

# Rubinstein's Similarity Consistent Preferences: A Complete Characterization\*

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## Abstract

This paper provides a complete characterization of utility functions that are consistent with similarity relations considered in Rubinstein (1988).

## 1 Introduction

Consider the following decision making problem. There are two divisible goods. The objective of a decision maker (DM) is to choose a bundle of goods  $(x_1^*, x_2^*)$  in a set of feasible consumption bundles which survives the following criteria: first, there is no feasible bundle  $(x_1, x_2)$  which gives a strictly larger amount of each good than  $(x_1^*, x_2^*)$ ; secondly, if a bundle  $(x_1, x_2)$  is feasible and  $x_1$  is larger than but not similar to  $x_1^*$ , then  $x_2^*$  is larger than but not similar to  $x_2$ ; similarly, if  $x_2$  is larger than but not similar to  $x_2^*$ , then  $x_1^*$  is larger than but not similar to  $x_1$ . This decision making process appears to be plausible, at least in a descriptive sense. Interesting questions are

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\*After completing an earlier version of this paper, I noticed a paper by X. Vila entitled "Decision bajo riesgo y relaciones de similitud: observaciones sobre la conjetura de Rubinstein", *Investigaciones Económicas XIII* (1989). The English summary of the paper indicates that it contains a result very similar to the result reported in this paper.

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(i) when can we think of this DM as if he is maximizing a utility function  $u(x_1, x_2)$ ?; (ii) given that the DM is maximizing a utility function, when can we regard the DM as if he is making choices according to the criteria above?

In his stimulating paper, given a set of assumptions, Rubinstein (1988) shows that for any given pair of similarity relations for each good, there is a separable utility function  $v_1(x_1)v_2(x_2)$  such that the DM following the decision scheme described above appears as if he is maximizing this utility function. He also shows that if the DM is maximizing a utility function and if his choices are consistent with the criteria for some similarity relations, then the utility function must be close to a separable utility function. The latter result is striking since it means that the decision criteria has a profound implication on utility theory. Notice that Rubinstein's results do not exclude the possibility that only separable utility functions can be consistent with the criteria.

The purpose of this paper is to sharpen these results by providing a complete characterization of the set of utility functions that are consistent with the criteria given similarity relations. The result shows exactly what type of utility functions are consistent, and moreover that there are a variety of non-separable utility functions that are consistent.

## 2 Setup

A similarity relation on  $[0, \infty)$  is represented by a continuous, increasing function  $J$  on  $[0, \infty)$  such that  $J(x) > x$  for  $x > 0$ . We say that  $x$  is similar to  $y$  (under  $J$ ) if  $J(x) \geq y$  and  $J(y) \geq x$ , or equivalently:  $J^{-1}(x) \leq y \leq J(x)$ ; or,  $J^{-1}(y) \leq x \leq J(y)$ . Thus  $J(x)$  is the largest element that is similar to  $x$ .  $J^{-1}(x)$  is the smallest element that is similar to  $x$ . If we write  $x \sim y$  when  $x$  and  $y$  are similar, then it is straightforward to check that the binary relation  $\sim$  satisfies Rubinstein's axioms (S-1) to (S-4) and (S-6), which are:

**(S-1)** for all  $x$ ,  $x \sim x$ ;

**(S-2)** if  $x \sim y$  then  $y \sim x$ ;

**(S-3)** the graph of  $\sim$  is closed;

**(S-4)** if  $x \leq y \leq z \leq w$  and  $x \sim w$ , then  $y \sim z$ .

(S-5) needs to be modified appropriately since Rubinstein consider a compact interval.

**(S-5)\*** (1) For any  $x > 0$ , there are  $\underline{x}$  and  $\bar{x}$  such that  $\underline{x} < x < \bar{x}$  with  $\underline{x} \sim x$  and  $x \sim \bar{x}$ : (2) there is  $x > 0$  such that  $0 \sim x$  does not hold.

Conversely, if the binary relation  $\sim$  satisfies these axioms, then set  $J(x) = \sup\{x' : x' \sim x\}$ , and it is readily verified that  $J$  has the required properties and represents  $\sim$ .

From now on, we shall always require that a similarity relation satisfies  $J(0) = 0$ ; that is, it is always assumed that 0 is similar only to itself. It will become clear that our characterization result can be modified in a straightforward way in absence of this assumption.

Write  $X = [0, \infty) \times [0, \infty)$ , and  $X_+ = (0, \infty) \times (0, \infty)$ . Fix a pair of similarity relations  $(J_1, J_2)$ . For  $(x_1, x_2), (x'_1, x'_2) \in X$ , write  $(x_1, x_2) \mathbf{P} (x'_1, x'_2)$  if (a)  $x_1 > x'_1$ , and  $x_2 > x'_2$ , or (b)  $x_1 > J_1(x'_1)$  and  $x_2 \geq J_2^{-1}(x'_2)$ , or (c)  $x_1 \geq J_1^{-1}(x'_1)$  and  $x_2 > J_2(x'_2)$ . For instance, case (b) can be interpreted as  $x_1$  is significantly larger than  $x'_1$  and  $x_2$  is not significantly smaller than  $x'_2$ . Note that because of monotonicity, (b) holds if and only if  $J_1^{-1}(x_1) > x'_1$  and  $J_2(x_2) \geq x'_2$ .

**Definition 1** A function  $V$  on  $C \subseteq X$  is said to be consistent with  $(J_1, J_2)$  on set  $C$  if  $V((x_1, x_2)) > V((x'_1, x'_2))$  holds whenever  $(x_1, x_2) \mathbf{P} (x'_1, x'_2)$  and  $(x_1, x_2), (x'_1, x'_2) \in C$ .

In particular, if  $V$  is consistent on  $X$ , then  $V$  is weakly increasing.

### 3 Main Result

We shall characterize the class of increasing functions  $V$  that is consistent with given  $(J_1, J_2)$ . So fix a pair of similarity relations  $(J_1, J_2)$  throughout. The following is a restatement of Lemma 2 of Rubinstein (1988), which gives a necessary condition for consistency:

**Lemma 2** Suppose  $V$  is continuous and consistent on  $X$ . Then for any  $(x_1, x_2) \in X_+$ , (i)  $V((x_1, x_2)) = V\left(\left(J_1(x_1), J_2^{-1}(x_2)\right)\right)$ ; (ii)  $V((x_1, x_2)) = V\left(\left(J_1^{-1}(x_1), J_2(x_2)\right)\right)$ .

In words, if you replace one of the two elements with the largest element similar it and the other element with the smallest element similar to it, then the value does not change.

**Proof.** We shall prove (i) only since (ii) is completely symmetric. Since  $x_2 > 0, J_2^{-1}(x_2) > 0$ . For any small enough  $\varepsilon > 0, (x_1, x_2) \mathbf{P} \left(J_1(x_1), J_2^{-1}(x_2) - \varepsilon\right)$ ,

since  $J_2 \left( J_2^{-1}(x_2) - \varepsilon \right) < x_2$ . So by consistency,  $V((x_1, x_2)) > V((J_1(x_1), J_2^{-1}(x_2) - \varepsilon))$ , hence  $V((x_1, x_2)) \geq V\left(\left(J_1(x_1), J_2^{-1}(x_2)\right)\right)$  by the continuity of  $V$ . Suppose  $V((x_1, x_2)) > V\left(\left(J_1(x_1), J_2^{-1}(x_2)\right)\right)$ . Then since  $x_1 > 0$ , by the continuity of  $V$ , there is  $\varepsilon > 0$  such that  $V((x_1 - \varepsilon, x_2)) > V\left(\left(J_1(x_1), J_2^{-1}(x_2)\right)\right)$ , but on the other hand  $\left(J_1(x_1), J_2^{-1}(x_2)\right) \mathbf{P}(x_1 - \varepsilon, x_2)$ , a contradiction to consistency.  $\square$

**Definition 3** A function  $V$  is said to be minimally consistent with  $(J_1, J_2)$  if it is increasing, and for any  $(x_1, x_2) \in X_+$ , (i)  $V((x_1, x_2)) = V\left(\left(J_1(x_1), J_2^{-1}(x_2)\right)\right)$ ; (ii)  $V((x_1, x_2)) = V\left(\left(J_1^{-1}(x_1), J_2(x_2)\right)\right)$ .

By Lemma 2, the minimal consistency is necessary for consistency. Our main result will show that it is in fact sufficient.

For each  $t > 0$ , consider the set  $B(t) = \left[ (t, t), \left( J_1(t), J_2^{-1}(t) \right) \right]$ , the line segment connecting  $(t, t)$  and  $\left( J_1(t), J_2^{-1}(t) \right)$ , and set  $B = \cup_{t \in (0, 1]} B(t)$ . So  $B$  is a closed subset of  $X$  and the diagonal subset of  $X$  is on the boundary of  $B$ . Moreover, if  $V$  is consistent, by Lemma 2, for any  $t \in (0, 1]$ ,  $V((t, t)) = V\left(\left(J_1(t), J_2^{-1}(t)\right)\right)$  holds, where point  $\left( J_1(t), J_2^{-1}(t) \right)$  is also on the boundary of  $B$ . Note that the set  $\left\{ \left( J_1(t), J_2^{-1}(t) \right) : t > 0 \right\}$  is an upward sloping curve by the continuity and monotonicity of  $J$ . Similarly, construct the set  $A$ , by the rule  $A = \cup_{t \in (0, 1]} A(t)$  where  $A(t) = \left[ \left( J_1^{-1}(t), J_2(t) \right), (t, t) \right]$ .

Note that if we restrict  $V$  on  $B$ , then the minimal consistency places no restriction on the behavior of  $u$  other than continuity and monotonicity. Thus in particular, the minimal consistency does not require that a monotonic transformation of  $V$  is separable. However, it does give a rich enough structure on the set  $B$  as the following result shows.

**Lemma 4** Let  $u$  be continuous, increasing function on  $B$ . Suppose  $u((t, t)) = u\left(\left(J_1(t), J_2^{-1}(t)\right)\right)$  for any  $t > 0$ . Then  $u$  is consistent on  $B$ .

**Proof.** Say  $(x_1, x_2) \mathbf{P}(x'_1, x'_2)$  and  $(x_1, x_2), (x'_1, x'_2) \in B$ . We want to show  $u(x_1, x_2) > u(x'_1, x'_2)$ . If  $(x_1, x_2) > (x'_1, x'_2)$ , then it follows by monotonicity. So first consider the case  $x_1 > J_1(x'_1)$  and  $x_2 < x'_2 \leq J_2(x_2)$ . Pick the unique  $\bar{t} > 0$  with  $u((\bar{t}, \bar{t})) = u((x'_1, x'_2))$ , thus  $u(x'_1, x'_2) = u\left(J_1(\bar{t}), J_2^{-1}(\bar{t})\right)$  by the minimal consistency. Since  $(x'_1, x'_2) \in B$  so  $x'_1 \geq x'_2$ , we have

$x'_2 \leq \bar{t} \leq x'_1$  by monotonicity, and hence  $J_1(\bar{t}) \leq J_1(x'_1) < x_1$ . Since the curve  $\left\{ \left( J_1(t), J_2^{-1}(t) \right) : t > 0 \right\}$  is upward sloping,  $J_1(\bar{t}) < x_1$  and  $(x_1, x_2) \in B$  imply that  $J_2^{-1}(\bar{t}) \leq x_2$ . Thus by monotonicity,  $u(x_1, x_2) > u\left(J_1(\bar{t}), J_2^{-1}(\bar{t})\right)$ . So  $u(x_1, x_2) > u(x'_1, x'_2)$ .

If  $J_2^{-1}(x_2) > x'_2$  and  $x_1 < x'_1 \leq J_1(x_1)$ , pick the unique  $\bar{t} > 0$  with  $u((\bar{t}, \bar{t})) = u((x_1, x_2))$ . Since  $x_1 \geq x_2$ , by monotonicity,  $x_2 \leq \bar{t} \leq x_1$ , and hence  $x'_2 < J_2^{-1}(x_2) \leq J_2^{-1}(\bar{t})$ . Since the curve  $\left\{ \left( J_1(t), J_2^{-1}(t) \right) : t > 0 \right\}$  is upward sloping and  $(x'_1, x'_2) \in B$ , it must be the case  $J_1(\bar{t}) \geq x'_1$ . The rest is the same as in the previous case.  $\square$

Define a relation  $\phi_A$  for  $x_2 > x_1 > 0$  by the rule:

$$\phi_A(x_1, x_2) = \left( J_1(x_1), J_2^{-1}(x_2) \right),$$

and for any  $x_1 > x_2 > 0$ , define  $\phi_B$  by the rule:

$$\phi_B(x_1, x_2) = \left( J_1^{-1}(x_1), J_2(x_2) \right).$$

By Lemma 2, if  $V$  is continuous and consistent, then for any  $x_2 > x_1 > 0$ ,  $V((x_1, x_2)) = V(\phi_A^n(x_1, x_2))$  for all  $n = 0, 1, 2, \dots$ , where  $\phi_A^n(x_1, x_2) = \phi_A \circ \dots \circ \phi_A(x_1, x_2)$ , as long as it is well defined. Similarly, for any  $x_1 > x_2 > 0$ ,  $V((x_1, x_2)) = V(\phi_B^n(x_1, x_2))$  for all  $n = 1, 2, \dots$ .

**Lemma 5** (i) For any  $x_2 > x_1 > 0$ , there is an  $n$  such that  $\phi_A^n(x_1, x_2) \in A$ .  
(ii) For any  $x_1 > x_2 > 0$ , there is an  $n$  such that  $\phi_B^n(x_1, x_2) \in B$ .

**Proof.** We shall prove (i). (ii) is analogous. Assume  $x_1 < x_2$ , thus  $\phi_A(x_1, x_2) = \left( J_1(x_1), J_2^{-1}(x_2) \right)$ . Since  $(J_1)^n(x) > 1$  for large enough  $n$  and  $\left( J_2^{-1} \right)^n(x_2) \rightarrow 0$  as  $n \rightarrow \infty$ , there is an  $n$  such that  $(J_1)^n(x_1) < \left( J_2^{-1} \right)^n(x_2)$  and  $(J_1)^{n+1}(x_1) \geq \left( J_2^{-1} \right)^{n+1}(x_2)$ . Thus the line segment connecting  $\left( (J_1)^n(x_1), \left( J_2^{-1} \right)^n(x_2) \right)$  and  $\left( (J_1)^{n+1}(x_1), \left( J_2^{-1} \right)^{n+1}(x_2) \right)$  intersects the diagonal set of  $X$ , and denote by  $(\bar{t}, \bar{t})$  be the intersection point. Then by construction  $(J_1)^{n+1}(x_1) \geq \bar{t} > (J_1)^n(x_1)$  and  $\left( J_2^{-1} \right)^{n+1}(x_2) \leq \bar{t} < \left( J_2^{-1} \right)^n(x_2)$ , thus the segment  $\left[ (\bar{t}, \bar{t}), \left( J_1(\bar{t}), J_2^{-1}(\bar{t}) \right) \right]$  contains  $\phi_A^{n+1}(x_1, x_2)$ .  
 $\square$

So for any  $(x_1, x_2) \in X_+$ , define  $\Psi(x_1, x_2)$  by the rule  $\Psi(x_1, x_2) = \phi_A^n(x_1, x_2)$  if  $x_1 < x_2$ , where  $n$  is the first number such that  $\phi_A^n(x_1, x_2) \in A$ ;  $\Psi(x_1, x_2) = \phi_B^n(x_1, x_2)$  if  $x_1 > x_2$ , where  $n$  is the first number such that

$\phi_B^n(x_1, x_2) \in B$ ; and  $\Psi(x_1, x_2) = (x_1, x_2)$  if  $x_1 = x_2$ . Then the following holds.

**Lemma 6** *Suppose that  $V$  is minimally consistent, and let  $u$  be the restriction of  $V$  on  $B$ . Then  $V(x_1, x_2) = u(\Psi(x_1, x_2))$ . Conversely, for any continuous increasing function  $u$  on  $B$ , if  $u((t, t)) = u\left(\left(J_1(t), J_2^{-1}(t)\right)\right)$  for any  $t > 0$ , then the function  $V$  defined by  $V(x_1, x_2) = u(\Psi(x_1, x_2))$  is consistent on  $X$ .*

**Proof.** The first part follows from Lemma 2 and by the definition of  $\Psi$ . The second part follows from Lemma 4 and by the definition of  $\Psi$ .  $\square$

Note that the second part of Lemma 6 implies that a minimally consistent function does not necessarily have a separable representation.

Now we are ready to present the main result of this paper.

**Proposition 7**  *$V$  is continuous, increasing and consistent on  $X_+$  if and only if  $V$  is minimally consistent.*

**Proof.** The only if part is immediate from Lemma 2. To establish “only if” part, fix any minimally consistent function  $V$ , and let  $u$  be its restriction on  $B$ . Thus  $V(x_1, x_2) = u(\Psi(x_1, x_2))$  by Lemma 6. Define a series of disjoint sets  $A^k$ ,  $k = 0, 1, \dots$  by the rule  $A^k = \cup_{t \in (0, 1]} A^k(t)$  where  $A^k(t)$  is the line segment connecting  $\left(\left(J_1^{-1}\right)^{k+1}(t), \left(J_2\right)^{k+1}(t)\right)$  and  $\left(\left(J_1^{-1}\right)^k(t), \left(J_2\right)^k(t)\right)$ , excluding  $\left(\left(J_1^{-1}\right)^k(t), \left(J_2\right)^k(t)\right)$ . So  $A^k$  is the area between two upward sloping lines  $\left\{\left(\left(J_1^{-1}\right)^{k+1}(t), \left(J_2\right)^{k+1}(t)\right) : t \geq 0\right\}$  and  $\left\{\left(\left(J_1^{-1}\right)^k(t), \left(J_2\right)^k(t)\right) : t \geq 0\right\}$ . Similarly, define a series of sets  $B^k$ ,  $k = 1, 2, \dots$  by the rule  $B^k(t) = \left[\left(\left(J_1\right)^k(t), \left(J_2^{-1}\right)^k(t)\right), \left(\left(J_1\right)^{k+1}(t), \left(J_2^{-1}\right)^{k+1}(t)\right)\right]$  and  $B^k = \cup_{t \in (0, 1]} B^k(t)$ . Finally, define an increasing series of sets  $D^n$  iteratively by the rule: set  $D^0 = B$ , and for  $n = 1, 2, \dots$ ,  $D^n = D^{n-1} \cup A^{(n-1)/2}$  if  $n$  is odd, and  $D^n = D^{n-1} \cup B^{n/2}$  if  $n$  is even. So  $\{D^n\}$  is obtained by adding  $A^k$  and  $B^k$  alternately. It is straightforward to check that  $\cup_{n=0}^{\infty} D^n = X_+$ , so it suffices to show the following claim: for any  $n$ ,  $V$  is continuous, increasing and consistent on  $D^n$ , and  $V((x_1, x_2)) = V\left(\left(J_1(x_1), J_2^{-1}(x_2)\right)\right) = V\left(\left(J_1^{-1}(x_1), J_2(x_2)\right)\right)$  whenever the relevant points are in  $D^n$ .

We proceed by induction. First when  $n = 0$ , the statement holds by Lemma 4. So assume it holds up to  $n$ . Since argument is symmetric, we shall consider the case where  $n$  is even, thus  $D^{n+1} = A^{n/2} \cup D^n$ . Writing  $k = n/2$ , the boundary of sets  $A^k$  and  $D^n$  is the curve  $\left\{ \left( (J_1^{-1})^k(t), (J_2)^k(t) \right) : t > 1 \right\}$ .

First note that by the construction of  $V$ ,  $V((x_1, x_2)) = V\left(\left(J_1(x_1), J_2^{-1}(x_2)\right)\right) = V\left(\left(J_1^{-1}(x_1), J_2(x_2)\right)\right)$  whenever the relevant points are in  $D^{n+1} = A^k \cup D^n$ .

Claim.  $V$  is continuous on  $D^{n+1}$ : Choose any sequence  $(x_1^l, x_2^l)$ ,  $l = 1, 2, \dots$ , that converges to  $(\bar{x}_1, \bar{x}_2)$ . We want to show that  $\lim V((x_1^l, x_2^l)) = V((\bar{x}_1, \bar{x}_2))$ . Since  $V$  is continuous on  $D^n$  by hypothesis, it is enough to consider the case  $(x_1^l, x_2^l) \in A^k$  for all  $l$ . Then we have  $\lim V((x_1^l, x_2^l)) = \lim V(\phi_A(x_1^l, x_2^l)) = V(\lim \phi_A(x_1^l, x_2^l)) = V(\phi_A(\bar{x}_1, \bar{x}_2))$ . So if  $(\bar{x}_1, \bar{x}_2) \in A^k$ ,  $V(\phi_A(\bar{x}_1, \bar{x}_2)) = V(\bar{x}_1, \bar{x}_2)$  by construction. If  $(\bar{x}_1, \bar{x}_2) \in D^n$ , then  $V(\phi_A(\bar{x}_1, \bar{x}_2)) = V(\bar{x}_1, \bar{x}_2)$  holds since  $V$  is consistent on  $D^n$  by hypothesis.

Claim.  $V$  is increasing on  $D^{n+1}$ : Suppose  $(x_1, x_2) \geq (y_1, y_2)$  but not  $(x_1, x_2) = (y_1, y_2)$ . If both  $(x_1, x_2)$  and  $(y_1, y_2)$  are in  $D^n$ , it follows by the hypothesis. If both are in  $A$ , it follows since  $\phi_A(x_1, x_2) \geq \phi_A(y_1, y_2)$ , and  $\phi_A(x_1, x_2)$  and  $\phi_A(y_1, y_2)$  are in  $D^n$ . Suppose  $(x_1, x_2) \in A$ , and  $(y_1, y_2) \in D^n$ . Pick the unique  $\bar{t}$  with  $V\left(\left(\left(J_1^{-1}\right)^k(\bar{t}), (J_2)^k(\bar{t})\right)\right) = V((y_1, y_2))$ . Recall that  $\left(\left(J_1^{-1}\right)^k(\bar{t}), (J_2)^k(\bar{t})\right)$  is on the boundary of  $D^n$ , so since  $V$  is increasing on  $D^n$ ,  $\left(J_1^{-1}\right)^k(\bar{t}) \leq y_1$  and  $(J_2)^k(\bar{t}) \geq y_2$ . Hence  $J_1(x_1) \geq J_1(y_1) \geq \left(J_1^{-1}\right)^{k-1}(\bar{t})$ , and so  $J_2^{-1}(x_2) \geq \left(J_2^{-1}\right)^{m-1}(\bar{t})$  since  $\left(J_1(x_1), J_2^{-1}(x_2)\right) \in D^n$  and the curve  $\left\{ \left( \left(J_1^{-1}\right)^{k-1}(t), (J_2)^{k-1}(t) \right) : t > 0 \right\}$  is upward sloping. Therefore,  $\phi_A(x_1, x_2) > (y_1, y_2)$ , and so by hypothesis  $V(x_1, x_2) = V(\phi_A(x_1, x_2)) > V(y_1, y_2)$ . Finally, suppose  $(x_1, x_2) \in D^n$ , and  $(y_1, y_2) \in A$ . Pick the unique  $\bar{t}$  with  $V\left(\left(\left(J_1^{-1}\right)^k(\bar{t}), (J_2)^k(\bar{t})\right)\right) = V((x_1, x_2))$ , so  $\left(J_1^{-1}\right)^k(\bar{t}) \leq x_1$  and  $(J_2)^k(\bar{t}) \geq x_2$ . Hence  $J_2(y_2) \leq J_2(x_2) \leq \left(J_2^{-1}\right)^{k-1}(\bar{t})$ , and thus arguing as in the previous case,  $J_1(y_1) \leq \left(J_1\right)^{k-1}(\bar{t})$  and  $V(x_1, x_2) > V(\phi_A(y_1, y_2)) = V(y_1, y_2)$ .

Claim.  $V$  is consistent on  $D^{n+1}$ . Suppose  $(x_1, x_2) \mathbf{P} (y_1, y_2)$  and  $(x_1, x_2), (y_1, y_2) \in D^{n+1}$ . We want to show  $V((x_1, x_2)) > V((y_1, y_2))$ . If both

$(x_1, x_2)$  and  $(y_1, y_2)$  are in  $D^n$ ,  $(\cup_{k=0}^m A^k) \cup (\cup_{k=0}^{m-1} B^k)$ , it follows by hypothesis. So first consider the case where both  $(x_1, x_2)$  and  $(y_1, y_2)$  are in  $A$ . If  $(x_1, x_2) \gg (y_1, y_2)$ , then it follows since we have shown that  $V$  is increasing on  $D^{n+1}$ . If  $x_1 > J_1(y_1)$  and  $J_2(x_2) \geq y_2$ , then  $\phi_A(x_1, x_2) \mathbf{P} \phi_A(y_1, y_2)$  so  $V(x_1, x_2) = V(\phi_A(x_1, x_2)) > V(\phi_A(y_1, y_2)) = V((y_1, y_2))$ . The case where  $J_1(x_1) \geq (y_1)$  and  $x_2 > J_2(y_2)$  is analogous.

Suppose  $(x_1, x_2) \in A$ , but  $(y_1, y_2) \in D^n$ . If  $(x_1, x_2) \gg (y_1, y_2)$ , it follows by monotonicity. If  $x_1 > J_1(y_1)$ , then  $x_2 > y_2$  since  $(x_1, x_2) \in A$  and  $(y_1, y_2) \in D^n$ , thus it follows by monotonicity. If  $J_1(x_1) \geq (y_1)$  and  $x_2 > J_2(y_2)$ , then  $J_2^{-1}(x_2) > y_2$ , and so  $\phi_A(x_1, x_2) = (J_1(x_1), J_2^{-1}(x_2)) > (y_1, y_2)$ , so it follows by monotonicity.

Finally suppose  $(x_1, x_2) \in D^n$ , and  $(y_1, y_2) \in A$ . If  $(x_1, x_2) \gg (y_1, y_2)$ , it follows by monotonicity. If  $x_1 > J_1(y_1)$  and  $J_2(x_2) \geq y_2$ , then  $(x_1, x_2) > (J_1(y_1), J_2^{-1}(y_2)) = \phi_A(y_1, y_2)$ . So again it follows by monotonicity. Finally, if  $x_2 > J_2(y_2)$ , then  $x_1 > y_1$  since  $(x_1, x_2) \in D^n$ , and it follows by monotonicity.  $\square$

## 4 Remarks

Proposition 7 says that the class of utility functions that are consistent with the given pair of similarity relation can be parameterized by the set of minimally consistent functions on  $B$ . To relate this to Rubinstein's results, we shall provide a result that summarizes Proposition 2 and 3 of Rubinstein (1988).<sup>1</sup>

**Corollary 8** *If  $V$  and  $V'$  are consistent with  $(J_1, J_2)$ , then for any  $(x_1, x_2), (y_1, y_2) \in X_+$ ,  $V'(x_1, x_2) > V'(x'_1, x'_2)$  implies  $V(x_1, J_2(x_2)) > V(x'_1, x'_2)$  or  $V(J_1(x_1), x_2) > V(x'_1, x'_2)$ , thus in particular  $V(J_1(x_1), J_2(x_2)) > V(x'_1, x'_2)$ .*

So any consistent functions must be close to each other in the sense above. Note that by symmetry, if  $V'(x_1, x_2) > V'(x'_1, x'_2)$ ,  $V(x_1, x_2) > V(J_1^{-1}(x'_1), x'_2)$  or  $V(x_1, x_2) > V(x'_1, J_2^{-1}(x'_2))$  must also hold.

**Proof.** By Proposition 7, we can write  $V((x_1, x_2)) = u(\Psi(x_1, x_2))$  and  $V'((x_1, x_2)) = u'(\Psi(x_1, x_2))$  where  $u$  and  $u'$  are minimally consistent. So writing  $(y_1, y_2)$  for  $\Psi(x_1, x_2)$  and  $(z_1, z_2)$  for  $\Psi(x'_1, x'_2)$ , suppose  $u'(y_1, y_2) > u'(z_1, z_2)$ . So  $(y_1, y_2) < (z_1, z_2)$  cannot occur. It is enough to show  $V(J_1(y_1), y_2) > u(z_1, z_2)$  or  $V(y_1, J_2(y_2)) > u(z_1, z_2)$  holds. If  $(y_1, y_2) > (z_1, z_2)$ , then

<sup>1</sup>Slightly improved versions of Rubinstein's results are shown in Aizpurua - Ichiishi - Nieto - Uriarte (1993).

the result follows by monotonicity. So suppose  $y_1 > z_1$ , and  $y_2 < z_2$ . If  $J_1(z_1) \geq y_1$ , then  $J_2(y_2) \geq z_2$ , otherwise  $(z_1, z_2) \mathbf{P}(y_1, y_2)$ , a contradiction to the consistency of  $u'$ . So  $(y_1, J_2(y_2)) > (z_1, z_2)$ , and hence  $V(J_1(x_1), J_2(x_2)) > V(x_1, J_2(x_2)) = V(y_1, J_2(y_2)) > V(z_1, z_2)$ . Recall that  $(y_1, y_2), (z_1, z_2) \in B$  thus  $J_1^{-1}(y_1) < J_2(y_2)$  and  $z_1 \geq z_2$ . So if  $J_1(z_1) < y_1$ , then  $z_2 \leq z_1 \leq J_1^{-1}(y_1) < J_2(y_2)$ , so  $(y_1, y_2) \mathbf{P}(z_1, z_2)$  and hence  $u(y_1, y_2) > u(z_1, z_2)$ .  $\square$

The choice of the set  $B$  is arbitrary in the following sense. It is straight forward to check that we could pick any upward sloping curve as a starting point instead of the diagonal set in constructing the series of the sets  $D^n$ .

If  $J(0) > 0$  holds, we could still proceed analogously, except that the shape of function  $V$  in the region of points which are similar to zero will become indeterminate.

## References

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- [2] A. Rubinstein (1988), Similarity and Decision-making under Risk (Is There a Utility Theory Resolution to the Allais Paradox?). *Journal of Economic Theory* **46**, 145-153.