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A New Control Variate Estimator for an Asian Option

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Abstract. There exist several estimators for valuing the Asian option on the arithmetic mean. Among all variance reduction estimators, the one with the control variate derived from the geometric mean has been shown by Boyle et al. (1997) to perform best so far. In this paper, a new improved control variate estimator for this type of Asian option is proposed and investigated. Simulation results confirm that it does perform better than the control variate derived from the geometric mean. The improvement becomes more significant as the volatility increases and/or as the time to expiration lengthens.

Key words: control variate estimator, variance reduction technique, Monte-Carlo simulation, option pricing

1. Introduction

Several variance reduction methods such as the antithetic variate (AV) method, the control variate (CV) method, and the moment matching (MM) method have been proposed for valuing derivatives. With an extensive overview, Boyle et al. (1997) (shortened as BBG henceforth) compared the performances of these variance reduction methods to that of the traditional Monte Carlo (MC) method. Among other findings, they showed by simulation that the CV method is the most efficient for valuing an Asian call option, C^A, which is difficult to value analytically in practice. This is deduced from the observation that the sample variance under CV method is the smallest among the four sample variances shown in Table 2 of BBG. The CV method considered in BBG is to improve upon the MC estimator (MCE) for the Asian call by choosing the analytical variate of the call on the geometric mean, C^L, as the control variate (see Turnbull and Wakeman (1991) for a closed-form solution for this price). This call on the geometric mean yields a formula similar to the Black-Scholes formula for the European call, and thus can be computed for each volatility, interest rate and time to expiration. Most relevant references are found in BBG.[1]

In this paper, we propose a new optimal CV estimator (CVE) for the Asian call. We show, by both theoretical arguments and simulation results, that it is better than the CV estimator treated in BBG. Our estimator is motivated by the fact,

$$C_T^L < C_T^A \quad \text{at time } T, \quad \text{and hence} \quad C_0^L < C_0^A \quad \text{at time } 0. \tag{1.1}$$

Here $C_t^\dagger$ indicates the value of the call C of type † at time t, and T is the time of expiration. In other words, the value C_0^L of the call on the geometric mean at 0 gives a lower bound for the value C_0^A of the Asian call at 0. Hence we employ an additional variate C^U such that $C_T^L < C_T^A < C_T^U$, and use two control variates C^L and C^U to improve upon the CVE with C^L alone. We show that using two control variates C^L and C^U is much better than using C^L alone. The performance of an estimator is measured here in terms of its sample standard deviation. The smaller standard deviation of an estimator implies more stability and credibility of the estimator under each simulation run. We observe that the improvement of our estimator is far more significant for certain values of the time to expiration T, volatility σ and moneyness S_0/K. Further, for comparison, we consider a suboptimal CVE with C^U alone and the CVE with the optimal combination of C^L and C^U as a control variate estimator. The latter is better than the CVE with C^L alone, while the former does not necessarily outperform it.

2. Optimal CVE

To describe our CVE, we begin by introducing some notations and background material. We denote the payoff of the Asian call at T by

$$C_N^A = \max\{\bar{S}_N - K, 0\}, \tag{2.1}$$

where $\bar{S}_N = \frac{1}{N}\sum_{n=1}^N S_n$, and K is the strike price. Here $S_n = S(nh)$ is the n-th price observed in time frequency h (year), $Nh = T$, and $S(t)$ is the price at the time t of an underlying asset (e.g., stock). The price $S(t)$ is assumed to follow the geometric Brownian motion, i.e.

$$S(t) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right], \tag{2.2}$$

(under the risk neutral measure), where $W(t)$ is a Wiener process, r is the interest rate and σ is the volatility. Both r and σ are assumed to be constants. Note that (2.2) implies

$$S_n = S_{n-1}\exp\left[\left(r - \frac{1}{2}\sigma^2\right)h + \sigma\sqrt{h}\,\epsilon_n\right], \tag{2.3}$$

where $\epsilon_n \sim i.i.d.\mathrm{N}(0, 1)$. Consequently, the Asian call with payoff (2.1) at $T = Nh$ is valued at $t = nh$ by

$$C_n^A = \exp[-r(N-n)h]\mathrm{E}_n\big[C_N^A\big], \tag{2.4}$$

where $\mathrm{E}_n(Q)$ is the conditional expectation of Q under the model (2.3).

Without loss of generality, we treat only the case $n = 0$ in valuing C_n^A. The treatment applies to any given n. Let θ denote the value of C_0^A, i.e.,

$$\theta \equiv \theta(\boldsymbol{Z}) = \exp(-rNh)\mathrm{E}_0\big[C_N^A\big], \tag{2.5}$$

with

$$\boldsymbol{Z} = (S_0/K, \sigma, N, r). \tag{2.6}$$

Since it is impossible to evaluate θ analytically for a given $\boldsymbol{Z}$, θ is often estimated by the MC method. We denote this estimator by $\hat{\theta}^{MC}$. The so-called CVE, which often improves upon $\hat{\theta}^{MC}$ in the sense of reducing the sample variance, is an estimator of the following form

$$\hat{\theta}^{CV} = \hat{\theta}^{MC} + \lambda(\hat{\theta}^* - \theta^*). \tag{2.7}$$

Here $\theta^* = \mathrm{E}_0[f(S_1, \dots, S_N)]$ for some function f which is a known analytically computable function of $\boldsymbol{Z}$, and $\hat{\theta}^*$ is the MCE of θ^* computed by the same run as the run of $\hat{\theta}^{MC}$. Since in each path of simulation, $\hat{\theta}^*$ is an unbiased estimator for θ^*, i.e., $\mathrm{E}_0[\hat{\theta}^*] = \theta^*$, $\hat{\theta}^{CV}$ in (2.7) is also an unbiased estimator of θ, i.e.

$$\mathrm{E}_0[\hat{\theta}^{CV}] = \mathrm{E}_0[\hat{\theta}^{MC}] = \theta.$$

Consider now the variance of $\hat{\theta}^{CV}$. Evidently, it is minimized when

$$\lambda = -\mathrm{Cov}(\hat{\theta}^{MC}, \hat{\theta}^*)/\mathrm{Var}(\hat{\theta}^*) = -\frac{\mathrm{Cov}(\hat{\theta}^{MC}, \hat{\theta}^*)}{\sqrt{\mathrm{Var}(\hat{\theta}^{MC})\mathrm{Var}(\hat{\theta}^*)}}\sqrt{\frac{\mathrm{Var}(\hat{\theta}^{MC})}{\mathrm{Var}(\hat{\theta}^*)}}.$$

In this case, the attained minimum variance of $\hat{\theta}^{CV}$, denoted by $\mathrm{Var}(\hat{\theta}^{CV})$, turns out to be $\mathrm{Var}(\hat{\theta}^{MC})[1-\rho^2]$, where ρ is the correlation coefficient between $\hat{\theta}^{MC}$ and $\hat{\theta}^*$. Since $0 \le \rho^2 \le 1$, we obtain $\mathrm{Var}(\hat{\theta}^{CV}) \le \mathrm{Var}(\hat{\theta}^{MC})$. Note that λ is also analytically intractable as a function of $\boldsymbol{Z}$, and is often chosen to be -1 in applications. Clearly, the choice $\lambda = -1$ is particularly appropriate if ρ is close to 1 and $\mathrm{Var}(\hat{\theta}^{MC})$ is close to $\mathrm{Var}(\hat{\theta}^*)$, which turns out to be our case here. Under such a circumstance, $\mathrm{Var}(\hat{\theta}^{CV}) < \mathrm{Var}(\hat{\theta}^{MC})$ remains true provided that $\rho > 0$.

In the class of CVE's in (2.7), BBG considered the control variate θ^* to be the value at time 0 of the call on the geometric mean. The payoff of this call at N is given by

$$C_N^L = \max\{S_N^* - K, 0\},$$

where

$$S_N^* = \left[\prod_{n=1}^{N} S_n\right]^{1/N}. \tag{2.8}$$

Specifically, the control variate θ^* is taken to be the θ^L defined as follows,

$$\theta^L \equiv \theta^L(\mathbf{Z}) = \exp(-rNh)\mathrm{E}_0\big[C_N^L\big]. \tag{2.9}$$

In this case, the CV estimator in (2.7) can be identified as

$$\hat{\theta}^{BBG} = \hat{\theta}^{MC} - (\hat{\theta}^{LMC} - \theta^L). \tag{2.10}$$

The estimator $\hat{\theta}^{BBG}$ is shown in BBG (1997) to be the best among all estimators considered in that paper (cf. Table 2 of BBG).

Before we describe our estimator, we note the following for later reference:

$$\begin{aligned}\theta^L = S_0 \exp\left\{-\frac{1}{2}\left[r + \frac{\sigma^2}{6}\left(1+\frac{1}{N}\right)\right]Nh\left(1-\frac{1}{N}\right)\right\}\Phi(d)\\ -Ke^{-rNh}\Phi(d-b),\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}d &= \frac{1}{b}\left[\log(S_0/K) + \frac{1}{2}\left(r - \frac{\sigma^2}{2}\right)Nh\left(1+\frac{1}{N}\right)\right.\\ &\quad\left.+\frac{1}{3}\sigma^2 Nh\left(1+\frac{1}{N}\right)\left(1+\frac{1}{2N}\right)\right],\\ b &= \sigma\left[\frac{Nh}{3}\left(1+\frac{1}{N}\right)\left(1+\frac{1}{2N}\right)\right]^{1/2}.\end{aligned}$$

We also note that $C_N^L < C_N^A$, and thus $\theta^L < \theta$ with probability 1, implying that θ^L forms a lower bound of θ.

To improve upon $\hat{\theta}^{BBG}$, we begin with choosing an upper bound θ^U of θ which is a function of $\boldsymbol{Z}$ in (2.6). Since $f(x) = \max\{x, 0\}$ is a convex function of x, by Jensen's inequality,

$$C_N^A < C_N^U \equiv \frac{1}{N}\sum_{n=1}^{N} \max\{S_n - K, 0\}, \tag{2.12}$$

which implies

$$\begin{aligned} \theta < \theta^U &\equiv \exp(-rNh)\mathrm{E}_0\big[C_N^U\big], \\ &= \exp(-rNh)\frac{1}{N}\sum_{n=1}^{N} BS(nh)\exp(rnh), \end{aligned} \tag{2.13}$$

where $BS(nh)$ is the Black-Scholes values of the European call with the strike price K and the time to expiration nh. We choose θ^U in (2.13) as our analytically tractable upper bound for θ and form the following control variate estimator:

$$\hat{\theta}^{OPT}(\boldsymbol{\beta}) = \hat{\theta}^{MC} - \beta^L(\hat{\theta}^{LMC} - \theta^L) - \beta^U(\hat{\theta}^{UMC} - \theta^U),$$

where $\boldsymbol{\beta} = [\beta^L, \beta^U]'$ and $\hat{\theta}^{UMC}$ is the MCE of θ^U. Clearly this is a general form of a CVE with two control variates θ^L and θ^U which includes θ^{CV} in (2.7) as the case with $\boldsymbol{\beta} = [-\lambda, 0]'$ or $\boldsymbol{\beta} = [0, -\lambda]'$ and $\hat{\theta}^{BBG}$ in (2.10) as the case with $\boldsymbol{\beta} = [1, 0]'$. Therefore when $\boldsymbol{\beta}$ is optimally chosen, say $\boldsymbol{\beta}^*$, $\hat{\theta}^{OPT}(\boldsymbol{\beta}^*)$ should be better than $\hat{\theta}^{BBG}$. To choose $\boldsymbol{\beta}$ optimally, we minimize the variance of $\hat{\theta}^{OPT}(\boldsymbol{\beta})$ with respect to $\boldsymbol{\beta}$ to obtain

$$\boldsymbol{\beta}^* = \begin{bmatrix} \mathrm{Cov}(\hat{\theta}^{LMC}, \hat{\theta}^{LMC}) & \mathrm{Cov}(\hat{\theta}^{LMC}, \hat{\theta}^{UMC}) \\ \mathrm{Cov}(\hat{\theta}^{UMC}, \hat{\theta}^{LMC}) & \mathrm{Cov}(\hat{\theta}^{UMC}, \hat{\theta}^{UMC}) \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{Cov}(\hat{\theta}^{LMC}, \hat{\theta}^{MC}) \\ \mathrm{Cov}(\hat{\theta}^{UMC}, \hat{\theta}^{MC}) \end{bmatrix}. \tag{2.14}$$

Because $\boldsymbol{\beta}^*$ is analytically tractable, we estimate it in advance by generating $\hat{\theta}^{MC}$, $\hat{\theta}^{LMC}$, and $\hat{\theta}^{UMC}$ as we will discuss the details below.

Further for comparisons, we consider the following CVE's as special cases of $\hat{\theta}^{OPT}(\boldsymbol{\beta})$:

$$\hat{\theta}^{CVU} = \hat{\theta}^{MC} - (\hat{\theta}^{UMC} - \theta^U), \tag{2.15}$$

$$\hat{\theta}^{SUB}(\alpha) = \hat{\theta}^{MC} - \{\hat{\theta}^{LU}(\alpha) - \theta^{LU}(\alpha)\}, \tag{2.16}$$

where

$$\hat{\theta}^{LU}(\alpha) = \alpha\hat{\theta}^{LMC} + (1-\alpha)\hat{\theta}^{UMC} \quad \text{and} \quad \theta^{LU}(\alpha) = \alpha\theta^L + (1-\alpha)\theta^U.$$

Clearly $\hat{\theta}^{CVU}$ is the CVE with the upper control variate. The reason why we consider (2.16) is to demonstrate the role of the upper control variate for an improvements on BBG. In fact, for a given α, $\hat{\theta}^{LU}(\alpha)$ is regarded as a single control variate for the MCE $\hat{\theta}^{MC}$ though it is a combination of $\hat{\theta}^{LMC}$ and $\hat{\theta}^{UMC}$ and it includes the control variate $\hat{\theta}^{LMC}$ for $\hat{\theta}^{BBG}$ when $\alpha = 1$. Hence when α is optimally chosen, say α^*, then $\hat{\theta}^{SUB}(\alpha^*)$ is better than $\hat{\theta}^{BBG}$ and the improvement corresponds to the role of $\hat{\theta}^{UMC}$. The optimal α is obtained by minimizing the variance of $\hat{\theta}^{SUB}(\alpha)$ with respect to α and is given as

$$\alpha^* = (D_2 + D_3)\,/\,(D_1 + 2D_2 + D_3)\,, \tag{2.17}$$

where

$$D_1 = \mathrm{Var}(\hat{\theta}^{MC} - \hat{\theta}^{LMC}),$$
$$D_2 = \mathrm{Cov}(\hat{\theta}^{MC} - \hat{\theta}^{LMC}, \hat{\theta}^{UMC} - \hat{\theta}^{MC}),$$
$$D_3 = \mathrm{Var}(\hat{\theta}^{UMC} - \hat{\theta}^{MC}).$$

The minimized variance of $\hat{\theta}^{SUB}(\alpha)$ is

$$\mathrm{Var}[\hat{\theta}^{SUB}(\alpha^*)] = \big(D_1D_3 - D_2^2\big)/(D_1 + 2D_2 + D_3). \tag{2.18}$$

Note that $\mathrm{Var}[\hat{\theta}^{SUB}(\alpha^*)]$ decreases if and only if $D_2^2/(D_1D_3)$ increases, where $D_2^2/(D_1D_3)$ is just the squared correlation between $(\hat{\theta}^{UMC} - \hat{\theta}^{MC})$ and $(\hat{\theta}^{MC} - \hat{\theta}^{LMC})$. Clearly, $\hat{\theta}^{SUB}(1) = \hat{\theta}^{BBG}$. Otherwise, the smaller α^* is, the more improvement $\hat{\theta}^{SUB}(\alpha^*)$ achieves over $\hat{\theta}^{BBG}$. Since α^* is difficult to evaluate analytically, we again estimate it in advance and carry out the analysis based on this prefixed value.

Now we discuss on how to estimate α^* and $\boldsymbol{\beta}^*$ in advance. They are estimated based on J_0 independent samples of

$$\hat{\boldsymbol{\theta}}_{(j)} = \big(\hat{\theta}_{(j)}^{LMC}, \hat{\theta}_{(j)}^{MC}, \hat{\theta}_{(j)}^{UMC}\big), \quad \text{for } j = 1, \ldots, J_0, \tag{2.19}$$

and each $\hat{\boldsymbol{\theta}}_{(j)}$ is computed on I_0 common paths

$$\{S_{1(i)}, S_{2(i)}, \ldots, S_{N(i)}\}, \quad \text{for } i = 1, \ldots, I_0,$$

where the paths are generated by the MC method. Clearly the covariances in (2.14) and D_i $(i = 1, 2, 3)$ in (2.17) are estimated with the simulated samples in (2.19) in the usual manner. Note that all the samples to estimate α^* and $\boldsymbol{\beta}^*$ should be independent of the MC analysis in the sequel; otherwise, the validity of (2.14) or (2.17) is no longer guaranteed. Once we estimate α^* and $\boldsymbol{\beta}^*$ in advance, we can reuse them as many times as we wish. Therefore the computational time for estimation of α^* and $\boldsymbol{\beta}^*$ is a one-time cost. Readers may worry that errors in estimated α^* or $\boldsymbol{\beta}^*$ reduce the accuracy of $\hat{\theta}^{OPT}$ or $\hat{\theta}^{SUB}$. However, as we show in the next section, $\hat{\theta}^{OPT}$ and $\hat{\theta}^{SUB}$ are still better than the other MC estimators even though they have the extra source of error. In any cases, we can improve the accuracy of α^* and $\boldsymbol{\beta}^*$ by increasing J_0 and I_0 as much as possible.

Finally, we proceed to compare the five estimators $\hat{\theta}^{MC}$, $\hat{\theta}^{BBG}$, $\hat{\theta}^{CVU}$, $\hat{\theta}^{SUB}$, and $\hat{\theta}^{OPT}$ in terms of their sample standard deviations. Let d be an indicator of one of these five estimators. We examine the following

$$S^d = \sqrt{\frac{1}{J_1}\sum_{j=1}^{J_1}\left(\hat{\theta}^d_{(j)} - \bar{\hat{\theta}}^d\right)^2}, \quad (d = MC, BBG, CVU, SUB, OPT)$$

where $\hat{\theta}^d_{(j)}$ is the type d estimator under the j-th run and computed on I_1 sample paths which are generated by the MC method and fixed for $j = 1, \ldots, J_1$. In order to provide a fair comparison, the same set of the I_1 sample paths should be used to generate $\{\hat{\theta}^d_{(j)}\}_{j=1}^{J_1}$ for all estimators. The simulation results are summarized in the next section.

3. Simulation Results

In estimating α^* and $\boldsymbol{\beta}^*$ in advance, we take $J_0 = 10{,}000$ and $I_0 = 500$. In the subsequent Monte Carlo experiments, we take $J_1 = 10{,}000$, and $I_1 = 500$ or $1{,}000$, and $r = 0.05$ or 0.10. Here we report only the simulation results for the case $I_1 = 500$ and $r = 0.05$, since the results are almost identical in the case $I_1 = 1{,}000$ or $r = 0.10$.

Tables I and II contain respectively the estimated values of α^* and $\boldsymbol{\beta}^* = [\beta^{L^*}, \beta^{U^*}]'$ for many settings determined by the three factors below,

$$\begin{aligned} S_0/K \text{ (moneyness)} &= 1.1,\ 1.0,\ 0.9 \\ N \text{ (days to expiration)} &= 30,\ 72,\ 90,\ 180,\ 270 \\ \sigma \text{ (volatility)} &= 0.1,\ 0.2,\ 0.4,\ 0.6,\ 0.9,\ 1.0 \end{aligned}$$

Recall that the optimal α^* and $\boldsymbol{\beta}^*$ are respectively computed following (2.17) and (2.14). They are functions of the moneyness S_0/K, the volatility σ, days to expiration N and the interest rate r, namely, $\alpha^* = \alpha^*(S_0/K, \sigma, N, r)$ and $\boldsymbol{\beta}^* = \boldsymbol{\beta}^*(S_0/K, \sigma, N, r)$.

Table III(A)–(C) summarize the comparison results among $\hat{\theta}^{MC}$, $\hat{\theta}^{BBG}$, $\hat{\theta}^{SUB}$, and $\hat{\theta}^{OPT}$ under three different types of moneyness, $S_0/K = 1.1, 1.0, 0.9$. In the second column of Table III, the sample standard deviation of $\hat{\theta}^{MC}$, S^{MC} is reported. From the third column to the sixth, the percentage ratios of S^d $(d = BBG, CVU, SUB, OPT)$ to S^{MC} are shown. The smaller the ratio is, the better

Table I. Optimal α's (10,000 runs for each 500-path-based estimate)

N	30	72	90	180	270
σ			**A.** $S_0/K = 1.1$		
0.1	0.14615	0.82099	0.86862	0.92544	0.93420
0.2	0.92919	0.96439	0.96856	0.97367	0.96355
0.4	0.97777	0.96588	0.95220	0.84881	0.72233
0.6	0.96826	0.88711	0.84099	0.58656	0.41524
0.9	0.89543	0.65135	0.55415	0.27697	0.15471
1.0	0.85970	0.56564	0.47496	0.20994	0.10837
			B. $S_0/K = 1.0$		
0.1	0.99684	0.99444	0.99505	0.98533	0.97865
0.2	0.98889	0.97613	0.97160	0.93926	0.90636
0.4	0.96206	0.91572	0.89324	0.77488	0.67580
0.6	0.91745	0.80542	0.76337	0.56880	0.42995
0.9	0.82798	0.61700	0.55085	0.30156	0.18792
1.0	0.79617	0.54673	0.47615	0.24656	0.14385
			C. $S_0/K = 0.9$		
0.1	N.A.[a]	0.99479	0.99103	0.96971	0.95295
0.2	0.98426	0.95610	0.94590	0.90601	0.87518
0.4	0.93497	0.87924	0.85949	0.77750	0.70079
0.6	0.88800	0.79810	0.76985	0.61310	0.48959
0.9	0.81283	0.64948	0.58808	0.37547	0.24157
1.0	0.77956	0.60132	0.53222	0.30202	0.18591

[a] S^{MC} is zero.

the estimator is. In these columns, the sample mean of simulated Asian call option prices, $\bar{\hat{\theta}}^d$, is also given in parentheses. In Table III(A)–(C), we observe the following:

(i) $\hat{\theta}^{OPT}$ dominates the others in the sense that $S^{OPT} \leq S^d$ ($d = MC, BBG, CVU, SUB$) in all cases;
(ii) $\hat{\theta}^{SUB}$ dominates $\hat{\theta}^{MC}$, $\hat{\theta}^{BBG}$, and $\hat{\theta}^{CVU}$;
(iii) $\hat{\theta}^{BBG}$ uniformly outperforms $\hat{\theta}^{MC}$ but is beaten by $\hat{\theta}^{CVU}$ when the volatility (σ) is high and days to expiration (N) is long;
(iv) The percentage ratio of S^{CVU} to S^{MC} is more than 100% when $S_0/K = 0.9$ and σ is low, i.e., $\hat{\theta}^{CVU}$ is worse than $\hat{\theta}^{MC}$ in those cases.

Figure 1(A)–(C) show normal probability plots of simulated $\hat{\theta}^{BBG}$, $\hat{\theta}^{SUB}$, and $\hat{\theta}^{OPT}$ under $S_0/K = 1.1, 1.0, 0.9$ with $(\sigma, N, r) = (0.4, 30, 0.05)$. In Figure 1(A)–(C), a steeper plot means a more accurate estimator. We omit $\hat{\theta}^{MC}$ and $\hat{\theta}^{CVU}$ because

Table II. Optimal **β**'s (10,000 runs for each 500-path-based estimate)

σ		30	72	90	180	270
			A. $S_0/K = 1.1$			
0.1	β^{L^*}	0.15096	0.81495	0.85762	0.90039	0.89616
	β^{U^*}	0.84917	0.18677	0.14470	0.10502	0.11246
	ν^a	1.00013	1.00172	1.00232	1.00541	1.00862
0.2	β^{L^*}	0.91669	0.93105	0.92759	0.90151	0.87961
	β^{U^*}	0.08555	0.07579	0.08126	0.11519	0.14267
	ν	1.00223	1.00683	1.00885	1.01670	1.02228
0.4	β^{L^*}	0.93076	0.88973	0.87725	0.81207	0.76498
	β^{U^*}	0.07979	0.13224	0.14816	0.22747	0.28351
	ν	1.01055	1.02197	1.02541	1.03955	1.04850
0.6	β^{L^*}	0.89637	0.84098	0.81421	0.70995	0.63098
	β^{U^*}	0.12390	0.19507	0.22614	0.34758	0.43521
	ν	1.02027	1.03605	1.04035	1.05753	1.06620
0.9	β^{L^*}	0.84546	0.74413	0.70233	0.54910	0.43923
	β^{U^*}	0.18834	0.30959	0.35769	0.52335	0.63667
	ν	1.03380	1.05372	1.06002	1.07245	1.07590
1.0	β^{L^*}	0.82501	0.70353	0.66530	0.49483	0.38278
	β^{U^*}	0.21284	0.35507	0.39956	0.58053	0.69388
	ν	1.03785	1.05861	1.06486	1.07536	1.07666
			B. $S_0/K = 1.0$			
0.1	β^{L^*}	0.98992	0.98228	0.97973	0.96632	0.95296
	β^{U^*}	0.01496	0.02575	0.02928	0.04696	0.06358
	ν^a	1.00489	1.00802	1.00901	1.01328	1.01654
0.2	β^{L^*}	0.97995	0.96497	0.96077	0.93572	0.91483
	β^{U^*}	0.02966	0.05046	0.05645	0.08904	0.11564
	ν	1.00961	1.01544	1.01722	1.02476	1.03046
0.4	β^{L^*}	0.95651	0.92593	0.91395	0.86059	0.80949
	β^{U^*}	0.06259	0.10430	0.11951	0.18591	0.24560
	ν	1.01910	1.03023	1.03346	1.04650	1.05509
0.6	β^{L^*}	0.93100	0.87805	0.85376	0.76444	0.69198
	β^{U^*}	0.09710	0.16601	0.19434	0.30052	0.38278
	ν	1.02809	1.04406	1.04810	1.06496	1.07476
0.9	β^{L^*}	0.88397	0.78996	0.75576	0.60830	0.48500
	β^{U^*}	0.15674	0.27244	0.31049	0.47412	0.59847
	ν	1.04071	1.06240	1.06624	1.08242	1.08347
1.0	β^{L^*}	0.86575	0.75891	0.71793	0.54518	0.42590
	β^{U^*}	0.17930	0.30825	0.35310	0.53747	0.65798
	ν	1.04505	1.06716	1.07102	1.08265	1.08388

[a] $\nu = \beta^{L^*} + \beta^{U^*}$.

(Continued on next page)

Table II. (*Continued*)

σ		30	72	90	180	270
				C. $S_0/K = 0.9$		
0.1	β^{L^*}	N.A.[b]	1.12408	1.10390	1.04963	1.03016
	β^{U^*}	N.A.	0.00283	0.00490	0.01731	0.02729
	ν^a	N.A.	1.12691	1.10880	1.06694	1.05745
0.2	β^{L^*}	1.06895	1.02970	1.02027	0.99492	0.97727
	β^{U^*}	0.00909	0.02574	0.03195	0.05508	0.07510
	ν	1.07804	1.05544	1.05221	1.05000	1.05237
0.4	β^{L^*}	1.00803	0.97729	0.96784	0.92564	0.88669
	β^{U^*}	0.03926	0.07389	0.08540	0.13754	0.18389
	ν	1.04728	1.05118	1.05324	1.06318	1.07058
0.6	β^{L^*}	0.97826	0.93593	0.91929	0.83754	0.76708
	β^{U^*}	0.07052	0.12498	0.14558	0.24129	0.32105
	ν	1.04878	1.06091	1.06487	1.07883	1.08813
0.9	β^{L^*}	0.93781	0.85845	0.82316	0.68085	0.56050
	β^{U^*}	0.11971	0.21816	0.25756	0.41263	0.53761
	ν	1.05752	1.07661	1.08072	1.09348	1.09811
1.0	β^{L^*}	0.91921	0.82932	0.78695	0.61745	0.49167
	β^{U^*}	0.14142	0.25084	0.29730	0.47885	0.60568
	ν	1.06063	1.08016	1.08424	1.09630	1.09735

[a] $\nu = \beta^{L^*} + \beta^{U^*}$. [b] S^{MC} is zero.

the normal probability plots of these two estimators are much flatter than the others and it is not informative to plot them in the same frame. The normal probability plots in Figure 1(A)–(C) indicate that the simulated Asian call option prices seem normally distributed. They also illustrate the substantial improvement of the OPT estimator over the BBG and SUB estimator. In particular, in cases of at-the-money ($S_0/K = 1.0$) and in-the-money ($S_0/K = 1.1$), the BBG and SUB estimators are almost identical but the OPT estimator still outperforms both estimators.

To give an insight into the results on Table III, we first observe the values of α^*'s under $r = 0.05$ in Table I and β^* in Table II, and compare performance of the five estimators under different moneyness, $S_0/K = 1.1, 1.0, 0.9$.

A. *In-the-money* ($S_0/K = 1.1$).

First, we examine estimated α^*'s in part (A) of Table I. In the three-dimensional space (σ, N, α^*), the function $\alpha^*(1.1, \sigma, N, 0.05)$ seems to form a mountain chain with maximum points (peaks) crossing nearly horizontally at $(\sigma, N) = \{(0.4, 30), (0.4, 72), (0.2, 90), (0.2, 180), (0.2, 270)\}$, as shown in the (A) part of Table I. Away from these peak points, the larger the deviation is, the smaller $\alpha^*(1.1, \sigma, N, 0.05)$ is. More specifically, α^* decreases towards either the north-west or the south-east

Table III. Comparisons among estimators

	A. $S_0/K = 1.1$				
		Ratio of the sample standard deviations in %			
σ	S^{MC}	S^{BBG}/S^{MC}	S^{CVU}/S^{MC}	S^{SUB}/S^{MC}	S^{OPT}/S^{MC}
			$N = 30$		
0.1	0.08385	0.38 (10.19192)	0.17 (10.19191)	0.15 (10.19191)	0.15 (10.19191)
0.2	0.16953	0.73 (10.19380)	4.31 (10.19380)	0.66 (10.19380)	0.62 (10.19381)
0.4	0.31882	1.43 (10.42395)	9.60 (10.42475)	1.43 (10.42397)	1.12 (10.42387)
0.6	0.43413	2.45 (11.12905)	10.09 (11.12870)	2.42 (11.12904)	1.69 (11.12906)
0.9	0.62847	3.98 (12.66971)	9.80 (12.67158)	3.87 (12.66991)	2.37 (12.67081)
1.0	0.67503	4.70 (13.25161)	10.24 (13.24953)	4.46 (13.25132)	2.74 (13.25087)
			$N = 72$		
0.1	0.12635	0.59 (10.44830)	1.55 (10.44827)	0.51 (10.44829)	0.48 (10.44829)
0.2	0.25580	1.09 (10.49983)	7.55 (10.49943)	1.06 (10.49982)	0.91 (10.49978)
0.4	0.45856	2.56 (11.37938)	10.01 (11.37849)	2.55 (11.37935)	1.67 (11.37940)
0.6	0.61765	4.39 (12.92983)	9.92 (12.92866)	4.24 (12.92970)	2.47 (12.93114)
0.9	0.91612	7.63 (15.68985)	9.72 (15.68501)	6.23 (15.68816)	3.51 (15.68935)
1.0	1.04687	8.41 (16.65207)	9.20 (16.65824)	6.49 (16.65475)	3.68 (16.65448)
			$N = 90$		
0.1	0.14738	0.65 (10.55765)	2.14 (10.55760)	0.58 (10.55764)	0.52 (10.55766)
0.2	0.28416	1.24 (10.64884)	8.00 (10.64954)	1.22 (10.64886)	0.98 (10.64891)
0.4	0.48553	3.06 (11.78442)	9.72 (11.78326)	3.01 (11.78437)	1.87 (11.78392)
0.6	0.71700	5.01 (13.60878)	9.15 (13.60814)	4.70 (13.60868)	2.71 (13.60765)
0.9	1.03316	8.29 (16.75143)	9.34 (16.75108)	6.72 (16.75128)	3.83 (16.75236)
1.0	1.18678	9.62 (17.84447)	9.27 (17.84515)	6.93 (17.84483)	4.04 (17.84450)
			$N = 180$		
0.1	0.20153	0.96 (11.10287)	4.40 (11.10275)	0.90 (11.10286)	0.74 (11.10286)
0.2	0.36535	2.01 (11.44992)	9.30 (11.44972)	1.99 (11.44991)	1.34 (11.45006)
0.4	0.65112	4.72 (13.60342)	9.20 (13.60879)	4.45 (13.60423)	2.58 (13.60397)
0.6	0.96528	8.04 (16.44210)	9.52 (16.43900)	6.61 (16.44082)	3.61 (16.44032)
0.9	1.56137	13.72 (21.01606)	8.25 (21.00751)	7.50 (21.00988)	4.74 (21.01100)
1.0	1.75135	15.00 (22.56039)	8.21 (22.55572)	7.42 (22.55670)	4.97 (22.55497)
			$N = 270$		
0.1	0.23986	1.30 (11.64724)	5.58 (11.64653)	1.26 (11.64720)	0.98 (11.64727)
0.2	0.44549	2.73 (12.25239)	8.66 (12.25135)	2.70 (12.25235)	1.75 (12.25254)
0.4	0.80737	6.20 (15.15205)	9.23 (15.15372)	5.62 (15.15251)	3.24 (15.15219)
0.6	1.24619	10.13 (18.70720)	8.66 (18.70856)	7.15 (18.70799)	4.21 (18.70742)
0.9	1.93563	17.41 (24.30104)	7.95 (24.29656)	7.42 (24.29726)	5.21 (24.29039)
1.0	2.28595	18.87 (26.14479)	7.69 (26.16213)	7.33 (26.16025)	5.25 (26.15673)

(Continued on next page)

Table III. (*Continued*)

	B. $S_0/K = 1.0$				
		Ratio of the sample standard deviations in %			
σ	S^{MC}	S^{BBG}/S^{MC}	S^{CVU}/S^{MC}	S^{SUB}/S^{MC}	S^{OPT}/S^{MC}
			$N = 30$		
0.1	0.04580	0.58 (0.78611)	12.58 (0.78645)	0.58 (0.78611)	0.35 (0.78610)
0.2	0.09371	1.12 (1.45834)	12.05 (1.45844)	1.11 (1.45834)	0.65 (1.45831)
0.4	0.18991	2.30 (2.80618)	11.53 (2.80553)	2.24 (2.80616)	1.29 (2.80622)
0.6	0.28947	3.57 (4.15382)	11.88 (4.15369)	3.36 (4.15381)	1.88 (4.15381)
0.9	0.44699	5.29 (6.17156)	11.58 (6.17040)	4.86 (6.17136)	2.76 (6.17091)
1.0	0.51312	5.99 (6.84223)	11.15 (6.83884)	5.22 (6.84154)	2.90 (6.84186)
			$N = 72$		
0.1	0.07272	0.97 (1.29421)	11.93 (1.29465)	0.97 (1.29421)	0.57 (1.29420)
0.2	0.14925	1.81 (2.31112)	12.25 (2.31119)	1.78 (2.31112)	0.99 (2.31116)
0.4	0.29683	3.64 (4.35692)	11.90 (4.35587)	3.48 (4.35683)	2.00 (4.35701)
0.6	0.46885	5.40 (6.40134)	11.83 (6.40351)	4.97 (6.40176)	2.77 (6.40261)
0.9	0.70655	8.67 (9.45723)	11.36 (9.45767)	6.86 (9.45740)	3.99 (9.45698)
1.0	0.84255	9.66 (10.46994)	10.92 (10.46791)	7.07 (10.46902)	4.14 (10.46936)
			$N = 90$		
0.1	0.08550	1.05 (1.47925)	11.17 (1.47963)	1.05 (1.47925)	0.59 (1.47924)
0.2	0.16542	2.10 (2.60933)	12.20 (2.61025)	2.06 (2.60935)	1.15 (2.60935)
0.4	0.32262	4.13 (4.88650)	11.79 (4.88577)	3.91 (4.88642)	2.24 (4.88665)
0.6	0.52581	6.05 (7.16229)	11.24 (7.16070)	5.39 (7.16192)	3.05 (7.16160)
0.9	0.83203	9.57 (10.55737)	10.83 (10.55877)	7.23 (10.55800)	4.16 (10.55737)
1.0	0.94825	10.67 (11.68166)	10.50 (11.68246)	7.52 (11.68208)	4.50 (11.68062)
			$N = 180$		
0.1	0.12210	1.55 (2.28066)	11.43 (2.28096)	1.54 (2.28066)	0.87 (2.28068)
0.2	0.23200	3.03 (3.84012)	11.81 (3.84029)	2.93 (3.84013)	1.70 (3.84017)
0.4	0.48246	5.86 (7.00650)	11.29 (7.00489)	5.21 (7.00614)	2.87 (7.00643)
0.6	0.78939	9.11 (10.16415)	10.39 (10.17078)	6.75 (10.16701)	3.88 (10.16543)
0.9	1.22953	14.63 (14.85258)	9.48 (14.85773)	7.75 (14.85618)	5.37 (14.85950)
1.0	1.44703	16.35 (16.40835)	9.12 (16.40498)	7.91 (16.40581)	5.53 (16.40490)
			$N = 270$		
0.1	0.15574	1.94 (2.97597)	10.74 (2.97568)	1.93 (2.97596)	1.06 (2.97602)
0.2	0.31012	3.64 (4.84515)	10.56 (4.84504)	3.47 (4.84514)	1.81 (4.84498)
0.4	0.61657	7.55 (8.66916)	11.00 (8.66783)	6.13 (8.66873)	3.52 (8.66834)
0.6	0.98284	11.77 (12.48020)	9.99 (12.47570)	7.72 (12.47764)	4.92 (12.47862)
0.9	1.67111	18.27 (18.10212)	8.95 (18.11593)	8.00 (18.11334)	5.74 (18.11293)
1.0	1.96254	20.67 (19.94938)	8.54 (19.96686)	7.96 (19.96434)	5.86 (19.96580)

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Table III. (*Continued*)

	C. $S_0/K = 0.9$				
		Ratio of the sample standard deviations in %			
σ	S^{MC}	S^{BBG}/S^{MC}	S^{CVU}/S^{MC}	S^{SUB}/S^{MC}	S^{OPT}/S^{MC}
			$N = 30$		
0.1	0.00000	N.A. (0.00000)	N.A. (0.00000)	N.A. (0.00000)	N.A. (0.00000)
0.2	0.00191	10.96 (0.00118)	329.26 (0.00101)	10.12 (0.00117)	7.03 (0.00118)
0.4	0.04394	6.93 (0.18695)	64.06 (0.18732)	5.23 (0.18697)	3.41 (0.18694)
0.6	0.12446	6.72 (0.80174)	32.55 (0.80194)	5.16 (0.80176)	3.02 (0.80185)
0.9	0.27146	7.97 (2.18716)	21.04 (2.18755)	6.23 (2.18723)	3.48 (2.18698)
1.0	0.31695	8.25 (2.71164)	19.54 (2.71125)	6.76 (2.71156)	3.82 (2.71140)
			$N = 72$		
0.1	0.00022	21.91 (0.00004)	1091.60 (0.00005)	20.99 (0.00004)	16.10 (0.00004)
0.2	0.02101	7.73 (0.05230)	96.42 (0.05340)	6.10 (0.05234)	3.91 (0.05229)
0.4	0.12780	7.13 (0.88813)	33.08 (0.88947)	5.54 (0.88829)	3.33 (0.88818)
0.6	0.27890	8.57 (2.33762)	21.55 (2.33025)	6.49 (2.33614)	3.73 (2.33571)
0.9	0.48994	11.07 (4.89411)	17.14 (4.89083)	8.04 (4.89296)	4.98 (4.89050)
1.0	0.58952	11.89 (5.78246)	15.57 (5.79126)	8.35 (5.78597)	5.10 (5.78553)
			$N = 90$		
0.1	0.00084	14.81 (0.00025)	510.83 (0.00024)	13.58 (0.00025)	9.44 (0.00025)
0.2	0.02886	7.68 (0.10141)	87.13 (0.10113)	5.95 (0.10140)	3.88 (0.10146)
0.4	0.16512	7.29 (1.21068)	28.48 (1.20980)	5.59 (1.21056)	3.25 (1.21043)
0.6	0.32761	8.99 (2.93188)	20.42 (2.92894)	6.81 (2.93120)	3.96 (2.93050)
0.9	0.61742	11.96 (5.85667)	15.02 (5.85665)	8.47 (5.85666)	4.96 (5.85564)
1.0	0.68584	13.09 (6.87290)	14.05 (6.86798)	8.65 (6.87060)	5.41 (6.86923)
			$N = 180$		
0.1	0.00908	9.43 (0.01595)	168.14 (0.01617)	7.76 (0.01595)	5.12 (0.01598)
0.2	0.08226	7.09 (0.49796)	44.84 (0.49745)	5.48 (0.49792)	3.18 (0.49785)
0.4	0.30249	8.79 (2.73506)	20.50 (2.73784)	6.71 (2.73568)	3.89 (2.73542)
0.6	0.55621	11.71 (5.46450)	15.43 (5.46534)	8.16 (5.46483)	4.90 (5.46410)
0.9	0.99666	15.99 (9.74556)	12.46 (9.76206)	9.34 (9.75586)	6.09 (9.75713)
1.0	1.15893	18.10 (11.19925)	11.90 (11.19781)	9.30 (11.19824)	6.72 (11.19996)
			$N = 270$		
0.1	0.02430	8.35 (0.08116)	100.61 (0.08167)	6.49 (0.08118)	4.16 (0.08115)
0.2	0.13077	7.34 (1.00440)	33.99 (1.00514)	5.65 (1.00449)	3.35 (1.00429)
0.4	0.41543	9.80 (4.08422)	17.70 (4.08132)	7.66 (4.08335)	4.45 (4.08317)
0.6	0.75933	14.20 (7.54205)	13.65 (7.52813)	8.84 (7.53494)	5.56 (7.53045)
0.9	1.35916	19.83 (12.77782)	10.88 (12.79065)	9.29 (12.78755)	6.56 (12.78615)
1.0	1.59635	22.37 (14.51234)	10.24 (14.53329)	8.96 (14.52940)	6.76 (14.53332)

Note. "N.A." means that the ratio is not computed because S^{MC} is zero.
The number in parentheses is the sample mean of simulated Asian call option prices.

corners of the table. Thus the two smallest values occur at the north-west and south-east corners of the table and they are 0.14615 and 0.10837 respectively, while the largest value is 0.97777 at $(\sigma, N) = (0.4, 30)$.

When we look into part (A) of Table I and Table III(A) carefully, we find that α^* is greater than 0.5 if S^{BBG} is less than S^{CVU}, but it is less than 0.5 if S^{BBG} is greater than S^{CVU}. This implies that in the control variate for $\hat{\theta}^{SUB}$, $\hat{\theta}^{LU}$ of (2.16), $\hat{\hat{\theta}}^{BBG}$ is given more weight than $\hat{\theta}^{CVU}$ if it is better; otherwise, it is given less weight.

Second, we examine estimated β^{L^*} and β^{U^*} in Table II(A). As functions of $(S_0/K, \sigma, N, R)$, the optimal values of $\boldsymbol{\beta}$'s estimated are of a similar structure as those of α^*'s. Since the optimal α^*'s, as we expected, lie in the interval [0, 1], β^{L^*} and β^{U^*} are positive as

$$\hat{\theta}^{OPT}(\boldsymbol{\beta}^*) = \hat{\theta}^{MC} - \nu[\delta(\hat{\theta}^{LMC} - \theta^L) + (1 - \delta)(\hat{\theta}^{UMC} - \theta^U)].$$

In fact, $\nu = \beta^{L^*} + \beta^{U^*}$ and $\delta = \beta^{L^*}/(\beta^{L^*} - \beta^{U^*})$. Table II shows that (i) all the $\boldsymbol{\beta}^*$'s are positive, (ii) ν's obtained from $\boldsymbol{\beta}^*$'s are all greater than 1, and (iii) the structure of δ relative to $(S_0/K, \sigma, N, R)$ is similar to that of α^*'s. More specifically we observe

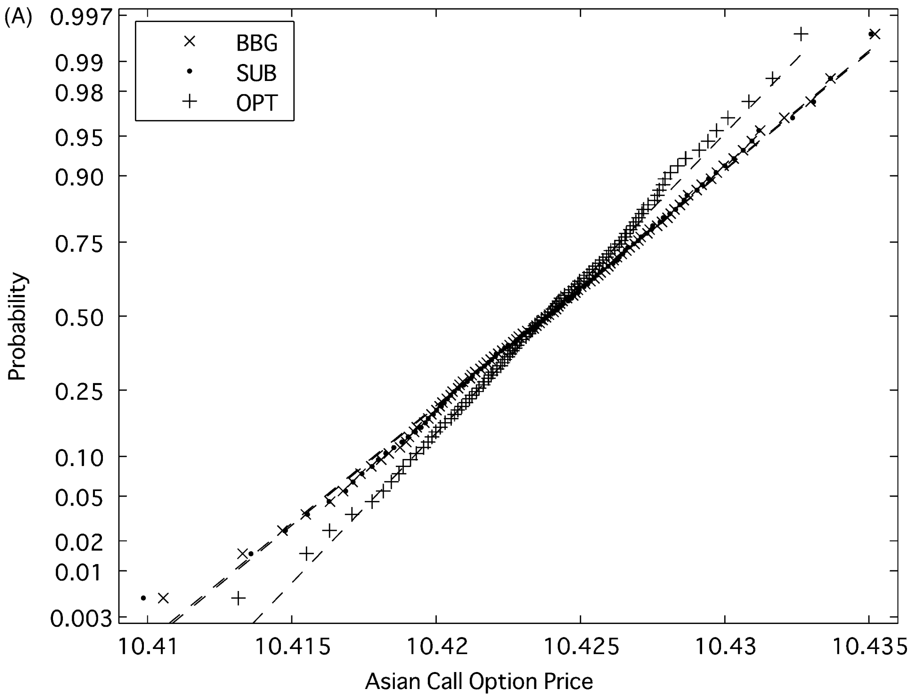

Figure 1. Normal probability plots of Asian call option prices (A) $(S_0/K, \sigma, N, r) = (1.1, 0.4, 30, 0.05)$. (B) $(S_0/K, \sigma, N, r) = (1.0, 0.4, 30, 0.05)$. (C) $(S_0/K, \sigma, N, r) = (0.9, 0.4, 30, 0.05)$.

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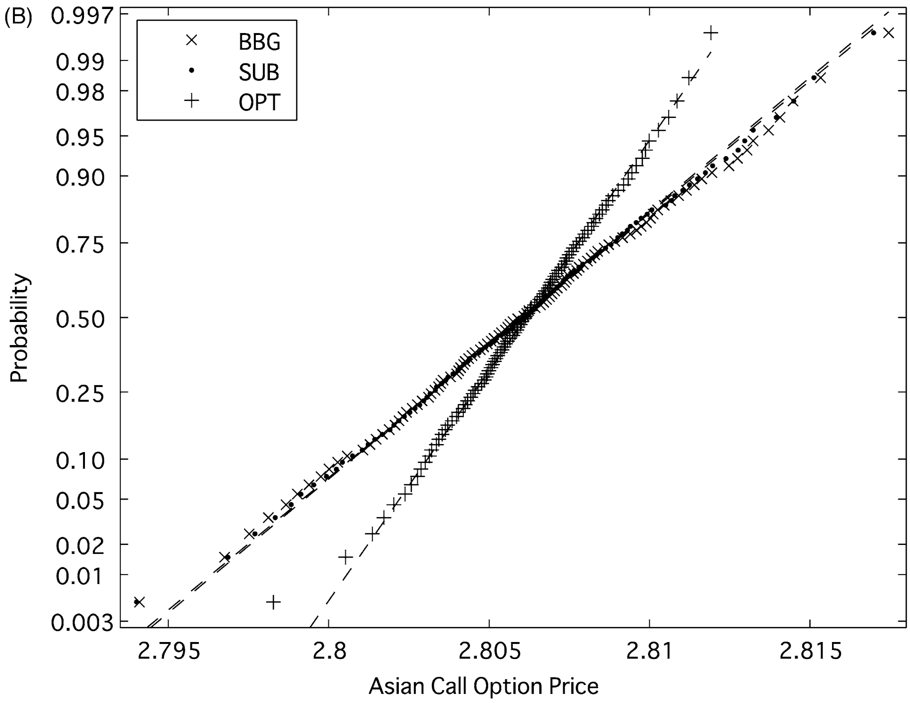

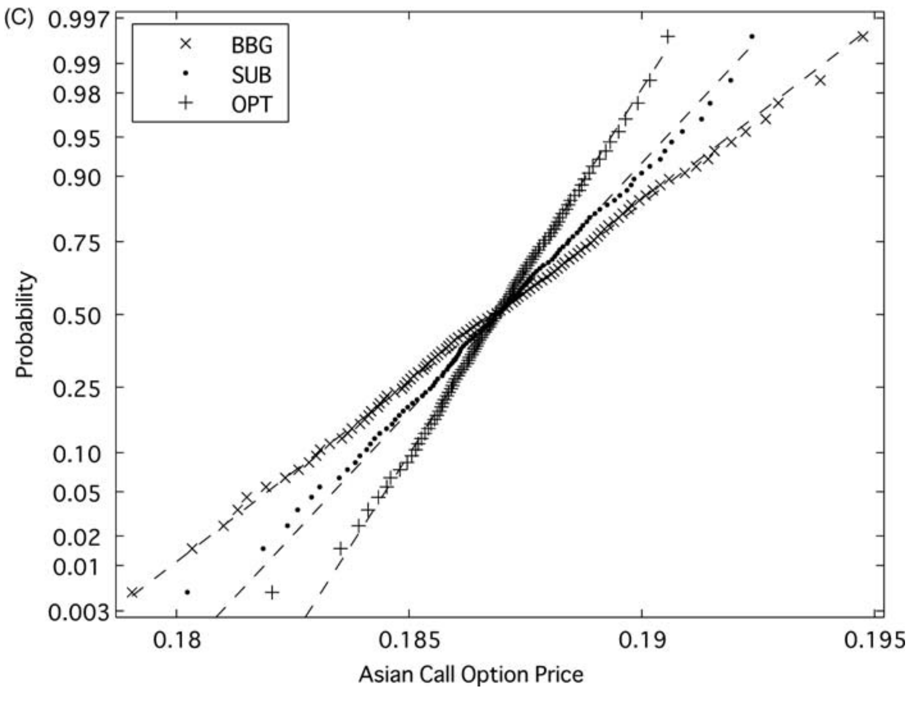

Figure 1. (Continued)

that if $S_0/K = 1.1$ the maximum β^{L^*} or equivalently δ occurs at $(\sigma, N) = (0.4, 30)$, $(0.2, 72)$, $(0.2, 90)$, $(0.2, 180)$, $(0.1, 270)$, at which β^{U^*} also takes its minimum values. This is slightly different from the result of case A in α^*, because α^* takes its maximum not at $(\sigma, N) = (0.2, 72)$ but at $(0.4, 72)$.

The sum of β^{L^*} and β^{U^*}, ν, is greater than 1.0, and it increases as σ increases for each fixed N and also increases as N increases for each fixed σ. Thus ν takes its maximum when $(\sigma, N) = (1.0, 270)$ and its minimum when $(\sigma, N) = (0.1, 30)$. Because $\hat{\theta}^{OPT}$ dominates $\hat{\theta}^{SUB}$ in Table III(A), allowing the sum of weights in $\hat{\theta}^{LU}$ of (2.16) to be greater than 1.0 seems to improve accuracy of the estimator.

In Table III(A) of $\boldsymbol{\beta}^*$ and Table II(A), we observe a similar relationship among β^{L^*}, β^{U^*}, S^{BBG}, and S^{CVU} as α^*. β^{L^*} is greater than β^{U^*} if $S^{BBG} < S^{CVU}$ and vice versa except for $(\sigma, N) = (1.0, 90)$, $(0.9, 180)$, $(0.6, 270)$.

Finally, we compare the five estimators when $r = 0.05$ and $S_0/K = 1.1$. For $\hat{\theta}^{BBG}$, $\hat{\theta}^{SUB}$, and $\hat{\theta}^{OPT}$, the improvement over $\hat{\theta}^{MC}$ becomes greater as σ decreases for fixed N and as N decreases for fixed σ. For $\hat{\theta}^{CVU}$, however, we do not see any obvious patterns of the improvement in the (σ, N) space.

To compare $\hat{\theta}^{OPT}$ and $\hat{\theta}^{BBG}$, we compute the percentage ratios of S^{OPT}/S^{BBG} for each (σ, N), which are reported in Table IV. In part (A) of Table IV, the maximal improvement of $\hat{\theta}^{OPT}$ over $\hat{\theta}^{BBG}$ occurs at $(\sigma, N) = (1.0, 270)$ where S^{OPT} is 27.82% of S^{BBG}. The minimal improvement occurs at $(\sigma, N) = (0.2, 30)$ where S^{OPT} is 84.91% of S^{BBG}. The improvement of $\hat{\theta}^{OPT}$ over $\hat{\theta}^{BBG}$ becomes more significant as σ increases for each fixed N or as N increases for each fixed σ except for $(\sigma, N) = (0.1, 30)$.

B. *At-the-money* ($S_0/K = 1.0$).

In part (B) of Table I, the largest observed value of $\alpha^*(1.0, \sigma, N, 0.05)$ is 0.99684, which occurs at the north-west corner point $(\sigma, N) = (0.1, 30)$, and the smallest observed value is 0.14385, which occurs at the south-east corner point $(\sigma, N) = (1.0, 270)$. α^* decreases as (σ, N) deviates from $(0.1, 30)$ in any directions.

In Table II(B), the largest observed value of β^{L^*} is 0.98992, which occurs at $(\sigma, N) = (0.1, 30)$ and its smallest value occurs at $(\sigma, N) = (1.0, 270)$ with value 0.42590. β^{L^*} decreases as (σ, N) deviates from $(0.1, 30)$ in any directions. In the same table, the largest observed value of β^{U^*} is 0.65798, which occurs at $(\sigma, N) = (1.0, 270)$ and the smallest value occurs at $(\sigma, N) = (0.1, 30)$ with value 0.01496. β^{U^*} increases as (σ, N) deviates from $(0.1, 30)$ in any directions. The sum of β^{L^*} and β^{U^*}, ν, is always greater than 1.0, and its largest observed value occurs at $(\sigma, N) = (1.0, 270)$ with value 1.08388 and its smallest value occurs at $(\sigma, N) = (0.1, 30)$ with value 1.00489. ν also increases as (σ, N) deviates from $(0.1, 30)$ in any directions.

We find the same relationship among the value of α^*, S^{BBG}, and S^{CVU} in part (B) of Table I as in case A, i.e., $\alpha^* > 0.5$ if $S^{BBG} > S^{SUB}$; otherwise, $\alpha^* < 0.5$. In Table II(B), the relationship among the value of $(\beta^{L^*}, \beta^{U^*})$, S^{BBG}, and S^{CVU} is not so clear as in case of α^*. But β^{L^*} tends to be larger than β^{U^*} if $S^{BBG} > S^{SUB}$ and vice versa.

Table IV. Comparisons between OPT and BBG: S^{OPT}/S^{BBG} in %

N	30	72	90	180	270
σ			**A.** $S_0/K = 1.1$		
0.1	40.70	80.74	80.54	77.13	75.98
0.2	84.91	83.06	78.73	66.65	64.13
0.4	78.37	65.11	61.02	54.54	52.22
0.6	68.79	56.31	54.19	44.94	41.60
0.9	59.51	45.96	46.23	34.58	29.92
1.0	58.37	43.73	42.03	33.12	27.82
			B. $S_0/K = 1.0$		
0.1	60.96	58.97	56.03	56.00	54.63
0.2	58.10	54.54	54.81	55.97	49.58
0.4	55.98	54.94	54.20	48.91	46.68
0.6	52.64	51.17	50.35	42.58	41.79
0.9	52.21	45.98	43.50	36.71	31.40
1.0	48.33	42.88	42.16	33.85	28.34
			C. $S_0/K = 0.9$		
0.1	N.A.[a]	73.48	63.78	54.29	49.79
0.2	64.09	50.65	50.53	44.92	45.70
0.4	49.16	46.66	44.67	44.30	45.42
0.6	44.86	43.50	44.04	41.83	39.13
0.9	43.68	45.01	41.50	38.12	33.06
1.0	46.26	42.90	41.37	37.14	30.23

[a] S^{BBG} is zero.

The improvement over the MC estimator in case B has a similar structure as in case A. From Table III(B), we observe that for $\hat{\theta}^{BBG}$, $\hat{\theta}^{SUB}$, and $\hat{\theta}^{OPT}$, the improvement over $\hat{\theta}^{MC}$ becomes more significant as σ decreases for each fixed N or as N decreases for each fixed σ. For $\hat{\theta}^{CVU}$, however, we do not see any obvious patterns of the improvement in the (σ, N) space.

The improvement of $\hat{\theta}^{OPT}$ over $\hat{\theta}^{BBG}$ in case B also has a similar structure as in case A. In part (B) of Table IV, the maximal improvement of $\hat{\theta}^{OPT}$ over $\hat{\theta}^{BBG}$ occurs at $(\sigma, N) = (1.0, 270)$ where S^{OPT} is 28.34% of S^{BBG}. The minimal improvement occurs at $(\sigma, N) = (0.1, 30)$ where S^{OPT} is 60.96% of S^{BBG}. The improvement of $\hat{\theta}^{OPT}$ over $\hat{\theta}^{BBG}$ becomes more significant as (σ, N) increases for each fixed N or as N increases for each fixed σ except for $(\sigma, N) = (0.2, 72)$.

C. *Out-of-the-money* ($S_0/K = 0.9$).
At $(\sigma, N) = (0.1, 30)$, $\hat{\theta}^{MC}$ and $\hat{\theta}^{LMC}$ are all zero in simulation. We ignore this case in our analysis. In part (C) of Table I, the maximal observed value of α^* occurs at $(\sigma, N) = (0.1, 72)$, with value 0.99479, while the minimum occurs at $(\sigma, N) = (1.0, 270)$, with value 0.18591. α^* decreases as (σ, N) deviates from $(\sigma, N) = (0.1, 72)$ in any directions. This is similar to case B.

In Table II(C), the largest observed value of β^{L^*} is 1.12408, which occurs at $(\sigma, N) = (0.1, 72)$ and its smallest occurs at $(\sigma, N) = (1.0, 270)$ with value 0.49167. β^{L^*} decreases as (σ, N) deviates from $(0.1, 72)$ in any directions. In the same table, the largest observed value of β^{U^*} is 0.60568, which occurs at $(\sigma, N) = (1.0, 270)$ and its smallest value occurs at $(\sigma, N) = (0.1, 72)$ with value 0.00283. β^{U^*} increases as (σ, N) deviates from $(0.1, 72)$ in any directions.

In the three-dimensional space (σ, N, ν), ν seems to form a valley with minimum points crossing nearly horizontally at $(\sigma, N) = \{(0.4, 30), (0.4, 72), (0.2, 90), (0.2, 180), (0.2, 270)\}$, as shown in Table II(C). Away from these minimum points, the larger the deviation is , the larger ν is. More specifically, ν^* increases towards either the north-west or the south-east corners of the table.

From Table III (C), we observe that for $\hat{\theta}^{BBG}$, $\hat{\theta}^{SUB}$, and $\hat{\theta}^{OPT}$, the improvement over $\hat{\theta}^{MC}$ becomes greater as σ decreases for each fixed N and as N decreases for each fixed σ. For $\hat{\theta}^{CVU}$, on the other hand, the improvement over $\hat{\theta}^{MC}$ becomes more significant as σ increases for each fixed N and as N increases for each fixed σ except for $N = 30$. This is a little different from case A and B.

Finally, we compare $\hat{\theta}^{OPT}$ with $\hat{\theta}^{BBG}$ in case C. In part (C) of Table IV, the maximal improvement of $\hat{\theta}^{OPT}$ over $\hat{\theta}^{BBG}$ occurs at $(\sigma, N) = (1.0, 270)$ where S^{OPT} is 30.23% of S^{BBG}. The minimal improvement occurs at $(\sigma, N) = (0.1, 72)$ where S^{OPT} is 73.48% of S^{BBG}. The improvement of $\hat{\theta}^{OPT}$ over $\hat{\theta}^{BBG}$ becomes more substantial as σ increases for each fixed N or as N increases for each fixed σ except for $(\sigma, N) = (1.0, 30), (0.9, 72)$.

4. Concluding Remarks

In this paper, we propose a new control variate estimator (CVE) of an Asian option price. The results of Monte Carlo simulations show that this new CVE is uniformly far superior than the plain MC estimator as well as the BBG estimator, with very substantial improvement in all settings considered. This improvement is greater as the volatility increases and/or as days to expiration lengthens.

Note

1. One of the referees points out that He and Takahashi (2000) propose a variance reduction technique for pricing the Asian option based on a change-of-numeraire method.

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