Lecture Notes on General Equilibrium Theory

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Overlapping Generation Model

- Generations = consumers $i = 1, 2, \ldots$
  Born at the beginning of period i and dies at the end of period $i+1$, having utility functions $u_i(x_i) = u_i(x_{i,i}, x_{i+1,i}) = x_{i,i} + x_{i+1,i}$.

- The aggregate initial endowment vector $\bar{\omega} = (1, 1, \ldots)$.

- The following $(x_1^*, x_2^*, x_3^*, \ldots)$ and p constitute a price equilibrium:

$$
\begin{aligned}
x_1^* &= (1, 0, 0, 0, \ldots) \\
x_2^* &= (0, 1, 0, 0, \ldots) \\
x_3^* &= (0, 0, 1, 0, \ldots) \\
&\vdots
\end{aligned}
$$

$$p = (p_1, p_2, p_3, \ldots) \text{ with } p_1 \leq p_2 \leq p_3 \leq \cdots$$

Note that $p \cdot \bar{\omega} = p_1 + p_2 + p_3 + \cdots = \infty$.

- The allocation $(x_1^*, x_2^*, x_3^*, \ldots)$ is **Pareto-dominated** by

$$x_1 = (1, 1, 0, 0, 0, \ldots)$$
$$x_2 = (0, 0, 1, 0, 0, \ldots)$$
$$x_3 = (0, 0, 0, 1, 0, \ldots)$$
$$\vdots$$

Exercise: Prove that $(x_1, x_2, x_3, \ldots)$ is Pareto-efficient.

- If $p' = (p'_1, p'_2, p'_3, \ldots)$ satisfies $p'_1 = p'_2 \geq p'_3 \geq \cdots$, then $(x_1, x_2, x_3, \ldots)$ and p' constitute a price equilibrium. In particular, we can take

$$p'_1 = p'_2 = 1, p'_3 = 1/2, p'_4 = 1/4, \ldots, p'_i = 1/2^{i-2}, \ldots.$$

Then $p \cdot \bar{\omega} = 1 + 1 + 1/2 + 1/4 + \cdots = 3 < \infty$.

- In fact, the first welfare theorem is valid whenever $p \cdot \bar{\omega} < \infty$.

Second Welfare Theorem

- The quasi-utility maximization condition: $p \cdot x_i < p \cdot x_i^* \Rightarrow x_i^* \succsim_i x_i$.
  Implied by the utility max condition: $p \cdot x_i \leq p \cdot x_i^* \Rightarrow x_i^* \succsim_i x_i$.
  Implied by the cost min condition: $x_i \succsim_i x_i^* \Rightarrow p \cdot x_i \geq p \cdot x_i^*$ if $\succsim_i$ is complete.

- A subset A of $\boldsymbol{R}^L$ is said to be **convex** if $ta + (1-t)a' \in A$ for all $a \in A$, $a' \in A$, and $t \in [0, 1]$.

- Message of the second welfare theorem
  - If wealth (income) transfers are possible in a lump-sum manner, then every Pareto-efficient allocation can be realized as a price quasi-equilibrium.
  - "There is a dichotomy between equity and efficiency."

Proof of the Second Welfare Theorem

1. Let $(x_1^*, \dots x_I^*, y_1^*, \dots y_J^*)$ be a Pareto-efficient allocation. Define

$$\forall i : A_i = \{x_i \in X_i : x_i \succ_i x_i^*\},$$

$$A = \sum_i A_i = \left\{ \sum_i x_i \mid \forall i : x_i \in A_i \right\},$$

$$B = \left\{ \sum_j y_j + \bar{\omega} \mid \forall j : y_j \in Y_j \right\}.$$

   Then A and B are convex subsets of $\boldsymbol{R}^L$ and $A \cap B = \varnothing$.

2. By the separating hyperplane theorem, there exist a $p \in \boldsymbol{R}^L \setminus \{0\}$ and $c \in \boldsymbol{R}$ such that

$$\forall a \in A \, \forall b \in B : p \cdot a \geq c \geq p \cdot b. \tag{1}$$

3. By the local non-satiation condition, if $(x_1, \ldots x_I)$ satisfies $x_i \succsim_i x_i^*$ for every i, then

$$p \cdot \left(\sum_i x_i \right) \geq c. \tag{2}$$

Thus, in particular, $p \cdot \left(\sum_i x_i^* \right) \geq c$.

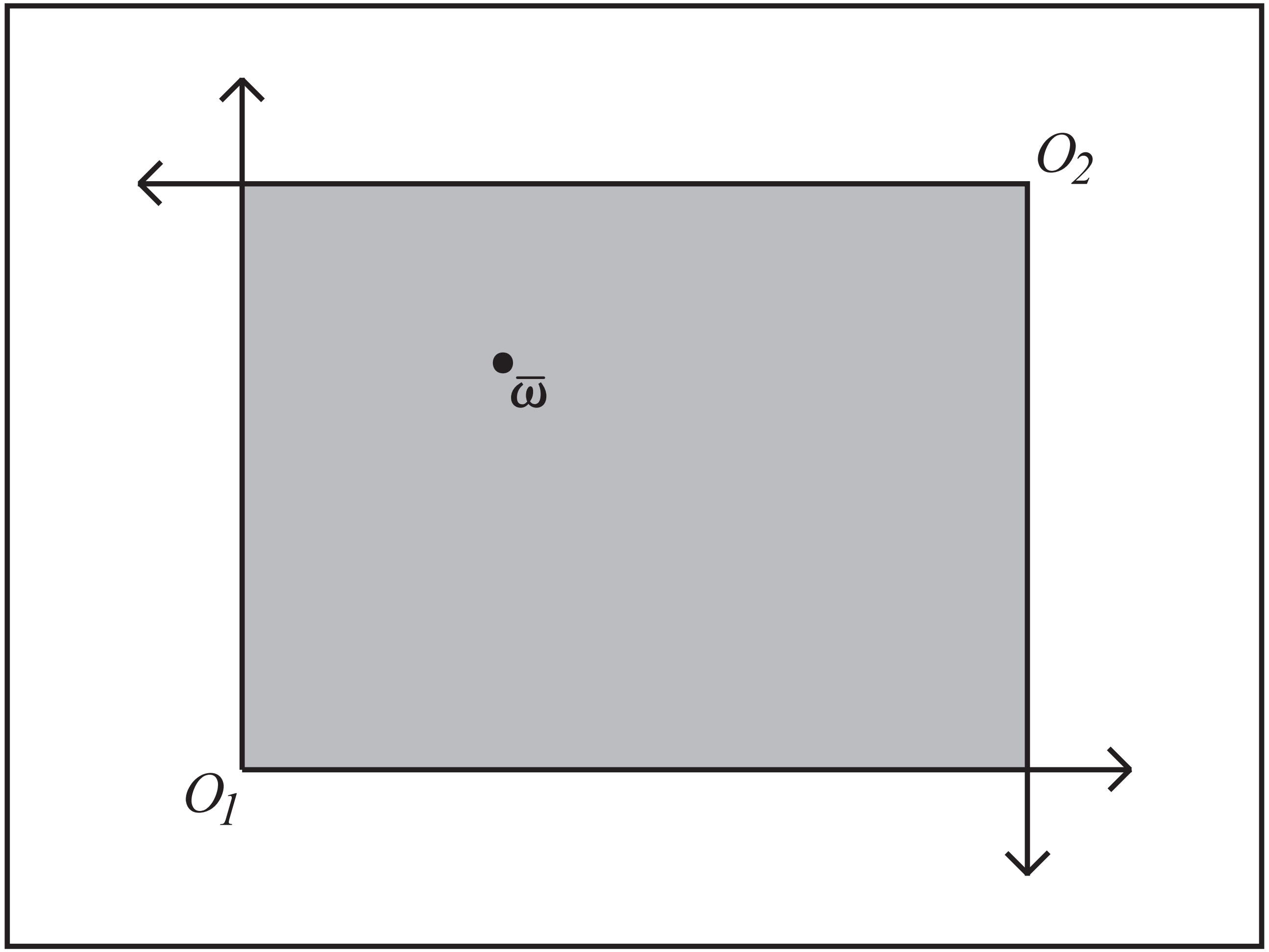
4. Since $\sum_i x_i^* = \sum_j y_j^* + \bar{\omega}$ and $\sum_j y_j^* + \bar{\omega} \in B$, (1) implies that $p \cdot (\sum_i x_i^*) \leq c$. Thus

$$p \cdot \left(\sum_i x_i^* \right) = p \cdot \left(\sum_j y_j^* + \bar{\omega} \right) = c. \tag{3}$$

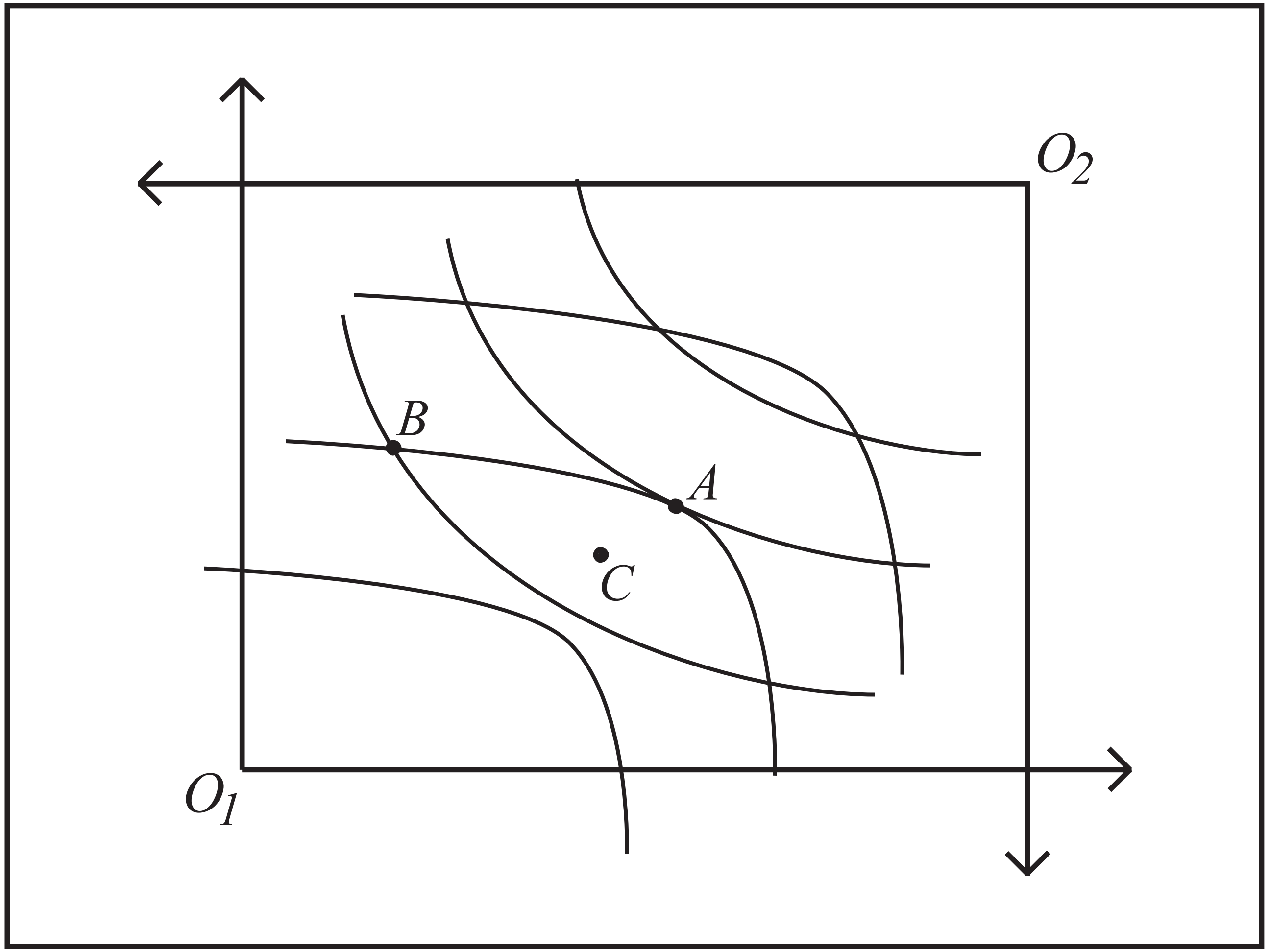
5. Profit maximization condition follows from (1) and (3).

6. By (2), the cost minimization condition holds.
Hence the quasi-utility maximization condition holds.

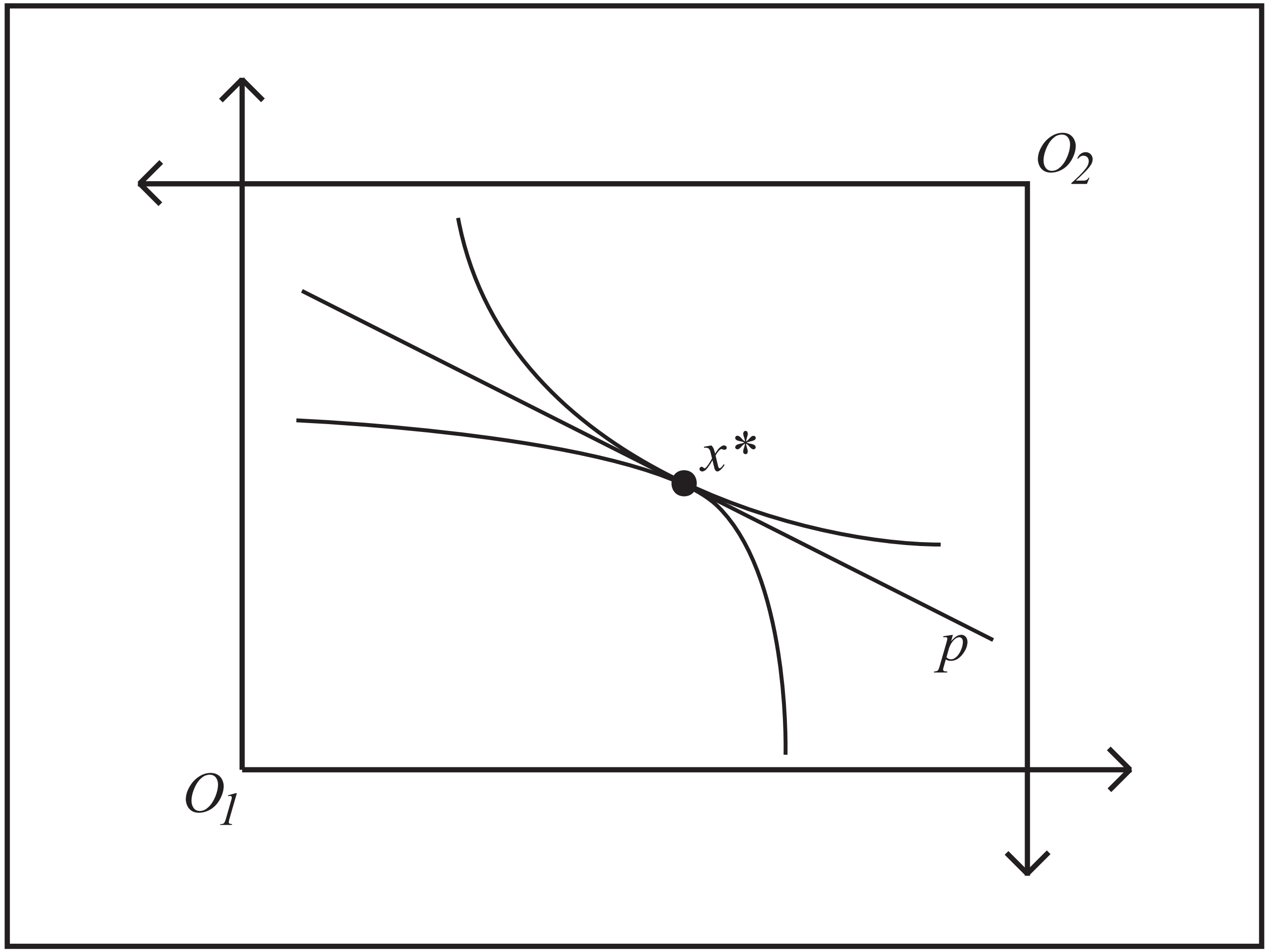
Edgeworth Box



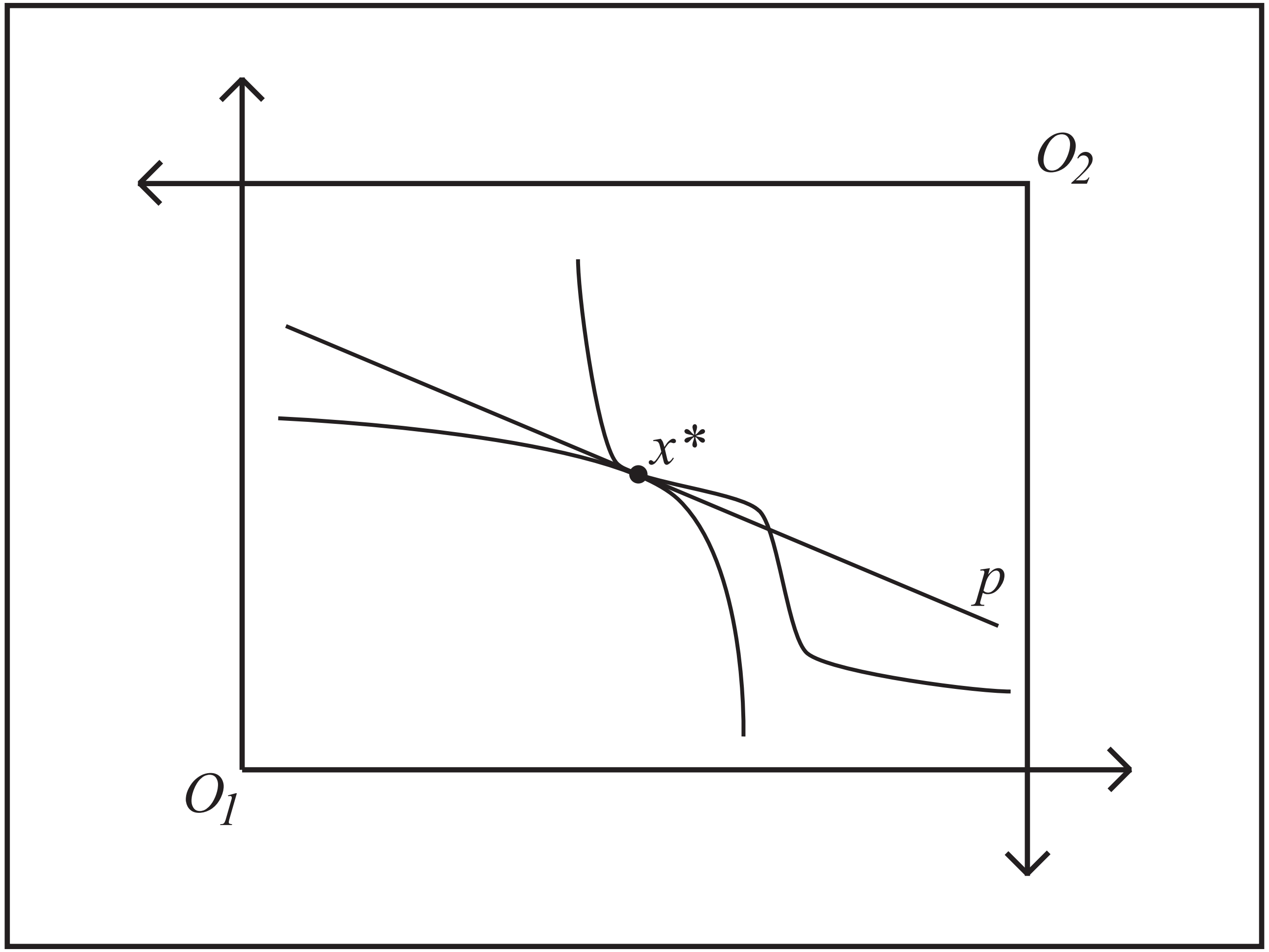
Pareto-Efficient Allocations in the Edgeworth Box



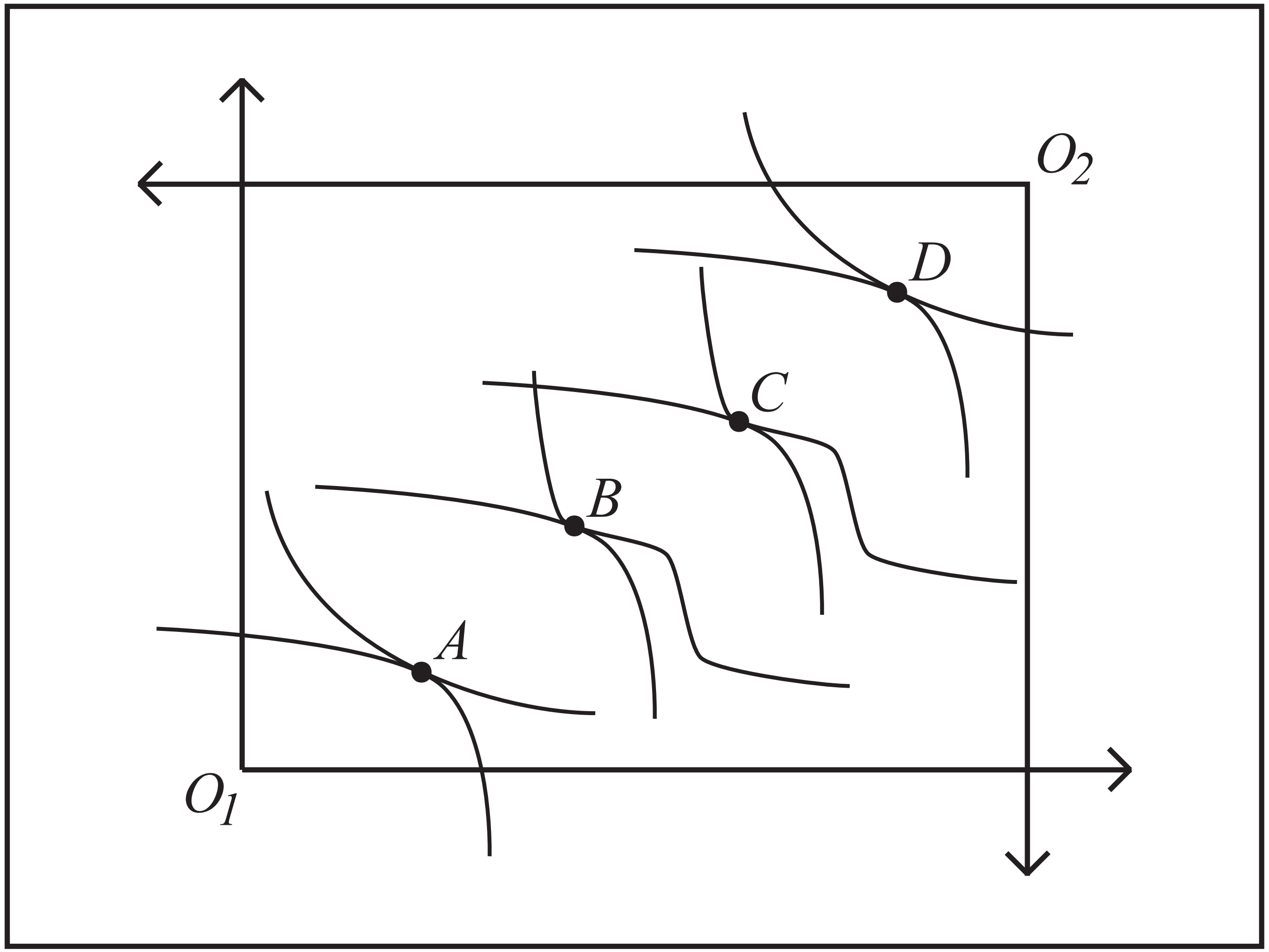
Second Welfare Theorem in the Edgeworth Box



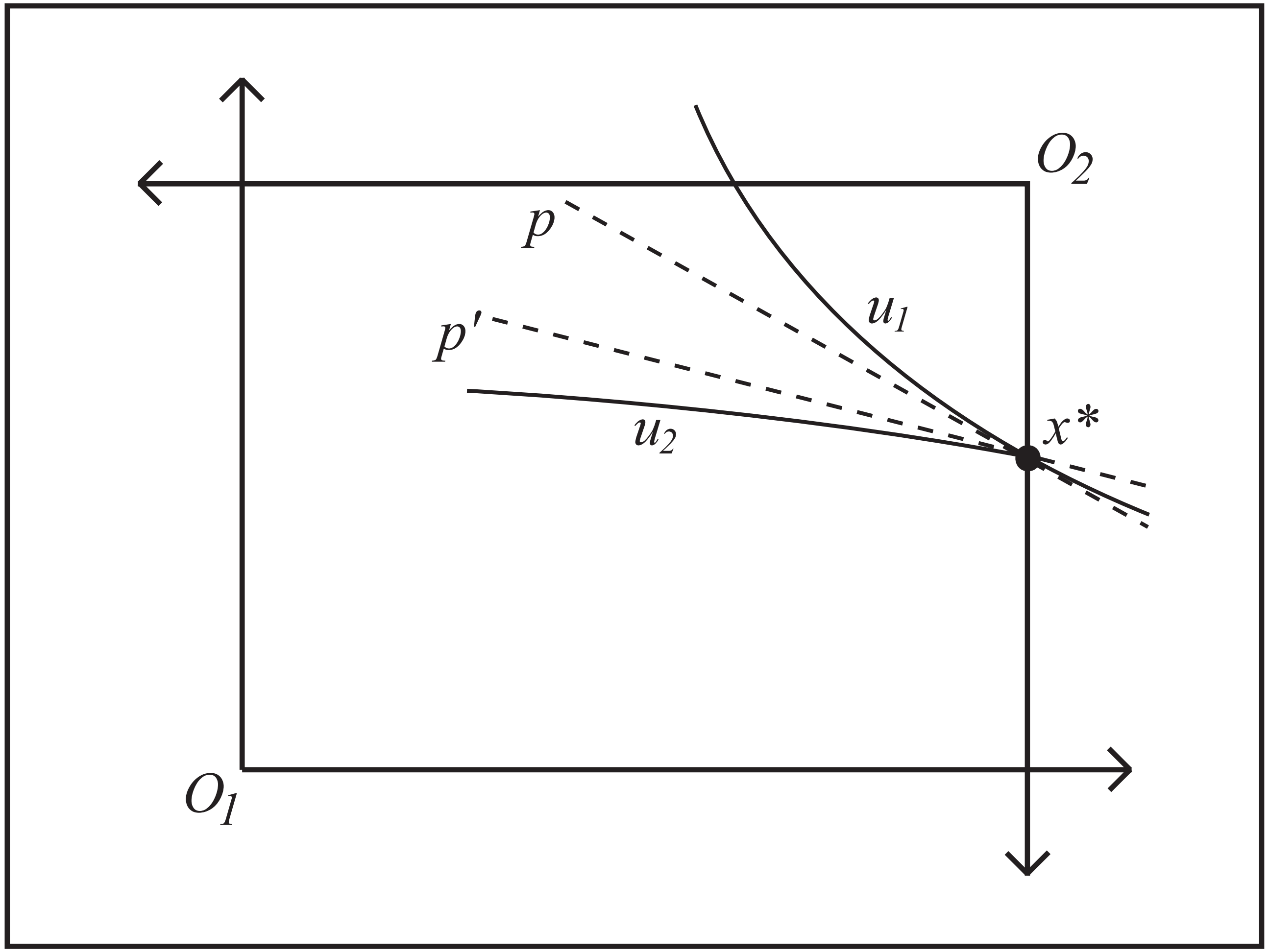
Failure of the Second Welfare Theorem with a Non-Convex Preference Relation



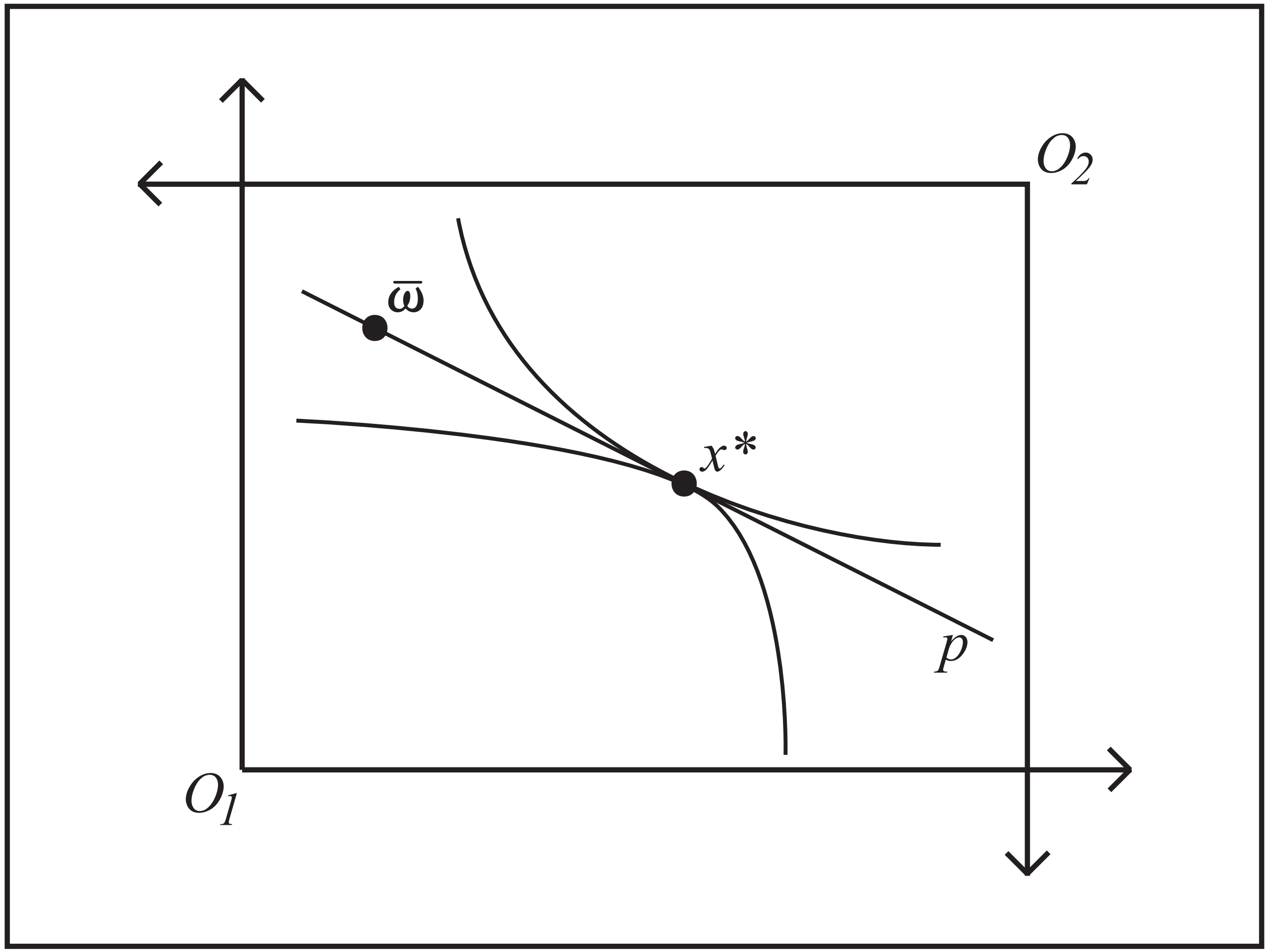
Trade-off Between Equity and Efficiency



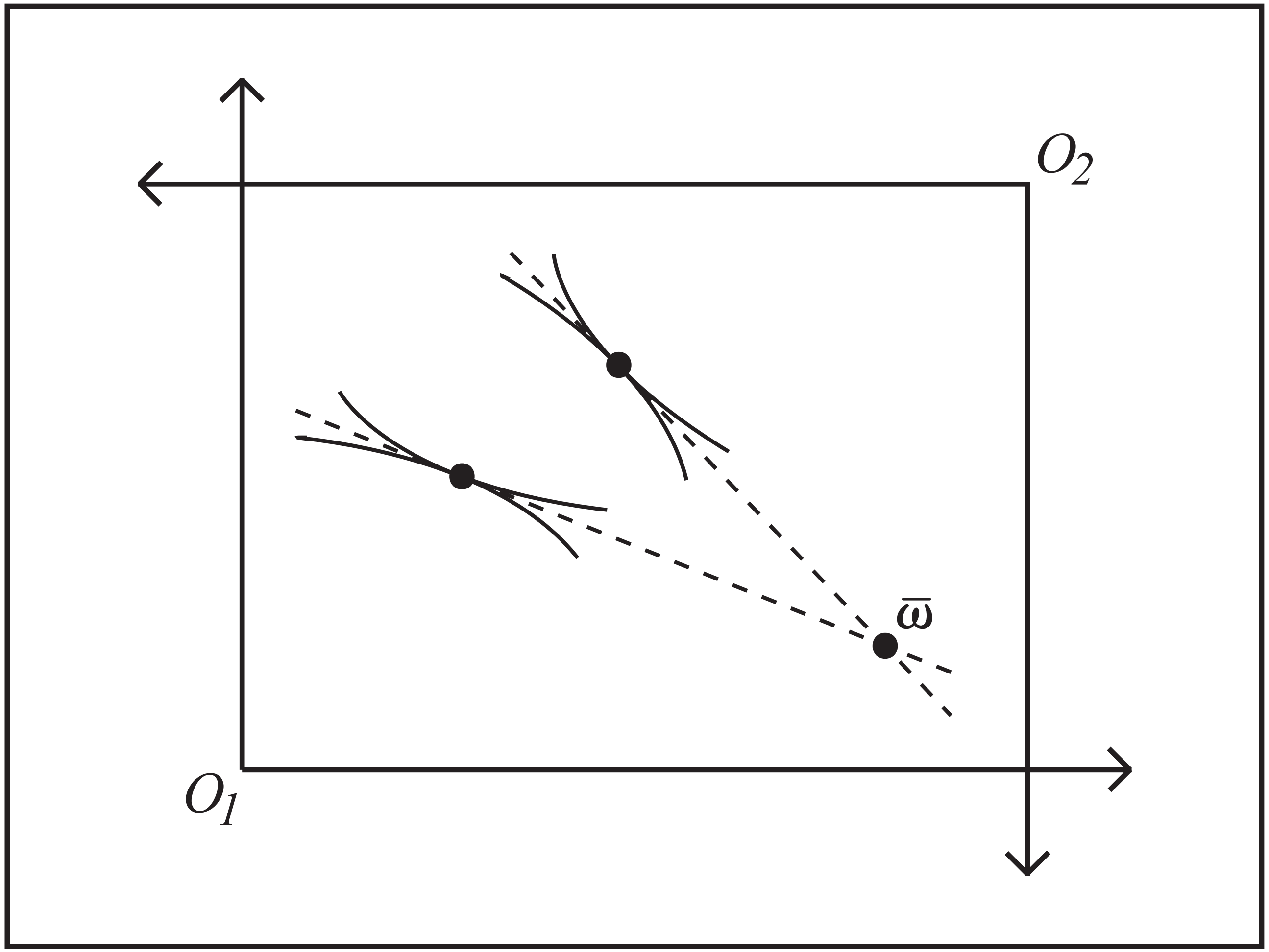
Equilibrium with Boundary Consumptions



Walrasian Equilibrium

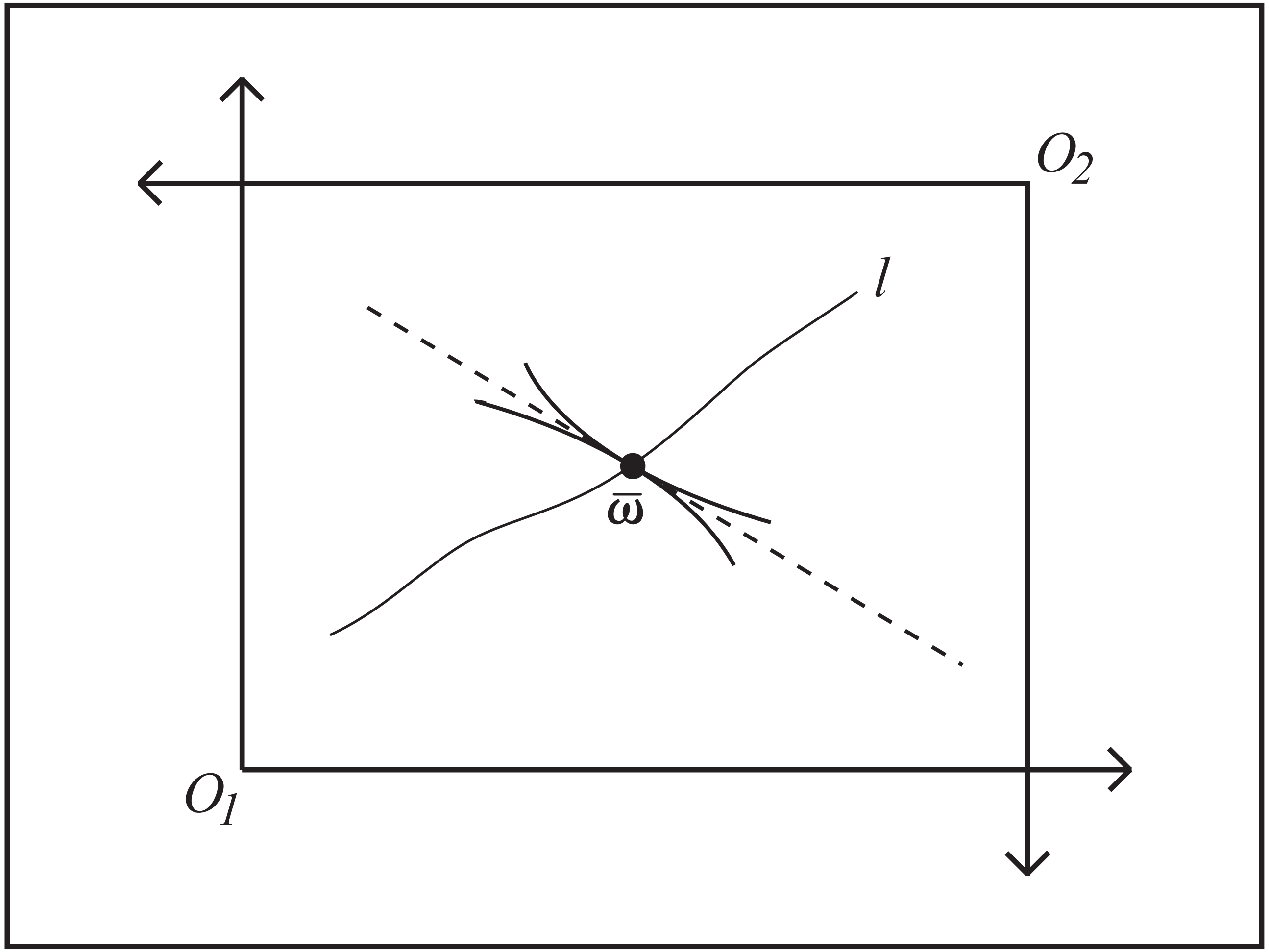


Multiple Walrasian Equilibria

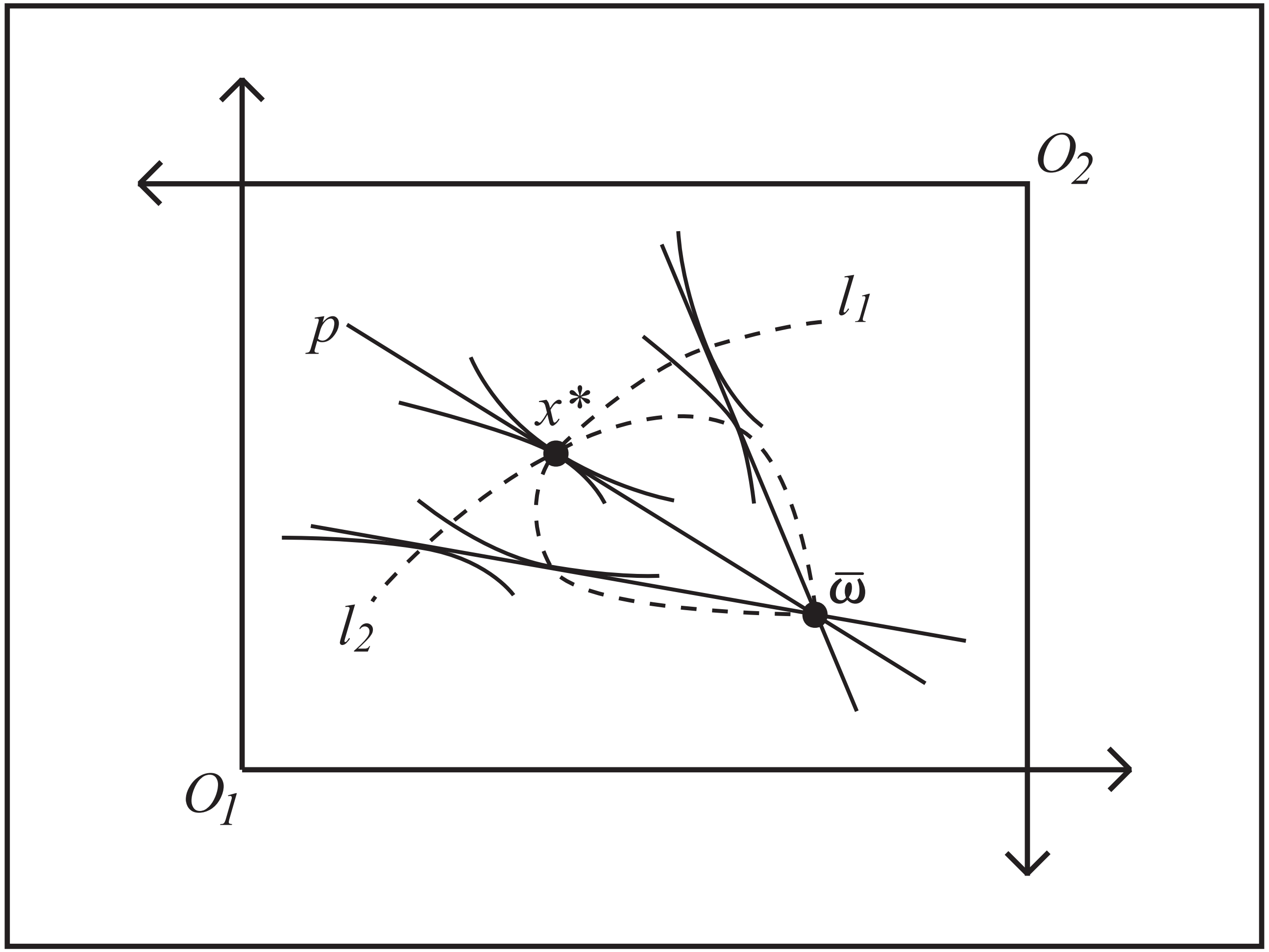


Unique Walrasian Equilibrium

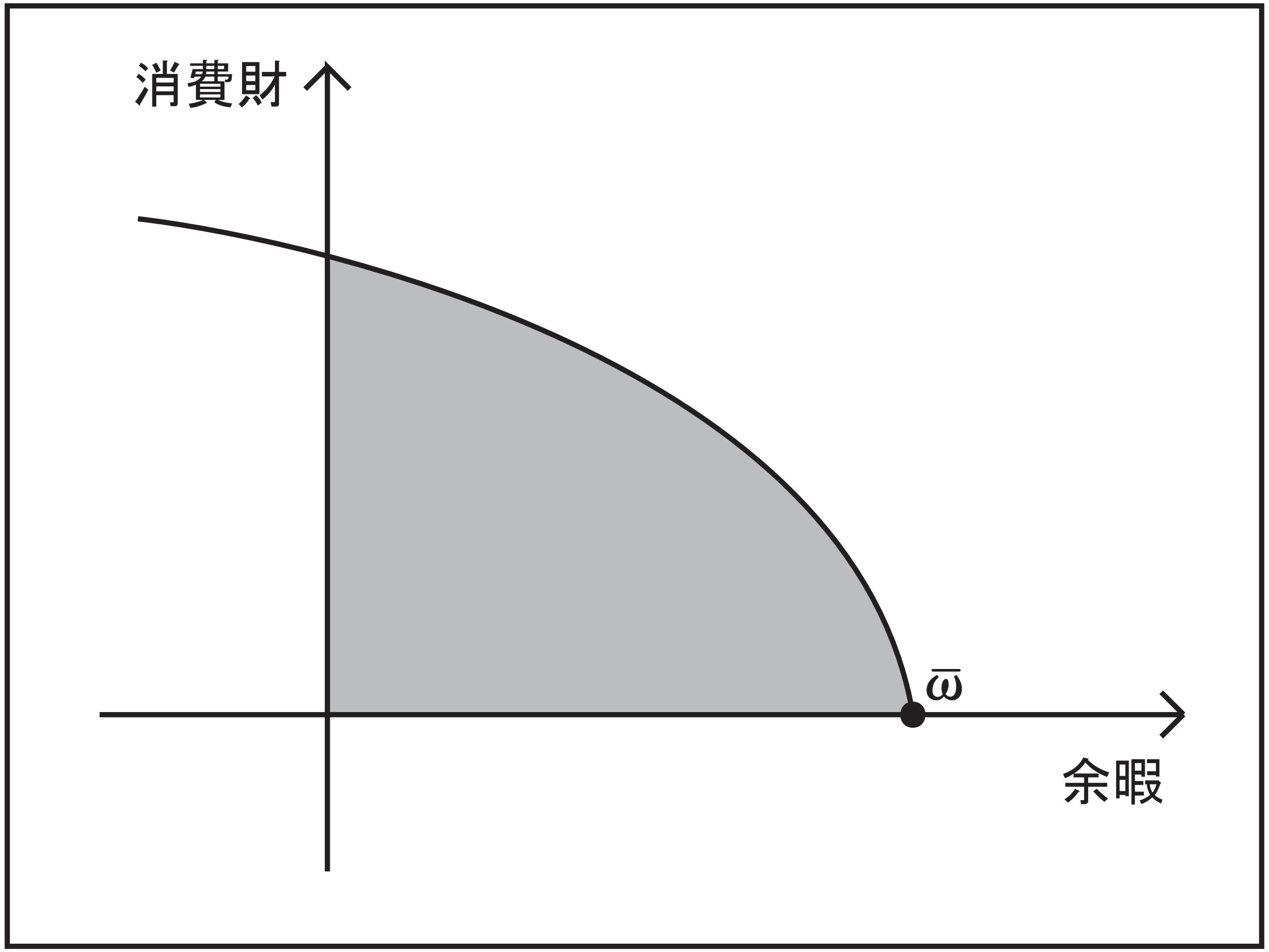
The initial endowment allocation is Pareto-efficient



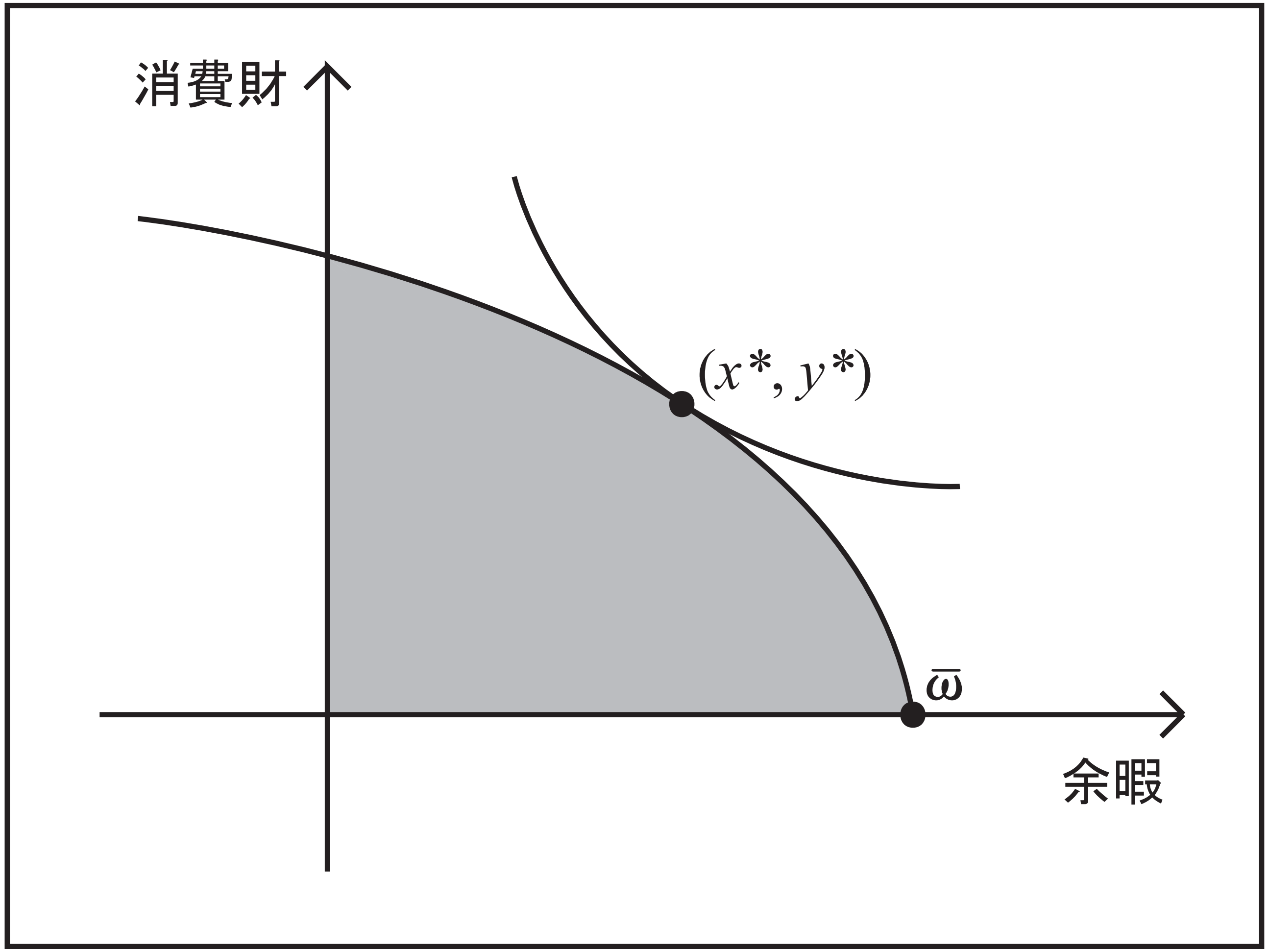
Walrasian Equilibrium and Offer Curves



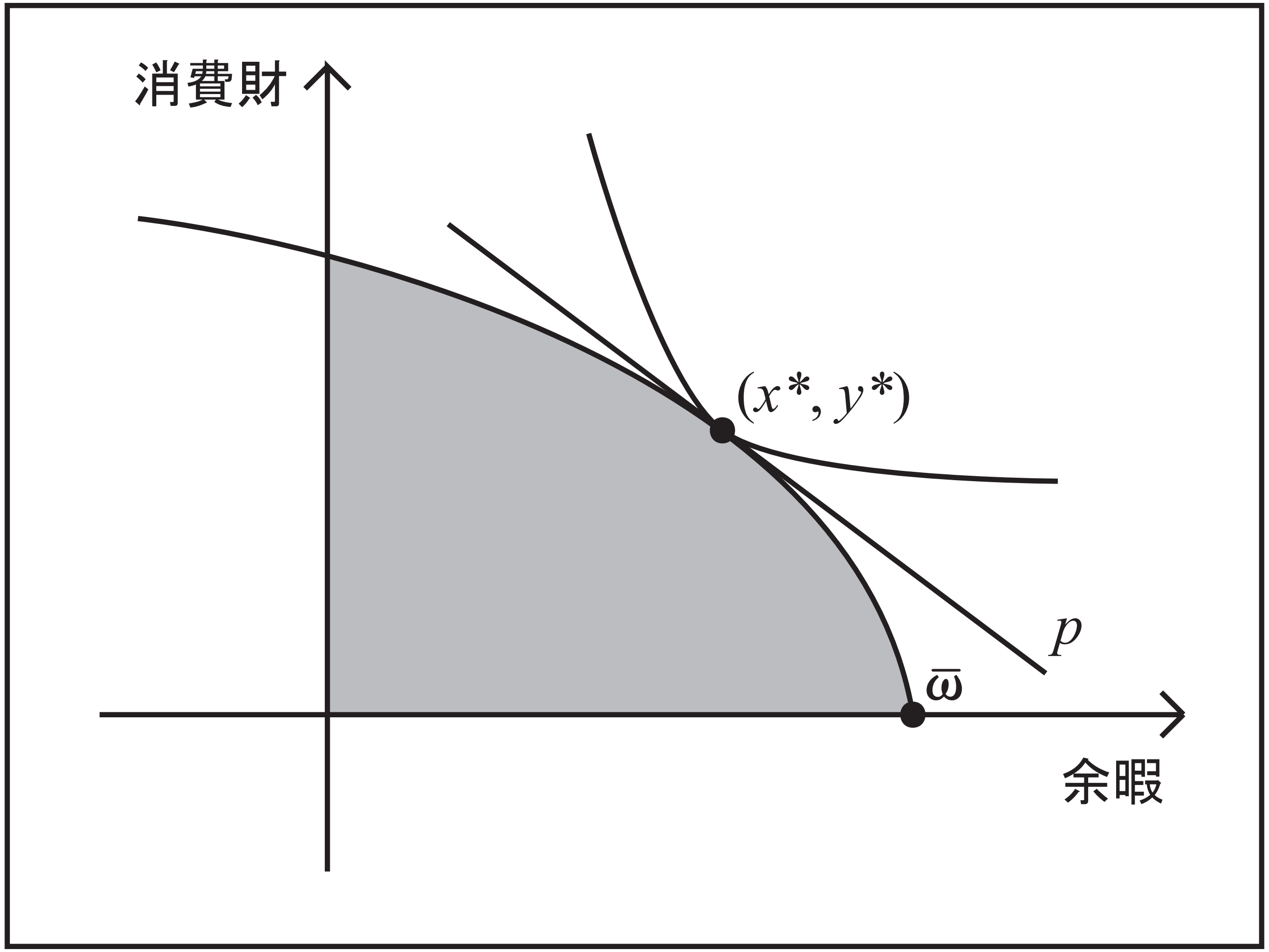
Robinson Crusoe Economy



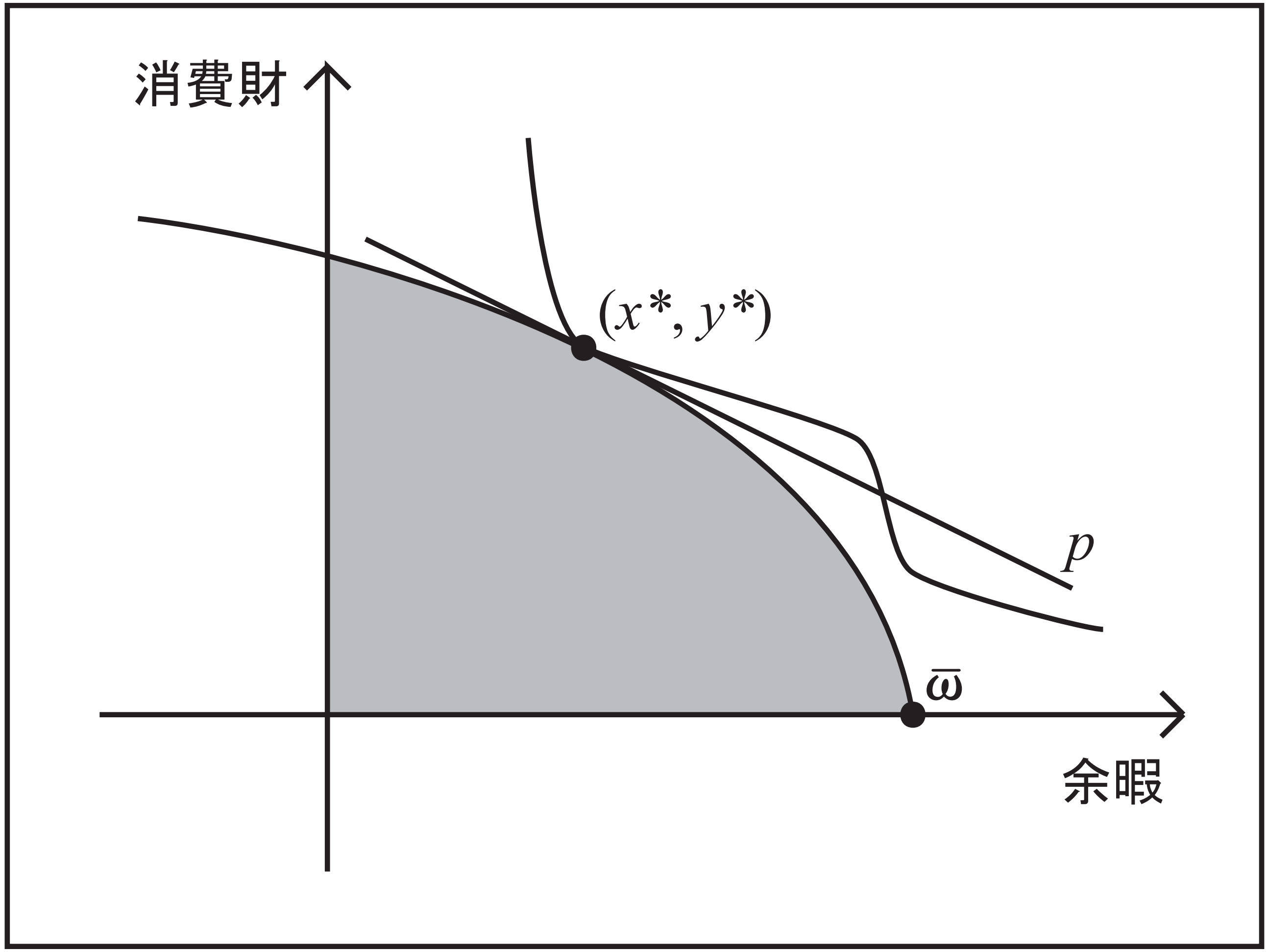
Efficient Allocation in the Robinson Crusoe Economy



Welfare Theorems
in the Robinson Crusoe Economy



Failure of the Second Welfare Theorem in the Robinson Crusoe Economy



Excess Demand Function

- The demand function of consumer i is denoted by $x_i : \boldsymbol{R}^L_{++} \times \boldsymbol{R}_+ \to \boldsymbol{R}^L_+$. His **excess demand function** is defined by

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i.$$

  Then $z(p) = \sum_i z_i(p)$.

- For every $p \in \boldsymbol{R}^L_{++}$, p is a Walrasian equilibrium price vector if and only if $z(p) = 0$.

  The equilibrium allocation is $(x_1(p, p \cdot \omega_1), \ldots, x_I(p, p \cdot \omega_I))$.

- **Continuity:** Because each x_i is continuous.

- **Homogeneity of Degree Zero:** Since each x_i is homogeneous of degree zero,

$$z_i(\alpha p) = x_i(\alpha p, (\alpha p)\cdot\omega_i) - \omega_i = x_i(\alpha p, \alpha(p\cdot\omega_i)) - \omega_i = x_i(p, p\cdot\omega_i) - \omega_i = z_i(p).$$

- **Walras' Law:** Since $p \cdot x_i(p, w_i) = w_i$ for each i,

$$p \cdot z_i(p) = p \cdot (x_i(p, p \cdot \omega_i) - \omega_i) = p \cdot \omega_i - p \cdot \omega_i = 0.$$

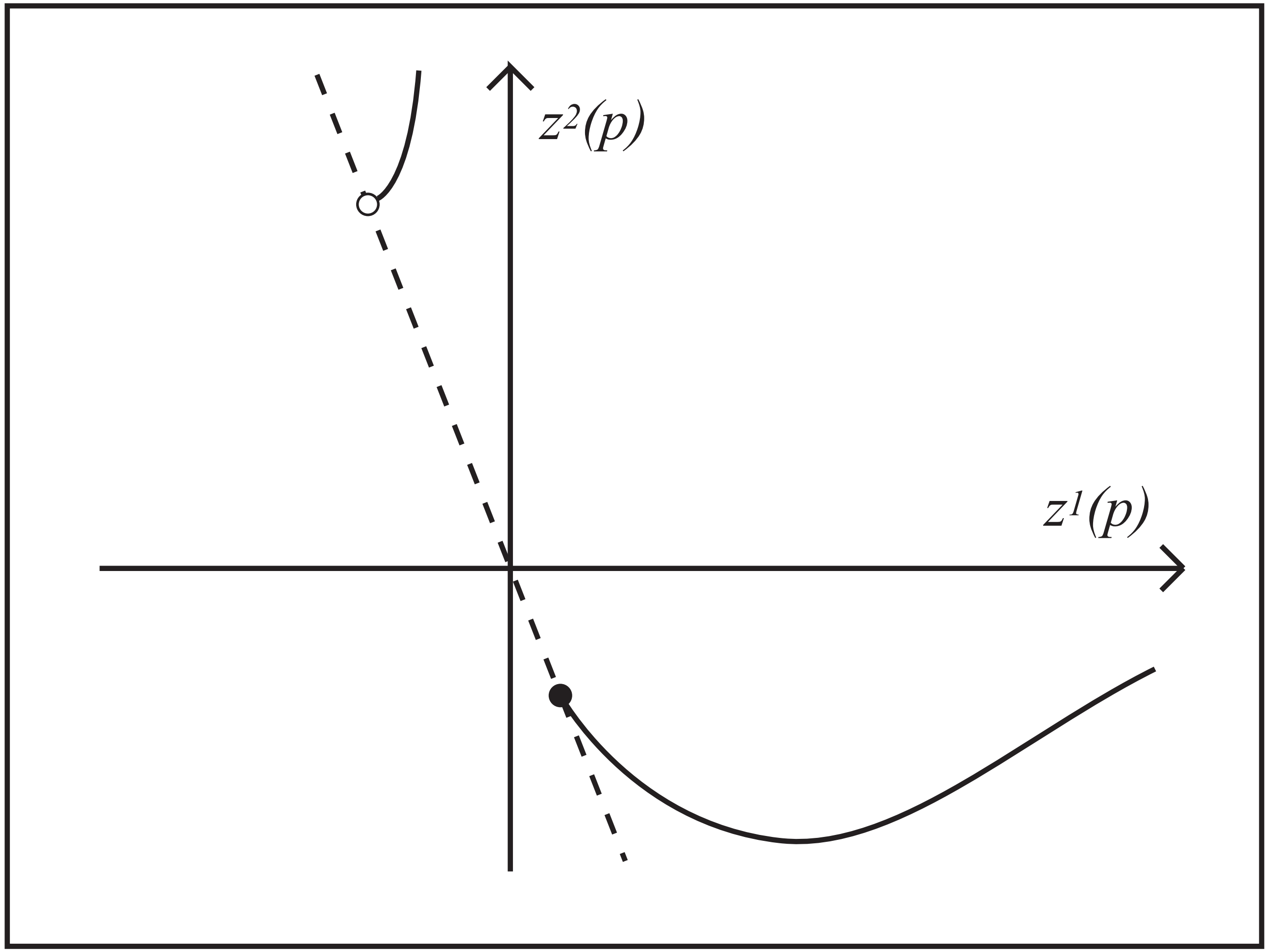
- **Bounded From Below:** Since $x_i(p, w_i) \geq 0$,

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i \geq -\omega_i$$

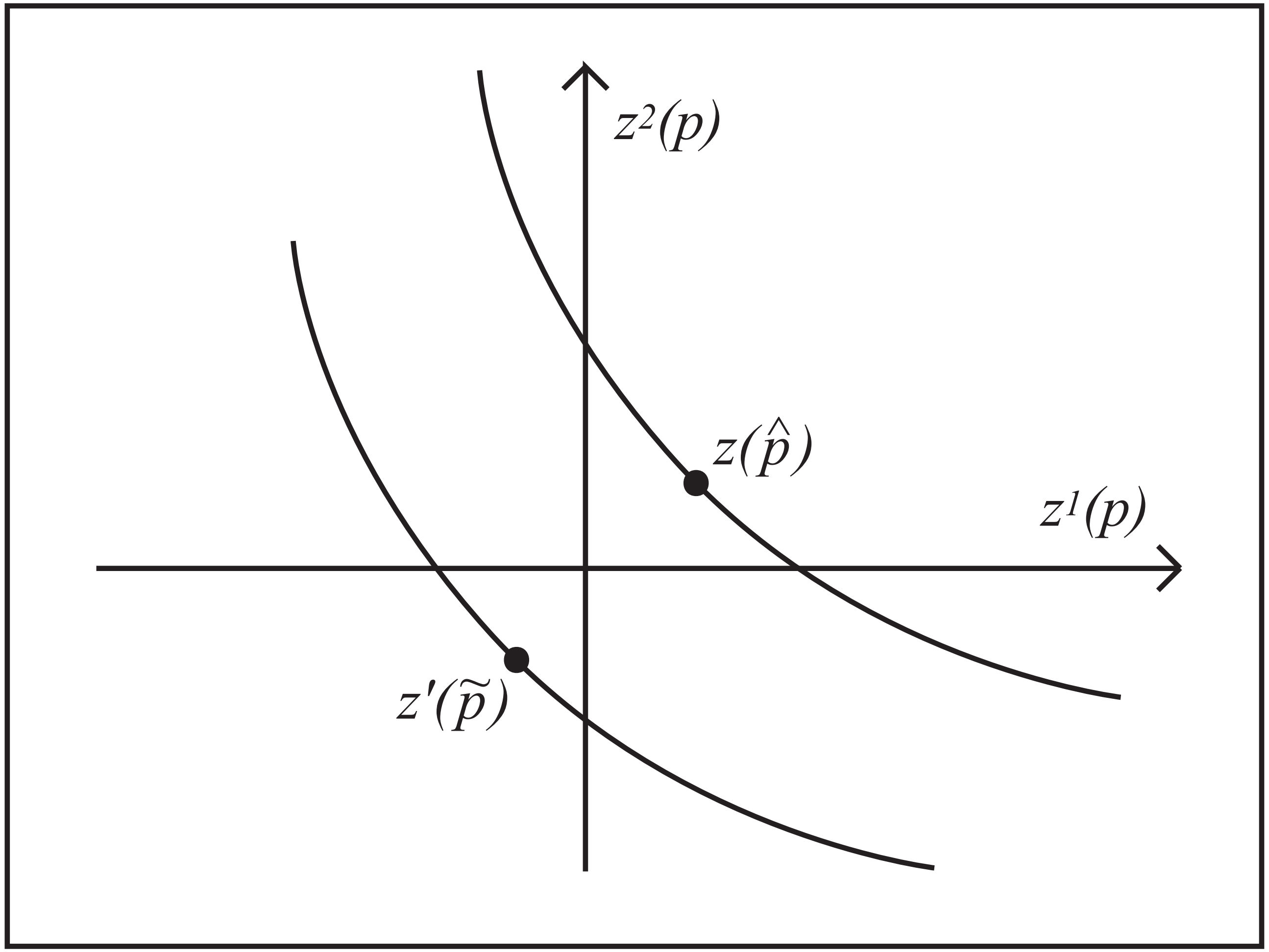
and hence $z(p) \geq -\sum_i \omega_i = -\bar{\omega}$.

- **Boundary Behavior** Follows from Exercise 7.

Continuity Eliminates the Following Case



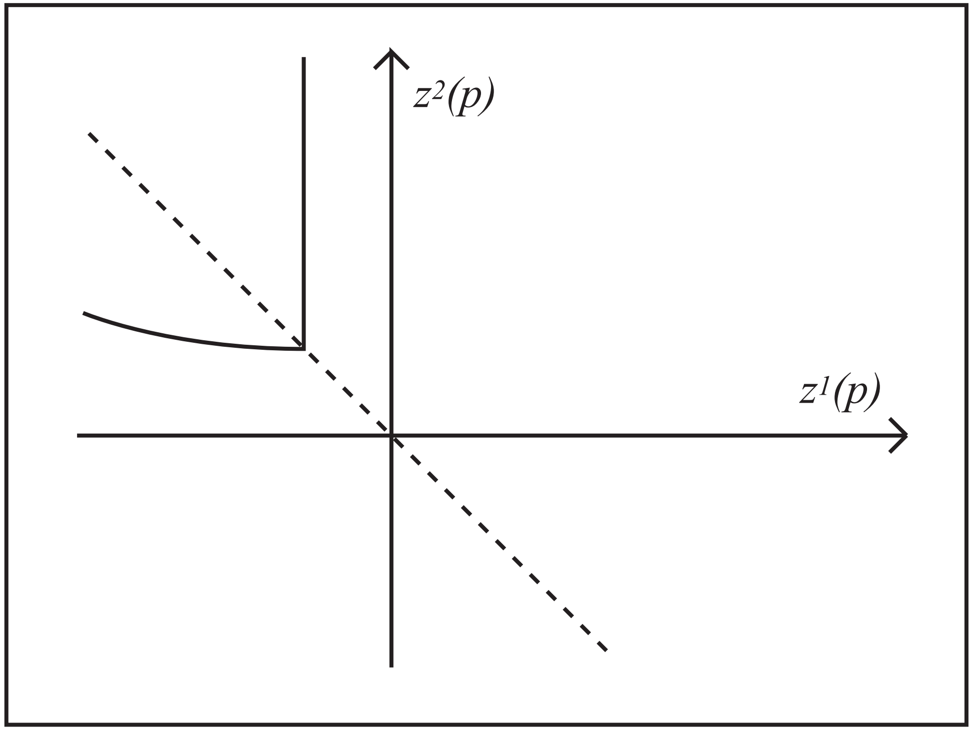
Walras' Law Eliminates the Following Case



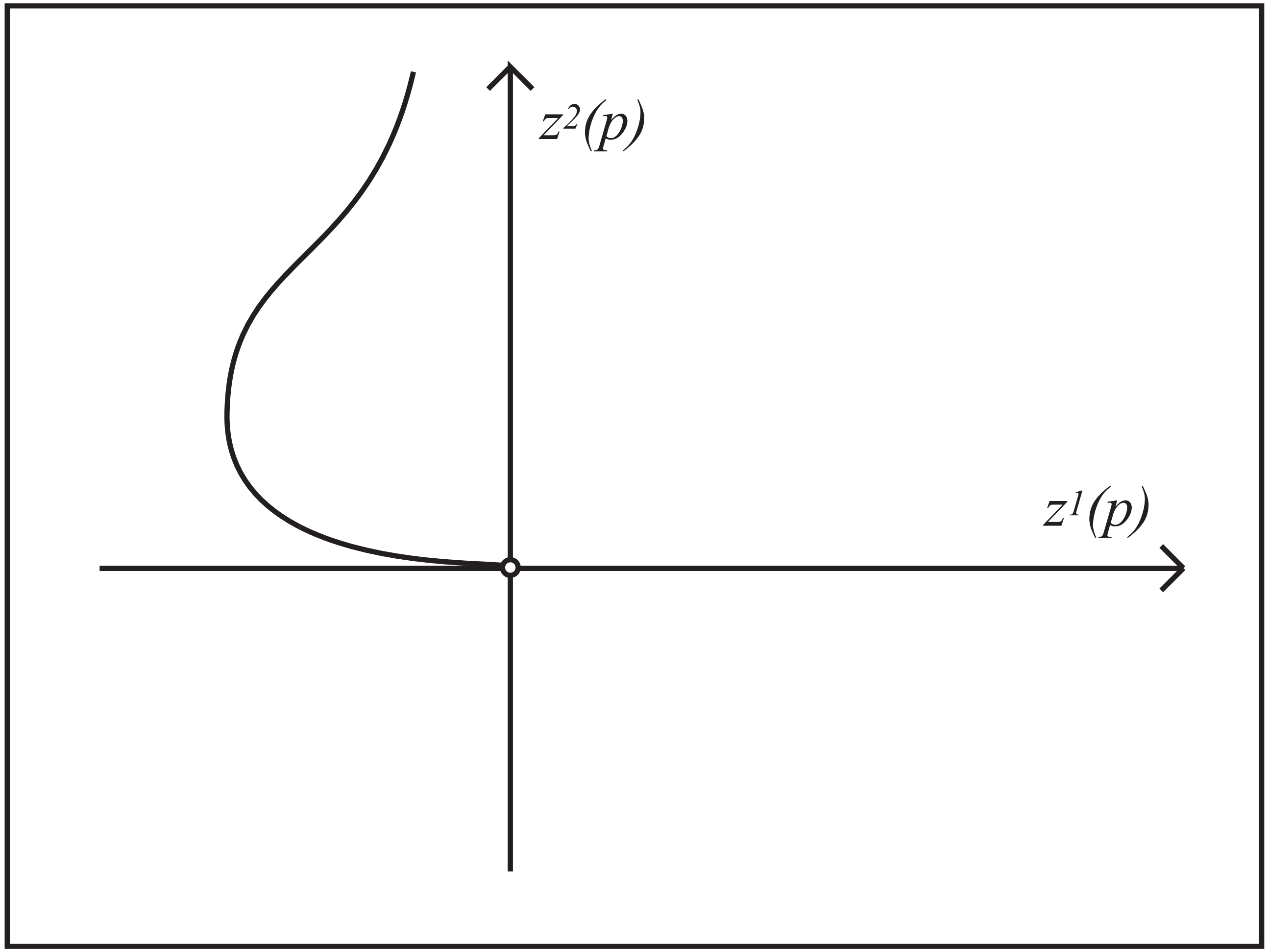
Boundedness from Below Eliminates the Following Case

Below is the graph of z defined by

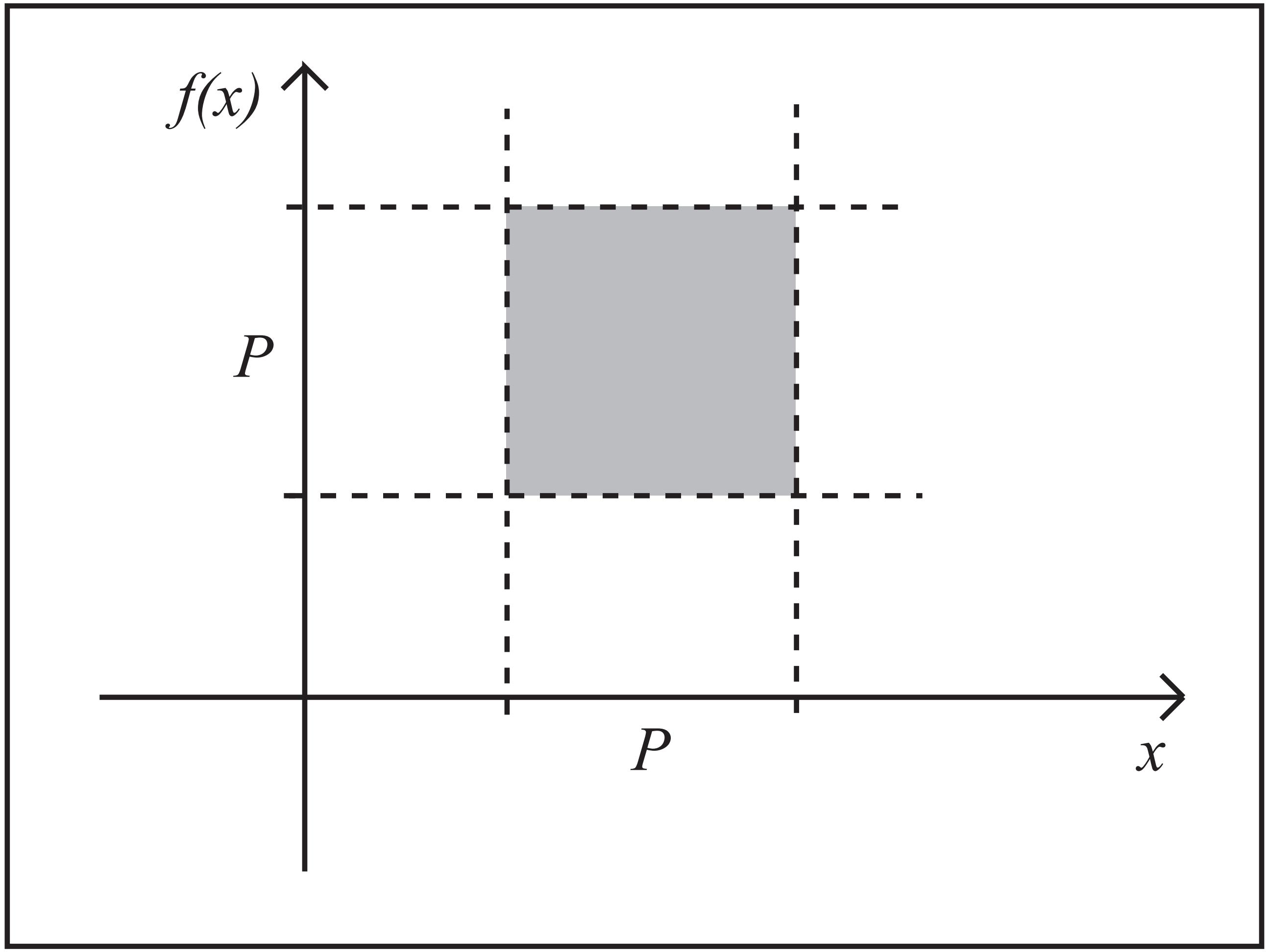
$$z_2(p) = \frac{\max\{p_1, p_2\}}{\min\{p_1, p_2\}}, \quad z_1(p) = -\frac{p_2}{p_1} z_2(p)$$



Boundary Behavior Eliminates the Following Case

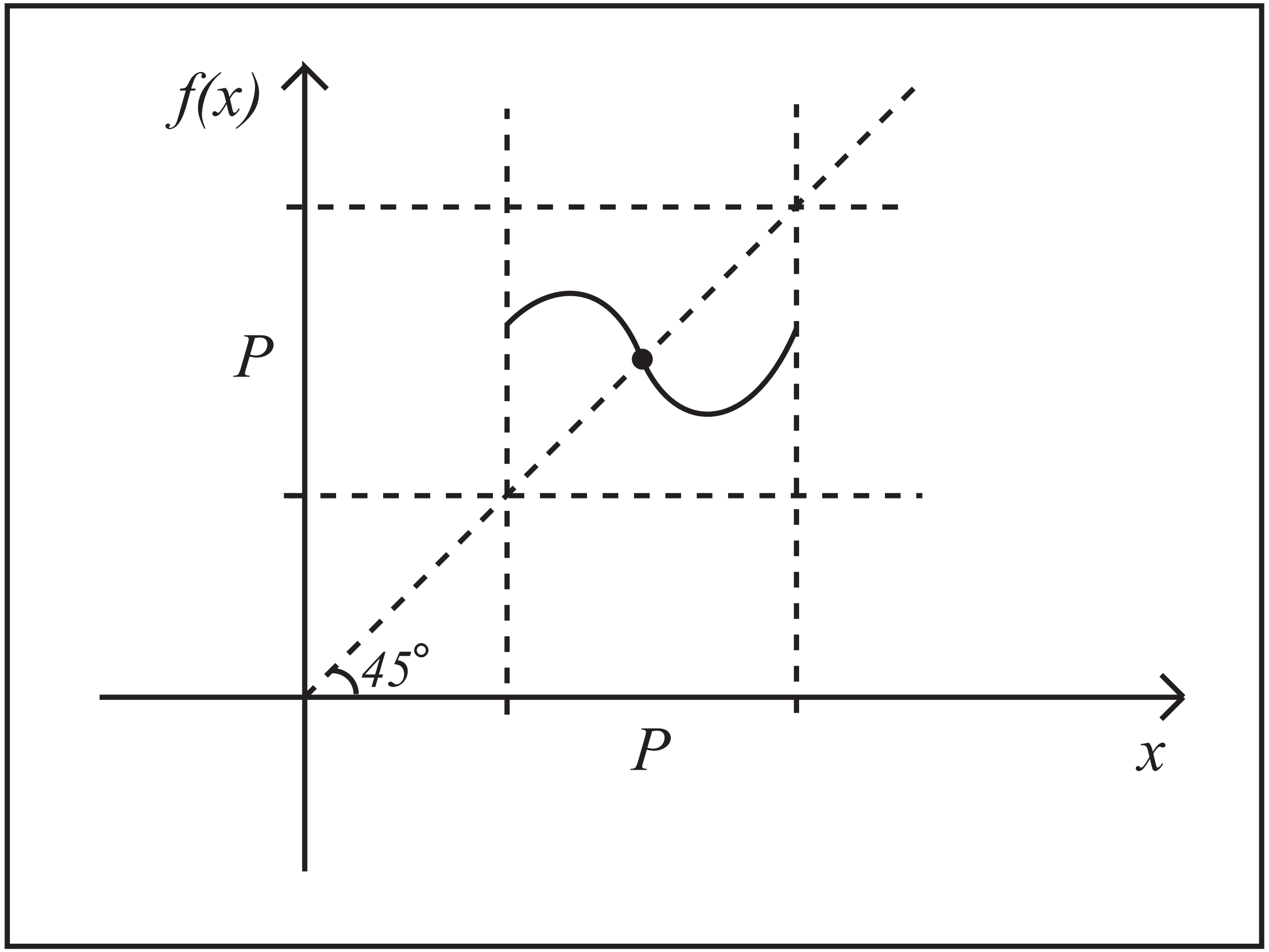


Kakutani's Fixed Point Theorem when $L = 1$



Kakutani's Fixed Point Theorem when $L = 1$

Intermediate value theorem if z is a mapping (function).



Proof of the Equilibrium Existence Theorem

The excess demand function is $z : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$, while Kakutani's fixed point theorem is applied to a correspondence $f : P \to P$, where P is convex and compact (closed and bounded).

We cannot directly apply the theorem to z because:

- $\boldsymbol{R}^L_{++}$ and $\boldsymbol{R}^L$ are different. In fact, $p \neq z(p)$ for every $p \in \boldsymbol{R}^L_{++}$;
- $\boldsymbol{R}^L_{++}$ is not bounded; and
- $\boldsymbol{R}^L_{++}$ is not closed.

We construct a new correspondence from z to conform the setting of the fixed point theorem.

Construction of the Fixed Point Correspondence

Let $P = \{p \in \boldsymbol{R}^L_+ \mid p_1 + \cdots + p_L = 1\}$ and define $f : P \to P$ by

$$f(p) = \begin{cases} \{q \in P \mid \forall \ell : z_\ell(p) < \max\limits_{k=1,\dots,L} z_k(p) \Rightarrow q_\ell = 0\} & \text{if } p \in \boldsymbol{R}^L_{++} \\ \{q \in P \mid \forall \ell : p_\ell > 0 \Rightarrow q_\ell = 0\} & \text{if } p \notin \boldsymbol{R}^L_{++} \end{cases}$$

- By Walras' law, if $p \in \boldsymbol{R}^L_{++}$ and $p \in f(p)$, then $z(p) = 0$.
- For every $p \in P \setminus \boldsymbol{R}^L_{++}$, $p \notin f(p)$.

Thus, if $p \in P$ is a fixed point of f, then it is an equilibrium price vector.

It is sufficient to prove that the assumptions of Kakutani's fixed point theorem are satisfied.

Steps of Proof of the Equilibrium Existence Theorem

1. P is non-empty, convex, and compact.

   This is easy to prove.

2. $f(p)$ is non-empty, convex, and compact for every $p \in P$.

   This is also easy to prove.

3. The graph of f is a closed subset of $P \times P$.

   Let $(p^1, q^1), (p^2, q^2), \ldots$ be a sequence in the graph of f and assume that $(p^n, q^n) \to (p, q) \in P \times P$ as $n \to \infty$.

   We need to show that (p, q) belongs to the graph of f; that is, if

   $$p^n \to p,\ q^n \in f(p^n),\ \text{and}\ q^n \to q$$

   then $q \in f(p)$.

4. We consider three cases

(a) $p \in \boldsymbol{R}^L_{++}$.

This is an easy case.

(b) $p \notin \boldsymbol{R}^L_{++}$ and there is an N such that for every $n > N$, $p^n \notin \boldsymbol{R}^L_{++}$.

This is also an easy case.

(c) $p \notin \boldsymbol{R}^L_{++}$ and for every N there exists an $n > N$ such that $p^n \in \boldsymbol{R}^L_{++}$.

This is the difficult case. To show that (p, q) belongs to the graph of f, we need to use Walras' law, boundedness from below, and the boundary behavior.

SMD Theorem: Compact Domain and Exercise 10

- In SMD Theorem, since the **candidate** excess demand function is restricted on a compact set $C \subset \boldsymbol{R}^L_{++}$, the "Boundedness from Below" and "Boundary Behavior" conditions are vacuously met.

- **Exercise 10, Part 2:** If $L = 2$, then $f : \boldsymbol{R}_{++} \to \boldsymbol{R}$. Then define $z : \boldsymbol{R}^2_{++} \to \boldsymbol{R}^2$ so that

$$z_1(p) = f\left(\frac{p_1}{p_2}\right),$$

$$z_2(p) = -\frac{p_1}{p_2} z_1(p) = -\frac{p_2}{p_1} f\left(\frac{p_1}{p_2}\right).$$

Then z satisfies continuity, homogeneity, and Walras' law.

SMD Theorem: Slutsky Decomposition

Why does the number of consumers matter for SMD Theorem?

- Let $x_i : \boldsymbol{R}^L_{++} \times \boldsymbol{R}_+ \to \boldsymbol{R}^L_+$ be the demand function of consumer i. Define the Slutsky substitution matrix $S_i(p, w_i) \in \boldsymbol{R}^{L \times L}$ by

$$S_i(p, w_i) = D_p x_i(p, w_i) + D_{w_i} x_i(p, w_i) x_i(p, w_i)^\top$$

  Then $S_i(p, w_i)$ is symmetric and negative semi-definite, that is, for every $v \in \boldsymbol{R}^L$,

$$v^\top S_i(p, w_i) v \leq 0.$$

- Let $z_i : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ be excess demand function. Then

$$Dz_i(p) = S_i(p, p \cdot \omega_i) - D_{w_i} x_i(p, p \cdot \omega_i) z_i(p)^\top.$$

- Thus, if $v \in \boldsymbol{R}^L$ and $z_i(p) \cdot v = 0$, then

$$\begin{aligned} v^\top Dz_i(p)v &= v^\top S_i(p, p \cdot \omega_i)v - v^\top D_{w_i}x_i(p, p \cdot \omega_i)z_i(p)^\top v \\ &= v^\top S_i(p, p \cdot \omega_i)v - (v \cdot D_{w_i}x_i(p, p \cdot \omega_i))(z_i(p) \cdot v) \\ &= v^\top S_i(p, p \cdot \omega_i)v \leq 0 \end{aligned}$$

Let $q = p + v$, then $v = q - p$ and $Dz_i(p)v \approx z_i(q) - z_i(p)$.

Hence $v^\top Dz_i(p)v \approx (q - p) \cdot (z_i(q) - z_i(p))$.

Thus $v^\top Dz_i(p)v \leq 0$ if and only if the change in prices and the change in excess demands are in the opposite directions.

- If $z_1(p) \cdot v = \cdots = z_I(p) \cdot v = 0$, then

$$v^\top Dz(p)v = v^\top \left(\sum_i Dz_i(p) \right) v = \sum_i v^\top Dz_i(p)v \leq 0$$

This can be an additional restriction on z.

- Thus there can be an additional restriction on z whenever there is a v such that $z_1(p) \cdot v = \cdots = z_I(p) \cdot v = 0$.

  Of course, $v = 0$ and $v = p$ are two possibilities. We are thus interested in the existence of a **non-trivial** solution to the system of linear equations

$$\begin{cases} v \cdot z_1(p) = 0 \\ \vdots \\ v \cdot z_I(p) = 0 \\ v \cdot p = 0 \end{cases} \tag{4}$$

  with $I + 1$ equations and L unknowns.

  This has a non-trivial solution if $L > I + 1$.

- If p is an equilibrium price vector, then $z_I(p) = -\sum_{i<I} z_i(p)$.

  Thus, if $v \cdot z_1(p) = \cdots = v \cdot z_{I-1}(p) = 0$, then $v \cdot z_I(p) = 0$.

  Hence (4) has a non-trivial solution if $L > I$.

Regular Equilibrium

Let $z : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ be the excess demand function of an economy.

- By differentiating both sides of $z(\alpha p) = z(p)$ with respect to α and evaluate the derivatives at $\alpha = 1$, we obtain $Dz(p)p = 0$. Thus

$$\operatorname{rank} Dz(p) = L - \dim(\operatorname{Ker} Dz(p)) \leq L - 1.$$

- **Exercise 11, Part 1:** By differentiate both sides of $p \cdot z(p) = 0$ with respect to p, we obtain $p^\top Dz(p) + z(p) = 0$. Thus, if $z(p) = 0$, then $p^\top Dz(p) = 0$ and hence

$$\operatorname{Col} Dz(p) \subseteq \{v \in \boldsymbol{R}^L \mid p \cdot v = 0\}.$$

- **Exercise 11, Part 2:** If $L = 1$, then $\hat{z}(\hat{p}) = z_1(p_1, 1)$ and hence

$$D\hat{z}(\hat{p}) = \frac{\partial z_1}{\partial p_1}(p_1, 1).$$

Parameterized Excess Demand Function

- The excess demand function of the economy $q \in Q$ is $z(\cdot, q) : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$.

  A price vector $p \in \boldsymbol{R}^L_{++}$ is an equilibrium price vector of q if $z(p, q) = 0$.

  It is a regular equilibrium price vector if $\text{rank}\, D_p z(p, q) = L - 1$, where $D_p z\,(p, q)$ is an $L \times L$ matrix.

- **Definition 9.5:** Q is a **regular parameterization** if

$$\text{rank}\, Dz\,(p, q) = \text{rank}\, [D_p z\,(p, q) \;\; D_q z\,(p, q)] = L - 1$$

  whenever $z\,(p, q) = 0$.

  Note that $Dz\,(p, q)$ is an $L \times (L + S)$ matrix, $D_p z\,(p, q)$ is an $L \times L$ matrix, and $D_q z\,(p, q)$ is an $L \times S$ matrix.

- **Exercise 12:** Define $\widehat{z} : \boldsymbol{R}^{L-1}_{++} \times Q \to \boldsymbol{R}^{L-1}$ by

$$\widehat{z}\left(\widehat{p}, q\right) = \left(z_1\left(\left(\widehat{p}, 1\right), q\right), \ldots, z_{L-1}\left(\left(\widehat{p}, 1\right), q\right)\right),$$

where $\widehat{p} \in \boldsymbol{R}^{L-1}_{++}$. Then, Q is a regular parameterization if and only if

$$\text{rank}\, D\widehat{z}\left(\widehat{p}, q\right) = \text{rank}\left[D_{\widehat{p}}\widehat{z}\left(\widehat{p}, q\right) \;\; D_q\widehat{z}\left(\widehat{p}, q\right)\right] = L - 1$$

whenever $\widehat{z}\left(\widehat{p}, q\right) = 0$.

Note that $D\widehat{z}\left(\widehat{p}, q\right)$ is an $(L-1) \times (L-1+S)$ matrix, $D_{\widehat{p}}\widehat{z}\left(\widehat{p}, q\right)$ is an $(L-1) \times (L-1)$ matrix, and $D_q\widehat{z}\left(\widehat{p}, q\right)$ is an $(L-1) \times S$ matrix.

Regularity of Example 9.3

- Let $z_1 : \boldsymbol{R}^L_{++} \times \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ be the parameterized excess demand function of consumer 1, with parameter $\omega_1 \in \boldsymbol{R}^L_{++}$.

  Let $z_i : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ be the demand function of consumer $i \geq 2$.

- Let $z : \boldsymbol{R}^L_{++} \times \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ be the parameterized excess demand function, then

$$z(p, \omega_1) = z_1(p, \omega_1) + \sum_{i \geq 2} z_i(p).$$

  To show that Q is a regular parameterization, it is sufficient to show that $\operatorname{rank} D\widehat{z}_1(\widehat{p}, \omega_1) = L - 1$.

  For this, it is sufficient to prove that $\operatorname{rank} D_{\omega_1}\widehat{z}_1(\widehat{p}, \omega_1) = L - 1$.

- For this, it is sufficient to prove that for every $\widehat{v} \in \boldsymbol{R}^{L-1}$,

$$\widehat{z}_1\left(\widehat{p}, \omega_1 + (-\widehat{v}, \widehat{p} \cdot \widehat{v})\right) = \widehat{z}_1\left(\widehat{p}, \omega_1\right) + \widehat{v},$$

because by differentiating both sides with respect to $\widehat{v}$, we obtain

$$D_{\omega_1}\widehat{z}_1(\widehat{p}, \omega_1)\begin{bmatrix} -I_{(L-1)\times(L-1)} \\ \widehat{p}^\top \end{bmatrix} = I_{(L-1)\times(L-1)},$$

where $I_{(L-1)\times(L-1)}$ is the $(L-1)\times(L-1)$ identity matrix.

- Indeed, since $(\widehat{p}, 1) \cdot (-\widehat{v}, \widehat{p} \cdot \widehat{v}) = -\widehat{p} \cdot \widehat{v} + \widehat{p} \cdot \widehat{v} = 0$,

$$x_1\left((\widehat{p}, 1), (\widehat{p}, 1) \cdot (\omega_1 + (-\widehat{v}, \widehat{p} \cdot \widehat{v}))\right) = x_1\left((\widehat{p}, 1), (\widehat{p}, 1) \cdot \omega_1\right).$$

Thus

$$\begin{aligned} &\widehat{z}_1\left(\widehat{p}, \omega_1 + (-\widehat{v}, \widehat{p} \cdot \widehat{v})\right) \\ =&\widehat{x}_1\left(\widehat{p}, (\widehat{p}, 1) \cdot (\omega_1 + (-\widehat{v}, \widehat{p} \cdot \widehat{v}))\right) - (\widehat{\omega}_1 - \widehat{v}) \\ =&\widehat{x}_1\left(\widehat{p}, (\widehat{p}, 1) \cdot \omega_1\right) - \widehat{\omega}_1 + \widehat{v} \\ =&\widehat{z}_1\left(\widehat{p}, \omega_1\right) + \widehat{v}. \end{aligned}$$

Transversality Theorem

- $P \subseteq \boldsymbol{R}^M$, $Q \subseteq \boldsymbol{R}^N$, and $f : P \times Q \to \boldsymbol{R}^L$.

- For Theorem 9.6, let $P = \boldsymbol{R}^{L-1}_{++}$, $Q = Q$, $L = L - 1$, and $f = \hat{z}$.

- In the case of Example 9.3, the theorem implies that whatever $\succsim_1, \dots, \succsim_I$ and $\omega_2, \dots, \omega_I$ are, for almost every $\omega_1 \in \boldsymbol{R}^L_{++}$, all the Walrasian equilibria of the economy determined by ω_1 are regular.

- In other applications, $M < L$. Of course, then, $\operatorname{rank} D_p f(p, q) = L$ can never hold, because $D_p f(p, q)$ is an $L \times M$ matrix and hence $\operatorname{rank} D_p f(p, q) \leq \min\{M, L\} = M < L$.

  In this case, the theorem is used to show that for almost every $q \in Q$, there is **no** $p \in P$ such that $f(p, q) = 0$.

Arrow-Debreu Equilibrium and Assets

- **Remark 10.1:** Let $S = 2$ and $L = 1$.

$$\text{In the first period } p_0 = 11 \left\langle \begin{array}{l} p_1 = 4 \text{ in state 1} \\ \\ p_2 = 6 \text{ in state 2} \end{array} \right.$$

- Let $S = 2$ and $L = 1$. Examples of assets:

$$\left\langle \begin{array}{l} 1 \text{ in state 1} \\ \\ 1 \text{ in state 2} \end{array} \right. \quad \left\langle \begin{array}{l} 1 \text{ in state 1} \\ \\ 0 \text{ in state 2} \end{array} \right. \quad \left\langle \begin{array}{l} 0 \text{ in state 1} \\ \\ 1 \text{ in state 2} \end{array} \right. \quad \left\langle \begin{array}{l} \bar{\omega}_1 \text{ in state 1} \\ \\ \bar{\omega}_2 \text{ in state 2} \end{array} \right.$$

The first asset is the **risk-free** bond; the next two are **Arrow assets**; and the last one is the **market portfolio**.

Remark 11.2

- If $(p, q) = (p_0, p_1, \ldots, p_S, q)$ is a Radner equilibrium price vector, for any $(\alpha_0, \alpha_1, \ldots, \alpha_S) \in \boldsymbol{R}^{1+S}_{++}$, $(\alpha_0 p_0, \alpha_1 p_1, \ldots, \alpha_S p_S, \alpha_0 q)$ is also a Radner equilibrium price vector.

  The essential number of prices is reduced by $1 + S$.

- By adding budget constraints over all consumers, we obtain

$$p_0 \cdot \left(\sum_i x_{0i} \right) + q \cdot \left(\sum_i z_i \right) = p_0 \cdot \left(\sum_i \omega_{0i} \right),$$

$$p_s \cdot \left(\sum_i x_{si} \right) = p_s \cdot \left(\sum_i \omega_{si} \right) + \sum_{j=1}^{J} (p_s \cdot a_{sj}) \left(\sum_i z_{ji} \right).$$

  If $L = 1$, p_0 and the p_s are all positive numbers. Thus if $\sum_i z_i = 0$, then $\sum_i x_{0i} = \sum_i \omega_{0i}$ and $\sum_i x_{si} = \sum_i \omega_{si}$ for all $s \geq 1$.

  Hence the market-clearing conditions are reduced by $1 + S$.

Let $L \geq 2$. We can apply Walras' law $(1+S)$ times:

$$\sum_i x_{10i} - \sum_i \omega_{10i} = -\frac{1}{p_{10}} \left(\sum_{\ell \geq 2} p_{\ell 0} \left(\sum_i x_{\ell 0 i} - \sum_i \omega_{\ell 0 i} \right) + q \cdot \left(\sum_i z_i \right) \right),$$

$$\sum_i x_{1si} - \sum_i \omega_{1si}$$

$$= -\frac{1}{p_{s0}} \left(\sum_{\ell \geq 2} p_{\ell s} \left(\sum_i x_{\ell s i} - \sum_i \omega_{\ell s i} \right) - \sum_{j=1}^{J} (p_s \cdot a_{sj}) \left(\sum_i z_{ji} \right) \right).$$

Again, the market-clearing conditions are reduced by $1+S$.

Asset Payoff and Incomplete Markets

- **Exercise 15:**

$$V(p) = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix}$$

Markets are complete if and only if $\det V(p) = p_{11}p_{22} - p_{21}p_{12} \neq 0$.

- **Theorem 12.2, Part 1:** The AD equilibrium price vector p is also the Radner equilibrium price vector with $q_j = \sum_s p_s \cdot a_{sj}$.

- **Theorem 12.2, Part 2:** The Radner equilibrium price vector $p = (p_0, p_1, \ldots, p_S)$ is, in general, not the AD equilibrium price vector. But there exists an $(\alpha_1, \ldots, \alpha_S) \in \boldsymbol{R}^S_{++}$ such that $(p_0, \alpha_1 p_1, \ldots, \alpha_S p_S, q)$ is the AD equilibrium price vector.