

Auctions with Options for Re-auction*

Abstract

We examine the role of seller bidding and reserve prices in an infinitely repeated independent-private-value (IPV) auction. The seller has a single object that she values at zero. At the end of any round of the auction, she may either accept the highest bid or pass-in the object for re-auctioning in the next round. New bidders are drawn randomly in each round. The ability to re-auction motivates a notion of reserve price as the option value of retaining the object for subsequent sale in a later round. Even in the absence of commitment to a reserve price, the optimal “secret” reserve is shown to exceed zero. Moreover, despite the infinite repetition, there may be significant value to the seller from a commitment to a reserve price in each round, with the optimal binding reserve price typically higher than the optimal “secret” reserve. However, we show that the ability of the seller to submit “phantom” bids during the auction is revenue-equivalent to a commitment to a public reserve price.

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1 Introduction

We examine the role of seller bidding and reserve prices in a repeated Independent Private Values (IPV) auction model. There is a single object for sale, and an auction may be held in each of an infinite number of periods. The seller may, at the end of any period, accept the highest bid, or pass-in the object for re-auctioning in the next period. The ability to re-auction sustains an optimal reserve price that exceeds the immediate consumption value of the good to the seller (normalized to zero), as it incorporates the option value of retaining the object for subsequent re-sale. This repeated auction structure captures some of the features of internet auctions for items such as used computer parts, software, etc. This structure allows us to investigate the role of (secret and public) reserve prices and seller bidding.

Within an IPV framework, it was established by Myerson (1981) and Riley and Samuelson (1981) that standard auctions (such as first- or second-price sealed-bid auctions) with a reserve above the seller's valuation maximise the seller's expected revenue within a wide class of mechanisms. In particular, the seller can achieve strictly higher expected revenue by imposing a reserve, and therefore failing to sell the object with some positive probability, than by holding a "no reserve" auction, even if she values the object less than the reserve price.

These results assume an exogenous, and commonly known, number of bidders, n . However, the optimal reserve is independent of n when all bidder valuations are drawn from the same distribution (the case considered here). Moreover, the second-price auction with reserve remains optimal even if *bidders* are uncertain of n (McAfee and McMillan (1987-a)). Indeed, provided bidders have symmetric posteriors over the number of rivals they face, these authors show that the first-price auction with reserve will also remain optimal.¹

These results rely on the credibility of two implicit commitments on the part of the seller: first, the commitment to reject bids below the reserve; and second, the commitment not to re-auction

¹ However, a "no reserve" auction may be optimal if the number of bidders is determined by endogenous entry in the presence of participation costs incurred before bidders learn their values for the object: see McAfee and McMillan (1987-b), Engelbrecht-Wiggans (1987,1993), and Levin and Smith (1994). In these models, the private gain from further entry corresponds to that of society (and the seller). Since the seller has no reason to discourage entry, the reserve price loses its only appeal. Menezes and Monteiro (2000) show that the value of a reserve is restored when bidders learn their value for the object before making their entry decision. In particular they show that the optimal auction again involves setting a reserve price above the seller's valuation.

the object later if no bids above the reserve are received. It is arguable that the first commitment might be implementable at negligible cost in many circumstances² ; but the second is unlikely to be plausible. By examining a structure which explicitly allows for re-auctioning we investigate the relationship between seller bidding and the value of commitment to a reserve price. In particular, we assume that sellers cannot commit not to re-auction the object.

Binding *public* reserves are usually more valuable to a seller than a non-binding secret reserve price, much as being a first-mover in a Stackelberg duopoly game is preferable to playing the Cournot game. This remains so in our setting, as we demonstrate below. However, in practice, reserve prices are sometimes kept secret. For example, Horstmann and LaCasse (1997) argue that a common feature of many auctions, namely the seller's refusal to sell to the high-bidder, might be interpreted as evidence of the existence of a secret reserve.³ These authors explain the existence of a secret reserve price as a signalling device about the (common) value of the object being sold. Such information could not be credibly transmitted to buyers by a publicly announced reserve price because of adverse selection. Instead the seller uses costly delay as a signalling device.⁴

We offer an alternative rationalisation for the rejection of high bids. A rejected high bid may indicate that the high bidder was in fact the seller. In the absence of any other commitment technology, "phantom" bidding allows the seller to replicate the effect of a binding public reserve. Indeed, we show that seller bidding (or "shilling") results in the *same expected revenue* as a binding public reserve. This equivalence may at first seem surprising, since shilling gives the seller far greater strategic flexibility than a public reserve. In effect, shilling allows the seller to re-set the reserve price at any stage during the auction. We show that this added flexibility, however, adds no value for the seller. This contrasts with a view held by many that "dummy bidders" or phantom bidding helps the seller. Indeed, the New South Wales State Government in Australia

² Of course the implementability of an optimal auction involving a positive reserve requires neither the ability to commit to it nor its observability; it suffices that the seller uses a second-price auction where it is a (weakly) dominant strategy to bid truthfully irrespective of the announced reserve price. (See, for example, Wolfstetter (1999), Chapter 8).

³ Ashenfelter (1989) reports significant percentages of high-bid rejections in wine and fine art auctions. Jones, Menezes and Vella (1996) report high-bid rejection rates for wool auctions in Australia.

⁴ Vincent (1995) argues that a secret reserve price might be used to increase bidder participation at the auction in a common value environment. His model, however, cannot explain why high bids are often rejected once an auction takes place.

has moved to end phantom bidding by requiring bidders to register and by limiting to one the number of bids a Vendor is allowed to make.

In the sequel we also consider the more general problem in which the seller chooses the duration of each round of the auction as well as the reserve price for each round. Unsurprisingly, if publicly announced reserve prices are not binding then we show that the optimal length of a round of the auction is finite and strictly positive. For the public-reserve auction, we identify two countervailing forces at work. A shorter length of a round of the auction increases the frequency in which the seller may decide whether or not to sell to the current standing bidder (and to revise her reserve price). However, the shorter the round, the lower is the probability of two or more bidders arriving during that round, and hence the less fierce will be the bidding within any given round. Indeed, in the limit as the length of round goes to zero, the likelihood that the object will be sold at the reserve goes to one, that is, the auction collapses to a fixed-price sale. We do not have any general result but we do show in our simulations for the case of Uniformly distributed values that the auction with a round of length “zero” (i.e. a fixed-price sale) is optimal with a public reserve.

Related Literature

There is also an existing literature that examines the role of publicly announced reserve prices in the context of repeated auctions. This literature has focussed on the case of long-lived bidders. For example, McAfee and Vincent (1997) consider an infinitely repeated, second-price independent-private-value model, in which the same n bidders participate in each round. As discount rates converge to unity, a version of the Coase conjecture takes hold, and the equilibrium reserve prices converge to zero in each period. With bidder valuations distributed uniformly on $[0, 1]$, they show that the value of commitment to a reserve price is seriously undermined by the inability to commit not to re-auction .

Burguet and Sákovics (1996) obtain a contrary result. They show that optimal reserves may be high (relative to the utility of the object to the seller) and valuable in a two-stage, first-price IPV model when buyers face participation costs, even with no discounting. They offer no justification, however, for their assumption that the seller can credibly commit not to re-auction the object after the second stage. Moreover, the seller is further assumed not to impose any reserve in the

second-stage auction, despite having had access to a suitable commitment technology in the first period to implement a first-period reserve.

Haile (2000) also has a two period model where re-auctioning is assumed not to be possible after the second stage. Instead, a re-sale market opens in the second period, in which the seller commits not to participate. Since buyers are uncertain about their true valuations when they bid, this re-sale market is always active with some positive probability. Haile's main results concern the effect of the re-sale market on optimal bidding strategies in the (second-price) auction, but he also shows that high reserves may be valuable.

Therefore, when bidders are (infinitely) long-lived, and the seller can never credibly commit not to re-auction, it seems that reserve prices lose much of their value to the seller. However, the case of short-lived bidders – where each round attracts a new cohort of participants – has, to the best of our knowledge, not been studied. As we demonstrate below, with short-lived bidders, the Coasian logic vanishes and optimal reserves are typically even higher than in the one-shot case. This optimal reserve, in the absence of a suitable commitment technology, can be implemented by seller bidding, but seller bidding cannot improve on the optimal public reserve auction.

Thus, our model dispenses with the untenable assumption of a seller commitment not to re-auction the object, but (a) implies a high value of commitment to a reserve price; and (b) provides a possible explanation for the phenomena of objects being passed-in in the absence an announced reserve, even when a costless technology exists to bind the seller to such a reserve.

The model is presented in the next section. Some general results are presented in section 3, and in section 4 we derive (by simulation) the optimal secret and public reserve prices when bidder valuations are Uniformly distributed, as well as the optimal length of each auction round. Proofs of most results appear in the appendix.

2 The model

An auction may be held in each of an infinite number of discrete periods. Within any given round, there is one unit of time, with elapsed time described (in the obvious fashion) by the Uniform

density on $[0, 1]$.⁵ We let t denote the elapsed time in the current round. No time elapses between the end of one auction round and the start of the next. Although we ignore any fixed monetary cost of conducting an auction, there is one cost to re-auctioning in our model: any bidder present at time $t = 1$ in some round is assumed no longer to be present at time $t = 0$ of the next.

There are infinitely many potential buyers. Bidders arrive according to a Poisson process with rate $\lambda > 0$, so that, within any given round, the probability of n arrivals in the interval $[s, t] \subseteq [0, 1]$ is

$$e^{-\lambda(t-s)} \frac{\lambda^n (t-s)^n}{n!}.$$

In this setting, buyers arrive one at a time within any given auction period. Buyers' values are determined by independent draws from a common distribution F with positive density f supported on $[0, 1]$. A buyer's own value is private information; but F is common knowledge.

We assume that bidders play “myopically” (e.g. that each auction round attracts a different pool of bidders). The seller, however, has regard for the future beyond the current round, and a (continuous) rate of time preference $\rho > 0$. The seller values the object at zero.

We consider cases where a publicly announced reserve may or may not be enforceable, and the seller may or may not be able to submit phantom bids. This yields the following four alternative auction mechanisms:

1. ‘Cheap-talk’ reserve price (CTRP) – announced reserves are *not* enforceable, and phantom bids are precluded.
2. Public (binding) reserve price (PRP) – announced reserves are enforced, but phantom bids are precluded.
3. Seller Phantom Bidding (with cheap-talk reserve) – announced reserves are *not* enforceable, but phantom bids are allowed.
4. Seller Phantom Bidding and public binding reserve price – publicly announced reserves are enforced, and phantom bids are allowed.

⁵ We shall relax this assumption, by making the duration of auction rounds endogenous, in section 3.6.

Descriptions of the operation of these alternate mechanisms, the corresponding strategy choices available to the seller, and the strategies of the bidders who arrive during the course of the auction are outlined in the subsequent subsections.

2.1 Cheap-talk-reserve-price

At time $t = 0$ the standing bid is automatically set to zero. The seller may, if she wishes, announce a price at which she purports to be willing to sell the object if the standing bid at the end of the current period has reached that amount. But this announcement in no way restricts her later choice of action.⁶ Seller (phantom) bids are precluded.

When the first bidder arrives he may post a bid to become the new standing bidder. As each subsequent bidder arrives, she engages in an open ascending auction with the standing bidder until one drops out and the other becomes the new standing bidder. Bidders may not re-enter the auction once they have dropped out.⁷

We assume that the bidding process between the current standing bidder and the new arrival consumes zero time. Thus, our framework captures an environment in which bilateral bidding contests consume amounts of time that are negligible relative to the length of each auction round, and also relative to the arrival rate of bidders.⁸

At the end of each period ($t = 1$), the seller may either accept the current standing bid and sell the object immediately to the high-bidder, thereby ending the repeated auction; or reject the standing bid and pass-in the object for re-auctioning. We preclude seller negotiations with bidders at the conclusion of an auction.

⁶ Porter (1995) reports that 12.7% of U.S. off-shore oil and gas leases auctioned between 1970 and 1979 were passed in, despite the announced reserve being exceeded.

⁷ This assumption is innocuous. See Section 3.

⁸ An example of an environment that fits this description is “proxy bidding” in the popular eBay on-line auction. Arriving bidders submit their maximum bids to automated “proxy bidders”. These proxy bidders maintain the bidder in the standing bidder position until the maximum bid level is exceeded. They also implement the bidding strategy at light-speed.

2.2 Public-(binding-)reserve-price

At time $t = 0$ the seller chooses the initial standing bid which is taken to be an *enforceable* reserve price. The standing bid changes only if a bidder arrives who is prepared to bid above this reserve. The auction otherwise continues as in the cheap-talk-reserve-price scenario.

2.3 Seller-phantom-bidding

At time $t = 0$ the standing bid is set to zero. The auction proceeds as in the cheap-talk-reserve-price scenario, except that at any time (*including* $t = 0$) the seller may imitate the arrival of a new buyer and bid on her own account. Since negotiation between bidders and seller are precluded by assumption, if a phantom bid stands at the end of the auction round, the object is automatically passed-in for re-auction next period.

2.4 Seller-Phantom-Bidding with public binding reserve price

This scenario combines a binding public reserve with phantom bidding by the seller. It is redundant, however, as the ability to submit phantom bids effectively undermines the enforceability of the reserve price: the seller can always trump a standing bid to force a re-auction. Hence, this scenario is equivalent to the seller-phantom-bidding (without an enforceable public reserve) scenario. We shall therefore confine attention to the preceding three auction formats in the rest of the paper.

3 General results

3.1 Bidding strategies

Because bidders have independent private values, participate in at most one auction round, and the auctions have a second-price structure, optimal bidding strategies are straightforward to determine. There is a weakly dominant strategy which is to bid up to one's valuation. For example, bidders could submit maximum bids equal to their values to proxy bidders in an eBay auction.

Let us be a little more precise. The optimal bidding strategy is not affected by the presence or otherwise of the seller as a phantom bidder. Phantom bids have the same consequences for bidders as “genuine” rival bids: if a phantom bid stands at $t = 1$, the object is passed in for re-auctioning.

Suppose that a new bidder arrives and finds a standing bidder in residence. Since re-entry is precluded, it is optimal for the arriving bidder to remain in the bidding war with the standing bidder unless her (the new arrival’s) valuation is exceeded.⁹ In fact, even if re-entry within the current round were allowed, there would be nothing for the arrival to gain by doing otherwise. She might drop out of the current bidding war “early”, but will always find it optimal to re-enter before $t = 1$ and bid until her valuation is reached (unless it has already been exceeded in the meantime). The assumption that bidders may not re-enter avoids the unnecessary complication of these timing issues.¹⁰

Next, consider the decision of an arrival in the absence of a standing bidder. If a public reserve has been posted, this is the *de facto* standing bid, so the new bidder will either depart (if her value is no greater than this), or bid the reserve (plus epsilon) to become the standing bidder.

In a cheap-talk-reserve-price auction, if no standing bidder is in residence, the effective standing bid is zero. However, if bidders anticipate that the seller has a non-zero secret reserve r , a first arrival will optimally bid the secret reserve (plus epsilon) if her value exceeds r . If she did not, she will expect to “lose” the auction, and fail to acquire the object at a price less than her valuation. If her value is no greater than r , then she has several optimal courses of action. She may choose not to bid, or to submit any bid less than r . None of our results depends upon which option the bidder chooses in this situation, as will be clear.

Finally, consider an arriving bidder who wins a bidding war with the standing bidder in a cheap-talk-reserve-price auction. If the bidder believes that the seller has a secret reserve of r , and the previous standing bidder dropped out of the bidding war before bidding reached r , then,

⁹ If the new arrival’s value is no greater than the standing bid, she departs without bidding.

¹⁰ Some authors, such as Roth and Ockenfels (2001), have observed a tendency for “sniping”, or last-minute bidding, in on-line auctions. However, the use of proxy bidding in eBay auctions avoids this. Other on-line auctions employ “soft” endings to the same effect. Auctions may be extended for short periods (say, 5 minutes) until no further activity is recorded; a sort of virtual “going...going..gone!”. This, too, restores the usual second-price logic, and eliminates any incentive to “snipe”. See Lucking-Reiley (2000).

as in the previous case, the new bidder will submit one further bid equal to r (plus epsilon). This may happen, for example, if the previous standing bidder was the first arrival, and had a value less than r (see the previous paragraph).

In summary, let r denote the reserve price anticipated by bidders: r is the public reserve for auctions in which public reserves are posted and binding; or the anticipated secret reserve in a cheap-talk-reserve-price format. A bidder with value $v \in [r, 1]$ will submit bids in $[r, v]$ just high enough to meet the reserve and remain the standing bidder. If bidding exceeds v they will drop out of the auction. Bidders with values in $[0, r)$ expect zero surplus from participation in the auction. They may choose not to bid, or else submit a bid in $[0, r)$.

3.2 The seller's decision problem

Observe that the seller's problem is stationary: it looks the same at the start of each auction round. Leaving aside seller bidding for the moment, the seller's problem is to choose a reserve price, or acceptance rule, for each auction period. Because of the stationarity of the problem, we assume that the same reserve price is applied in each period.

In a cheap-talk-reserve-price auction, the reserve is secret, so we may think of the seller deciding on her acceptance rule at $t = 1$: she may accept or reject the standing bid at that time.¹¹ With a public reserve price, the acceptance rule is set at $t = 0$. Moreover, the reserve will affect the bidders' strategies, as described above. In each case, however, the institutional assumptions we have made impose the restriction that the acceptance rule will take the form of a cut-off (reserve) price, r , such that the standing bid at $t = 1$ is accepted iff it is at least r . This restriction is innocuous.

If seller bidding is allowed, the seller has a much more complex strategy space. In addition to setting a (secret) reserve, she also chooses a bid function that may depend on the elapsed time in the current round, the reserve, and the level of the last genuine bid submitted. By stationarity, we may assume that the seller employs the same bidding strategy in any round, but the range of potential bid functions is still very large. Fortunately, there exists an optimal seller bidding

¹¹ There is also no advantage from separately announcing the reserve, as it has no signalling value in the IPV context, and is thus ignored by bidders: it is "cheap talk".

strategy with a very simple form. We derive this strategy in subsection 3.5.

3.3 The value of a public reserve

Let us first compare, from the seller's point of view, the cheap-talk-reserve-price format to an auction with a public-(binding-)reserve-price. Is there value to the seller from committing to a (public) reserve price?

For a cheap-talk-reserve-price auction, the seller's strategy involves choosing an acceptance price (or reserve), r , being the lowest price at which she is prepared to sell the object in any round. This choice will depend on the bidders' strategies, which in turn depend on their expectation of the reserve price, which we denote r^e .¹² Let $V_0(r, r^e)$ be the expected continuation value of the auction game to the seller, from $t = 0$ in some round, as a function of the actual and anticipated reserve, given the bidder strategies described in subsection 3.1.

We impose the usual equilibrium conditions that players maximise their expected payoffs at all points in the game, and beliefs about rival strategies are correct. Given our construction of bidder strategies, this is equivalent to imposing the following two additional conditions:

$$r = r^e \tag{1}$$

$$r = V_0(r, r^e) \tag{2}$$

Condition (1) says that bidder expectations of the reserve are correct. Condition (2) ensures that the seller sets the reserve optimally: she plans to accept the standing bid at $t = 1$ iff it is at least as high as the discounted expected value of re-auctioning. That is, condition (2) is the seller's best response equation (where r^e is given) determining r .

Defining $v_0(r) \equiv V_0(r, r)$, which is the discounted expected value from committing to a reserve, we may summarise (1) and (2) in the single condition

$$r = v_0(r) \tag{3}$$

¹² In equilibrium, expectations will be correct, so here we assume that all bidders form the same expectation.

Equation (3) says that the seller’s optimal reserve is equal to the value of re-starting the auction game, given the bidders’ optimal responses to this reserve. If bidding fails to reach v_0 , it is better to pass in the object; and accepting a bid above v_0 is preferable to re-auctioning the object.

Proposition 1 *There exists an equilibrium secret reserve, and all solutions to (3) are interior to $[0, 1]$.*

The formal proof appears in the appendix but to sketch the main idea of the proof we first note that it is intuitive from the definitions that $v_0(0) > 0$ and $v_0(1) = 0$. That is, the seller expects a non-zero surplus in a “no reserve” auction; and setting a reserve of $r = 1$ ensures that the probability of achieving a sale in any given round is zero. Since v_0 is continuous, the result follows by application of the intermediate value theorem to the function $r - v_0(r)$.

For the public-(binding-)reserve-price scenario, the seller’s strategy again consists of choosing an acceptance price (reserve) r . However, r must now be stated publicly, and committed to, at the start of the auction round. The optimal reserve price is therefore found by maximising $v_0(r)$ with respect to r . Thus, an optimal reserve solves

$$\max_{r \in [0,1]} v_0(r) \tag{4}$$

Continuity of v_0 ensures the existence of a solution to (4).

The seller can clearly do no worse in the public-(binding-)reserve-price auction, since the value of the cheap-talk-reserve-price auction is just $v_0(r)$ evaluated at an r that satisfies (3). The seller would rather choose the *best anticipated* reserve, than choose a reserve that is *anticipated* and best given this *fixed bidder expectation*. Hence we have:

Proposition 2 *From the seller’s point of view, the public-reserve-price scenario is preferable to the cheap-talk-reserve-price scenario.*

In section 4 we shall verify that the public-reserve-price format is *strictly* preferable when bidder valuations are Uniformly distributed. We shall also be able to quantify the value to the seller from commitment to a public reserve.

In general, the optimal public reserve will be higher than the equilibrium secret reserve price. Raising a reserve is more beneficial if done publicly, since it induces higher bids from high-value arrivals. Conversely, reductions in reserve are best kept secret, to raise the chances of a sale without encouraging bid reductions from sole high-value bidders. This creates a tendency for reserves to be higher when public and binding, than when they are secret. More precisely:

Proposition 3 *Suppose that $v_0(r)$ is “uni-modal” in the following sense: there exists some $\hat{r} \in [0, 1]$ such that v_0 is weakly increasing on $[0, \hat{r}]$ and weakly decreasing on $[\hat{r}, 1]$. Then, if r^* is an optimal public reserve and r^{**} is an equilibrium secret reserve price, $r^* > r^{**}$.*

Although the formal proof of Proposition 3 appears in the appendix, it is instructive to ‘verify’ the result algebraically. For this purpose, let

$$N(r) = \lambda[1 - F(r)]r + \lambda^2 \int_r^1 z f(z) [1 - F(z)] e^{\lambda[F(z) - F(r)]} dz$$

and

$$D(r) = e^{\rho + \lambda[1 - F(r)]} - 1.$$

In the proof of Proposition 1, we show that $v_0(r) = N(r)/D(r)$ (see equation (26) in the appendix). An optimal public reserve must satisfy the necessary condition

$$\frac{N(r)}{D(r)} = \frac{N'(r)}{D'(r)}.$$

By direct calculation:

$$\frac{N'(r)}{D'(r)} = \frac{N(r) + r - \frac{[1 - F(r)]}{f(r)}}{D(r) + 1}$$

Therefore

$$v_0(r^*) = r^* - \frac{[1 - F(r^*)]}{f(r^*)} \tag{5}$$

(where r^* denotes an optimal public reserve).¹³ Since

$$r - \frac{[1 - F(r)]}{f(r)} \leq r \quad (6)$$

uni-modality of v_0 implies that the optimal public reserve must (weakly) exceed the optimal secret reserve. In fact, r^* must *strictly* exceed r^{**} . For $r^* = r^{**}$ we need the common value to be 1. This follows from (3), (5) and the fact that F is strictly increasing. But $r^{**} = 1$ is ruled out by Proposition 1.

Since Propositions 1 and 3 further imply that $r^* > 0$, we see that any solution to (4) will lie in $(0, 1)$ under the “uni-modality” assumption. In fact, since it is obvious that $v_0(r^*) > 0$, (5) implies

$$r^* > \frac{[1 - F(r^*)]}{f(r^*)} \quad (7)$$

Given information about F , this may be used to impose even tighter bounds on r^* (see Proposition 9 below). Summarising the discussion:

Corollary 4 *When v_0 is uni-modal in the sense of Proposition 3, any optimal public reserve r^* is interior to $[0, 1]$ and satisfies (7).*

Finally, let us consider the dependence of the optimal public reserve on λ , the average number of bidders per round. Observe that

$$V(r) \equiv \lim_{\rho \rightarrow \infty} e^\rho v_0(r) = \frac{\lambda [1 - F(r)] r + \lambda^2 \int_r^1 z f(z) [1 - F(z)] e^{\lambda[F(z) - F(r)]} dz}{e^{\lambda[1 - F(r)]}}$$

is the (undiscounted) expected revenue from a *one-shot* auction with public reserve r . Some simple algebra gives

$$V'(r) = 0 \iff r = \frac{1 - F(r)}{f(r)} \quad (8)$$

¹³ Re-arranging (5) gives

$$r^* = v_0(r^*) + \frac{[1 - F(r^*)]}{f(r^*)}$$

This matches the familiar expression for the optimal reserve in a one-shot IPV auction – see, for example, Riley and Samuelson (1981, Proposition 3) – except that the seller’s valuation of the object, $v_0(r^*)$, is now endogenous.

Therefore, the optimal public reserve in a one-shot auction is independent of λ . In fact, this should come as no surprise. It is well known that the optimal reserve in a one-shot, IPV auction with n bidders is equal to (8) for any n (Myerson (1981) and Riley and Samuelson (1981)).

However, when the seller is non-myopic (finite ρ), the optimal public reserve will, in general, depend on λ .¹⁴

3.4 The surplus maximising reserve price

In a one-shot auction, any non-zero reserve price is surplus reducing, since there is some chance that no trade occurs. Hence, privately optimal reserve price commitments are necessarily too high from a social efficiency point of view. With a repeated auction, this is no longer the case. Unless the reserve price is set at unity – which is precluded by Propositions 1 and 4, at least in the uni-modal case – the object will sell with probability one. The efficiency issue concerns whether the sale occurs too quickly or too slowly.

On the one hand, as in the one-shot case, sellers have an incentive to set inefficiently high reserves to increase their share of surplus. However, there is also a countervailing incentive. Because sellers anticipate only a fraction of any future surplus, they are less patient than a Social Planner. The value of refusing to allocate the object to the highest value arrival in the current round is greater for a Social Planner, who anticipates the entire expected future surplus, than for a seller, who expects to give some of the surplus away to a future buyer. Therefore, privately optimal reserves may be too high or too low.

To be more precise, let us consider a Social Planner who must respect all the exogenous constraints of the auction mechanism, but not the informational constraints on allocation. The Social Planner discounts at the same rate as the seller. At the end of any period, the Planner may therefore choose to allocate the object to any one of the arrivals during that period, or else wait one more round. Which allocation rule maximises expected total surplus?

As usual, stationarity implies that the Planner's rule will consist of a cut-off, r , such that the good is allocated to the highest value arrival during the current period if and only if the highest

¹⁴ For the Uniform case, see Figure 1.

value exceeds r . Note, therefore, that the Social Planner's cut-off is a *valuation*, while the seller's is a *bid level*. However, in each case, the good is allocated in the current round if and only if a bidder arrives whose valuation exceeds the cut-off. In this sense, they are directly comparable for the purposes of determining the *allocative efficiency* of the auction.

Let $v_S(r)$ denotes the Planner's value function. The Social Planner problem is of the "optimal stopping" variety. In each period, a maximum surplus is drawn at random according to some density h on $[0, 1]$, with associated cumulative distribution function H .¹⁵ The Planner's problem is to determine a cut-off level of surplus at which to stop the process. In general:

$$\frac{1}{\delta}v_S(r) = [1 - H(r)]\mathbb{E}[z | z \geq r] + H(r)v_S(r) \quad (9)$$

Thus, using the fact that

$$\frac{d}{dr}\mathbb{E}[z | z \geq r] = \frac{-h(r)}{1 - H(r)}\{r - \mathbb{E}[z | z \geq r]\},$$

we have

$$\frac{1}{\delta}v'_S(r) = -h(r)r + H(r)v'_S(r) + h(r)v_S(r).$$

Therefore, $v'_S(r) = 0$ iff $v_S(r) = r$. This, too, is as one would expect. The Planner should allocate the object whenever the highest value arrival exceeds the continuation value, and this cut-off should also maximise the value function.

In the present situation, one may show that¹⁶

$$v_S(r) = \frac{\delta \left[1 - re^{-\lambda[1-F(r)]} - \int_r^1 e^{-\lambda[1-F(z)]} dz \right]}{1 - \delta e^{-\lambda[1-F(r)]}} \quad (10)$$

If we let $\hat{N}(r)$ denote the numerator of (10) and $\hat{D}(r)$ the denominator, one easily shows that

$$\frac{\hat{N}'(r)}{\hat{D}'(r)} = r$$

¹⁵ In the following calculations we treat these as continuous distributions for simplicity. Of course, in the auction example they are not. In particular:

$$h(z) = \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} g_1^n(z) = \lambda f(z) e^{-\lambda[1-F(z)]}$$

for $z > 0$, with an atom of size

$$\Pi_1(0) = e^{-\lambda}$$

at $z = 0$. However, we shall shortly verify that the same results obtain in this case.

¹⁶ Details are in the Appendix:

so

$$\frac{\hat{N}'(r)}{\hat{D}'(r)} = \frac{\hat{N}(r)}{\hat{D}(r)} \quad \text{iff} \quad \frac{\hat{N}(r)}{\hat{D}(r)} = r$$

That is, \hat{r} maximises $v_S(r)$ if and only if \hat{r} solves $v_S(r) = r$. Hence, the presence of atoms in the distribution of maximum surplus does not affect the basic optimal stopping logic.

We may immediately conclude that the optimal secret reserve is inefficiently low. It solves $v_0(r) = r$. Since the socially optimal reserve satisfies $v_S(r) = r$, and $v_S(r) \geq v_0(r)$ for all r , the optimal secret reserve is below the socially optimal reserve. The reason is easy to see. Consider an auction round that ends with a high bid equal to r (which in turn is equal to the high bidder's valuation). Accepting the current high bid earns r for the seller and the Social Planner, while passing-in earns the former $v_0(r)$ and the latter $v_S(r)$. Since the Social Planner anticipates the whole expected surplus from future trade, while the seller only anticipates a fraction, the Social Planner is more patient. The CTRP auction sells too quickly.

It is not so clear whether the optimal *public* reserve is too high or too low. Note that

$$v'_S(r) \propto [1 - \delta\Pi_1(r)] \delta S'(r) + \delta^2\Pi'_1(r)S(r)$$

$$v'_0(r) \propto [1 - \delta\Pi_1(r)] \delta R'(r) + \delta^2\Pi'_1(r)R(r)$$

Since $\Pi'_1(r) \geq 0$, and $S(r) \geq R(r)$, we see that

$$\delta^2\Pi'_1(r)S(r) \geq \delta^2\Pi'_1(r)R(r) \geq 0$$

This reflects the extra patience of the Social Planner: she has greater incentive to increase r in order to delay allocation. On the other hand:

$$[1 - \delta\Pi_1(r)] \delta S'(r) \leq 0$$

and

$$[1 - \delta\Pi_1(r)] \delta S'(r) \leq [1 - \delta\Pi_1(r)] \delta R'(r)$$

That is, raising r lowers $S(r)$, since it lowers the chances of an allocation being made in any period, without affecting the expected value of the highest-value arrival. In fact:

$$S'(r) = -\lambda f(r) r e^{-\lambda[1-F(r)]} \leq 0$$

For the seller, there are gains as well as losses from raising r . First, there is the reduced probability of sale, as for the Social Planner. However, since the seller does not always get all the surplus from a sale, the expected losses are smaller than for the Social Planner. Second, raising r increases the seller's portion of surplus when exactly one bidder arrives with a value above the reserve. By direct calculation:

$$R'(r) = S'(r) + \lambda[1 - F(r)][1 - (1 - r)\lambda f(r)] e^{-\lambda[1 - F(r)]}.$$

It is therefore not clear in general whether the optimal public reserve price is too high or too low from a surplus-maximisation point of view. In section 4, we examine the Uniform case.

3.5 Public reserves versus seller bids

A seller bid submitted at time t has the same effect on bidder behaviour as imposing a public reserve at t equal to the phantom bid. The ability to make phantom bids is therefore equivalent to the ability to set a public reserve at $t = 0$ and alter it continuously throughout the auction. Hence, the seller-phantom-bidding format is the best of the three, and it follows that the seller is happy to forego access to a reserve price commitment technology provided she can bid in the auction.

Of course, many auctions, including on-line eBay auctions, have rules against seller bidding. The extent to which these rules are enforceable, especially in the context of virtual auctions, is highly debatable. Surprisingly, however, in our framework these regulations are entirely redundant anyway, as the seller is *completely indifferent* between the seller-phantom-bidding and public-reserve-price formats. The additional strategic freedom afforded by seller bidding – essentially, the freedom to use a “flexible reserve” – adds no additional value.

Graham, Marshall and Richard (1990) showed that in a one-shot, second price, IPV auction, if the seller has the opportunity to bid once all genuine bidding activity has ceased, her optimal bid is independent of the standing bid. We extend this result to our framework. The incentive to phantom bid at t is shown to be independent of both the current standing bid and t , even though additional bidding activity may occur in $(t, 1]$ and there is the possibility of re-auctioning the object.

Before stating this result formally, let us first define a suitable value function for a seller-phantom-bidding auction. Once again, by the stationarity of the problem, the value function will not depend on the period, but it will depend on the time t that has elapsed in the current period. Therefore, if t units of time have elapsed in the current round, and there is a standing bid b , we let $\hat{v}_t(r, b)$ denote the seller's expected continuation value when she nominates a public reserve price of r at t and uses an optimal continuation strategy thereafter.

We shall make use of the following “regularity assumption”:

Assumption 5 *The value function $\hat{v}_t(r, b)$ is differentiable in r for any (t, b) and any $r \in (b, 1)$. Moreover,*

$$\frac{\partial \hat{v}_t(\hat{r}, b)}{\partial r} = 0$$

is a sufficient condition for $\hat{r} \in (b, 1)$ to maximise $\hat{v}_t(\cdot, b)$ on $(b, 1]$.

In particular, $\hat{v}_t(r, b)$ is continuous in r , so the *ex ante* value of the seller-phantom-bidding auction is

$$\max_{r \in [0, 1]} \hat{v}_0(r, 0) \tag{11}$$

Under this regularity condition (which in the next section we show holds for the case of uniformly distributed values) we obtain the following equivalence result.

Proposition 6 *Under Assumption 1, the seller-phantom-bidding and public-reserve-price auction formats generate the same ex ante expected revenue for the seller. That is:*

$$\max_{r \in [0, 1]} \hat{v}_0(r, 0) = \max_{r \in [0, 1]} v_0(r) \tag{12}$$

3.6 Optimal auction length

Until now, we have normalised the length of auctions to one unit of time. However, on-line auctions often allow sellers to choose their duration, at least within some bounds.¹⁷ It is therefore of interest to endogenise the length of auction rounds.

Let T denote this parameter. The seller's value function becomes

$$v_0(r, T) = \frac{\lambda T [1 - F(r)] r + (\lambda T)^2 \int_r^1 z f(z) [1 - F(z)] e^{\lambda T [F(z) - F(r)]} dz}{e^{\{\rho + \lambda [1 - F(r)]\} T} - 1} \tag{13}$$

¹⁷ Sellers who use eBay have the option of 3,5,7 or 10 day auctions.

It is clear that all of the preceding results apply (*pari passu*) for any given T . The question we address here is what value of T will the seller choose under each of the cheap-talk-reserve-price and public-reserve-price formats?

By familiar logic, a seller who uses a public reserve will choose (r, T) to maximise $v_0(r, T)$. With a secret reserve, for any T , r must satisfy

$$r = v_0(r, T) \tag{14}$$

Assuming that this has a unique solution $r(T)$ for each T , T will be chosen to maximise $v_0(r(T), T)$.

We shall allow T to be chosen from \mathbb{R}_+ . That is, duration is infinitely divisible, and $T = 0$ is a possible choice. With regard to the latter assumption, what does it mean to run an auction lasting zero units of time? Observe that the value function (13) is not even defined when $T = 0$.

Formally, we may simply define $v_0(r, 0)$ as follows

$$v_0(r, 0) \equiv \lim_{T \rightarrow 0} v_0(r, T) \tag{15}$$

However, an alternative, more intuitive, definition is the following. Since the probability of two or more bidders arriving at the same instant in time is zero, instantaneous re-auctioning ($T = 0$) means that the object is offered for sale to the first person willing to pay the reserve price, r . The object will therefore be sold to the first arriving bidder whose value exceeds r , and the sale price will be exactly r . Thus:

$$v_0(r, 0) = \int_0^\infty e^{-\rho t} r \mathcal{P}_r(t) dt$$

where $\mathcal{P}_r(t)$ is the density associated with the arrival time of the first bidder with a valuation that exceeds r . Since bidders with values in excess of r arrive according to a Poisson process with mean $\lambda [1 - F(r)]$, it is well known that arrival times will have the exponential density

$$\mathcal{P}_r(t) = \lambda [1 - F(r)] e^{-\lambda t [1 - F(r)]}.$$

Therefore:

$$\begin{aligned}
v_0(r, 0) &= r\lambda [1 - F(r)] \int_0^\infty e^{-\rho t - \lambda[1 - F(r)]t} dt \\
&= r\lambda [1 - F(r)] \left[\frac{e^{-\rho t - \lambda[1 - F(r)]t}}{-(\rho + \lambda[1 - F(r)])} \right]_{t=0}^\infty \\
&= \frac{r\lambda [1 - F(r)]}{\rho + \lambda[1 - F(r)]} \tag{16}
\end{aligned}$$

Re-writing (16), we have:

$$\rho v_0(r, 0) = \lambda [1 - F(r)] [r - v_0(r, 0)] \tag{17}$$

This expression is a Bellman equation familiar from the literature on optimal search.¹⁸ The imputed instantaneous rate of income accumulation, $\rho v_0(r, 0)$, is equal to the instantaneous arrival rate, $\lambda [1 - F(r)]$, of bidders with value above r , multiplied by the net benefit from such an arrival, $r - v_0(r, 0)$.

Let us first observe that definitions (15) and (16) are equivalent. Using definition (16) for $v_0(r, 0)$, we may write

$$\begin{aligned}
\frac{v_0(r, T)}{v_0(r, 0)} &= \\
&\frac{\lambda [1 - F(r)] r + T\lambda^2 \int_r^1 z f(z) [1 - F(z)] e^{\lambda T[F(z) - F(r)]} dz}{r\lambda [1 - F(r)]} \times \left[\frac{\{\rho + \lambda [1 - F(r)]\} T}{e^{\{\rho + \lambda [1 - F(r)]\} T} - 1} \right]
\end{aligned}$$

Since the second term goes to 1 as $T \rightarrow 0$ (by L'Hôpital's rule), we see that

$$\begin{aligned}
\lim_{T \rightarrow 0} \frac{v_0(r, T)}{v_0(r, 0)} &= \lim_{T \rightarrow 0} \frac{\lambda [1 - F(r)] r + T\lambda^2 \int_r^1 z f(z) [1 - F(z)] e^{\lambda T[F(z) - F(r)]} dz}{r\lambda [1 - F(r)]} \\
&= 1 + \lim_{T \rightarrow 0} \frac{T\lambda^2 \int_r^1 z f(z) [1 - F(z)] e^{\lambda T[F(z) - F(r)]} dz}{r\lambda [1 - F(r)]} \\
&= 1.
\end{aligned}$$

¹⁸ See, for example, Mortensen (1986). We thank Guillaume Rocheteau for this observation.

Now that we have defined the value function $v_0(r, T)$ on $[0, 1] \times \mathbb{R}_+$ we may address the question of the optimal auction duration. For the case of a cheap-talk-reserve-price auction it is clear that $T = 0$ cannot be optimal:

Proposition 7 *For a cheap-talk-reserve-price auction, some $T > 0$ will be chosen.*

Proof. When $T = 0$, the equilibrium condition (14) is

$$r = \frac{r\lambda[1 - F(r)]}{\rho + \lambda[1 - F(r)]},$$

which is satisfied if and only if $r = 0$. Thus

$$v_0(r(0), 0) = 0.$$

Since we already know that $v_0(r(1), 1) > 0$, the result follows. \square

However, for a public-reserve auction, matters are not so clear. There are advantages and disadvantages to shortening the length of each auction round. On the one hand, lower T values increase the frequency with which the seller can decide whether to sell to the current standing bidder or review her reserve price. Shorter rounds therefore convey the benefit of increased flexibility. On the other hand, the shorter the rounds, the lower the probability of two or more arrivals, and hence the less fierce will be the competition amongst buyers. In the limit, when $T = 0$, competition vanishes altogether: the object is sold at the reserve price with certainty. It is not clear a priori which T will provide an optimal balance between these two effects. However, the case in which F is the Uniform distribution shows that zero-length auctions may be optimal with a public reserve (see section 4).

4 The case of Uniformly distributed values

Our objective in the present section is to identify the optimal secret and public reserve prices when F is the Uniform distribution. We may then determine the value (to the seller) from a commitment to a public reserve.

For the case of Uniformly distributed bidder valuations, $f(z) = 1$, $F(z) = z$. Hence, as we

show in the appendix, $v_0(r)$ may be expressed as:

$$v_0(r) = \frac{e^{\lambda(1-r)} \left(1 - \frac{2}{\lambda}\right) + \left(1 - 2r + \frac{2}{\lambda}\right)}{e^{\rho + \lambda(1-r)} - 1} \quad (18)$$

Straightforward calculations reveal that

$$v'_0(r) \geq 0$$

$$\Leftrightarrow 2\delta e^{-\lambda(1-r)} \geq [2\delta - \lambda(1 + \delta)] + 2\lambda r \quad (19)$$

(where $\delta \equiv \exp(-\rho)$). From this we may conclude:

Proposition 8 *When F is Uniform, v_0 is uni-modal, with $v'_0(0) > 0$ and $v'_0(1) < 0$ for any $(\lambda, \delta) \in \mathbb{R}_{++} \times (0, 1)$.*

Proof. Substituting $r = 1$ into (19) and observing that $\lambda(1 + \delta) < 2\lambda$ for any $(\lambda, \delta) \in \mathbb{R}_{++} \times (0, 1)$, we have $v'_0(1) < 0$.

Similarly, by substituting $r = 0$ into (19), we see that $v'_0(0) > 0$ provided

$$\begin{aligned} 2\delta e^{-\lambda} &> 2\delta - \lambda(1 + \delta) \\ \Leftrightarrow e^{-\lambda} &> 1 - \lambda \frac{(1 + \delta)}{2\delta}. \end{aligned}$$

This will be true for any $\delta \in (0, 1)$ if $e^{-\lambda} \geq 1 - \lambda$. But this in turn is true for any $\lambda > 0$ (as one easily sees by graphing each side of the expression). Hence, $v'_0(0) > 0$ for any $(\lambda, \delta) \in \mathbb{R}_{++} \times (0, 1)$.

Finally, since the left-hand side of (19) is convex in r , and the right-hand side is linear, uni-modality follows. \square

4.1 Optimal public reserve

Consider the public-reserve-price scenario. Corollary 4 and Proposition 8 imply that there exists an interior maximum for $v_0(r)$. In fact:

Proposition 9 *There is a unique optimal reserve in the public-reserve-price auction for any $(\lambda, \delta) \in \mathbb{R}_{++} \times (0, 1)$. This optimal reserve lies in $(0.5, 1)$, converging to 0.5 as $\lambda \rightarrow 0$ or $\delta \rightarrow 0$.*

Proof. Uniqueness of the optimal public reserve follows from the facts that the left-hand side of (19) is convex in r , while the right-hand side is linear.

The lower bound on the optimal reserve comes from Corollary 4, which implies

$$r^* > 1 - r^*$$

and hence $r^* > 0.5$.

Letting $r = 0.5$ in (19) gives

$$\frac{2\delta}{e^{\lambda/2}} \geq (2 - \lambda)\delta \tag{20}$$

Both sides of this expression converge to 2δ as $\lambda \rightarrow 0$, or to 0 as $\delta \rightarrow 0$. This proves that $r = 0.5$ is optimal in each of these limiting cases, since we have already established the existence of a unique interior maximiser of $v_0(r)$. \square

In a one-period version of this auction game, it is well-known that $r = 0.5$ is the optimal (enforceable) reserve for any $\lambda > 0$ (i.e. for any number of bidders). Hence, if $\delta \rightarrow 0$ the optimal reserve must converge to $r = 0.5$. A similar result obtains in a model in which the same bidders vie for the object in each round – see McAfee and Vincent (1997, p.250). However, in their model, the optimal reserve in period 1 converges to 0.5 *from below* as $\delta \rightarrow 0$ (*ibid.*, Figure 1(b)). When facing the same set of bidders each period, the inability to commit *not* to re-auction the object places downward Coasian pressure on the reserve.

If $\delta > 0$ and re-auction is possible, the seller has an incentive to raise her reserve above $r = 0.5$, since she receives a positive value even if the object is passed in. However, as the expected number of bidders in any given round goes to zero, the value of this re-auction option becomes negligible, so the optimal reserve again converges on $r = 0.5$.

Conversely:

Proposition 10 *For any $\delta \in (0, 1)$, the optimal public reserve converges to $r = (1 + \delta)/2$ as $\lambda \rightarrow \infty$; while for any $\lambda > 0$, the optimal public reserve converges to $r = 1$ as $\delta \rightarrow 1$.*

Proof. The left-hand side of (19) goes to zero as $\lambda \rightarrow \infty$. The right-hand side will only go to zero if $r \rightarrow (1 + \delta)/2$.¹⁹

As $\delta \rightarrow 1$, (19) converges to the condition

$$e^{\lambda r} \geq e^{\lambda} [1 - \lambda(1 - r)].$$

The left-hand side of this expression is strictly increasing and strictly convex, rising to e^{λ} as $r \rightarrow 1$. The right-hand side is linear and strictly increasing in r , also rising to e^{λ} as $r \rightarrow 1$. When $r = 1$ both sides have slope λe^{λ} . Therefore, the two sides are equal if and only if $r = 1$. \square

Contrast this result with McAfee and Vincent (1997, p.251 and Figure 1(d)). In their model, with values drawn from the Uniform distribution on $[0, 1]$, the optimal first period reserve also rises with the number of bidders, n , but converges to the static solution, $r = 0.5$. In our model, a higher arrival rate λ augments the incentive to pass-in the object and search for higher value bidders. This reduces the pressure on the seller to sell in any given round, and pushes the reserve further and further above its optimal one-shot value. In the McAfee and Vincent model, with the same bidders each period, increasing the number of bidders raises the reserve for a quite different reason. Higher n raises the pressure on bidders to bid early, allowing the seller to raise her reserve. In the limit, the competitive pressure on bidders completely overcomes the Coasian dynamic, and the optimal one-shot reserve is achieved.

In contrast to our Proposition 10, McAfee and Vincent (1997, p.251 and Figure 1(b)) obtain that the optimal reserve declines with increases in δ in their model.

4.2 Optimal secret reserve

Let us now turn to properties of the cheap-talk-reserve-price auction. The first observation is the following:

Proposition 11 *There exists a unique equilibrium reserve in the cheap-talk-reserve-price auction for any $(\lambda, \delta) \in \mathbb{R}_{++} \times (0, 1)$.*

¹⁹ This puts an upper bound on the optimal reserve price (for given δ), since it is intuitive that r is increasing in λ . Indeed, if $r > (1 + \delta)/2$, then the right-hand side of (19) strictly exceeds 2δ (for any $\lambda > 0$), while the left-hand side is no greater than 2δ for any $r \leq 1$ and $\lambda > 0$.

Proof. Use $v_0 = r$ in (18) and re-arrange the resulting expression to get

$$e^\lambda [z + 2\delta - \delta\lambda] = \delta e^z [2 + \lambda - z] \quad (21)$$

where $z = \lambda r$. The left-hand side of (21) is linear and strictly increasing in z , while the right-hand side is strictly increasing and strictly convex. Hence there will be at most two solutions to (21) in $z \in [0, \lambda]$. A necessary and sufficient condition for two solutions is that the LHS \geq RHS at $z = 0$ and $z = \lambda$. The latter is always the case, but the former is so if and only if

$$e^\lambda \geq \frac{2 + \lambda}{2 - \lambda} \quad (22)$$

In fact, one may verify that (22) fails for any $\lambda > 0$. Hence, the equilibrium secret reserve is unique. \square

Propositions 3 and 8 imply that the optimal secret reserve will be strictly smaller than the optimal public reserve. Other simple properties of the former are established in the following propositions.

First, it is immediate from (21) that:

Proposition 12 *For any $\lambda > 0$, the equilibrium secret reserve $r \rightarrow 0$ as $\delta \rightarrow 0$.*

This is just convergence to the optimal secret reserve in the one-shot auction.

4.3 Simulation results

4.3.1 The case $T = 1$

Although explicit solutions for the optimal secret and public reserves are elusive, we may easily obtain them through numerical simulation. Figures 1 and 2 give the optimal reserves for a wide range of (δ, λ) values.²⁰

²⁰ Recall that $\delta = e^{-\rho}$.

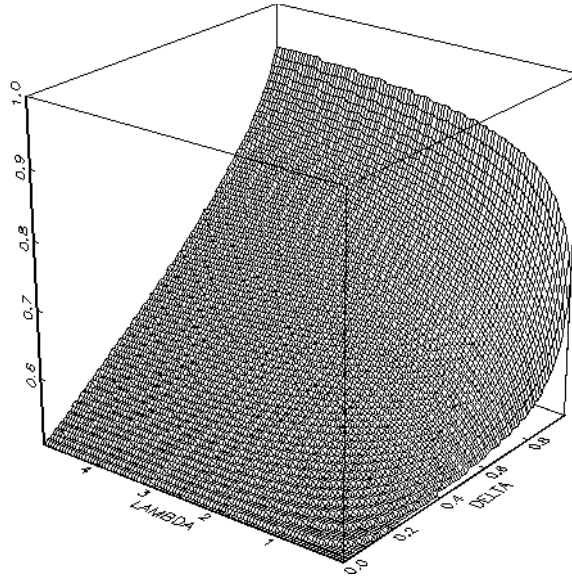


Figure 1: Optimal public reserve price

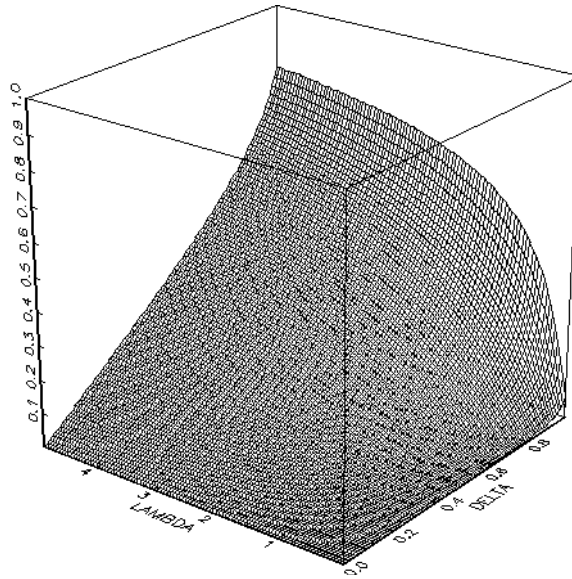


Figure 2: Equilibrium secret reserve price

The figures illustrate the behaviour of the reserves for limiting values of the parameters. In relation to Proposition 9, we observe that convergence of the optimal public reserve to 0.5 as $\lambda \rightarrow 0$ is comparatively slow. Figure 1 uses λ values that are bounded below by $\lambda = 0.1$. One verifies Proposition 9 by taking λ values down to $\lambda = 0.005$, as in Figure 3.

Similarly, Proposition 10 is also obscured by the scale Figure 1. Indeed, it is clear that Propositions 9 and 10 imply “extreme” behaviour of the surface near $(\delta = 1, \lambda = 0)$, with convergence to a non-smooth contour.

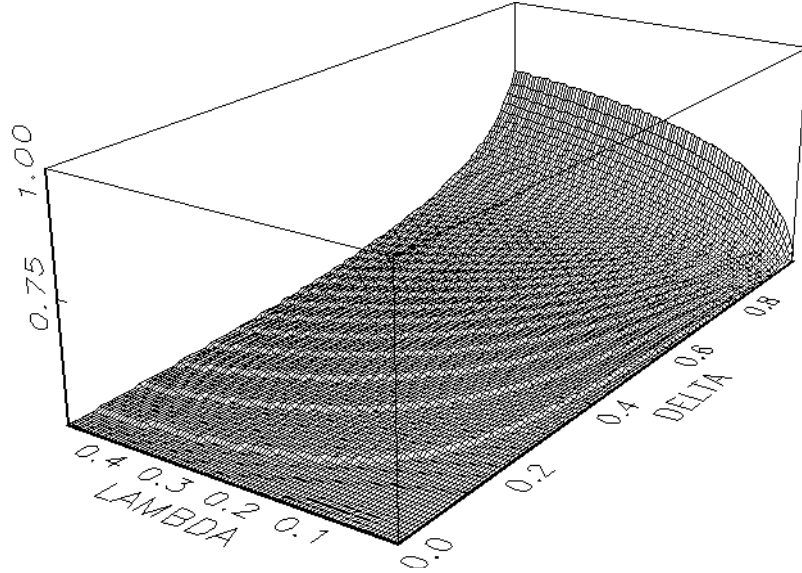


Figure 3: Optimal public reserve for low λ values

Figure 4 plots $r^{PRP} - r^{CTRP}$, where r^{PRP} denotes the optimal public reserve, and r^{CTRP} is the equilibrium secret reserve price. This is the increment in the reserve as a result of public announcements binding the seller.

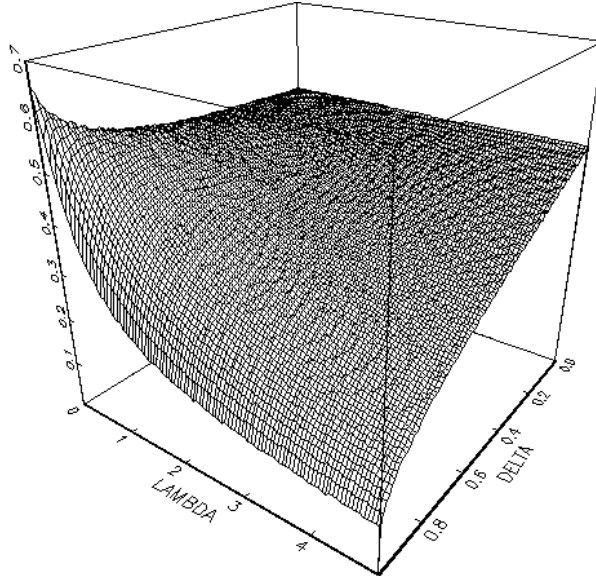


Figure 4: Public reserve less secret reserve

Figure 5 illustrates the value of this commitment to a public reserve. It plots the gain from commitment, as a percentage of the secret reserve value. That is:

$$100 \left[\frac{v_0(r^{PRP}) - v_0(r^{CTRP})}{v_0(r^{CTRP})} \right],$$

We observe that this value falls sharply as the average number of arrivals per period, λ , rises. The percentage gain is below 10% (for any δ) before λ reaches 5. Gains also tend to diminish with rises in δ , though much more slowly. In this sense, the inability to commit not to re-auction the object does devalue the commitment to a reserve price, but we do not see the dramatic decline in value observed by McAfee and Vincent (1997) in the case of long-lived bidders.

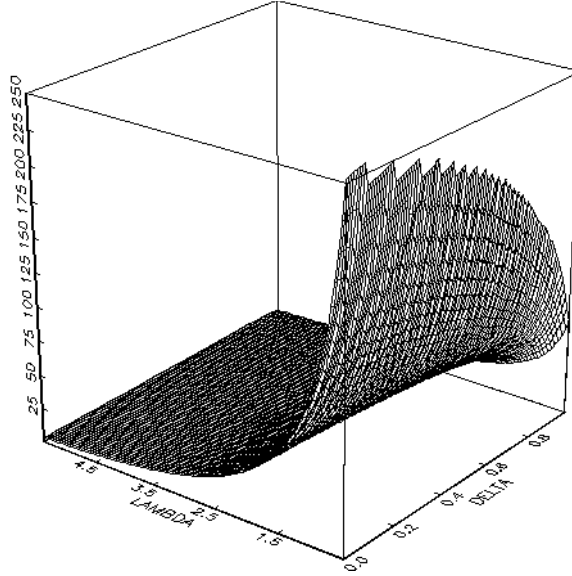


Figure 5: Value of commitment

Since public reserves are higher than secret reserves, on average it will take longer to sell an object using the public-reserve-price format. The probability of concluding a sale in any given period is $1 - \Pi_1(r)$, which is equal to $1 - e^{-\lambda(1-r)}$ in the Uniform case. Let $p(r, \lambda)$ denote this quantity. It is decreasing in r for any λ , as one would expect. Figure 6 indicates the increment in this probability from using a cheap-talk-reserve-price auction: $p(r^{CTRP}, \lambda) - p(r^{PRP}, \lambda)$.²¹

²¹ This difference depends on δ , as well as λ , since r^{CTRP} and r^{PRP} depend on both parameters.

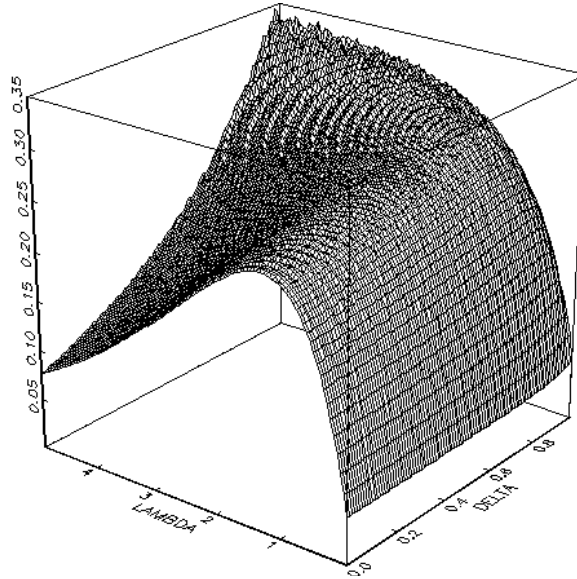


Figure 6: Extra per period sale probability from a secret reserve

The expected number of periods until a sale is achieved is $p(r, \lambda)^{-1}$. Figure 7 indicates the expected extra delay (in numbers of periods) from using the public-reserve-price format rather than a cheap-talk-reserve-price auction: $p(r^{PRP}, \lambda)^{-1} - p(r^{CTRP}, \lambda)^{-1}$.

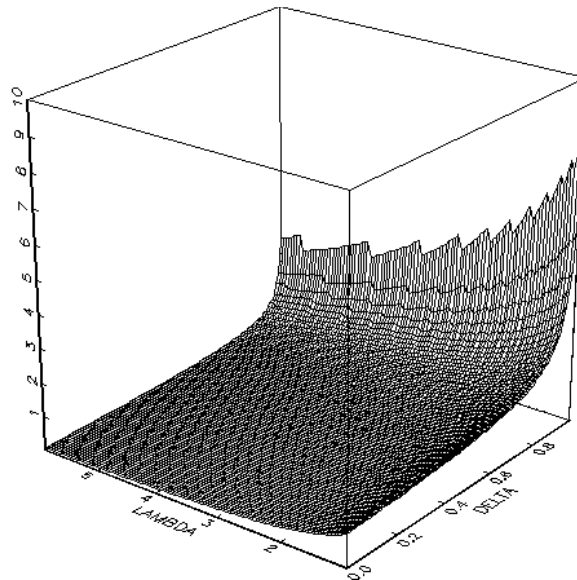


Figure 7: Expected extra delay from a public reserve

Finally, let us compare the optimal public reserve to the socially optimal reserve price. When F is Uniform on $[0, 1]$, the Social Planner's value function (10) becomes

$$v_S(r) = \frac{\delta \{1 - re^{-\lambda(1-r)} - e^{-\lambda(1-r)}(r - \lambda^{-1})\}}{1 - \delta e^{-\lambda(1-r)}} \quad (23)$$

Figure 8 reveals that the optimal public reserve is too high. It graphs the difference between the optimal public reserve and the maximiser of (23). Figure 9 illustrates the loss in efficiency from the PRP auction mechanism. It plots the difference between $v_S(r)$ evaluated at the socially optimal reserve and its value at r^{PRP} , expressed as a percentage of the latter.

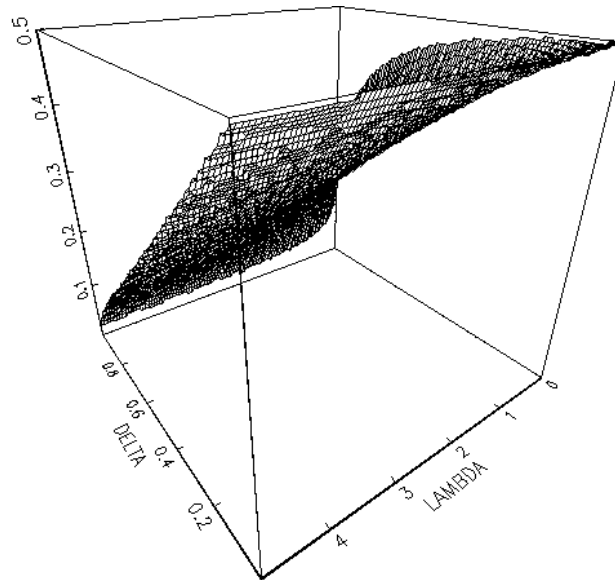


Figure 8: Optimal public reserve less socially optimal reserve.

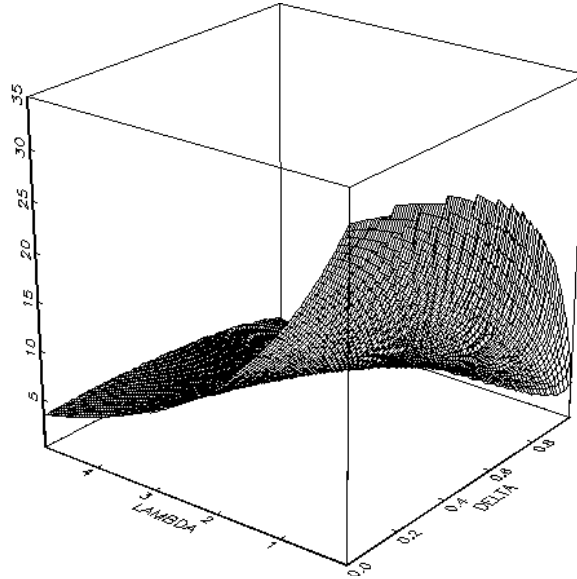


Figure 9: Percentage efficiency loss from the PRP mechanism.

4.3.2 Endogenous T

For the special case in which F is Uniform, Figure 10 plots the value function $v_0(r, T)$ when $\rho = 0.1$ and $\lambda = 3$, using (13).

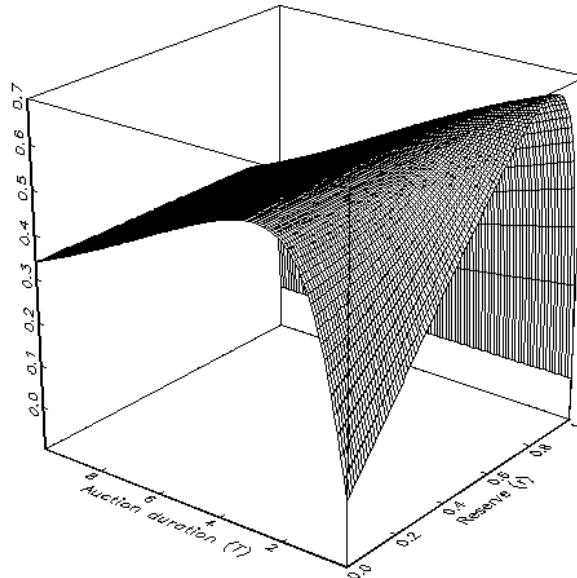


Figure 10: $v_0(r, T)$ when F Uniform, $\rho = 0.1$ and $\lambda = 3$

It is easy to observe that the optimal choice of (r, T) will have $T = 0$.²² In fact, if we plot

$$\max_{r \in [0,1]} v_0(r, T)$$

as a function of T (again assuming that values are Uniformly distributed, $\rho = 0.1$ and $\lambda = 3$) we obtain Figure 11. Furthermore, straightforward calculations reveal that $v_0(r, 0)$ is maximised when

$$r = (1 + \theta) - \sqrt{\theta(1 + \theta)} \quad (25)$$

(where $\theta = \rho/\lambda$). Hence, with Uniformly distributed valuations, the optimal public-reserve-price auction has $T = 0$ and reserve price (25). We therefore have a complete characterisation of the optimal auction (within the scenarios considered here) for the Uniform case.

The optimality of $T = 0$ in a public-reserve-price-auction should be interpreted with care. It relies on our assumption that there is no delay *between* auction rounds, and no other fixed (monetary) cost to running another auction. In reality, at least one, and probably both, of these assumptions will be violated. For example, eBay charges sellers a small fee for each auction they run. Time or other costs of running many auctions will provide pressure to increase T .

²² We obtain the same conclusion for all other parameter values that we have tried. In fact, one may show that

$$\left. \frac{dv_0(r^*(T), T)}{dT} \right|_{T \downarrow 0} < 0 \quad (24)$$

where $r^*(T)$ denotes the optimal public reserve given T . To do so, we first calculate

$$\begin{aligned} \frac{\partial v_0(r, T)}{\partial r} &= 0 \\ \Leftrightarrow e^{-[\rho + \lambda(1-r)]T} &= e^{-\rho T} + \frac{\lambda T}{2} [2r - 1 - e^{-\rho T}] \end{aligned}$$

Now substitute this into the expression resulting from the calculation

$$\frac{\partial v_0(r, T)}{\partial T} \geq 0,$$

divide through by T^2 and simplify to obtain

$$\begin{aligned} -[\rho + \lambda(1-r)] \left[\lambda + \lambda(1-2r)e^{-\lambda T(1-r)} + \frac{2(e^{-\lambda T(1-r)} - 1)}{T} \right] &\geq \\ \left[\frac{e^{\rho T} - 1}{T} - \frac{\lambda}{2} e^{-\rho T} + \lambda r \right] \left[\lambda(1-r)(\lambda T - 2) - \frac{2(e^{-\lambda T(1-r)} - 1)}{T} \right] &\geq \end{aligned}$$

Taking limits as $T \rightarrow 0$ we get

$$-[\rho + \lambda(1-r)] 2\lambda \geq 0$$

Since $\rho + \lambda(1-r) > 0$, (24) follows.

Figure 12 illustrates $v_0(r(T), T)$, the value of a cheap-talk-reserve-price auction as a function of T . This has a maximum at an auction length around 2.

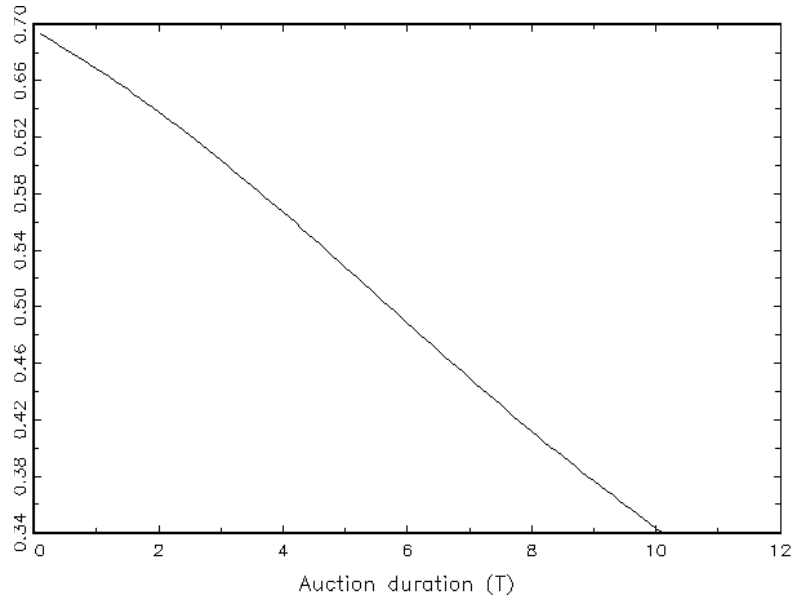


Figure 11: Auction value with a public reserve

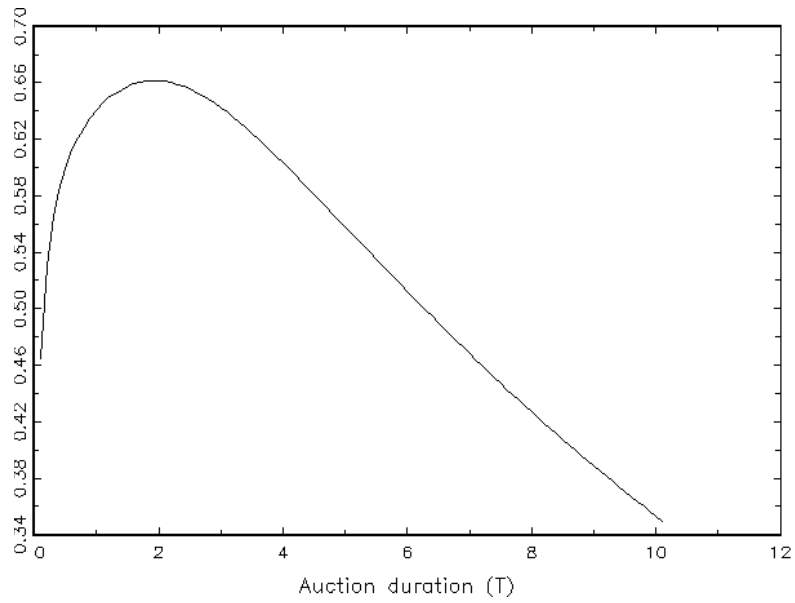


Figure 12: Auction value with a secret reserve

Appendix

Proof of Proposition 1. From the formal definition of $v_0(r)$ it follows that we may express $v_0(r)$ by $\exp(-\rho) R(r) / (1 - \exp(-\rho) \Pi_1(r))$ where $\Pi_1(r)$ is the probability that, in any given period, the object is passed in, and $R(r)$ is the expected revenue from any given auction round. Thus, $\Pi_1(r)$ is equal to the probability that no bidder arrives with a value in $[r, 1]$. The process of arrival of bidders with values in $[r, 1]$ is easily shown to be Poisson with mean $\lambda[1 - F(r)]$, so $\Pi_1(r) = \exp(-\lambda[1 - F(r)])$. The expected revenue per round is:

$$R(r) = \Pi_2(r)r + \int_r^1 \left(\sum_{n=2}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} g_2^n(z) \right) z dz,$$

where $\Pi_2(r) = \lambda[1 - F(r)] \exp(-\lambda[1 - F(r)])$ is the probability that the auction yields r – which is just the probability that exactly one bidder arrives with a valuation in $[r, 1]$. The function g_2^n is the density of the second order statistic from n random draws from the distribution F . In particular:

$$g_2^n(z) = n(n-1)f(z)F^{n-2}(z)[1 - F(z)].$$

Therefore, $\int_r^1 (\sum_{n=2}^{\infty} \exp(-\lambda) \lambda^n g_2^n(z) / n!) z dz$ is the expected value of the second-highest arrival with a value above r (the random variable defaulting to zero along sample paths for which there do not exist two such arrivals). The bracketed term in this expression may be simplified as follows:

$$\begin{aligned} \sum_{n=2}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} g_2^n(z) &= f(z)[1 - F(z)] \sum_{n=2}^{\infty} e^{-\lambda} \frac{\lambda^n}{(n-2)!} F^{n-2}(z) \\ &= f(z)[1 - F(z)] \lambda^2 \sum_{m=0}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!} F^m(z) \\ &= f(z)[1 - F(z)] \lambda^2 e^{-\lambda[1 - F(z)]} \sum_{m=0}^{\infty} e^{-\lambda F(z)} \frac{[\lambda F(z)]^m}{m!} \\ &= \lambda^2 f(z)[1 - F(z)] e^{-\lambda[1 - F(z)]} \end{aligned}$$

Hence:

$$v_0(r) = \frac{\lambda[1 - F(r)]r + \lambda^2 \int_r^1 z f(z)[1 - F(z)] e^{\lambda[F(z) - F(r)]} dz}{e^{\rho + \lambda[1 - F(r)]} - 1} \quad (26)$$

Notice that $v_0(0) > 0$, $v_0(1) = 0$ and $v_0(r)$ is a continuous function. Hence from the intermediate value theorem it follows there exists an $\hat{r} \in (0, 1)$ for which $\hat{r} - v_0(\hat{r}) = 0$ as required. \square

Proof of Proposition 3.²³ Suppose, contrary to what is to be shown, that $r^* \leq r^{**}$. As per the intuition given prior to the statement of the proposition, we claim that $h(r) \equiv V_0(r, r^{**})$ cuts $v_0(r)$ from above at $r = r^{**}$. Since $r^{**} \in (0, 1)$ (Proposition 1), the result follows once this claim is proved: v_0 is weakly decreasing at r^{**} (by uni-modality), so $V_0(r, r^{**})$ is *strictly* decreasing in r at $r = r^{**}$, and hence r^{**} cannot be an equilibrium secret reserve (the seller can do strictly better by lowering the reserve).

Consider some $r' \in (r^{**}, 1)$. Note that $r^{**} = v_0(r^{**}) \geq v_0(r')$ since v_0 is weakly decreasing at r^{**} . Hence, $r' > v_0(r')$, so sales at r' are desirable to the seller. The net gain to the seller is strictly greater if bidders know that the reserve has increased than if the increase is kept secret. To see this, first note that the level of the reserve matters only in situations in which exactly one bidder arrives whose valuation exceeds the reserve price. Thus, consider a round in which exactly one bidder arrives with a value in $[r', 1]$. Such rounds occur with strictly positive probability (since $r' < 1$). A binding announcement will cause these bidders to increase their bids from r^{**} to r' . If the increased reserve were secret, they would continue to bid r^{**} . This proves that

$$V_0(r', r^{**}) < v_0(r') \tag{27}$$

Conversely, if the reserve is lowered below r^{**} to some $r'' < r^{**}$, it is better to lower it in secret, at least for “small” reductions. Once again, we need only consider a round in which exactly one bidder arrives with a value $v \in [r'', 1]$, an event with strictly positive probability. If $v \in [r^{**}, 1]$, then the bidder will bid r'' if the new reserve is public, but will continue to bid r^{**} if it is secret. Hence, with respect to these bidders, there is a strict gain to the seller from a secret reduction relative to a public one. If $v \in (r'', r^{**})$, then r'' will be bid if the new reserve is public, but if the reduction in reserve is secret, the bidder’s behaviour is not uniquely determined. Believing that the reserve exceeds her value, she may choose not to bid at all, or to bid any amount less than r^{**} . Hence, the relative impact on seller expected revenue is ambiguous. However, we can always make $r^{**} - r''$ small enough that the behaviour of types in $[r^{**}, 1]$ have the dominant effect on

²³ We thank Rhema Vaithianathan for correcting an error in a previous “proof.”

seller revenue, and hence

$$V_0(r'', r^{**}) > v_0(r'') \quad (28)$$

Combining (27) and (28) verifies the claim made at the start of the proof, and so we are done.

□

Derivation of (10). Familiar reasoning²⁴ determines that

$$v_S(r) = \frac{\delta S(r)}{1 - \delta \Pi_1(r)}$$

where

$$\Pi_1(r) = e^{-\lambda[1-F(r)]}$$

is the probability that there is no arrival with a value above r during a period of length 1, and $S(r)$ is the expected total surplus generated in the current round (given that the object has not yet been allocated). The latter is

$$S(r) = \int_r^1 \left[\sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} g_1^n(z) \right] z dz$$

where $g_1^n(z) = n f(z) F^{n-1}(z)$ is the pdf of the highest of n random draws from distribution F .

We observe that $S(r) \geq R(r)$, so $v_S(r) \geq v_0(r)$, as one would expect. The difference represents the buyers' expected surplus from an auction with reserve r .

To evaluate $v_S(r)$, we first note that the square-bracketed term in the definition of $S(r)$ is

$$\sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} n f(z) F^{n-1}(z) = \lambda f(z) e^{-\lambda[1-F(z)]}$$

Therefore:

$$\begin{aligned} S(r) &= \int_r^1 \lambda f(z) e^{-\lambda[1-F(z)]} z dz \\ &= 1 - r e^{-\lambda[1-F(r)]} - \int_r^1 e^{-\lambda[1-F(z)]} dz \end{aligned}$$

Proof of Proposition 6. Define

$$h(r | b, t) = \frac{\partial \hat{v}_t(r, b)}{\partial r}$$

²⁴ See the proof of Proposition 1.

to be the marginal incentive to raise the phantom bid from r to $r + dr$ at time t when there is a (genuine) standing bid of $b \leq r$. Then

$$h(r | b, t) = \left[\frac{1 - F(r)}{1 - F(b)} \right] h(r | r, t) + \left[\frac{F(r) - F(b)}{1 - F(b)} \right] \int_t^1 \mathcal{P}_r(t' | t) h(r | r, t') dt'$$

where $\mathcal{P}_r(t + x | t)$ is the density describing the random variable x , being the elapsed time until the arrival of the first bidder with a value in excess of r , given no such arrivals by t . The arrival process of bidders with values above r is Poisson with mean $\lambda[1 - F(r)]$, so:

$$\mathcal{P}_r(t + x | t) = \lambda[1 - F(r)] e^{-\lambda[1 - F(r)]x}.$$

Therefore:

$$h(r | b, t) = \left[\frac{1 - F(r)}{1 - F(b)} \right] h(r | r, t) + \frac{\lambda[1 - F(r)][F(r) - F(b)]}{[1 - F(b)]} \int_t^1 e^{-\lambda[1 - F(r)](t' - t)} h(r | r, t') dt'$$

Defining $g_t(r) = h(r | r, t)$, we have $h(r | b, t) = 0$ iff

$$g_t(r) + \lambda[F(r) - F(b)] \int_t^1 e^{-\lambda[1 - F(r)](t' - t)} g_{t'}(r) dt' = 0 \quad (29)$$

Let r^* denote the optimal public reserve in a public-reserve-price auction. We claim that an optimal seller bidding strategy is to counter-bid r^* at any t at which there is a standing bid $b < r^*$. To verify our claim, it suffices to show that $g_t(r^*) = 0$ for all t when the seller adopts this strategy. It then follows that the optimal seller bidding strategy is equivalent to conducting an optimal public-reserve-price auction.²⁵

However, it is natural to suspect that $g_t(r^*)$ might strictly decrease with t . If there is less time remaining in the current round, the seller may wish to bid less aggressively. Nevertheless,

²⁵ In particular, the calculations here are based on the assumption that the seller makes a *single counter-bid* for each (b, t) , rather than engaging in a bidding war with the standing bidder. The independence of the optimal counter-bid with respect to the standing bid b implies that nothing is lost by excluding this alternative form of seller bidding strategy.

one may show that in fact $g_t(r^*)$ is independent of t . In particular, given a standing bid of r^* at $t = 1$, the optimal counter-bid solves

$$\max_{r \geq r^*} \left[\frac{1 - F(r)}{1 - F(r^*)} \right] r + \left[\frac{F(r) - F(r^*)}{1 - F(r^*)} \right] v_0(r^*)$$

since the seller's strategy ensures a continuation value of $v_0(r^*)$ from the start of the next round.

The first-order condition for this problem is

$$v_0(r^*) = r - \frac{[1 - F(r)]}{f(r)}$$

which is satisfied iff $r = r^*$ (recall equation (5)). Therefore, the optimal public reserve remains the optimal counter-bid at $t = 1$. So $g_t(r^*)$ does not decline strictly with t .

More generally, $g_t(r^*)$ may be decomposed as follows:

$$g_t(r^*) = \Gamma_t(r^*) - e^{-\{\rho + \lambda[1 - F(r^*)]\}(1-t)} \frac{f(r^*)}{[1 - F(r^*)]} v_0(r^*).$$

The first term, $\Gamma_t(r^*)$, represents the change in the expected revenue from the current round, while the second reflects the change in the probability that the object will not be sold in the current round. In the event of no sale, the seller's continuation strategy ensures a continuation value of $v_0(r^*)$. To determine $\Gamma_t(r^*)$, note that revenue is unaffected provided at least one new bidder arrives in $(t, 1]$ with a value above r^* . The impact on revenue is only non-zero if there are no such arrivals. Thus:

$$\Gamma_t(r^*) = e^{-\{\rho + \lambda[1 - F(r^*)]\}(1-t)} \left\{ \frac{f(r^*)r^*}{[1 - F(r^*)]} - 1 \right\}.$$

Substituting this into the expression for $g_t(r^*)$ and implies that $g_t(r^*) = 0$ iff

$$v_0(r^*) = r^* - \frac{[1 - F(r^*)]}{f(r^*)}.$$

Hence, the condition $g_t(r^*) = 0$ is independent of t , and is satisfied when r^* is the optimal public reserve (recall equation (5)).

Hence, an optimal strategy for the seller is to submit an optimal phantom bid at time $t = 0$, and play no further part in that round. This implies (12) and completes the proof. \square

Derivation of Equation 18. For the case of Uniformly distributed bidder valuations, $f(z) = 1$, $F(z) = z$. Hence, (26) becomes:

$$v_0(r) = \frac{\lambda(1-r)r + \lambda^2 \int_r^1 z(1-z) e^{\lambda(z-r)} dz}{e^{\rho+\lambda(1-r)} - 1}$$

The integral in the numerator may be evaluated using integration by parts:

$$\begin{aligned} \lambda^2 \int_r^1 z(1-z) e^{\lambda(z-r)} dz &= \lambda \int_r^1 z(1-z) \lambda e^{\lambda(z-r)} dz \\ &= \lambda \left[z(1-z) e^{\lambda(z-r)} \right]_r^1 - \int_r^1 (1-2z) \lambda e^{\lambda(z-r)} dz \\ &= -\lambda r(1-r) - \left[(1-2z) e^{\lambda(z-r)} \right]_r^1 - 2 \int_r^1 e^{\lambda(z-r)} dz \\ &= -\lambda r(1-r) + e^{\lambda(1-r)} \left[1 - \frac{2}{\lambda} \right] + \left(1 - 2r + \frac{2}{\lambda} \right) \end{aligned}$$

Therefore

$$v_0(r) = \frac{e^{\lambda(1-r)} \left(1 - \frac{2}{\lambda} \right) + \left(1 - 2r + \frac{2}{\lambda} \right)}{e^{\rho+\lambda(1-r)} - 1}$$

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