+2} \geq B(x_{t+2})$, by (25),

$$y_{t+4} = L(y_{t+2}) \text{ and } x_{t+4} = A(y_{t+2}).$$

If, instead, that $y_{t+2} < B(x_{t+2})$, $R(y_t) < B(A(y_t))$. By Condition 3, therefore, $y_{t+3} < B(x_{t+3})$ and $y_{t+4} < B(x_{t+4})$. Thus, by Lemma 1, $(x_{t+4}, y_{t+4}) = (A(y_{t+2}), T(y_{t+2}))$.

Finally since $y_C \leq B(x_{t+4}) \leq y_{t+4} < y_H$ by (26), the above process repeats. Thus, $(x_{t+2\tau}, y_{t+2\tau}) = (S^\tau(y_t), T^\tau(y_t))$ for all τ if $(x_0, y_0) = (x, y)$ and $y_C \leq B(x) \leq y < y_H$. ■

3.2 Linear Example

Think of the following example:

$$f : x_{t+1} = y_t \text{ and } y_{t+1} = \max\{ay_t, -bx_t - dy_t + e\}. \tag{35}$$

Then, functions B, r, R, G, L and l are given as follows:

$$B : y = -\frac{b}{d+a}x + \frac{e}{d+a}; \tag{36}$$

$$r : y_{t+1} = -(d + \frac{b}{a})y_t + e; \tag{37}$$

$$R : y_{t+1} = -a(d + \frac{b}{a})y_{t-1} + e; \tag{38}$$

$$G : y_{t+2} = -\left(d - \frac{ba}{b+da}\right) y_{t+1} - ba\left(\frac{e}{b+da}\right) + e; \tag{39}$$

$$L : y_{t+3} = \left[d\left(d - \frac{ba}{b+da}\right) - b\right] y_{t+1} + dba\left(\frac{e}{b+da}\right) - de + e; \tag{40}$$

$$l : y_{t+3} = -\left[d - \frac{b}{d - \frac{ba}{b+da}}\right] y_{t+2} + e\left[\left(\frac{ba}{b+da}\right) - 1\right]\left[\frac{b}{d - \frac{ba}{b+da}}\right] + e. \tag{41}$$

Suppose this example satisfies the conditions of Theorem 1. Then, functions B, r, R, G, L, and l have graphs shown in Figure 2. Let $y_C \leq y_{t-1} \leq y_{\max}$ and $y_{t-1} > B(y_{t-2})$, which implies $y_t = ay_{t-1}$. Since $y_t = ay_{t-1} \leq B(y_{t-1})$, $y_{t+1} = r(y_t)$. If $y_C \leq y_{t-1} < y_E$, $y_{t+1} = r(y_t) > B(y_t)$; thus, $y_{t+2} = ay_{t+1} \leq B(y_{t+1})$, and $y_{t+3} = r(y_{t+2}) = R(y_{t+1})$. If $y_E \leq y_{t-1} \leq y_{\max}$, $y_{t+1} = r(y_t) \leq B(y_t)$, which implies $y_{t+2} = G(y_{t+1})$ and $y_{t+3} = l(y_{t+2}) = L(y_{t+1})$. In short, if $y_C \leq y_{t-1} \leq y_{\max}$ and $y_{t-1} > B(y_{t-2})$, $y_{t+1} = T(y_{t-1})$ and $y_{t+3} = T(y_{t+1})$. Since $L' > 1$ and $R' < -1$, T is expansive and unimodal. Since the graph of T lies above line B for $y_L \leq y \leq y_{\max}$, and since the graph of line A lies below line B for $y_L \leq y \leq y_{\max}$, the process repeats itself.

Theorem 1 captures this fact in a general setting to characterize our nonlinear system, (11).

4 Ergodically Chaotic Innovation Dynamics

In our model of innovation dynamics, double-period dynamics is described by specific functions: That is, with $\eta = (1-\theta)^{1/\theta - 1}$ and $\zeta = (1-\theta)^{1/\theta}$,

$$\begin{aligned} B_\theta(x) \quad = \quad & \frac{1}{2\alpha\zeta}\left\{-((1-\zeta)x - \frac{\bar{E}\beta\eta}{\kappa}\eta\alpha)\right. \\ & \left. + \sqrt{\left((1-\zeta)x - \frac{\bar{E}\beta\eta}{\kappa}\eta\alpha\right)^2 + 4\frac{\bar{E}\beta\eta}{\kappa}\zeta\alpha(1-\eta)x}\right\}; \end{aligned} \tag{42}$$

see Appendix B for a derivation. Moreover,

$$R_\theta(y) = -\frac{1}{\alpha^2}\left(\frac{1}{(\beta\theta+1)\eta} - 1\right) y + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta}{\kappa}}{\beta\theta+1}; \tag{43}$$

$$L_\theta(y) = \frac{1}{\alpha}G_\theta(y) + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}\frac{\eta(\alpha G_\theta(y)-y)+y}{\zeta(\alpha G_\theta(y)-y)+y} - G_\theta(y)}{\phi\beta\theta\eta\frac{\eta(\alpha G_\theta(y)-y)+y}{\zeta(\alpha G_\theta(y)-y)+y} + \eta}; \tag{44}$$

$$G_\theta(y) = \frac{1}{\alpha}y + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}\frac{\eta(\alpha y - \frac{1}{\alpha}R_\theta^{-1}(y)) + \frac{1}{\alpha}R_\theta^{-1}(y)}{\zeta\left(\alpha y - \frac{1}{\alpha}R_\theta^{-1}(y)\right) + \frac{1}{\alpha}R_\theta^{-1}(y)} - y}{\phi\beta\theta\eta\frac{\eta\left(\alpha y - \frac{1}{\alpha}R_\theta^{-1}(y)\right) + \frac{1}{\alpha}R_\theta^{-1}(y)}{\zeta\left(\alpha y - \frac{1}{\alpha}R_\theta^{-1}(y)\right) + \frac{1}{\alpha}R_\theta^{-1}(y)} + \eta}. \tag{45}$$

The initial condition is

$$\bar{x} = \bar{N}/\theta \text{ and } \bar{y} = \frac{1}{\alpha}(\bar{N}/\theta + \bar{Z}/\theta). \tag{46}$$

The equilibrium dynamical system, (11), has a unique fixed point,

$$y_S^\theta = \frac{\frac{\bar{E}\beta\eta}{\kappa}}{(\alpha-1)\beta\theta\eta + \zeta(\alpha-1) + 1}. \tag{47}$$

The next proposition implies that, due to Theorem 1, our model of innovation dynamics, (11), is ergodically chaotic if $\theta > 0$ is sufficiently small; see Appendix C for a proof.

Proposition 1 *Define T_θ and $y_{\max}^\theta$ as in Theorem 1. Let (x_t, y_t) be the solution to the two-dimensional equilibrium system (11) from $(x_0, y_0) = (\bar{x}, \bar{y})$ satisfying $y_C^\theta < \bar{y} < y_{\max}^\theta$ and $\bar{y} > B_\theta(\bar{x})$. There is $\theta' > 0$ such that $0 < \theta < \theta'$ implies that $T_\theta : [y_S^\theta, y_{\max}^\theta] \to [y_S^\theta, y_{\max}^\theta]$ is expansive and unimodal and that the double-period solution $y_{2\tau}$, $\tau = 0, 1, 2, ...,$ follows $y_{2(\tau+1)} = T_\theta(y_{2\tau})$ if and only if*

$$\sqrt{\frac{-(e-1)^2 + \sqrt{(e-1)^4 + 4(e-1)^3}}{2}} \approx 1.103 < \alpha < \sqrt{e-1} = 1.310 \tag{48}$$

4.1 Three Patterns of Take-offs

In the rest of this section, we focus on the case in which (48) holds and in which $\theta > 0$ is sufficiently small. In that case, the graph of T_θ in Proposition

1 has essentially the same structure as that in Figure 2. Drop subscripts and superscripts, θ, from functions and variables.

If $y_t = A(y_{t-1})$, no innovation occurs in period $t-1$. If, subsequently, $y_{t+1} = r(y_t) = R(y_{t-1})$, active innovation takes place in period t. This phase shift, from a period of no innovation (period $t-1$) to that of active innovation (period t), may be called an innovation take-off. Innovation take-offs emerge in three patterns of dynamics. Let $y_E = R^{-1}(y_C)$ and $y_D = R^{-1}(y_E)$.

Pattern 1: Let $y_C \leq y_{\tau_1} \leq y_D$. Then $y_{\tau_1+1} = A(y_{\tau_1})$, $y_{\tau_1+2} = r(y_{\tau_1+1})$.

Pattern 2: Let $y_E \leq y_{\tau_2} \leq y_{\max}$. Then $y_{\tau_2+1} = A(y_{\tau_2})$, $y_{\tau_2+2} = r(y_{\tau_2+1})$, $y_{\tau_2+3} = G(y_{\tau_2+2})$, $y_{\tau_2+4} = l(y_{\tau_2+3})$, $y_{\tau_2+5} = A(y_{\tau_2+4})$, and $y_{\tau_2+6} = r(y_{\tau_2+5})$.

Pattern 3: Let $y_D \leq y_{\tau_3} \leq y_E$. Then $y_{\tau_3+1} = A(y_{\tau_3})$, $y_{\tau_3+2} = r(y_{\tau_3+1})$, $y_{\tau_3+3} = A(y_{\tau_3+2})$, $y_{\tau_3+4} = r(y_{\tau_3+3})$, $y_{\tau_3+5} = A(y_{\tau_3+4})$ and $y_{\tau_3+6} = r(y_{\tau_3+5})$.

If $y_C \leq y_t \leq y_D$ for some $t = \tau_1$, a Pattern 1 take-off occurs in period $t+1$. The equilibrium path will immediately move to $y_E \leq y_{t+2} \leq y_{\max}$ in period $t+2 = \tau_2$, from which a Pattern 2 process starts. Six periods later, $t+7$, the equilibrium falls either $y_D \leq y_{t+7} \leq y_E$ or $y_C \leq y_{t+7} \leq y_D$. In the former case, a Pattern 3 process starts, in which a period of no innovation and that of positive innovation alternate every other periods for a certain number of times. It will eventually fall back into $y_C \leq y_{\tau_1} \leq y_D$, from which a Pattern 1 process restarts.

4.2 Magnitudes of Take-offs

The magnitude of a take-off in period t may be measured by the growth rate of differentiated products in period t, $g_t = N_{t+1}/N_t = \alpha y_t / y_{t-1}$. Since, by (10), $g_t = \alpha R(y_{t-1})/A(y_{t-1})$, g_t is decreasing in y_{t-1}. Moreover, since $y_C < y_D < y_E < y_{\max}$, take-offs in Pattern 1, $y_{\tau_1+2}/y_{\tau_1+1}$, are larger than those in Patterns 2 and 3, while second-period take-offs in Pattern 2, $y_{\tau_2+2}/y_{\tau_2+1}$, are smaller than those in Patterns 1 and 3.

4.3 Second Wave

As is shown below, a Pattern 1 take-off, $y_{\tau_1+2}/y_{\tau_1+1}$, may be followed by even a larger innovation wave four periods later, $y_{\tau_2+6}/y_{\tau_2+5}$.

Proposition 2 *Let $\gamma(y) = \alpha R(y)/A(y)$ and $g(y) = \alpha L(y)/G(y)$. Denote as y_F the y satisfying $g(R(R(y))) = \gamma(y)$. In addition to the hypothesis of*

Theorem 1, suppose $L'(y) > L(y)/y$. Then, $y_C < y_F < y_D$, and for any $y \in (y_F, y_D) \neq \phi$,

$$\gamma(y) < g(R(R(y))). \tag{49}$$

Proof. Since $R' < -1$, $\gamma' < 0$. Since $G' < 0$ and $L' > -1$, $g'R'R' > 0$. This implies (49). ■

5 Frequency of Industrial Revolution Cycles

Assume that an industrial revolution is captured in our model by a Pattern 1 take-off, which is larger than any other take-off in Patterns 2 and 3; a Pattern 1 take-off starts with $y_t \in [y_C, y_D]$. As is shown above, a Pattern 1 take-off, which is larger than any take-off in Patterns 2 and 3, starts with $y_t \in [y_C, y_D]$. Since T is ergodic chaos, by von-Neumann-Birkhoff's theorem, we may calculate the average frequency with which an equilibrium path will fall in this interval, $[y_C, y_D]$. Here, we will simulate this frequency for different parameter values. We will also simulate the probability with which a Pattern 1 take-off is followed by an even larger innovation wave.

The average long-run real interest rate may be identified with the steady state interest rate, which is

$$r_o = \frac{\alpha^{1/(n(1-\theta))}}{\beta^{1/n}} - 1,^{5} \tag{50}$$

per year, where n is the length (in years) of a single period in the model. Borio, Dysyatat, Jeselius, and Rungchaoenkitkul (2017) report that from the 1870s through 2010, except several brief periods, the long-run interest rate was about $r_o = 2.5\%$, although it has been lower since 2006 and was lower during the two world war periods, or about $r_o = 2\%$. These findings more or less agree with Schmelzing (2000), tracking the long-run interest rate since the early 14th century. For our simulation, we adopt $r_o \in [1.9\%, 2.6\%]$.

Parameter $\alpha_o = \alpha^{1/n}$ is the annual growth rate of labor productivity. According to Maddison (2010), the total Western European per capita GDP grew at about 1.6 percent annually ($\alpha_o = 1.016$) from 1820 through 2008.

In the steady state, by (50), (5) implies $X_{t+1}/X_t = \alpha^{1/(1-\theta)}$, which may be thought of as the growth rate of total factor productivity (TFP); the annual

[5] See Appendix D for a derivation.

rate is $\alpha_o^{1/(1-\theta)}$. Shackleton (2013) shows that in the U.S., the average annual TFP growth rate was between 1.6% to 1.8% ($\alpha_o^{1/(1-\theta)} \in [1.016, 1.018]$) over the period from 1870 through 2010.

If $\alpha_o = 1.016$ and $\alpha_o^{1/(1-\theta)} = 1.017$, $\theta = 0.056$. By (50), $\alpha_o^{1/(1-\theta)} = 1.017$ and $r_o = 1.024$ imply the annual discount factor is $\beta_o = \beta^{1/n} = 0.993$. The frequency of innovation waves do not respond much to a change in the discount factor, β.

With these considerations, we fix $\beta_o = 0.993$ and $\theta = 0.05$ and calculate the average frequency of Pattern 1 take-offs for $n = 4$ years through 20 years and $r_o \in [1.9\%, 2.6\%]$, which implies $\alpha_o = 1.011\%$ to 1.018%.

As Table 1 shows, the frequency is sensitive to a change in the length of a single period, n. If we set $n = 8$ years, a Pattern 1 take-off, $y_{\tau_1+2}/y_{\tau_1+1}$, has several key features of the past industrial revolutions.

First of all, a Pattern 1 take-off is larger than any other take-offs in Patterns 2 and 3. Moreover, the average frequency of Pattern 1 take-offs is about once in 116 years if $r_o = 1.9\%$, in 111 years if $r_o = 2\%$, and 104 years if $r_o = 2.6\%$. These are consistent with the facts that the past three industrial revolutions were about one hundred years apart.

Moreover, if $n = 8$, the second big wave and a Pattern 1 take-off, $y_{\tau_2+4}/y_{\tau_2+3} = y_{\tau_1+6}/y_{\tau_1+5}$ and $y_{\tau_1+2}/y_{\tau_1+1}$, are 32 years apart (or four periods). For the third industrial revolution, as is noted above, the first wave came in the middle of the 1980s whereas the second wave came, with the introduction of bigdata, IOT, AI, and blockchain, in the middle of the 2010s. For the second industrial revolution, the first wave came in the 1850s and the second wave, with the extensive use of electric power, in the 1890s.

By Proposition 2, we may calculate the probability with which the second wave, $y_{\tau_2+4}/y_{\tau_2+3}$, is larger than the first take-off, $y_{\tau_1+2}/y_{\tau_1+1}$.. Table 2 reports the results for the case in which $n = 8$ and $r_o \in [1.019, 1.026]$. The results show that the probability is between 22% to 34%.

6 Conclusion

Innovation dynamics is intrinsically multi-dimensional, as shown in our model. We have developed a new characterization for two-dimensional chaos and shown that the coexistence of industrial revolution cycles and shorter Kondratieff waves may be explained as a two-dimensional chaos. Provided that parameter values set consistently with the real-world average long-run interest rate, TFP growth rate, and labor productivity growth rate, a big industrial take-off emerges about every one hundred years if the monopolistic control of a new invention is assumed to last for above eight years.

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Appendix A: Derivation of (11).
In equilibrium, the cost of an invention, $w_t\kappa$, cannot exceed the present value of the invention, $P_{t+1}/(1+r_t)$, which gives rise to, by (7),

$$\kappa \geq \frac{\phi(p_{t+1}^M - \lambda w_{t+1})v_{t+1}^M}{(1+r_t)w_t} = \phi\beta\theta\frac{\left(\frac{\lambda w_{t+1}}{1-\theta}\right)^{1-1/\theta} X_t}{w_t} \tag{51}$$

where equality holds if $Z_t > 0$. Since

$$w_t = \frac{1}{\lambda}\left(N_{t-1} + \left(\frac{1}{1-\theta}\right)^{1-1/\theta} Z_{t-1}\right)^{\frac{\theta}{1-\theta}} \tag{52}$$

and

$$X_t = \frac{\bar{E}\alpha^t - \kappa Z_t}{\lambda^{1-1/\theta}\left(N_{t-1} + \left(\frac{1}{1-\theta}\right)^{-1/\theta} Z_{t-1}\right) w_t^{-1/\theta}}, \tag{53}$$

(51) may be written as a dynamical system of N_t and Z_t. This may be transformed into (11) by using (9) and (10). ■

Appendix B: Derivation of (42).
Note that $Y(x,y) = A(y)$ in Assumption 3 holds if and only if $b_\theta(x,y) = 0$, where

$$b_\theta(x,y) = Y(x,y) - A(y) \tag{54}$$

and, by (11) and (14),

$$b_\theta(x_t,y_t) = \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}\frac{\xi(\alpha y_t - x_t)+x_t}{\zeta(\alpha y_t - x_t)+x_t} - y_t}{\phi\beta\theta\eta\frac{\xi(\alpha y_t - x_t)+x_t}{\zeta(\alpha y_t - x_t)+x_t} + \xi}. \tag{55}$$

Thus, $b_\theta(x,y) = 0$ if and only if

$$x = x_1(y) + x_2(y) \tag{56}$$

where

$$x_1(y) = \frac{\alpha\left(\frac{\bar{E}\beta\eta}{\kappa}\right)^2\frac{\eta-\zeta}{1-\zeta}\frac{1-\eta}{1-\zeta}}{(1-\zeta)y - \frac{\bar{E}\beta\eta}{\kappa}(1-\eta)} \tag{57}$$

and

$$x_2(y) = -\frac{\alpha\zeta}{1-\zeta}y + \alpha\frac{\bar{E}\beta\eta}{\kappa}\frac{\eta-\zeta}{(1-\zeta)^2}. \tag{58}$$

This implies that, in the x-y space, the graph of $x = x_1(y) + x_2(y)$ is the horizontal sum of a downward sloping rectangular hyperbola, $x = x_1(y)$, and a downward sloping line, $x = x_2(y)$. Since $x = x_1(y)$ is downward sloping and asymptotic to the horizontal line $y = \frac{\bar{E}\beta\eta}{\kappa}\frac{1-\eta}{1-\zeta}$, for $x \geq 0$, the y satisfying (56) is uniquely determined. Therefore, $Y(x, y) = A(y)$ determines a unique y for $x \geq 0$, which is $y = B_\theta(x)$, given by (42). ■

Appendix C: Proof of Proposition 1.

Since this proposition is proved for a range of sufficiently small θ, it is desirable at the outset to extend the domain of $T_\theta : [y_S^\theta, y_{\max}^\theta] \to [y_S^\theta, y_{\max}^\theta]$ to an interval that is independent of θ and contains $[y_S^\theta, y_{\max}^\theta]$ for any sufficiently small θ. Towards this end, focus on θ, $0 < \theta \leq 1/2$. Note that, as $\theta \to 0$,

$$\eta \to 1/e \text{ and } \zeta \to 1/e, \tag{59}$$

where e is the base for natural logarithm. Moreover, $0 < \theta \leq 1/2$ implies

$$1/e < \eta \leq 1/2 \text{ and } 1/4 < \zeta \leq 1/e. \tag{60}$$

Let y_H^θ be the x-axis value of the intersection between curves A and $\bar{B}_\theta$; i.e., $\bar{B}_\theta(y_H^\theta) = A(y_H^\theta)$. It is easy to check

$$y_H^\theta = \frac{\bar{E}\alpha\beta\eta}{\kappa}. \tag{61}$$

We may prove the following: ■

Lemma 2

$$\bar{y}_S \equiv \frac{1}{e(e(\alpha-1)+1)}\frac{\bar{E}\beta}{\kappa} < y_S^\theta < y_H^\theta < \frac{\alpha}{2}\frac{\bar{E}\beta}{\kappa} \equiv \bar{y}_H. \tag{62}$$

Proof. The first and third inequality follows from (60). The second inequality follows from

$$A(y_S^\theta) < y_S^\theta < \bar{B}_\theta(y_S^\theta). \tag{63}$$

■

In what follows, we restrict the domain of T_θ to the closed interval $[\bar{y}_S, \bar{y}_H]$, which is independent of θ. In order to prove Theorem 1, by Theorem 1, it suffices to prove that Conditions 1, 2 and 3 of Theorem 1 is met.

In the next two lemmas, we will obtain conditions under which Condition 1 of Theorem 1 ($L'_\theta > 1$ and $R'_\theta < -1$) is satisfied.

Lemma 3 *There is $\theta' > 0$ such that $0 < \theta \leq \theta'$ implies $R'_\theta < -1$ on $[\bar{y}_S, \bar{y}_H]$ if and only if*

$$e - 1 > \alpha^2. \tag{64}$$

Proof. This follow from (43) and (59). ■

Lemma 4 *There is $\theta' > 0$ such that $0 < \theta \leq \theta'$ implies $G'_\theta < -1$ and $L'_\theta > 1$ on $[\bar{y}_S, \bar{y}_H]$ if and only if*

$$e - 1 > \alpha. \tag{65}$$

Proof. Let $y_{t+1} = \frac{1}{\alpha} R_\theta^{-1}(y_{t+2})$. Then, (43) implies

$$y_{t+1} = -\frac{\alpha\,(\beta\theta + 1)\,\eta}{1 - (\beta\theta + 1)\,\eta} y_{t+2} + \frac{\frac{\bar{E}\beta}{\kappa}}{1 - (\beta\theta + 1)}. \tag{66}$$

By definition, $y_{t+3} = G_\theta(y_{t+2})$ and $y_{t+4} = L_\theta(y_{t+2})$ are given by the the system of equation (66) and the following equations.

$$y_{t+3} = \frac{1}{\alpha} \frac{\frac{\bar{E}\beta\eta}{\kappa} z_{t+1} - y_{t+2}}{\beta\theta\eta z_{t+1} + \eta} + \frac{1}{\alpha} y_{t+2}; \tag{67}$$

$$y_{t+4} = \frac{1}{\alpha} \frac{\frac{\bar{E}\beta\eta}{\kappa} z_{t+2} - y_{t+3}}{\beta\theta\eta z_{t+2} + \eta} + \frac{1}{\alpha} y_{t+3}; \tag{68}$$

$$z_{t+1} = \frac{\eta\,(\alpha y_{t+2} - y_{t+1}) + y_{t+1}}{\zeta\,(\alpha y_{t+2} - y_{t+1}) + y_{t+1}}. \tag{69}$$

$$z_{t+2} = \frac{\eta\,(\alpha y_{t+3} - y_{t+2}) + y_{t+2}}{\zeta\,(\alpha y_{t+3} - y_{t+2}) + y_{t+2}}; \tag{70}$$

This implies y_{t+1}, y_{t+3} y_{t+4}, z_{t+2}, and z_{t+4} satisfy (66), (67), (68), (69), and (70) if and only if $y_{t+2} = R_\theta(y_t)$ and $y_{t+4} = L_\theta(y_{t+2})$.

Then, z_{t+1}, z_{t+2}, y_{t+3}, y_{t+4} and y_{t+1} may be thought of as functions of y_{t+2}. By differentiating (66) through (70) with respect to y_{t+2}, we obtain the following:

$$y'_{t+1} = -\frac{\alpha^2(\beta\theta + 1)\eta}{1 - (\beta\theta + 1)\eta};$$

$$y'_{t+3} = \frac{1}{\alpha} \frac{(\frac{\bar{E}\beta\eta}{\kappa} + \phi\beta\theta y_{t+2}) z'_{t+1}}{\eta(\beta\theta z_{t+1} + 1)^2} - \frac{1}{\alpha}\left(\frac{1}{\beta\theta\eta z_{t+1} + \eta} - 1\right);$$

$$
\begin{aligned}
y'_{t+4} &= \frac{1}{\alpha^2}\left(\frac{1}{\beta\theta\eta z_{t+2}+\eta}-1\right)^2 \\
&\quad -\frac{1}{\alpha^2}\left(\frac{1}{\beta\theta\eta z_{t+1}+\eta}-1\right)\frac{(\frac{\bar{E}\beta\eta}{\kappa}+\phi\beta\theta y_{t+2})}{\eta(\beta\theta z_{t+1}+1)^2}z'_{t+1} \\
&\quad +\frac{1}{\alpha}\frac{(\beta\theta y_{t+3}+\frac{\bar{E}\beta\eta}{\kappa})}{\eta(\beta\theta z_{t+2}+1)^2}z'_{t+2};
\end{aligned}
$$

$$
z'_{t+1} = \alpha(\eta-\zeta)\frac{y_{t+1}-y_{t+2}y'_{t+1}}{(\zeta(\alpha y_{t+2}-y_{t+1})+y_{t+1})^2};
$$

$$
z'_{t+2} = \alpha(\eta-\zeta)\frac{y_{t+2}y'_{t+3}-y_{t+3}}{(\zeta(\alpha y_{t+3}-y_{t+2})+y_{t+2})^2}.
$$

By $y_{t+2} \in [\bar{y}_S, \bar{y}_H]$, the lemma follows from (59). ■

In the next two lemmas, we will obtain conditions under which Condition 2 of Theorem 1 ($y_{\max} > y_C$ and $y_L < L(y_L)$) is satisfied.

Lemma 5 *There is $\theta' > 0$ such that if $0 < \theta < \theta'$, $y_C^\theta \in [\bar{y}_S, \bar{y}_H]$ and $y_{\max}^\theta \in [\bar{y}_S, \bar{y}_H]$.*

Proof. By Lemmas 4 and 3, y_C^θ is determined by $R_\theta(y_C^\theta) = L_\theta(y_C^\theta) = y_{\max}^\theta$. Then, by (20), (45), (44), and (59), $y_C^\theta \to \frac{E\beta}{e\kappa}$ and $y_{\max}^\theta \to \frac{E\beta}{e\kappa}(\alpha e - (e-1))$ as $\theta \to 0$. Thus, the lemma holds for y_C^θ and $y_{\max}^\theta$. ■

Lemma 6 *Suppose condition (64) is satisfied. Then, there is $\theta' > 0$ such that $0 < \theta \leq \theta'$ implies $L_\theta(y_L^\theta) > B_\theta(G_\theta(y_L^\theta)) > y_L^\theta$ if and only if*

$$
\alpha > \sqrt{\frac{-(e-1)^2+\sqrt{(e-1)^4+4(e-1)^3}}{2}}. \tag{71}
$$

Moreover, if (65) and (71) are met, there is $\theta' > 0$ such that $0 < \theta \leq \theta'$ implies $y_{\max}^\theta > y_C^\theta$.

Proof. By Lemmas 4 and 3, we may choose θ' in such a way that $0 < \theta \leq \theta'$ implies $L'_\theta > 1$, $G'_\theta < -1$ and $R'_\theta < -1$ on $[\bar{y}_S, \bar{y}_H]$. Thus, $y_C^\theta = y_{t+2}$, $R(y_2) = L_\theta(y_2) = y_{t+4}$, and $y_L^\theta = R_\theta(y_4) = y_{t+6}$ satisfy the following:

$$
y_{t+4} = -\frac{1}{\alpha^2}\left(\frac{1}{\beta\theta\eta+\eta}-1\right)y_{t+2}+\frac{1}{\alpha}\frac{\frac{\bar{E}\beta}{\kappa}}{\beta\theta+1}; \tag{72}
$$

$$y_{t+6} = -\frac{1}{\alpha^2}\left(\frac{1}{\beta\theta\eta+\eta}-1\right)y_{t+4} + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta}{\kappa}}{\beta\theta+1}. \tag{73}$$

Thus, y_{t+1}, y_{t+2}, y_{t+3}, y_{t+4}, y_{t+6}, z_{t+1} and z_{t+2} are determined by the system of equations (66), (67), (68), (69), (70), (72), and (73). Moreover, $L_\theta(y_L^\theta)$ is determined by

$$y_{t+8} = -\frac{1}{\alpha^2}\left(\frac{1}{\beta\theta\eta+\eta}-1\right)y_{t+6} + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta}{\kappa}}{\beta\theta+1}. \tag{74}$$

Boundary equation (56) with (57) and (58) shows that, as $\theta \to 0$, $B_\theta(x_t) \to \bar{B} = \frac{\bar{E}\beta}{e\kappa}$ uniformly in $x_t \in [\bar{y}_S, \bar{y}_H]$. Thus, if and only if there is $\varepsilon > 0$ such that $y_{t+8} = L_\theta(y_{t+6}) > \frac{\bar{E}\beta}{e\kappa} + \varepsilon$ and $\frac{\bar{E}\beta}{e\kappa} - \varepsilon > y_{t+6}$ for any θ, $0 < \theta \leq \theta'$, it holds that $L_\theta(y_{t+6}) > B(G_\theta(y_{t+6})) > y_{t+6}$.

In order to prove this, take the limit case of $\theta = 0$. Since $\xi = \zeta$ in the limit case, by (69) and (70), $z_{t+1} = z_{t+2} = 1$. By using this fact, we may solve the system of (68) and (72) to obtain $y_C = y_{t+2} = \frac{\bar{E}\beta}{e\kappa}$. By this together with (73) and (74), in $\theta = 0$, we have

$$y_{t+6} = \frac{\bar{E}\beta\eta}{\kappa}\left(\frac{1}{\alpha^4}\left(\frac{1}{\eta}-1\right)^2 - \frac{1}{\alpha^3}\left(\frac{1}{\eta}-1\right)\frac{1}{\eta} + \frac{1}{\alpha}\frac{1}{\eta}\right)$$

This implies $y_{t+6} - \frac{\bar{E}\beta\eta}{\kappa} < 0$, given (64) and (71). Moreover,

$$\begin{aligned} y_{t+8} &= \frac{1}{\alpha^2}\frac{\bar{E}\beta}{\kappa e}((e\alpha-(e-1)e) \\ &\quad +(e-1)^2\left(\frac{1}{\alpha^4}\left(e\alpha^3-(e-1)\,e\alpha+(e-1)^2\right)\right). \end{aligned}$$

Thus, $y_{t+8} > \frac{\bar{E}\beta}{e\kappa}$ if and only if

$$(\alpha-(e-1))\left(\alpha^4+(e-1)^2\alpha^2-(e-1)^3\right) < 0.$$

Given (65), this implies that there are $\theta' > 0$ and $\varepsilon > 0$ such that $y_{t+8} = L_\theta(y_{t+6}) > \frac{\bar{E}\beta}{e\kappa} + \varepsilon$ for any θ, $0 < \theta \leq \theta'$.

The latter half of the lemma directly follows from $L'_\theta > 1$, $L_\theta(y_L^\theta) > y_L^\theta$ and the definitions of y_C^θ and $y_{\max}^\theta$. This establishes the lemma. ■

Proof. We are now ready to complete the proof of Proposition 1. Note that if (48) is satisfied, (64), (65), and (71) are also satisfied. Since this implies that Conditions 1 and 2 are met for sufficiently small θ, it suffices to prove that Condition 3 is also met for sufficiently small θ.

Take an equilibrium path, (x_t, y_t), solving (11) from $(\bar{x}, \bar{y})$. Suppose $y_t \leq y^\theta_{\max}$ and $y_t > B_\theta(y_{t-1})$. This implies $y_{t+1} = \frac{1}{\alpha} y_t < y^\theta_{\max}$. Moreover, since $y_{t+1} = \frac{1}{\alpha} y_t$, $y_t < y^\theta_{\max} < y^\theta_H$ implies implies $y_{t+1} < B_\theta(y_t)$, which implies $y_{t+2} = r_\theta(y_{t+1})$. We will prove that if $y_{t+2} < B_{t+1}(y_{t+1})$, (25) and (26) are satisfied.

Let $y_{t+2} < B_\theta(y_{t+1})$. Then, $y_{t+3} = G_\theta(y_{t+2})$, as is shown in the proof of Theorem 1. Moreover, $y_{t+2} < b_\theta(y_{t+1})$ implies $y^\theta_L \leq y_{t+2} < R_\theta(y^\theta_D)$, where y^θ_D is given by $R_\theta(y^\theta_D) = B_\theta(\frac{1}{\alpha} y^\theta_D)$. Let $z^\theta_S = G_\theta(z^\theta_S)$. Then, by (20), (45) and (59), $\left| y^\theta_S - x^\theta_S \right| \to 0$, as $\theta \to 0$. Thus, $y^\theta_L \leq y_{t+2}$ implies $z^\theta_S < y_{t+2}$. Since $G'_\theta < -1$, this implies $G_\theta(y_{t+2}) < z^\theta_S$. Since $G'_\theta < -1$, the graphs of $y_{t+3} = B_\theta(y_{t+2})$ and $y_{t+3} = G_\theta(y_{t+2})$ intersects only once. These facts together with $y^\theta_L \leq y_{t+2} < R_\theta(y^\theta_D)$ implies that $G_\theta(y_{t+2}) = y_{t+3} < B_\theta(y_{t+2})$. Thus, condition (25) is satisfied.

Moreover, $y_{t+4} = l_\theta(y_{t+3}) = l_\theta(G_\theta(y_{t+2})) = L_\theta(y_{t+2})$. Since, by Lemma 6, $L_\theta(y^\theta_L) > B_\theta(G_\theta(y^\theta_L))$, and since $y_{t+4} \geq y^\theta_L$, $L_\theta(y) > B_\theta(G_\theta(y))$. Thus, condition (26) is satisfied. ■

Appendix D: Derivation of (50). This follows from (5), (52) and (53). ■

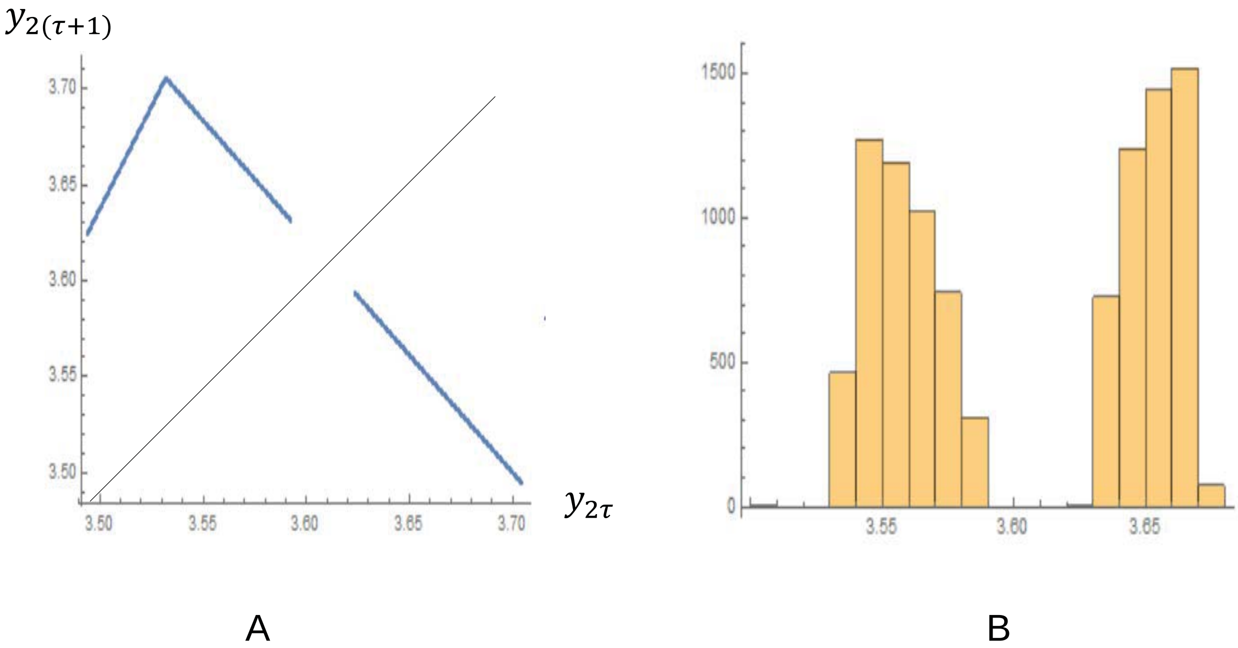

Figure 1: Chaos in Innovation Dynamics

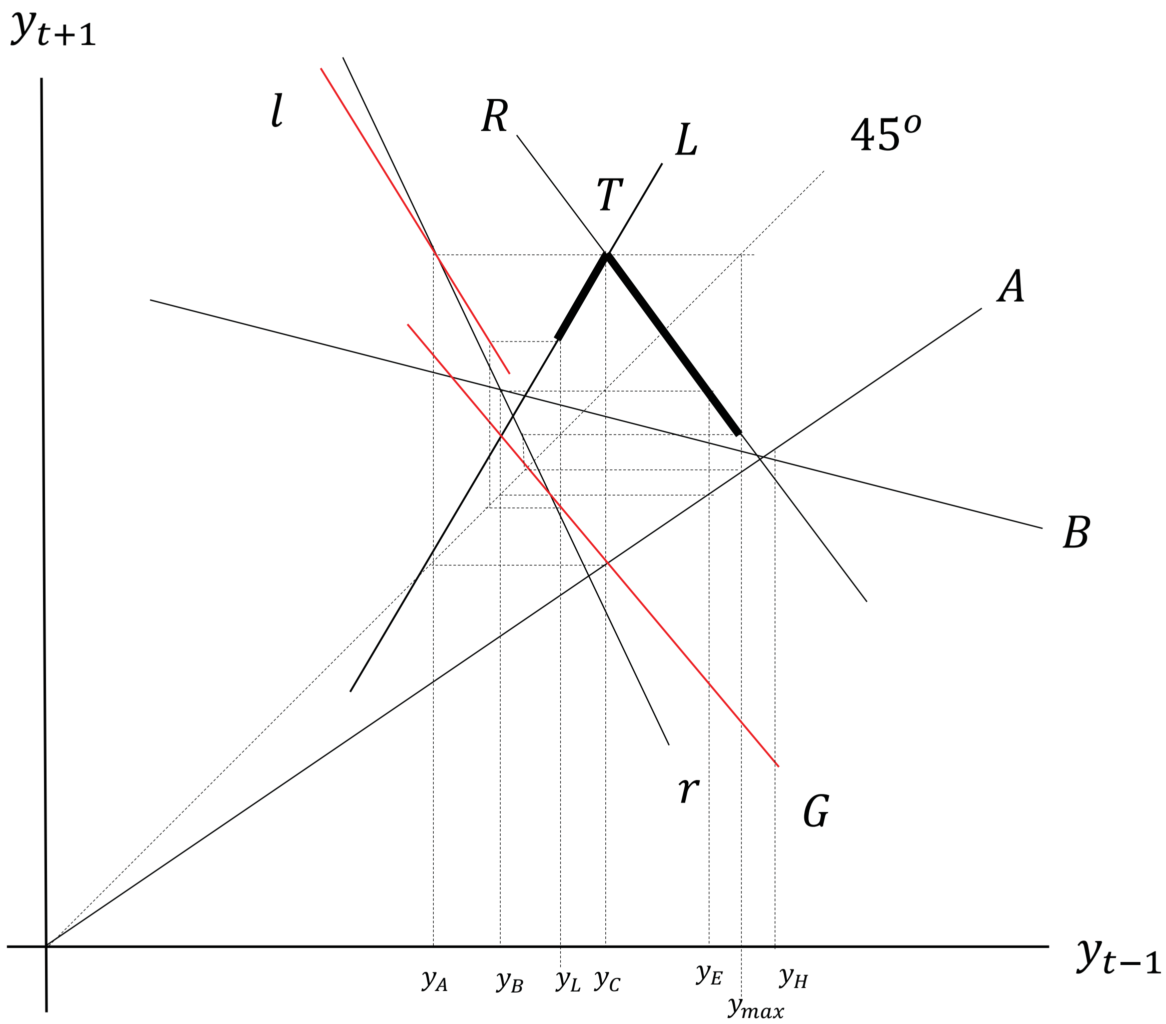

Figure 2: Two-Dimensional Constrained Chaos

			Length of a Single Period (years)								
Long-run Instest Rate (% per year)	(TFP%)	(α_o)	4	6	8	10	12	14	16	18	20
1.90%	1.19	1.011	X	103	116	135	151	193	252	212	S
				(1161)	(1380)	(1481)	(1580)	(1450)	(1268)	(1695)	
2.00%	1.29	1.012	X	102	110	130	134	203	196	210	S
				(1174)	(1458)	(1543)	(1786)	(1373)	(1636)	(1711)	
2.10%	1.39	1.013	X	93	107	122	158	209	161	S	S
				(1288)	(1494)	(1633)	(1517)	(1342)	(1980)		
2.20%	1.48	1.014	X	89	105	127	137	164	S	S	S
				(1341)	(1519)	(1572)	(1758)	(1706)			
2.30%	1.58	1.015	X	85	102	134	181	95	S	S	S
				(1409)	(1617)	(1496)	(1323)	(2924)			
2.40%	1.68	1.016	X	82	98	136	141	S	S	S	S
				(1472)	(1632)	(1467)	(1702)				
2.50%	1.78	1.017	X	80	104	121	126	S	S	S	S
				(1493)	(1536)	(1324)	(1908)				
2.60%	1.88	1.018	X	81	104	123	S	S	S	S	S
				(1480)	(1535)	(1306)					

$\theta = 0.05$, β_o=0.993

X: will gointo the negative phase.

Table 1. Frequency of Large Innovation Waves

Long-run Instest Rate (	(TFP%)	(α_o)	Probability
1.90%	1.19	1.011	0.3
2.00%	1.29	1.012	0.26
2.10%	1.39	1.013	0.29
2.20%	1.48	1.014	0.34
2.30%	1.58	1.015	0.22
2.40%	1.68	1.016	0.25
2.50%	1.78	1.017	0.3
2.60%	1.88	1.018	0.3

$\theta = 0.05$, β_o=0.993, ν=8

X: will gointo the negative phase.

Table 2. Probability of a Second Wave Larger than the First Big Take-off