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Shurojit Chatterji, Atsushi Kajii and Huaxia Zeng



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# Intertemporal efficiency does not imply a common price forecast


Shurojit Chatterji
Singapore Management University

Atsushi Kajii*
Kyoto University

Huaxia Zeng
Shanghai University of Finance and Economics


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**Abstract**

Do price forecasts of rational economic agents need to coincide in perfectly competitive complete markets? To address this question, we define an efficient temporary equilibrium (ETE) within the framework of a two period economy. Although an ETE allocation is intertemporally efficient and is obtained by perfect competition, it can arise without the agents forecasts being coordinated on a perfect foresight price. We show that there is a one dimensional set of such Pareto efficient allocations for generic endowments.

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## 1 Introduction

Do price forecasts of rational economic agents need to coincide in perfectly competitive complete markets? If not, is it rewarding to have a more accurate forecast than others? This classical but fundamental question does not appear to have received the attention it deserves. The pervasive approach *assumes* that they should coincide, perhaps with a justification that if the economic agents understand the market environment perfectly, they


*Visiting Professor, Singapore Management University. Kajii acknowledges support from the Grant-in-aid for scientific research (S)18H05217 and (A)16H02026.

must reach the same conclusion, and hence in particular, their forecasts must coincide. But it is against the spirit of perfectly competitive markets to require that the agents should understand the market environment beyond the market prices they commonly observe.

To address the issue precisely, let us consider a sequence of commodity markets with no uncertainty, where there is a riskless bond market in each period, and so the markets are complete. Finitely many households trade competitively in these markets. There are two approaches to the study of the functioning of such economies. The first, the classical temporary equilibrium approach (Grandmont, 1977), asks if there are market clearing prices in a particular period in question for arbitrarily given anticipated prices for the markets in the following periods. Since market clearing for subsequent periods is not required, this approach hardly explains how a sequence of market prices are determined over time and furthermore, it does not require that forecasts should be equalized. Also, being completely mute on welfare analysis for intertemporal allocations of goods, it cannot begin to address the benefits from accurate forecasts.

The second, and the aforementioned more pervasive approach, is the perfect foresight approach, which assumes that agents' price forecasts are all perfectly coordinated and correct. Here, a perfect foresight (more generally, a rational expectations) equilibrium (Radner, 1982) predicts a sequence of market prices in a determinate way, and the resulting allocation is Pareto efficient. This approach explains prices and addresses the welfare issue, but it incurs a serious cost in that perfect foresight is assumed, rather than derived. That the assumption of perfect foresight is extraordinarily strong is a view expressed by various scholars; a case in point is Radner's own critique of perfect foresight.[1] It goes without saying that this approach is absolutely inadequate for comparing the quality of price forecasts and explaining, among other issues, the use of policy

[1]On page 942, Radner (1982) writes "Although it is capable of describing a richer set of institutions and behaviour than is the Arrow-Debreu model, the perfect foresight approach is contrary to the spirit of much of competitive market theory in that it postulates that individual traders must be able to forecast, in some sense, the equilibrium prices that will prevail in the future under all alternative states of the environment. Even if one grants the extenuating circumstances mentioned in previous paragraphs, this approach still seems to require of the traders a capacity for imagination and computation far beyond what is realistic."

tools that seek to influence the forecasts of diverse subsets of agents. In spite of these obvious shortcomings, the pervasive use of this approach would appear to stem from the presumption that perfect foresight is indispensible to a theory that delivers efficient outcomes and retains some predictive power.

So the following classical question on price forecasts seems a very natural one to pose in this setup: First, require that all the spot markets clear in the temporary equilibrium sense. That is, even when the households traded anticipating wrong prices in the past, describe how they consume and save in every period so that one can address welfare issues. Secondly, suppose that the markets are so elaborated that the resulting sequence of consumption constitutes a Pareto efficient allocation, not only within each period but also intertemporally. Such an equilibrium is referred to as an efficient temporary equilibrium (henceforth, ETE). Although there is no prerequisite for price forecasts, market clearing and efficiency property of an ETE should rule out inadequate forecasts. The question we pose is, must an ETE necessarily be a perfect foresight equilibrium?

At first sight the answer might appear positive, under the standard set of assumptions on utility functions such as monotonicity, concavity, and differentiability. Intuitively, the dimension of Pareto efficient allocations should be one less than the number of the households, since it is in effect the set of wealth transfers across the households. On the other hand, at an ETE, since the final consumption bundle must be attained in markets, each household's consumption bundle must satisfy some budget constraint. By market clearing one of these budget constraints might be redundant, but still these create additional restrictions at least as many as the dimension of Pareto efficient allocations. Recall that the set of Arrow-Debreu equilibrium allocations can be found from Pareto efficient allocations and budget constraints by the second fundamental theorem of welfare economics, and Debreu's theorem shows that the set of Arrow-Debreu equilibria is zero dimensional generically. Therefore, the same logic seems to suggest that the set of ETE allocations is zero dimensional generically. Hence if an ETE which does not entail perfect foresight ever exists, it must be an isolated case relying on some coincidence.

The idea above is reinforced if one recalls "no trade results", which assert that with no private information the efficiency of an allocation implies that households' relative evaluations of goods must agree with each other. In our context, it would appear to rule out even an isolated example mentioned above.

The surprise, the aforementioned logic notwithstanding, is that this conjecture is incorrect. We illustrate this point in its simplest form, using a standard competitive two-period exchange economy with inside money. There is one perishable consumption good in each period to be traded, that is, we trivialize temporary markets to eliminate potential multiplicity caused by indeterminacy of absolute prices in those markets. In the first period, a bond which pays off one unit in units of account (dollar) in the second period is traded. We restrict attention to agents forming point forecasts as this allows for a more transparent comparison of our approach to the perfect foresight approach. This restriction not only makes our argument simpler, but also eliminates possible multiplicity caused by delicate coordination of randomly forecasted prices.

In this set up, an ETE is defined in a straightforward manner: in addition to requiring market clearing in each of the two periods, it requires the efficiency of the resulting two-period consumption bundle. We present a real indeterminacy result for ETE, which has been shorn of all complications arising from multiple goods and random forecasts so as to make the indeterminacy more striking, as our principal finding. More precisely, our main result shows the existence of a one dimensional set of ETE allocations around each Arrow-Debreu equilibrium allocation, generically in endowments. This result generalizes the two-agent Edgeworth box example of Chatterji et al. (2018) to arbitrary economies.

Curiously enough, the degree of real indeterminacy does not depend on the number of households, while the dimension of Pareto efficient allocations increases as explained above. Therefore, when the number of households is very large, which is a plausible circumstance for perfect competition, an ETE does require a very delicate coordination of price forecasts. If one conjectured, despite our intuitive illustration using budget constraints, that an ETE would hardly restrict price forecasts, then the invariance to the number of households should turn up as a surprising result.

Coming back to the question we posed above, our answer is that decentralized markets are able to deliver a significantly larger set of acceptable (Pareto efficient) outcomes under less restrictive assumptions on forecasts. Moreover, the extra degree of freedom is only one at least in our model, so the explanatory power is almost as strong as the perfect foresight approach. Therefore, we contend that the approach based on ETE has considerably greater descriptive appeal than believed erstwhile.

An ETE is a particular variant of a perfectly contracted equilibrium (Chatterji and Ghosal, 2013), where intertemporal exchanges are modeled using reduced form intertemporal (price-contingent) contracts which are required to satisfy a Pareto efficiency and an individual rationality requirement,[2] but are otherwise unstructured and allow for considerable differences in real interest rates across households; as a consequence, perfectly contracted equilibria generate a subset of Pareto efficient allocations whose dimension is one less than the number of households. In this paper on the other hand, we focus on a well specified and well studied class of intertemporal contracts; these arise from decentralized trade in a bond market with heterogenous forecasts. This puts restrictions on ways in which the real interest rates across households can diverge in an ETE, so that we have exactly one degree of freedom in specifying the resulting ETE allocations.

The remainder of the paper is organized as follows. Section 2 presents the model and the basic definitions. The general result is stated and proved in Section 3. Section 4 presents an explicitly computed example of ETE. Section 5 discusses some aspects of our formulation and findings, especially the welfare implication of forecasts. In particular, our analysis demonstrates that an accurate forecast is not necessarily rewarded. Finally, we note some directions for future research and conjectures pertaining to these.

## 2 The Model and Definition

We consider a standard competitive exchange economy with inside money. There are two periods, period 0 and 1, and there is one perishable consumption good in each period to be traded competitively.

There are $H \geq 1$ households, labelled by $h = 1, ..., H$. Abusing notation we use $H$ for the set of households as well. Household $h$ is endowed with $e_h^0$ units of good in the first period (period 0) and $e_h^1$ units in the second period (period 1). We write $e_h = \left(e_h^0, e_h^1\right)$.

[2]The particular forms of the Pareto efficiency and individual rationality requirements respectively in perfectly contracted equilibrium require time separable utilities. Svensson (1981) studies a more complicated market structure where *all* price contingent contracts are traded and showed that efficiency obtains when utilities are time separable and probabilistic forecasts are coordinated across agents. Our formulation of ETE and our result does not require time separable utilities or coordinated forecasts.

We simply call the good of the first period good 0 and the good of the second period good 1.

Household $h$'s consumption set is $\mathbb{R}^2_+$, and its preferences for consumption bundles are represented by a utility function $u_h : \mathbb{R}^2_+ \to \mathbb{R}$. Later, we will make assumptions on $u_h$ so that the consumption takes place in $\mathbb{R}^2_{++}$.

In the first period, a bond which pays off $1+r$ ($r > -1$) units in units of account (dollar) in the second period is traded competitively, i.e., a household takes the market interest rate $r$ as given to decide its saving. A negative saving corresponds to borrowing. There is no uncertainty, no limit for saving and borrowing, and default is not allowed. The net supply of the bond is zero, so it is inside money whose real return is determined in the markets.

Writing $z_h$ for the amount of saving of household $h$, and writing $p^0$ for the market price of the consumption good in period 0, the consumption $x^0_h$ of household $h$ in period 0 is therefore subject to $p^0 x^0_h \leq p^0 e^0_h - z_h$.

There is no futures market which might help predict the price of the good in the second period. Thus we assume that each household $h$ first anticipates the price $\hat{p}_h$ of the good in period 1 in order to decide consumption and saving/borrowing in period 0. Specifically, at the prevailing market interest rate $r$, household $h$ expects the period 1 budget $\hat{p}_h \left(x^1_h - e^1_h\right) \leq (1+r)\, z_h$ if his saving is $z_h$. Since there is no limit for saving/borrowing in our model, by eliminating $z_h$ household $h$ faces in effect the following budget constraint for consumption goods:

$$p^0 \left(x^0_h - e^0_h\right) + \frac{\hat{p}_h}{(1+r)} \left(x^1_h - e^1_h\right) \leq 0. \tag{1}$$

It is readily seen that if $\left(x^0_h, x^1_h\right) \in \mathbb{R}^2_+$ satisfies (1), then there is $z$ with which the budget is met in both periods. Note that the monotonicity of $u_h$ will assure that the equality will hold at the optimum.

We denote the market price of the good in period 1 by $p$. That is, in period 1, household $h$ is subject to the constraint $p\left(x^1_h - e^1_h\right) \leq (1+r)\, z_h$, i.e., the market value of the net consumption must be no greater than the nominal return from the saving. Notice that $z_h$ is already determined before satisfying the period 1 budget. So the realized consumption path $\left(x_h^0, x_h^1\right)$ must satisfy the following equation:

$$p^0\left(x_h^0 - e_h^0\right) + \frac{p}{(1+r)}\left(x_h^1 - e_h^1\right) \leq 0. \tag{2}$$

Note that although constraint (2) is not taken into account in period 0, household $h$ will spend all the income in period 1 at the market price, i.e., $p\left(x_h^1 - e_h^1\right) = (1+r)\, z_h$ will hold if $u_h$ is increasing, and then the equality holds for (2) at the optimum.

Now we shall define a dynamic temporary equilibrium: it is simply the standard classical temporary equilibrium notion applied for each period. Since all the budget inequalities are all homogeneous in prices, there is no loss of generality if $p^0 = 1$ is required and hence interest rate $r$ works as an equilibrating market price in period 0, and so we shall assume $p^0 = 1$ below to economize notation.

**Definition 1** *A temporary equilibrium is a tuple* $\left(x^*, r^*, (\hat{p}_h)_{h=1}^H, p^*\right) \in \left(\mathbb{R}^2_{++}\right)^H \times (-1,\infty) \times (\mathbb{R}_+)^H \times \mathbb{R}_+$ *such that:*
*(i)* $x^*$ *is a feasible allocation, i.e.,* $\sum_{h=1}^H x_h^* = \sum_{h=1}^H e_h$*;*
*(ii) for each* $h \in H$*, there exists* $\hat{x}_h^1$ *such that* $\left(x_h^{0*}, \hat{x}_h^1\right)$ *maximizes utility under budget (1) given* $(r^*, \hat{p}_h)$*;*
*(iii) for each* $h \in H$*,* $x_h^{1*}$ *maximizes* $u_h\left(x_h^{0*}, \cdot\right)$ *under constraint (2) at* $p = p^*$*,* $r = r^*$*, and* $x_h^0 = x_h^{0*}$*.*

Note that condition (i) implies that the total demand meets the total supply in both periods. Then, condition (ii) says that period 0 market is in *temporal equilibrium* given forecasts $(\hat{p}_h)_{h=1}^H$, and condition (iii) says that the period 1 market is also in temporal equilibrium, given the market interest rate and the consumption allocation in period 0. Since there is only one good, if $u_h$ is increasing, condition (iii) can be equivalently written as $p^*\left(x_h^{1*} - e_h^1\right) = (1+r^*)\left(e_h^0 - x_h^{0*}\right)$ for all $h$, i.e., the nominal income must be spent for consumption in the second period; that is, condition (iii) is equivalent to (2) holding with equality in equilibrium.

There is hardly any restriction on equilibrium forecasts, and hence there are many temporary equilibria, because one can solve the equilibrium condition sequentially in a trivial manner in this model of a single good. To illustrate this point, choose any $(\hat{p}_h)_{h=1}^H$. Given these forecasts, the aggregate demand for good 0 is a function of interest rate $r$, but it does not depend on the second period market price $p$. So there will be an interest rate $r^*$ which clears period 0 market, irrespective of $p$. Then the period 1 demand is derived by (2), where the first period variables are already fixed. So by an appropriate choice of $p$, the period 1 market will be in temporal equilibrium, too.

Since every market is in partial equilibrium, it is no surprise that a temporary equilibrium is weakly constrained efficient. However, since the equalization of the marginal rate of substitution of the two goods across agents is not warranted, a temporary equilibrium tends not to be Pareto efficient. But if one subscribes to the view that a perfect market structure would induce the households to trade until gains from trade vanish completely, it is natural to focus on an efficient temporary equilibrium.

**Definition 2** *An efficient temporary equilibrium (ETE) is a temporary equilibrium* $\left(x^*, r^*, (\hat{p}_h)_{h=1}^H, p^*\right)$ *where the consumption allocation* $x^*$ *is Pareto efficient.*

A hypothetical market transaction process justifying an ETE would rule out many forecasts which would allow unrealized gains from trade. The extreme case is a *perfect foresight equilibrium* (henceforth, PFE): by definition, a PFE is a particular temporal equilibrium $\left(x^*, r^*, (\hat{p}_h)_{h=1}^H, p^*\right)$ where $\hat{p}_h = p^*$ for all $h$. In this case, the two budget constraints (1) and (2) are identical, and each household's utility must be maximized within the common budget set. Hence a PFE is an Arrow-Debreu equilibrium where any contingent good can be traded, and vice versa. Needless to say, an Arrow-Debreu equilibrium is weakly efficient, and if utility functions are continuous and increasing, it is Pareto efficient. So under the standard assumptions, a PFE is an ETE.

Conversely, if forecasts $\hat{p}_h$, $h = 1, .., H$, coincide with each other in an ETE, then $\hat{p}_h = p^*$ must clear the period 1 market, i.e., the common forecast is correct ex post, constituting a PFE. To see this, observe that if the period 0 market clears with a common forecast, then by Walras law, the planned consumption allocation in period 1 must be feasible. Therefore, the common forecast is indeed a market clearing price.

It is then interesting to ask if an ETE is necessarily a PFE; is a common forecast required for efficiency? As mentioned in the introduction, under the standard set of assumptions on utility functions, there are reasons to expect that the set of ETE allocations is zero-dimensional generically, but we shall show that there is a one dimensional set of ETE allocation around each Arrow-Debreu equilibrium allocation generically in endowments, irrespective of the number of households. We will establish this result formally in section 3 below. Section 4 presents an explicitly computed example of ETE.

## 3 Generic Real Indeterminacy of ETE

In order to employ the standard technique of genericity analysis, we assume the following: for every household $h = 1, ..., H$,

- utility function $u_h$ is $C^2$ on $\mathbb{R}^2_{++}$, $\partial u_h \gg 0$, and differentiably strictly concave, and each indifference curve is closed in $\mathbb{R}^2$;
- initial endowments $e_h$ are strictly positive.

We fix utility functions throughout, and identify an economy with its initial endowments: so write $\mathcal{E} := \left(\mathbb{R}^2_{++}\right)^H$ and its generic element is denoted by $e = (\cdots, e_h, \cdots)$. We say a subset of $\mathcal{E}$ is *generic* if it is open and its complement has Lebesgue measure 0. The following is the main result:

**Proposition 3** *There is a generic set $\mathcal{E}^* \subset \mathcal{E}$ such that for each $e \in \mathcal{E}^*$, (i) there are finitely many PFE, and (ii) for each PFE allocation $\bar{x} \in \left(\mathbb{R}^2_{++}\right)^H$, there is a one dimensional $C^1$ manifold of ETE allocations containing $\bar{x}$.*

The method of our proof is to show that ETE allocations can be written as a regular system of equations, where the number of independent equations is one less than the unknowns, and the one degree of freedom in the system corresponds to the asserted real indeterminacy. But first, we begin with some back ground observations on the derived consumer demand function. We write $x_h^0\left(p^0, p^1, m\right)$ for the standard demand function for good 0 of household $h$ where $m > 0$ is the income level: that is, for a given positive price vector $\left(p^0, p^1\right)$, it is the unique solution to

$$\max_{x^0, x^1 \geq 0} u_h\left(x^0, x^1\right) \text{ subject to } p^0 x^0 + p^1 x^1 \leq m.$$

Under our assumptions, $x_h^0\left(p^0, p^1, m\right)$ is a $C^1$ function on $\mathbb{R}^3_{++}$, and the second order condition for utility maximization holds strictly: that is, at $x_h^0\left(p^0, p^1, m\right)$,

$$\frac{\partial^2 u_h}{\left(\partial x^0\right)^2}\left(p^1\right)^2 - 2\frac{\partial^2 u_h}{\partial x^0 \partial x^1} p^0 p^1 + \frac{\partial^2 u_h}{\left(\partial x^1\right)^2}\left(p^0\right)^2 < 0, \tag{3}$$

Notice that there are obvious nominal indeterminacies about ETE, because of the homogeneity of budget constraints (1) and (2). So we might as well normalize $p^0 = 1$ and $r = 0$ to establish the real indeterminacy result, that is, $p_h$ is the effective price for (1) and $p$ is the effective price (2). Taking this normalization into account, we write $x_h^0 (p_h; e_h)$ for $x_h^0 \left(1, p_h, e_h^0 + p_h e_h^1\right)$, which is a $C^1$ function on $\mathbb{R}_{++}^2$.

A PFE obtains if and only if $\sum_{h=1}^{H} \left[x_h^0 (\hat{p}_h; e_h) - e_h^0\right] = 0$ and $\hat{p}_h = p^*$ for all $h$, and is of course equivalent to a competitive equilibrium in the two good Arrow-Debreu economy. So we shall identify a PFE with its second period price $p$, and following the standard terminology, we say that a PFE $p^*$ is *regular* if the derivative of excess demand is non zero, i.e., $\sum_{h=1}^{H} \frac{\partial}{\partial \hat{p}_h} x_h^0 (p^*; e_h) \neq 0$. Also we say an economy is regular if all PFE are regular. As is well known, under our assumptions, the set of *regular* economies, $\mathcal{E}_R$, is generic, and each regular economy has finitely many PFE, each of which can be written as a $C^1$ function of $e$, locally[3].

Notice however that at a regular PFE $p^*$, $\frac{\partial}{\partial \hat{p}_h} x_h^0 (p^*; e_h) = 0$ is possible for some $h$. But it means that the substitution effect is offset precisely by the income effect, so it must be coincidental. We say that a PFE $p^*$ is *strongly regular* if it is regular and additionally $\frac{\partial}{\partial \hat{p}_h} x_h^0 (p^*; e_h) \neq 0$ hold for all $h$. An economy is said to be strongly regular if all the (finitely many) equilibria are strongly regular. Then the following result should appear plausible (a proof is provided in the Appendix).

**Lemma 4** *The set of strongly regular economies, $\mathcal{E}_{SR}$, is generic.*

We shall now write an ETE as a solution to a system of equations. Introduce an auxiliary variable $\lambda$, and consider the following system of $H + 1$ equations for $H + 2$ unknowns, $(\hat{p}_h)_{h=1}^H$, $p$, and $\lambda$:

$$\begin{aligned} \sum_h \left(x_h^0 (\hat{p}_h; e_h) - e_h^0\right) &= 0 \\ &\vdots \\ \lambda \frac{\partial u_h}{\partial x^0} \left(x_h^0 (\hat{p}_h; e_h), x_h^1\right) - \frac{\partial u_h}{\partial x^1} \left(x_h^0 (\hat{p}_h; e_h), x_h^1\right) &= 0 \\ &\vdots \end{aligned} \tag{4}$$

[3] See for instance, Section 5.5 of Mas-Colell (1985).

where for each $h$,

$$x_h^1 = e_h^1 - \frac{x_h^0\left(\hat{p}_h; e_h\right) - e_h^0}{p}. \tag{5}$$

If $(\hat{p}_h)_{h=1}^H$, $p$, and $\lambda$ satisfy equations (4) and (5), setting $x_h^* = \left(x_h^0\left(\hat{p}_h; e_h\right), x_h^1\right)$ for all $h$, $r^* = 0$, and $p^* = p$, a tuple $\left(x^*, r^*, (\hat{p}_h)_{h=1}^H, p^*\right)$ constitutes an ETE; recall Definition 1. The first equation of (4) and relation (5) imply that $x^*$ is a feasible allocation, referring to condition (i) of Definition 1. The first equation of (4) also shows that condition (ii) of Definition 1 is automatically met, and since $u_h$ is increasing, (5) implies that condition (iii) of Definition 1 is satisfied. The remaining $H$ equations in (4) assure that $x^*$ is Pareto efficient. Conversely, it can be readily confirmed that if $\left(x^*, r^*, (\hat{p}_h)_{h=1}^H, p^*\right)$ is an ETE, then $\left((\frac{\hat{p}_h}{1+r^*})_{h=1}^H, \frac{p^*}{1+r^*}, \lambda^*\right)$ where $\lambda^* = \frac{\partial u_h}{\partial x^1}\left(x_h^0(\frac{\hat{p}_h}{1+r^*}; e_h), x_h^1\right) \Big/ \frac{\partial u_h}{\partial x^0}\left(x_h^0(\frac{\hat{p}_h}{1+r^*}; e_h), x_h^1\right)$ and $x_h^1 = e_h^1 - \frac{x_h^0(\frac{\hat{p}_h}{1+r^*}; e_h) - e_h^0}{\frac{p^*}{1+r^*}}$ for all $h$ solve (4) and (5).

A PFE is a solution to (4) and (5) of the form $\left((\hat{p}_h)_{h=1}^H, p^*, \lambda^*\right)$ such that $\hat{p}_h = p^* = \lambda^* = \frac{\partial u_h}{\partial x^1}\left(x_h^0\left(p^*; e_h\right), x_h^1\right) / \frac{\partial u_h}{\partial x^0}\left(x_h^0\left(p^*; e_h\right), x_h^1\right)$ for all $h$. Since $\hat{p}_h$ and $\lambda^*$ are automatically constructed from $p^*$, we shall simply identify a PFE solution with its (normalized) realized price $p^*$ as we did earlier.

Next, we shall verify that the implicit function theorem can be applied at a PFE so that the equilibrium variables constitutes a one dimensional manifold about it. Fix $e \in \mathcal{E}^*$ and denote by $\Phi\left(p, \hat{p}_1, \cdots, \hat{p}_H, \lambda\right)$ the left hand side of (4). Then $\Phi$ is a $C^1$ function from $\mathbb{R}^{H+2}$ to $\mathbb{R}^{H+1}$, and we investigate its Jacobian at a PFE. By direct calculation we obtain the following result (a proof is provided in the Appendix).

**Lemma 5** *Evaluated at a PFE $p^*$, the derivatives of $\Phi$ with respect to $\hat{p}_1, \cdots, \hat{p}_H, \lambda$ has the following form:*

$$\partial\Phi/\partial\left(\cdots, \hat{p}_h, \cdots, \lambda\right) = \left.\begin{bmatrix} \cdots & \frac{\partial}{\partial \hat{p}_h} x_h^0\left(\hat{p}_h; e_h\right) & \cdots & 0 \\ \ddots & & 0 & \vdots \\ & \zeta_h & & \frac{\partial u_h}{\partial x^0}\left(x_h^0\left(\hat{p}_h; e_h\right), x_h^1\right) \\ 0 & & \ddots & \vdots \end{bmatrix}\right|_{\substack{p = p^* \\ \hat{p}_1 = \cdots = \hat{p}_H = p^* \\ \lambda = p^*}} \tag{6}$$

*where for each $h = 1, 2, ..., H$,*

$$\zeta_h = \left[p^* \frac{\partial^2 u_h}{\left(\partial x^0\right)^2} - 2\frac{\partial^2 u_h}{\partial x^0 \partial x^1} - \frac{1}{p^*}\frac{\partial^2 u_h}{\left(\partial x^1\right)^2}\right] \frac{\partial}{\partial \hat{p}_h} x_h^0\left(\hat{p}_h; e_h\right).$$

Now we are ready to prove Proposition 3. Let $\mathcal{E}^*$ be the set of all strongly regular economies such that the initial endowments are not Pareto efficient. We show that $\mathcal{E}^*$ has the properties we wanted.

Since the set of Pareto efficient allocations constitutes a closed set of a lower dimension, it can be readily verified by Lemma 4 that $\mathcal{E}^*$ is generic. Condition (i) of Proposition 3 is clearly met by the regularity. So it suffices to show that any PFE of $e \in \mathcal{E}^*$, which is locally unique by regularity, has an ETE with the desired manifold structure around it.

Fix a PFE $p^*$ of economy $e \in \mathcal{E}^*$. Then by Lemma 5 and the second order condition (3), $\zeta_h \neq 0$ holds for every $h$. Apply the following basic operations on the matrix: Multiply the $h$ column of (6) by

$$\frac{\frac{\partial u_h}{\partial x^0}\left(x_h^0\left(\hat{p}_h; e_h\right), x_h^1\right)}{\zeta_h}$$

and subtract it from the last column. then the resulting matrix is

$$\left.\begin{bmatrix} \cdots & \frac{\partial}{\partial \hat{p}_h} x_h^0\left(\hat{p}_h; e_h\right) & \cdots & \alpha \\ \ddots & & 0 & \vdots \\ & \zeta_h & & 0 \\ 0 & & \ddots & \vdots \end{bmatrix}\right|_{\substack{p=p^* \\ \hat{p}_1=\cdots=\hat{p}_H=p^* \\ \lambda=p^*}}$$

where

$$\begin{aligned} \alpha &= -\sum_{h=1}^{H} \frac{\frac{\partial u_h}{\partial x^0}\left(x_h^0\left(\hat{p}_h; e_h\right), x_h^1\right)}{\zeta_h} \frac{\partial}{\partial \hat{p}_h} x_h^0\left(\hat{p}_h; e_h\right) \\ &= -\sum_{h=1}^{H} \frac{\frac{\partial u_h}{\partial x^0}\left(x_h^0\left(\hat{p}_h; e_h\right), x_h^1\right)}{p^* \frac{\partial^2 u_h}{\left(\partial x^0\right)^2} - 2\frac{\partial^2 u_h}{\partial x^0 \partial x^1} - \frac{1}{p^*}\frac{\partial^2 u_h}{\left(\partial x^1\right)^2}}. \end{aligned}$$

Since $\frac{\partial u_h}{\partial x^0} > 0$ and the denominator is negative by (3) for every $h$, we conclude $\alpha \neq 0$. Hence the matrix (6) has a full rank of $H+1$, and thus by the implicit function theorem the solutions to (4) and (5) constitute a one dimensional manifold around the PFE. Moreover, equilibrium variables $\hat{p}_1, \ldots, \hat{p}_H$ and $\lambda$ are $C^1$ function of $p$ around $p^*$.

It therefore remains to show that this indeterminacy about prices is in fact real indeterminacy. If we can find $p$ arbitrarily close to $p^*$ but $p \neq p^*$, such that the corresponding equilibrium $\hat{p}_h$ differs from $p^*$ for some $h$, then by the strong regularity condition, such a household's (excess) demand for good 0 must be different from $x_h^0(p^*; e_h)$ for all $p$ close enough to $p^*$, exhibiting real indeterminacy. So suppose that the corresponding equilibrium is the same as $p^*$, that is, $\hat{p}_h$ is a constant function of $p$ for every $h$. Then for any $p$ close enough to $p^*$, every household $h$'s consumption of good 1 at the ETE is $e_h^1 - \frac{x_h^0 - e_h^0}{p}$ where $x_h^0 = x_h^0(\hat{p}_h; e_h)$ does not depend on $p$ since $\hat{p}_h$ does not depend on $p$. Since the initial allocation is not Pareto efficient, there must be some household $h$ who trades in the PFE, i.e., $x_h^0 - e_h^0 \neq 0$ at $p = p^*$. Such a household's period 1 consumption must change as $p$ changes, exhibiting real indeterminacy in this case as well. This completes the proof.

# 4 An Example

An earlier paper (Chatterji et al., 2018) provided a two-household example of ETE, but it is hard to tell from the example if the one dimensionality is independent of the number of households. Here we provide a class of three-household economies that describes better the one dimensionality of the ETE allocations. The following is the specification of the example.

- $H = \{1, 2, 3\}$.
- Endowments: $e_1 = (2 - \varepsilon, \varepsilon), e_2 = (\varepsilon', 2 - \varepsilon')$ and $e_3 = \big(1 + (\varepsilon - \varepsilon'), 1 + (\varepsilon - \varepsilon')\big)$, where $0 < \varepsilon, \varepsilon' < 1$ are given parameters. Note that $\sum_{h=1}^{H} e_h^0 = 3$ and $\sum_{h=1}^{H} e_h^1 = 3$.
- For each $h \in H$, $u_h(x_h^0, x_h^1) = \ln x_h^0 + \ln x_h^1$.

As in the main analysis, we set period-0 price equal to one and the interest rate equal to zero, and write $\hat{p}_1, \hat{p}_2, \hat{p}_3 > 0$ for the anticipated prices of the households.

Thus, for all those economies parameterized by $\varepsilon$ and $\varepsilon'$, the set of Pareto efficient allocations is

$$\mathcal{P} = \left\{ (x_h^0, x_h^1)_{h=1}^H \in \mathbb{R}_+^{2\times 3} \middle| (x_h^0, x_h^1) = \alpha_h(1,1), \alpha_h \geq 0 \text{ for all } h \in H, \text{ and } \sum\nolimits_{h=1}^{H} \alpha_h = 1 \right\}.$$

so in particular, the initial endowments are not Pareto efficient. It is readily seen that the unique Arrow-Debreu equilibrium, thus the unique PFE, occurs at $\hat{p}_h^* = p^* = 1$, with the allocation $x_h^* = (1, 1)$ for every $h \in H$.

For each $h \in H$, by utility maximization under the anticipated price of period 1, we get the demands in period 0:

$$x_1^0(\hat{p}_1; e_1) = \frac{1}{2}(2 - \varepsilon + \hat{p}_1\varepsilon), \tag{7}$$

$$x_2^0(\hat{p}_2; e_2) = \frac{1}{2}\left[\varepsilon' + \hat{p}_2(2 - \varepsilon')\right], \text{ and} \tag{8}$$

$$x_3^0(\hat{p}_3; e_3) = \frac{1}{2}\left[1 + (\varepsilon - \varepsilon') + \hat{p}_3(1 - (\varepsilon - \varepsilon'))\right]. \tag{9}$$

Thus, market clearing in period 0 requires

$$\varepsilon\hat{p}_1 + (2 - \varepsilon')\hat{p}_2 + [1 - (\varepsilon - \varepsilon')]\hat{p}_3 = 3. \tag{10}$$

Furthermore, an inspection of $\frac{d}{d\hat{p}_h}x_1^0(\hat{p}_h; e_1)_{|\hat{p}_h=1}$ for all $h$ reveals that all the economies are strongly regular. So it can be readily shown that each of these economies belongs to $\mathcal{E}^*$ in Proposition 3.

Given $(\hat{p}_h)_{h=1}^3$, Pareto efficiency requires that:

$$x_1^0 = x_2^1 = \frac{1}{2}(2 - \varepsilon + \hat{p}_1\varepsilon), \tag{11}$$

$$x_2^0 = x_2^1 = \frac{1}{2}\left[\varepsilon' + \hat{p}_2(2 - \varepsilon')\right], \text{ and} \tag{12}$$

$$x_3^0 = x_3^1 = \frac{1}{2}\left[1 + (\varepsilon - \varepsilon') + \hat{p}_3(1 - (\varepsilon - \varepsilon'))\right]. \tag{13}$$

Thus, if forecasts $(\hat{p}_h)_{h=1}^3$ satisfy (10), then the efficient allocation constructed as above is feasible in period 1, too.

To finish the construction of an ETE, we only need to find an actual price $p \geq 0$ of period 1 satisfying the realized budget for all (11), (12) and (13) which means the period 1 consumption is utility maximizing. That is, we need to find $p$ satisfying the followings:

$$\frac{1}{2}(2 - \varepsilon + \hat{p}_1\varepsilon)(1 + p) = 2 - \varepsilon + p\varepsilon,$$

$$\frac{1}{2}\left[\varepsilon' + \hat{p}_2(2 - \varepsilon')\right](1 + p) = \varepsilon' + p(2 - \varepsilon'), \text{ and}$$

$$\frac{1}{2}\left[1 + (\varepsilon - \varepsilon') + \hat{p}_3(1 - (\varepsilon - \varepsilon'))\right](1 + p) = 1 + (\varepsilon - \varepsilon') + p(1 - (\varepsilon - \varepsilon')).$$

Then, we have

$$\hat{p}_1 = \frac{2 - \varepsilon}{\varepsilon}\frac{1 - p}{1 + p} + \frac{2p}{1 + p}, \tag{14}$$

$$\hat{p}_2 = \frac{\varepsilon'}{2 - \varepsilon'}\frac{1 - p}{1 + p} + \frac{2p}{1 + p}, \text{ and} \tag{15}$$

$$\hat{p}_3 = \frac{1 + (\varepsilon - \varepsilon')}{1 - (\varepsilon - \varepsilon')}\frac{1 - p}{1 + p} + \frac{2p}{1 + p}. \tag{16}$$

Since $\hat{p}_1, \hat{p}_2, \hat{p}_3 > 0$ and $p \geq 0$, we have

$$0 \leq p < \frac{2-\varepsilon}{2-3\varepsilon} \quad \text{if } 0 < \varepsilon < \frac{2}{3} \text{ and } -1 < \varepsilon - \varepsilon' \leq \frac{1}{3}, \tag{17}$$

$$0 \leq p < \min\left(\frac{2-\varepsilon}{2-3\varepsilon}, \frac{1+(\varepsilon-\varepsilon')}{3(\varepsilon-\varepsilon')-1}\right) \quad \text{if } 0 < \varepsilon < \frac{2}{3} \text{ and } \frac{1}{3} < \varepsilon - \varepsilon' < 1, \tag{18}$$

$$p \geq 0 \quad \text{if } \frac{2}{3} < \varepsilon < 1 \text{ and } -1 < \varepsilon - \varepsilon' \leq \frac{1}{3}, \tag{19}$$

$$0 \leq p < \frac{1+(\varepsilon-\varepsilon')}{3(\varepsilon-\varepsilon')-1} \quad \text{if } \frac{2}{3} \leq \varepsilon < 1 \text{ and } \frac{1}{3} < \varepsilon - \varepsilon' < 1. \tag{20}$$

In conclusion, we have an ETE $\left((x_h^0, x_h^1)_{h=1}^H, (\hat{p}_h)_{h=1}^H, p\right)$ where the actual price of period 1, $p$, satisfies (17) - (20), the anticipated prices, $(\hat{p}_h)_{h=1}^H$, satisfy (14) - (16), and the ETE allocation, $(x_h^0, x_h^1)_{h=1}^H$, satisfies (11) - (13). In particular, we know that $(x_h^0, x_h^1)_{h=1}^H$ is the Arrow-Debreu allocation if and only if $\hat{p}_1 = \hat{p}_2 = \hat{p}_3 = p = 1$. This confirms that the Arrow-Debreu allocation can only be supported as a PFE. Note that the Arrow-Debreu equilibrium price $p^* = 1$ is included in each price interval of (17) - (20). Thus, combining (7) and (14), (8) and (15), and (9) and (16) respectively, for each price interval of (17) - (20), we have a one dimensional set of ETE allocations which includes the Arrow-Debreu allocation, corresponding to the PFE.

# 5 Concluding Remarks

## 5.1 Welfare gains and losses

Since an ETE is Pareto efficient, in comparison with a PFE in the vicinity, some households are better off while some households are worse off. It is then interesting to ask how the quality of price forecasts affects the welfare.

At first sight, it might appear plausible to expect that a household with a relatively accurate forecast should be better off than another household with an inaccurate forecast; if one's forecast turns out to be almost correct, his saving decision must be almost optimal, and if not, it would be clearly suboptimal. So in an ETE $\left(x^*, r^*, (\hat{p}_h)_{h=1}^H, p^*\right)$ where $p^*$ is close to a PFE, if $|p_h - p^*| < |p_{h'} - p^*|$, and if household $h'$ is better off than in the PFE, then household $h$ should also be better off.

But this is not the case in general: "better" ex post forecasts does not necessarily lead to better outcomes for the households. To understand the reason, imagine that each households' utility function has the form $v_h\left(x^0\right)+v_h\left(x^1\right)$ and the total endowments are the same in both periods. Then at an efficient allocation, the consumption must be the same in both periods. Then if a household happens to consume more than the amount in a PFE in the first period, then she is better off than in the PFE. So those who forecast high prices in the second period, and thereby consume more in the first period, tend to enjoy the advantage of this consumption smoothing effect. The accuracy of forecasts does not play any role in this logic, and one would "want" to forecast a high inflation rate, if possible.

Of course, since the dimension of ETE allocations is only one, the forecasts must be coordinated to a great extent accordingly, and so the implication of the consumption smoothing effect above is delicate and we are unable to deliver any general characterization result for welfare gains and losses. In order to confirm that accuracy does not necessarily imply welfare gain in a reasonable environment, we shall examine the example of Section 4 from this point of view.

Consider the special case where $\varepsilon=\varepsilon'=\frac{1}{2}$, and regard the ETE's as being parametrized by the second period market clearing price $p$. The PFE (the Arrow Debreu equilibrium) obtains at $p=p^*=1$. Recall that $\hat{p}_3=1$ at any ETE. By direct computation, we get from (14) that $\frac{d\hat{p}_1(p^*)}{dp}=-1$, and from (15) that $\frac{d\hat{p}_2(p^*)}{dp}=\frac{1}{3}>0$. That is, the ETE forecast of household 2 is less sensitive to $p$ than that of household 1, and so if $p$ is close to $p^*$, the former is closer to $p$ than the latter. On the other hand from (11) and (12), we have that $\frac{\partial x_h^{0*}}{\partial \hat{p}_h}>0$, $h=1,2$, conforming with the consumption smoothing effect. Then, if $p$ decreases slightly from $p^*$, then household 2 has a more accurate forecast $\hat{p}_2$, but since $\hat{p}_2$ is smaller than $p^*$, household 2 is worse off than in the PFE. On the other hand, $\hat{p}_1$ increases, and hence household 1 is better off than in the PFE. That is, household 1, whose forecast is farther away from $p$, is rewarded at the expense of agent 2. For instance, at $p=0.8$, $\hat{p}_1=\frac{11}{9}$, $\hat{p}_2=\frac{25}{27}$, $x_1^0=x_1^1=\frac{19}{18}$ and $x_2^0=x_2^1=\frac{51}{54}$, confirming the finding.

### 5.2 Source of the indeterminacy and the role of futures markets

The observations about welfare gains and losses shed light on the source of indeterminacy of ETE. The indeterminacy occurs since an incorrect price forecast yields an unanticipated income from saving, which may be positive or negative. For instance, if $\hat{p}_h > p$, i.e., household $h$ over estimates "inflation", then the realized real return on saving in period 1, $(1+r)/p$, is higher than the anticipated real return on saving, $(1+r)/\hat{p}_h$, and consequently household receives an unanticipated positive (resp. negative) income transfer in effect if he saved (resp. borrowed) in the first period. As ETE forecasts move away from a PFE, this set of unanticipated income transfers takes the households to a different efficient allocation.

As we have discussed earlier, a PFE is an ETE with a common forecast. We assumed that there is no direct market for future consumption, but if there is an institutional arrangement which forces the household to believe in a common forecast in ETE, then a PFE will occur, i.e., the common forecast will be correct a fortiori.

A complete set of futures markets together can serve as such an institution. In our simple framework of one good, a desired futures contract corresponds to a claim which promises to deliver the good. More explicitly, suppose that in addition to the saving market, there is another market in period 0 where the household can trade a security which promises to deliver one unit of the good in the second period. Under normalization of $p^0 = 1$ and $r = 0$, say that the market price of the security is $\pi$. Assume that households should not allow a free lunch for themselves, i.e., they may engage in speculative trade to take advantage of an arbitrage opportunity in the markets, but they should not create such an opportunity for themselves by simply forecasting outrageous prices. Then, each household $h$ must forecast $\hat{p}_h = \pi$ in equilibrium. Indeed, if $\hat{p}_h > \pi$, household $h$ can profit by borrowing to buy this security, and if $\hat{p}_h < \pi$ household $h$ can profit by selling this security to save. Consequently, every household must forecast $\hat{p}_h = \pi$ in order not to generate free lunch for themselves, and then we have a PFE.

But interestingly, at an PFE, the futures market is redundant, since the saving market alone is enough to achieve the PFE. That is, had we known that a PFE will result, there is no need to operate and maintain the futures market in addition to the saving market, since the markets are already complete in the Arrow-Debreu sense. So our indeterminacy result points out the importance of a market for price discovery, even if such a market might appear unnecessary.[4] Our result on the existence of a one-dimensional set of

[4] Kajii (1997) gives a similar observation on redundant security markets. In his model, an inefficent

ETE allocations opens up the possibility that a price discovery process, which does not necessarily culminate in full coordination of forecasts, may nonetheless be successful in achieving Pareto efficiency

### 5.3 Future research and conjectures

In our formulation of ETE, the forecasts have been fixed and held independent of the period 0 market clearing process. This has been done for simplicity. It is possible to incorporate forecast functions in our formulation at the cost of some additional complexity. A second and more vital restriction is the restriction to single good in the period 1 economy. We conjecture that the one dimensional real indeterminacy will continue to obtain generically for the case of multiple goods in both periods, especially in period 1.

When there are multiple goods, it would be interesting to investigate the structure of ETE when the asset is a real asset, to understand if our indeterminacy result is a consequence of the indeterminacy of the real value of the nominal money. For instance, suppose that there is one real asset, which pays off one unit of each good. Then the markets are complete and so a PFE is necessarily an Arrow Debreu equilibrium. As discussed earlier, the forecasts must agree on the implied value of this asset by no arbitrage argument, which means that the sum of forecasted prices is the same across the households. But this condition still allows differences in forecasts about relative prices, which would generate a consumption allocation different from any PFE in period 0. Hence we conjecture the indeterminacy of ETE obtains even when the asset is a real asset.

Finally, the case of multiple periods is obviously of great interest. The idea of ETE can be readily extended to accommodate a long sequence, even an infinite sequence of markets. On one hand, having multiple saving markets, there are more opportunities of unanticipated income transfers we mentioned before, which suggests that the degree of indeterminacy might increase with the number of time periods. On the other hand, since the dimension of efficient allocation does not change over time, the logic similar to the two period case might prevail, so that the degree of indeterminacy is still one.

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competitive equilibrium may arise because of sunspots, but a competitive equilibrium is efficient when there are options markets, which are redundant in equilibrium.

# Appendix

## Proof of Lemma 4:

Recall that the set of regular economies, $\mathcal{E}_R \subseteq \left(\mathbb{R}^2_{++}\right)^H$, is a generic set, and for each $e \in \mathcal{E}_R$, there are finitely many Arrow-Debreu equilibria, and the derivative of the market excess demand function is non zero at every equilibrium price. That is, for each $e \in \mathcal{E}$, if $\sum_{h=1}^{H} x_h^0\left(\hat{p}_h; e_h\right) = 0$ where $\hat{p}_h = p$ for all $h$, then $\sum_{h=1}^{H} \frac{\partial}{\partial \hat{p}_h} x_h^0\left(\hat{p}_h; e_h\right) \neq 0$.

We first establish the openness of $\mathcal{E}_{SR}$. Since $\mathcal{E}_R$ is open, it suffices to show that $\mathcal{E}_{SR}$ is open in $\mathcal{E}_R$. Suppose $e^n$ is a convergent sequence of regular economies such that $e^n \to \bar{e} \in \mathcal{E}_R$. Suppose $e^n \notin \mathcal{E}_{SR}$ holds for all sufficiently large $n$, i.e., there is an Arrow-Debreu equilibrium $p^n$ of $e^n$ which is not strongly regular. We want to show $\bar{e} \notin \mathcal{E}_{SR}$. Fix a neighborhood $V$ of $\bar{e}$ such that $V$'s closure in $\left(\mathbb{R}^2_{+}\right)^H$ is compact. By the standard properness argument, the set $\{(e,p) : p$ is an Arrow-Debreu equilibrium of $e\}$ is compact, so we might as well assume that $p^n$ converges to $\bar{p} > 0$. Since $p^n$ is not strongly regular, for some household $h$, $\frac{\partial}{\partial \hat{p}_h} x_h^0\left(\hat{p}_h; e_h^n\right) = 0$ at $\hat{p}_h = p^n$. Since there are only finitely many households, we can find a subsequence of $p^n$ for which there exists a household $h$ such that $\frac{\partial}{\partial \hat{p}_h} x_h^0\left(\hat{p}_h; e_h^n\right) = 0$ at $\hat{p}_h = p^n$ along the subsequence. Then by continuity $\frac{\partial}{\partial \hat{p}_h} x_h^0\left(\hat{p}_h; e_h\right) = 0$ must hold at $\hat{p}_h = \bar{p}$, so $\bar{p}$ is not a strongly regular Arrow-Debreu equilibrium of $\bar{e}$. Therefore, $\bar{e} \notin \mathcal{E}_{SR}$ as we wanted.

We show that $\mathcal{E}_R \backslash \mathcal{E}_{SR}$ is a measure zero set: Fix a household $h$, and consider the following system of equations:

$$\Psi_h\left(p; e\right) = \left[\begin{array}{c} \sum_{h=1}^{H}\left(x_h^0\left(p; e_h\right) - e_h^0\right) \\ \frac{\partial}{\partial \hat{p}_h} x_h^0\left(p; e_h\right) \end{array}\right] \in \mathbb{R}^2$$

and let

$$\bar{\mathcal{E}}_h^* = \left\{e \in \mathcal{E}_R : \text{ there is } p \text{ such that } \Psi_h\left(p; e\right) = 0\right\}.$$

Notice that $\mathcal{E}_R \backslash \mathcal{E}_{SR} = \cup_{h=1}^{H} \bar{\mathcal{E}}_h^*$, so we need to show that $\cup_{h=1}^{H} \bar{\mathcal{E}}_h^*$ is a measure zero set, for which it suffices to show $\bar{\mathcal{E}}_h^*$ is a measure zero subset of $\mathcal{E}$ for every $h$. Without loss of generality, we concentrate on the case $h = 1$.

Since $\Psi_1\left(\cdot; e\right)$ is a function of one variable, if $0 \in \mathbb{R}^2$ is a regular value of $\Psi_1\left(\cdot; e\right)$, then there cannot be any $p$ such that $\Psi_1\left(p; e\right) = 0$, i.e., such $e$ does not belong to $\bar{\mathcal{E}}_1^*$. By the transversality theorem,[5] if 0 is a regular value of $\Psi_1$, i.e., there is a $2 \times 2$ submatrix of $D\Psi_1$ whose rank is 2, then the set of $e \in \mathcal{E}_R$ for which 0 is not a regular value of $\Psi_1(\cdot; e)$ is of zero measure. The proof is therefore completed if we show that there is a $2 \times 2$ submatrix of $D\Psi_1$ of rank 2 whenever $\Psi_1(p; e) = 0$.

Now suppose $\Psi_1(p; e) = 0$. Since $e$ is a regular economy, $\sum_{h=1}^{H} \frac{\partial}{\partial \hat{p}_h} x_h^0(p; e_h) \neq 0$. So we need to show that there is a direction in the set of endowments such that the directional derivative of excess demand is zero while that of $\frac{\partial}{\partial \hat{p}_1} x_1^0(p; e_1)$ is non zero.

Specifically, choose a small number $t$ and consider $e_1(t) = \left(e_1^0 - tp, e_1^1 + t\right)$ and $e_2(t) = \left(e_2^0 + tp, e_2^1 - t\right)$, and $e_h(t) = e_h$ for other household $h$. By construction the total supplies are constant for all $t$. Notice that for any household $h$, the income level $e_h^0(t) + pe_h^1(t)$ is constant for all $t$, and so is its demand vector. Consequently, the market excess demand for good 0 is the same (i.e., equal to zero) for all $t$, and hence the directional derivative of excess demand is zero.

So it remains to verify $\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \hat{p}_1} x_1^0(p; e_1(t))\right) \neq 0$ evaluated at $t = 0$. First, notice that by construction $x_1^0(p; e_1) \equiv x_1^0(1, p, e_1^0 + pe_1^1)$ and the homogeneity of the demand function $x_1^0(p^0, p^1, m)$ at $(p^0, p^1, m) = (1, p, e_1^0 + pe_1^1)$, we have

$$\frac{\partial}{\partial \hat{p}_1} x_1^0(p; e_1) = \frac{\partial}{\partial p^1} x_1^0\left(1, p, e_1^0 + pe_1^1\right) + \frac{\partial}{\partial m} x_1^0\left(1, p, e_1^0 + pe_1^1\right) e_1^1, \text{ and} \tag{21}$$

$$1\frac{\partial}{\partial p^0} x_1^0(1, p, e_1^0 + pe_1^1) + p\frac{\partial}{\partial p^1} x_1^0(1, p, e_1^0 + pe_1^1) + (e_1^0 + pe_1^1)\frac{\partial}{\partial m} x_1^0(1, p, e_1^0 + pe_1^1) = 0. \tag{22}$$

Second, we claim $\frac{\partial}{\partial m} x_1^0\left(1, p, e_1^0 + pe_1^1\right) \neq 0$. Suppose not, i.e., $\frac{\partial}{\partial m} x_1^0\left(1, p, e_1^0 + pe_1^1\right) = 0$. As $\Psi_1(p; e) = 0$, we know $\frac{\partial}{\partial \hat{p}_1} x_1^0(p; e_1) = 0$. Then, (21) implies $\frac{\partial}{\partial p^1} x_1^0\left(1, p, e_1^0 + pe_1^1\right) = 0$. Consequently, (22) implies $\frac{\partial}{\partial p^0} x_1^0(1, p, e_1^0 + pe_1^1) = 0$. It then means that the derivative of $x_1^0$ vanishes completely, which is impossible under our assumptions about the utility function. Therefore, $\frac{\partial}{\partial m} x_1^0\left(1, p, e_1^0 + pe_1^1\right) \neq 0$.

Now, since the income is invariant of $t$, $\frac{\partial}{\partial p^1} x_1^0\left(1, p, e_1^0(t) + pe_1^1(t)\right) = \frac{\partial}{\partial p^1} x_1^0\left(1, p, e_1^0 + pe_1^1\right)$ and $\frac{\partial}{\partial m} x_1^0\left(1, p, e_1^0(t) + pe_1^1(t)\right) = \frac{\partial}{\partial m} x_1^0\left(1, p, e_1^0 + pe_1^1\right)$ holds for all $t$. These mean that

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial p^1} x_1^0\left(1, p, e_1^0(t) + pe_1^1(t)\right)\right) = 0, \text{ and} \tag{23}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial m} x_1^0\left(1, p, e_1^0(t) + pe_1^1(t)\right)\right) = 0. \tag{24}$$

[5] See, for instance, Chapter 1 of Mas-Colell's book.

Therefore,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \hat{p}_1}x_1^0\left(p;e_1(t)\right)\right)\\
&=\frac{\partial}{\partial t}\left(\frac{\partial}{\partial p^1}x_1^0\left(1,p,e_1^0\left(t\right)+pe_1^1\left(t\right)\right)+\frac{\partial}{\partial m}x_1^0\left(1,p,e_1^0\left(t\right)+pe_1^1\left(t\right)\right)e_1^1\left(t\right)\right) && \text{by (21)}\\
&=\frac{\partial}{\partial t}\left(\frac{\partial}{\partial m}x_1^0\left(1,p,e_1^0(t)+pe_1^1(t)\right)e_1^1(t)\right) && \text{by (23)}\\
&=\frac{\partial}{\partial t}\left(\frac{\partial}{\partial m}x_1^0\left(1,p,e_1^0\left(t\right)+pe_1^1\left(t\right)\right)\right)e_1^1\left(t\right)+\frac{\partial}{\partial m}x_1^0\left(1,p,e_1^0\left(t\right)+pe_1^1\left(t\right)\right)\left(\frac{\partial}{\partial t}e_1^1\left(t\right)\right)\\
&=\frac{\partial}{\partial m}x_1^0\left(1,p,e_1^0\left(t\right)+pe_1^1\left(t\right)\right)\left(\frac{\partial}{\partial t}e_1^1\left(t\right)\right) && \text{by (24)}\\
&=\frac{\partial}{\partial m}x_1^0\left(1,p,e_1^0+pe_1^1\right) && \left(\text{because } e_1^1\left(t\right)=e_1^1+t\right)\\
&\neq 0,
\end{aligned}
$$

which completes the proof.

## Proof of Lemma 5

Keep in mind $\frac{\partial x_h^0}{\partial p}=0$, since $x_h^0$ is determined by $\hat{p}_h$, independently of $p$. The realized consumption $x_h^1$ is determined by (5) and hence we always have

$$
\frac{\partial x_h^1}{\partial p}=\frac{x_h^0-e_h^0}{p^2} \quad \text{and} \quad \frac{\partial x_h^1}{\partial \hat{p}_h}=-\frac{1}{p}\frac{\partial}{\partial \hat{p}_h}x_h^0(\hat{p}_h;e_h).
$$

Using these relations, by direct computation, we obtain

$$
\partial\Phi/\partial\left(\cdots,\hat{p}_h,\cdots,\lambda\right)=\left.\begin{bmatrix}
\cdots & \frac{\partial}{\partial \hat{p}_h}x_h^0\left(\hat{p}_h;e_h\right) & \cdots & 0\\
\ddots & & 0 & \vdots\\
& \zeta_h & & \frac{\partial u_h}{\partial x^0}\left(x_h^0\left(\hat{p}_h;e_h\right),x_h^1\right)\\
0 & & \ddots & \vdots
\end{bmatrix}\right|_{\substack{p=p^*\\ \hat{p}_1=\cdots=\hat{p}_H=p^*\\ \lambda=p^*}}
$$

where

$$
\zeta_h=\frac{\partial}{\partial \hat{p}_h}\left[\lambda\frac{\partial u_h}{\partial x^0}\left(x_h^0\left(\hat{p}_h;e_h\right),x_h^1\right)-\frac{\partial u_h}{\partial x^1}\left(x_h^0\left(\hat{p}_h;e_h\right),x_h^1\right)\right]
$$

Now,

$$\zeta_h = \frac{\partial}{\partial \hat{p}_h}\left[\lambda \frac{\partial u_h}{\partial x^0}\left(x_h^0\left(\hat{p}_h; e_h\right), x_h^1\right) - \frac{\partial u_h}{\partial x^1}\left(x_h^0\left(\hat{p}_h; e_h\right), x_h^1\right)\right]_{|\, p=\hat{p}_h=\lambda=p^*}$$

$$= p^*\left[\frac{\partial^2 u_h}{(\partial x^0)^2}\frac{\partial}{\partial \hat{p}_h}x_h^0(\hat{p}_h; e_h) - \frac{1}{p^*}\frac{\partial}{\partial \hat{p}_h}x_h^0(\hat{p}_h; e_h)\frac{\partial^2 u_h}{\partial x^0 \partial x^1}\right]$$

$$- \left[\frac{\partial^2 u_h}{\partial x^1 \partial x^0}\frac{\partial}{\partial \hat{p}_h}x_h^0(\hat{p}_h; e_h) - \frac{1}{p^*}\frac{\partial}{\partial \hat{p}_h}x_h^0(\hat{p}_h; e_h)\frac{\partial^2 u_h}{(\partial x^1)^2}\right]$$

$$= \left[p^*\left(\frac{\partial^2 u_h}{(\partial x^0)^2} - \frac{1}{p^*}\frac{\partial^2 u_h}{\partial x^0 \partial x^1}\right) - \left(\frac{\partial^2 u_h}{\partial x^0 \partial x^1} - \frac{1}{p^*}\frac{\partial^2 u_h}{(\partial x^1)^2}\right)\right]\frac{\partial}{\partial \hat{p}_h}x_h^0(\hat{p}_h; e_h)$$

$$= \left[p^*\frac{\partial^2 u_h}{(\partial x^0)^2} - 2\frac{\partial^2 u_h}{\partial x^0 \partial x^1} - \frac{1}{p^*}\frac{\partial^2 u_h}{(\partial x^1)^2}\right]\frac{\partial}{\partial \hat{p}_h}x_h^0(\hat{p}_h; e_h),$$

and thus we have obtained the desired expression.

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