General Equilibrium Theory: Supplementary Note on Debreu's Theorem

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1 Introduction

The purpose of this note is to fill in a gap in the account by Mas-Colell (1985, page 193) on part of the proof by Debreu (1974), which is related to the strong axiom of revealed preference for excess demand functions.

2 Steps

2.1 Strong axiom of revealed preference

Let $z : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$.

2.1 Definition z satisfies the *strong axiom of revealed preference* (SARP for short) if the following condition is met: for every finite sequence $(p_1, p_2, \dots, p_N)$ in $\boldsymbol{R}^L_{++}$, if $p_1 \cdot z(p_2) \leq 0$, $p_2 \cdot z(p_3) \leq 0, \dots, p_{N-1} \cdot z(p_N) \leq 0$, and $p_N \cdot z(p_1) \leq 0$, then $z(p_1) = z(p_N)$.

Exercise 1 Prove that the above strong axiom of revealed preference is, indeed, equivalent to the strong axiom of revealed preference of Richter.

Exercise 2 Prove that SARP is equivalent to the following condition: for every finite sequence $(p_1, p_2, \dots, p_N)$ in $\boldsymbol{R}^L_{++}$, if $p_1 \cdot z(p_2) \leq 0$, $p_2 \cdot z(p_3) \leq 0, \dots, p_{N-1} \cdot z(p_N) \leq 0$, and $z(p_1) \neq z(p_N)$, then $p_N \cdot z(p_1) > 0$.

Exercise 3 Prove that SARP implies the following condition: for every finite sequence in $\boldsymbol{R}^L_{++}$, $p_1, p_2, \dots, p_N$, if $p_1 \cdot z(p_2) \leq 0$, $p_2 \cdot z(p_3) \leq 0, \dots, p_{N-1} \cdot z(p_N) \leq 0$, and $p_N \cdot z(p_1) \leq 0$, then $z(p_1) = z(p_2) = \cdots = z(p_N)$.

Exercise 4 Prove that if z satisfies SARP, then it is homogeneous of degree zero.

2.2 Definition Assume that z is homogeneous of degree zero. Then z satisfies the *very strong axiom of revealed preference* (VSARP for short) if the following condition is met: for every finite sequence $(p_1, p_2, \dots, p_N)$ in $\boldsymbol{R}^L_{++}$, if $p_1 \cdot z(p_2) \leq 0$, $p_2 \cdot z(p_3) \leq 0, \dots, p_{N-1} \cdot z(p_N) \leq 0$, and $p_N \cdot z(p_1) \leq 0$, then p_1 and p_N are proportional to each other (that is, there is no $\alpha > 0$ such that $p_1 = \alpha p_N$).

Exercise 5 Assume that z is homogeneous of degree zero. Prove that if z satisfies VSARP, then it satisfies SARP.

Exercise 6 Assume that if $p_1 \in \boldsymbol{R}^L_{++}$ and $p_2 \in \boldsymbol{R}^L_{++}$ are not proportional to each other, then $z(p_1) \neq z(p_2)$. Prove that z satisfies SARP if and only if it satisfies VSARP.

2.3 Proposition *Let $a : \boldsymbol{R}^L_{++} \to \boldsymbol{R}_{++}$. Define $az : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ by letting $az(p) = a(p)z(p)$ for every $p \in \boldsymbol{R}^L_{++}$. If z is homogeneous of degree zero and satisfies VSARP, and a is homogeneous of degree zero, then az is homogeneous of degree zero and satisfies VSARP.*

Proof of Proposition 2.3 This follows from the fact that $p_1 \cdot z(p_2) \leq 0$, $p_2 \cdot z(p_3) \leq 0, \ldots, p_{N-1} \cdot z(p_N) \leq 0$, and $p_N \cdot z(p_1) \leq 0$ if and only if $p_1 \cdot az(p_2) \leq 0$, $p_2 \cdot az(p_3) \leq 0, \ldots, p_{N-1} \cdot az(p_N) \leq 0$, and $p_N \cdot az(p_1) \leq 0$. ///

2.2 Projection

For each $\ell = 1, 2, \ldots, L$, denote by $e_\ell \in \boldsymbol{R}^L$ the ℓ-th unit vector, that is, the vector of which the ℓ-th coordinate is equal to one and the other coordinates are all equal to zero. Let $z_\ell(p)$ be the orthogonal projection of e_ℓ onto the hyperplane with normal p that goes through the origin. Then

$$z_\ell(p) = e_\ell - \frac{p^\ell}{\|p\|^2} p$$

for every $p \in \boldsymbol{R}^L_{++}$, where p^ℓ is the ℓ-th coordinate of p.

2.4 Proposition *The orthogonal projection $z_\ell : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ is homogeneous of degree zero and satisfies VSARP.*

Proof of Proposition 2.4 This proposition follows from the fact that z coincides with the demand function of a consumer whose consumption set coincides with the entire $\boldsymbol{R}^L$, preference relation is represented by $u_\ell(x) = -\|x - e_\ell\|^2$, and wealth level is equal to zero. ///

2.3 Decomposition of the aggregate excess demand

Let C be a compact subset of $\boldsymbol{R}^L_{++}$. Let $z : C \to \boldsymbol{R}^L$ be continuous and homogeneous and satisfy Walras' law (that is, $p \cdot z(p) = 0$ for every $p \in C$).

Exercise 7 Prove that there exists an $m \in \boldsymbol{R}$ such that

$$z(p) + \frac{m}{\|p\|} p \in \boldsymbol{R}^L_{++}$$

for every $p \in C$.

For each $\ell = 1, 2, \ldots, L$, denote by $a_\ell(p) \in \boldsymbol{R}^L$ the ℓ-th coordinate of $z(p) + (m/\|p\|)p$, where m is as in Exercise 7.

2.5 Proposition *For every $\ell = 1, 2, \ldots, L$, $a_\ell : C \to \boldsymbol{R}_{++}$ is homogeneous of degree zero. Moreover,*

$$z(p) = \sum_{\ell=1}^{L} a_\ell(p) z_\ell(p) \tag{1}$$

for every $p \in C$.

Proof of Proposition 2.5 The first part follows from the fact that $p \mapsto z(p) + (m/\|p\|)p$ is homogeneous of degree zero. As for the second, note that

$$z(p) + \frac{m}{\|p\|} p = \sum_{\ell=1}^{L} a_\ell(p) e_\ell.$$

Take the orthogonal projection of both sides onto the hyperplane with normal p that goes through the origin, then we obtain (1). ///

2.4 From excess demand to demand

Let C be a compact subset of $\boldsymbol{R}^L_{++}$. For each $\ell = 1, 2, \ldots, L$, let $z_\ell : C \to \boldsymbol{R}^L$ and $a_\ell : C \to \boldsymbol{R}_{++}$ be as in the previous subsection. Denote by $\succsim_\ell$ a complete and transitive binary relation (preference relation) on $\boldsymbol{R}^L$ that generates excess demand $a_\ell z_\ell : C \to \boldsymbol{R}^L$.

Exercise 8 Prove that there exists an $\omega_\ell \in \boldsymbol{R}^L_{++}$ such that $a_\ell z_\ell(p) + \omega_\ell \in \boldsymbol{R}^L_{++}$.

Define a binary relation $\succsim^*_\ell$ on $\boldsymbol{R}^L_+$ by letting, for all $x \in \boldsymbol{R}^L_+$ and $y \in \boldsymbol{R}^L_+$, $x \succsim^*_\ell y$ if and only if $(x - \omega_\ell) \succsim_\ell (y - \omega_\ell)$. Let $x^*_\ell : \boldsymbol{R}^L_{++} \times \boldsymbol{R}_+ \to \boldsymbol{R}^L_+$ be the demand function derived from $\succsim^*_\ell$. To be precise, however, $x^*_\ell(p, w)$ need not be defined for all $(p, w) \in \boldsymbol{R}^L_{++} \times \boldsymbol{R}_+$.

Exercise 9 Prove that for $p \in C$, $x^*_\ell(p, p \cdot \omega_\ell)$ is well defined and coincides with $a_\ell z_\ell(p) + \omega_\ell$.