Advanced Microeconomics
Lecture Note

Chiaki Hara
Institute of Economic Research, Kyoto University

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Chapter 1

Kuhn-Tucker Conditions

1.1 Introduction

In this section, we give Kuhn-Tucker necessary and sufficient conditions for a solution to a constrained maximization problem. Our conditions are more general than the standard ones in that they accommodate the case of multiple objective functions, so that the constrained maximization problem is a generalization of the problem of finding a Pareto-efficient allocation. Our proof is more elementary than the standard ones, in that they rely on the Minkowski-Farkas Lemma, which can be proved by an induction argument, but not on the separating hyperplane theorem, whose proof involves a topological argument.

1.2 Minkowski-Farkas Lemma

This materials owe much to Sections 2.1, 2.2, and 2.3 of Gale. Let J and L be two positive integers.

1.2.1 Lemma (Minkowski and Farkas) *Let $A \in \boldsymbol{R}^{J \times L}$ and $b \in \boldsymbol{R}^L$ (column vector). Then one and only one of the following two possibilities holds:*

1. *There exists a $z \in \boldsymbol{R}^J_+$ (column vector) such that $b^\top = z^\top A$.*
2. *There exists an $x \in \boldsymbol{R}^L$ (column vector) such that $Ax \in \boldsymbol{R}^J_+$ and $b^\top x < 0$.*

For each $a \in \boldsymbol{R}^L$, define $a^\perp = \{x \in \boldsymbol{R}^L | a \cdot x = 0\}$.

Let $a \in \boldsymbol{R}^L$ and $b \in \boldsymbol{R}^L$ and suppose that $a \cdot b \neq 0$. Then, for every $x \in \boldsymbol{R}^L$, there are a unique $v \in a^\perp$ and a unique $\lambda \in \boldsymbol{R}$ such that $x = v + \lambda b$. We say that v is the projection of x onto $a^\perp$ along b.

Exercise 1.2.1 Prove that $v = x - \dfrac{a \cdot x}{a \cdot b} b$.

Proof of Lemma 1.2.1 Let's first prove by a contradiction argument that the two possibilities do not hold simultaneously. So suppose there were z and x as in the two possibilities. Multiply x from the right to $b^\top = z^\top A$, then we obtain $b^\top x = z^\top Ax$. But, by Possibility 2 and $z \in \boldsymbol{R}R^J_+$, we have $b^\top x < 0$ and $z^\top Ax \geq 0$. Hence the equality could not be met. This is a contradiction. Hence the two possibilities do not hold simultaneously.

It is now sufficient to show that if Possibility 1 does not hold, then Possibility 2 must necessarily hold. We shall do so by an induction argument on J.

The case of $J = 1$ is left as an exercise.

Let $J \geq 2$ and assume that, for $J-1$, if Possibility 1 does not hold, then Possibility 2 holds. Write

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_{J-1} \\ a_J \end{bmatrix},$$

where $a_1, \ldots, a_{J-1}, a_J$ are row vectors in $\boldsymbol{R}^L$. Also write

$$A' = \begin{bmatrix} a_1 \\ \vdots \\ a_{J-1} \end{bmatrix} \in \boldsymbol{R}^{(J-1)\times L}.$$

Since Possibility 1 is not met by A, it is not met by A' either. That is, there is no $z' \in \boldsymbol{R}_+^{J-1}$ such that $b^\top = z'^\top A'$. Hence, by the induction hypothesis, there exists an $x' \in \boldsymbol{R}^L$ (column vector) such that $A'x' \in \boldsymbol{R}_+^{J-1}$ and $b^\top x' < 0$.

If $a_J \cdot x' \geq 0$, then x' is just as desired in Possibility 2 and the proof is completed. So assume that $a_J \cdot x' < 0$. Denote the projections of $a_1, \ldots, a_{J-1}, b$ onto $x'^\perp$ along a_J by $\widehat{a}_1, \ldots, \widehat{a}_{J-1}, \widehat{b}$, respectively. Define

$$\widehat{A} = \begin{bmatrix} \widehat{a}_1 \\ \vdots \\ \widehat{a}_{J-1} \end{bmatrix} \in \boldsymbol{R}^{(J-1)\times L}.$$

We shall now prove by a contradiction argument that there is no $w \in \boldsymbol{R}_+^{J-1}$ (column vector) such that $\widehat{b} = w^\top \widehat{A}$. So suppose there were such a

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_{J-1} \end{bmatrix},$$

then

$$\begin{aligned} \widehat{b}^\top &= \sum_{j=1}^{J-1} w_j \widehat{a}_j \\ &= \sum_{j=1}^{J-1} w_j \left(a_j - \frac{x' \cdot a_j}{x' \cdot a_J} a_J \right) \\ &= \sum_{j=1}^{J-1} w_j a_j - \left(\sum_{j=1}^{J-1} w_j \frac{x' \cdot a_j}{x' \cdot a_J} \right) a_J. \end{aligned}$$

Thus

$$b^\top = \sum_{j=1}^{J-1} w_j a_j + \left(\frac{x' \cdot b}{x' \cdot a_J} - \sum_{j=1}^{J-1} w_j \frac{x' \cdot a_j}{x' \cdot a_J} \right) a_J.$$

Since

$$\frac{x' \cdot b}{x' \cdot a_J} - \sum_{j=1}^{J-1} w_j \frac{x' \cdot a_j}{x' \cdot a_J} \geq 0,$$

this contradicts our initial hypothesis that Possibility 1 does not hold for A. Hence there is no $w \in \boldsymbol{R}_+^{J-1}$ such that $\widehat{b} = w^\top \widehat{A}$.

By our induction hypothesis, therefore, there exists an $\widehat{x} \in \boldsymbol{R}^L$ (column vector) such that $\widehat{A}\widehat{x} \in \mathbb{R}_+^{J-1}$ and $\widehat{b}^\top \widehat{x} < 0$. Denote by x the projection of $\widehat{x}$ onto $\widehat{a}_J^\perp$ along x'. We shall now prove that x is just as desired in Possibility 2 for A. Indeed, $a_J \cdot x = 0$ because $x \in a_J^\perp$. For each $j \leq J-1$, we have

$$a_j \cdot x = \widehat{a}_j \cdot x = \widehat{a}_j \cdot \widehat{x} \geq 0,$$

where the first equality follows from $\widehat{a}_j - a_j = \lambda a_J$ for some λ and $x \in \widehat{a}_J^\perp$; and the second equality follows from $\widehat{a}_j \in x'^\perp$ and $x - \widehat{x} = \lambda x'$ for some λ. We can similarly show that $b \cdot x < 0$. This completes the proof. ///

1.3 Separating Hyperplane Theorem

1.3.1 Definition A subset C of $\boldsymbol{R}^L$ is *convex* if $\lambda c + (1-\lambda)c' \in C$ for every $c \in C$, every $c' \in C$, and every $\lambda \in [0,1]$.

A subset C is *closed* if, roughly speaking, the boundary of C is completely included in C itself.

1.3.2 Theorem (Separating Hyperplane Theorem) *Let C be a closed, convex subset of $\boldsymbol{R}^L$, and $b \in \boldsymbol{R}^L \backslash C$. Then there exist $x \in \boldsymbol{R}^L$ and $d \in \mathbb{R}$ such that*

$$x \cdot c \geq d > x \cdot d$$

for every $c \in C$.

Exercise 1.3.1 Prove the Minkowski-Farkas Lemma by applying the Separating Hyperplane Theorem by following the steps below:

1. Prove that the cone spanned by the row vectors of A,

$$\left\{ \sum_{j=1}^{J} z_j a_j \in \boldsymbol{R}^N \mid z_1 \geq 0, \ldots, z_J \geq 0 \right\},$$

   is convex.

2. Denote the above cone by C. You can use the fact that C is closed without proof. Apply the Separating Hyperplane Theorem to C and b to show that there exist an $x \in \boldsymbol{R}^N$ such that $x \cdot c \geq 0$ for every $c \in C$ and $x \cdot b < 0$. (Hint: By the separating hyperplane theorem, there exist an $x \in \boldsymbol{R}^N$ and a $d \in \boldsymbol{R}$ such that $x \cdot c \geq d > x \cdot b$ for every $c \in C$. Show then that, for such x and d, we must necessarily have $d \leq 0$ and $x \cdot c \geq 0$ for every $c \in C$.)

3. Show that the result in the second step implies the Minkowski-Farkas Lemma.

1.4 Strict Supportability for a Pointed Cone

1.4.1 Lemma *Let $A \in \boldsymbol{R}^{J \times L}$. Then one and only one of the following two possibilities holds:*

1. *There exists a $z \in \boldsymbol{R}_+^J \setminus \{0\}$ (column vector) such that $z^\top A = 0$.*

2. *There exists an $x \in \boldsymbol{R}^L$ (column vector) such that $Ax \in \boldsymbol{R}_{++}^J$.*

Proof of Lemma 1.4.1 It is an exercise to show that Possibilities 1 and 2 do not hold simultaneously.

Define

$$e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \boldsymbol{R}^J$$

and define $\widetilde{A} = \begin{bmatrix} \widetilde{A} & e \end{bmatrix} \in \boldsymbol{R}^{J \times (L+1)}$. It is then an easy exercise to show that Possibility 1 is equivalent to saying that there exists a $z \in \boldsymbol{R}^J_+$ (column vector) such that

$$z^\top \widetilde{A} = (\underbrace{0, \dots, 0}_{L}, 1).$$

Thus, according to the Minkowski-Farkas Lemma, if this condition is not met, then there exists an $\widetilde{x} \in \boldsymbol{R}^{L+1}$ (column vector) such that $\widetilde{A}\widetilde{x} \in \boldsymbol{R}^J_+$ and $(0, \dots, 0, 1)\widetilde{x} < 0$. Write

$$\widetilde{x} = \begin{bmatrix} x \\ x_{L+1} \end{bmatrix},$$

where $x \in \boldsymbol{R}^L$ (column vector). Then $\widetilde{A}\widetilde{x} = Ax + x_{L+1}e$ and $(0, \dots, 0, 1)\widetilde{x} = x_{L+1}$. Thus $x_{L+1} < 0$ and $Ax \in \boldsymbol{R}^J_{++}$. Hence Possibility 2 holds. ///

1.5 Constrained Maximization Problem with Multiple Objective Functions

The following formulation of a constrained maximization problem with multiple objective functions is due to Smale.

Let N, M, and L be positive integers. A *constrained maximization problem* with N objective functions is defined by a collection of $(X, f_1, \dots, f_N, g_1, \dots, g_M)$, where X is a subset of $\boldsymbol{R}^L$ and the f_n $(n = 1, \dots, N)$ and the g_m $(m = 1, \dots, M)$ are real-valued functions defined on X. We call X to be the domain, the f_n to be the objective functions, and the g_m to be the constraint functions. This problem can somewhat heuristically be written as:

$$\begin{array}{cc} \max\limits_{x \in X} & (f_1(x), \dots, f_N(x)) \\ \text{subject to} & g_1(x) \geq 0, \\ & \vdots \\ & g_M(x) \geq 0. \end{array}$$

We say that an $x^* \in X$ is a *solution* if $g_m(x^*) \geq 0$ for every m and there is no $x \in X$ such that $g_m(x) \geq 0$ for every m, $f_n(x) \geq f_n(x^*)$ for every n, and $f_n(x) > f_n(x^*)$ for some n.

In the rest of this lecture note, we assume that X is open and the f_n and the g_m are continuously differentiable.

1.6 Kuhn-Tucker Necessary Condition

1.6.1 Theorem (Kuhn-Tucker Necessary Condition) *If $x^* \in X$ is a solution to the constrained maximization problem, then there exists a vector of $N+M$ non-negative numbers, $(\mu_1, \dots, \mu_N, \lambda_1, \dots, \lambda_M) \in \boldsymbol{R}^{N+M}_+$, such that:*

1. *At least one of the $N + M$ non-negative numbers is strictly positive;*

2. *For every m, if $g_m(x^*) > 0$, then $\lambda_m = 0$; and*

3. $\sum_{n=1}^{N} \mu_n \nabla f_n(x^*) + \sum_{m=1}^{M} \lambda_m \nabla g_m(x^*) = 0.$

Proof of Theorem 1.6.1 (Sketch)

1. By re-numbering the constraint functions if necessary, we can assume that the first K constraint functions are the binding ones, where $K \le M$. Then show by a contradiction argument that there is no $v \in \boldsymbol{R}^L$ such that $\nabla f_n(x^*) \cdot v > 0$ for every n and $\nabla g_m(x^*) \cdot v > 0$ for every $m \le K$.

2. By Lemma 1.4.1, this implies that there exists a vector of $N + K$ non-negative numbers, $(\mu_1, \ldots, \mu_N, \lambda_1, \ldots, \lambda_K) \in \boldsymbol{R}_+^{N+K}$, such that at least one of the $N + K$ non-negative numbers is strictly positive and that

$$\sum_{n=1}^{N} \mu_n \nabla f_n(x^*) + \sum_{m=1}^{K} \lambda_m \nabla g_m(x^*) = 0.$$

3. Finally, let $\lambda_m = 0$ for every $m > K$, then the vector $(\mu_1, \ldots, \mu_N, \lambda_1, \ldots, \lambda_M)$ of $N + M$ numbers has the three properties in the Kuhn-Tucker necessary condition.

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1.7 Kuhn-Tucker Sufficient Condition

1.7.1 Definition Let X be convex and $h : X \to \boldsymbol{R}$. We say that h is *quasi-concave* if the following condition is satisfied: For every $x \in X$, $y \in X$, and $\alpha \in [0, 1]$, if $h(y) \ge h(x)$, then $h(\alpha x + (1 - \alpha)y) \ge h(x)$.

1.7.2 Lemma *Let X be convex and $h : X \to \boldsymbol{R}$ be continuously differentiable. Then h is quasi-concave if and only if $\nabla h(x) \cdot (y - x) \ge 0$ whenever $x \in X$, $y \in X$, and $h(y) \ge h(x)$.*

1.7.3 Definition Let X be convex and $h : X \to \boldsymbol{R}$ be continuously differentiable. Then h is *pseudo-concave* if $\nabla h(x) \cdot (y - x) > 0$ whenever $x \in X$, $y \in X$, and $h(y) > h(x)$.

1.7.4 Lemma *Let X be convex, $h : X \to \boldsymbol{R}$ be continuously differentiable. If h is pseudo-concave, then it is quasi-concave.*

1.7.5 Lemma *Let X be convex, $h : X \to \boldsymbol{R}$ be continuously differentiable, and suppose that $\nabla h(x) \ne 0$ for every $x \in X$. Then h is quasi-concave if and only if it is pseudo-concave.*

1.7.6 Theorem (Kuhn-Tucker Sufficient Condition) *Suppose that X is convex, the f_n are pseudo-concave, and the g_m are quasi-concave. Suppose that $x^* \in X$ and that there exists a vector of $N + M$ non-negative numbers, $(\mu_1, \ldots, \mu_N, \lambda_1, \ldots, \lambda_M) \in \boldsymbol{R}_+^{N+M}$, such that:*

1. *$g_m(x^*) \ge 0$ for every m;*

2. *$(\mu_1, \ldots, \mu_N) \in \boldsymbol{R}_{++}^N$;*

3. *For every m, if $g_m(x^*) > 0$, then $\lambda_m = 0$; and*

4. $\sum_{n=1}^{N} \mu_n \nabla f_n(x^*) + \sum_{m=1}^{M} \lambda_m \nabla g_m(x^*) = 0.$

Then x^* is a solution.

Proof of Theorem 1.7.6 (Sketch)

1. By re-numbering the constraint functions if necessary, we can assume that the first K constraint functions are the binding ones, where $K \leq M$. Show then that if x^* were not a solution, there would exist a $v \in \boldsymbol{R}^L$ such that $\nabla f_n(x^*) \cdot v \geq 0$ for every n, $\nabla f_n(x^*) \cdot v > 0$ for some n, and $\nabla g_m(x^*) \cdot v \geq 0$ for every $m \leq K$.

2. Using Conditions 2 and 3 in the Kuhn-Tucker sufficient condition, show then that

$$\sum_{n=1}^{N} \mu_n \nabla f_n(x^*) \cdot v + \sum_{m=1}^{M} \lambda_m \nabla g_m(x^*) \cdot v > 0.$$

3. Since the left hand side of the above inequality equals

$$\left(\sum_{n=1}^{N} \mu_n \nabla f_n(x^*) + \sum_{m=1}^{M} \lambda_m \nabla g_m(x^*) \right) \cdot v,$$

this is a contradiction to Condition 4 in the Kuhn-Tucker sufficient condition.

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Exercise 1.7.1 Give examples to show that each of Conditions 2 and 3 in the Kuhn-Tucker Sufficient Condition is indispensable.

Exercise 1.7.2 Suppose that there are two goods. The price of each good is \$1. Consider a consumer with a utility function $u : \boldsymbol{R}^2_+ \to \boldsymbol{R}$ defined by $u(x_1, x_2) = x_1^{1/2} e^{x_2}$. His income is w, where $w > 0$. We shall consider a standard utility maximization problem under the budget constraint. Note that the domain of the utility function u is $\boldsymbol{R}^2_+$, which is not an open subset of $\boldsymbol{R}^2$, and u is not differentiable at every $x \in \boldsymbol{R}^2_+$ with $x_1 = 0$. In the chapter, we have maintained the assumptions that the domain of the objective function (and the constraint functions) is an open subset of $\boldsymbol{R}^L$ and that the objective and constraint functions are continuously differentiable. In this problem, we will show how to introduce additional constraint functions and modify the domain of the objective function to satisfy these maintained assumptions.

1. Prove that if $x \in \boldsymbol{R}^2_+$ is a solution to the original utility maximization problem, then $x_1 > 0$. This implies that we can restrict the domain to the subset $\{x \in \boldsymbol{R}^2 \mid x_1 > 0 \text{ and } x_2 \geq 0\}$ without altering solutions.

Note that the function u can be extended to the subset $X = \{x \in \boldsymbol{R}^2 | \, x_1 > 0\}$ because the function e^{x_2} can be defined for any values of x_2. Note also that X is an open subset of $\boldsymbol{R}^2$. Define $g_1 : X \to \boldsymbol{R}$ by $g_1(x) = w - x_1 - x_2$. This is simply the budget constraint.

2. Define a constraint function $g_2 : X \to \boldsymbol{R}$ such that the original utility maximization problem is equivalent to the following constrained maximization problem:

$$\begin{array}{ll} \max\limits_{x \in X} & u(x) \\ \text{s.t.} & g_1(x) \geq 0, \\ & g_2(x) \geq 0. \end{array}$$

   (Note that x is chosen from the new domain X.)

3. Apply the Kuhn-Tucker sufficient conditions to find a solution for different values of w.

1.8 Envelope Theorem

Let K, M, and L be positive integers. Let X be an open subset of $\boldsymbol{R}^L$ and P be an open subset of $\boldsymbol{R}^K$. Also let f and g_m ($m = 1, \ldots, M$) be twice continuously differentiable real-valued functions defined on $X \times P$. For each $p \in P$, we consider the following constrained maximization problem with a single objective function

$$\begin{array}{rc} \max\limits_{x \in X} & f(x,p) \\ \text{subject to} & g_1(x,p) \geq 0, \\ & \vdots \\ & g_M(x,p) \geq 0. \end{array}$$

By varying p over P, we can consider a class of constrained maximization problems. The set P is called the *parameter space* of this class of constrained maximization problems.

In the rest of this lecture note, we assume that for every $p \in P$ there exists a unique solution to the constrained maximization problem with parameter p. Denote the solution by $a(p) \in X$; this defines a mapping $a : P \to X$, often called the *policy function*. Define $b : P \to \boldsymbol{R}$ by $b(p) = f(a(p), p)$. This is called the *value function*.

1.8.1 Continuous Differentiability of the Policy Function

In the statement of the following proposition, we regard gradient vectors as row vectors.

1.8.1 Proposition *Let $(x^*, p^*) \in X \times P$ and suppose that x^* satisfies the Kuhn-Tucker sufficient condition with strictly positive multipliers $(1, \lambda_1, \ldots, \lambda_M) \in \boldsymbol{R}^{1+M}_{++}$ for parameter p^*. Suppose also that the $(L+M) \times (L+M)$ matrix*

$$\begin{bmatrix} \nabla_x^2 f(x^*,p^*) + \sum\limits_{m=1}^{M} \lambda_m \nabla_x^2 g_m(x^*,p^*) & \nabla_x g_1(x^*,p^*)^\top & \cdots & \nabla_x g_M(x^*,p^*)^\top \\ \nabla_x g_1(x^*,p^*) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \nabla_x g_M(x^*,p^*) & 0 & \cdots & 0 \end{bmatrix}$$

is invertible. Then there exists an open subset Q of P containing p^* such that the restrictions of a and b on Q are continuously differentiable.

The proof is a direct application of the implicit function theorem.

1.8.2 Envelope Theorem

1.8.2 Theorem (Envelope Theorem) *Let* $(x^*, p^*) \in X \times P$ *and suppose that* x^* *satisfies the Kuhn-Tucker sufficient condition with strictly positive multipliers* $(1, \lambda_1, \ldots, \lambda_M) \in \boldsymbol{R}^{1+M}_{++}$ *for parameter* p^**. Suppose also that* a *and* b *are continuously differentiable. Then*

$$\nabla b(p^*) = \nabla_p f(x^*, p^*) + \sum_{m=1}^{M} \lambda_m \nabla_p g_m(x^*, p^*).$$

Proof of Theorem 1.8.2 (Sketch)

1. By $g_m(a(p), p) = 0$ for every $p \in P$ and m, show that

$$\nabla_x g_m(x^*, p^*)\nabla a(p^*) + \nabla_p g_m(x^*, p^*) = 0.$$

2. Multiply λ_m to both sides of the above equality and apply the Kuhn-Tucker sufficient conditions to show that

$$-\nabla_x f(x^*, p^*)\nabla a(p^*) + \sum_{m=1}^{M} \lambda_m \nabla_p g_m(x^*, p^*) = 0.$$

3. By $b(p) = f(a(p), p)$ for every p, show that

$$\nabla b(p^*) = \nabla_x f(x^*, p^*)\nabla a(p^*) + \nabla_p f(x^*, p^*)$$

4. Combine the above two equalities to complete the proof.

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Chapter 2

Preference and Choice in an Abstract Setting

2.1 Introduction

In this chapter we explore basic properties of preference and choice in an abstract setting, which includes, but is not restricted to, the consumer's and producer's choice problems to be considered in the subsequent analysis. The materials of this section owe much to Chapter 1 of MWG. The strong axiom of revealed preference is dealt with in the setting of consumption choice in Section J of Chapter 3.

Let X be a nonempty set of *alternatives*. There is an agent (such as a consumer, a producer, an investor, and a social planner) who has some preferences (possibly embodying moral judgments and probability assessments) over these alternatives, and chooses over sets of alternatives (subsets of X).

2.2 Binary Relations

A *binary relation* (or, more simply, *relation*) R *on* X can be considered simply as a mathematical object such that, for every $x \in X$ and $y \in X$, whether the relation "xRy" holds or not has been unambiguously defined. Formally, R can be identified with a (unique) subset of $X \times X$ by regarding "xRy" as equivalent to "$(x, y) \in R$". Whenever it is untrue that xRy, we write $xR^{c}y$. Then, as subsets of $X \times X$, R^{c} is the complement of R.

2.2.1 Definition A binary relation R on X is:

1. *reflexive* if for every $x \in X$, xRx.
2. *irreflexive* if R^{c} is reflexive.
3. *complete* if for every $x \in X$ and every $y \in X$, xRy or yRx.
4. *symmetric* if for every $x \in X$ and every $y \in X$, yRx whenever xRy.
5. *asymmetric* if for every $x \in X$ and every $y \in X$, $yR^{c}x$ whenever xRy.
6. *acyclic* if there is no pair of $N \in \boldsymbol{N}$ and $(x_1, x_2, \dots, x_N) \in X^N$ such that $x_n R x_{n+1}$ for every $n \leq N - 1$ and $x_N R x_1$.
7. *transitive* if for every $x \in X$, every $y \in X$, and every $z \in X$, xRz whenever xRy and yRz.

8. *negatively transitive* if R^c is transitive.

Exercise 2.2.1 Prove that:

1. every complete relation is reflexive.
2. no reflexive relation is asymmetric.
3. every acyclic relation is asymmetric.
4. every asymmetric relation is irreflexive.
5. every asymmetric and negatively transitive relation is transitive.
6. every irreflexive and transitive relation is acyclic.

Exercise 2.2.2 Let R be a binary relation on X. Prove that:

1. R is negatively transitive if and only if for every $x \in X$ and $y \in X$, there exists a $z \in X$ such that xRz or zRy.
2. R is transitive if and only if it is negatively transitive, provided that R is complete.
3. R is asymmetric if and only if R^c is complete.
4. R is asymmetric and negatively transitive if and only if R^c is complete and transitive.

In economics, the interpretation of R depends, among other things, on whether R is reflexive or irreflexive. If R is reflexive, "xRy" is interpreted as "alternative x is at least as preferable as alternative y". In this case, we often use symbol $\succsim$ in place of R. If R is irreflexive, "xRy" is interpreted as "alternative x is more preferable than alternative y". In this case, we often use symbol $\succ$ in place of R.

Exercise 2.2.3

1. Let $X = \boldsymbol{R}^2$ and define a binary relation $\succsim$ by letting, for every $x \in X$ and every $y \in X$, $x \succsim y$ if and only if $x_1 \geq y_1$ and $x_2 \geq y_2$. Prove that $\succsim$ is transitive, and give an example to show that it is not complete.
2. Let $X = \boldsymbol{R}^3$ and define a binary relation $\succsim$ by letting, for every $x \in X$ and every $y \in X$, $x \succsim y$ if and only if $x_n \geq y_n$ for at least two $n \in \{1, 2, 3\}$. Prove that $\succsim$ is complete, and give an example to show that it is not transitive.

Let R be a binary relation on X. Define another binary relation R^s by letting xR^sy whenever xRy and yR^cx. Then R^s is the *strict* or *asymmetric* part of R. Define yet another binary relation R^i by letting xR^iy whenever xRy and yRx. Then R^i is the *indifference* or *symmetric* part of R. These definitions make sense for any binary relation R, but are useful when R is reflexive. In economics, when a reflexive relation $\succsim$ is given, $\succsim^s$ is often denoted by $\succ$ and $\succsim^i$ is denoted by $\sim$.

Exercise 2.2.4 Let R be a binary relation on X. Prove that:

1. R^i is symmetric. It is transitive if R is transitive.
2. R^s is asymmetric. It is transitive if R is transitive.

Exercise 2.2.5 Is it true that for every binary relation R, R is transitive if and only if R^s and R^i are transitive? If so, prove it. If not, present a counterexample and provide additional conditions on R under which this equivalence is guaranteed.

Exercise 2.2.6 For a binary relation R on X, we define another binary relation R^{t} by letting, for every $x \in X$ and every $y \in X$, $xR^{\mathrm{t}}y$ if and only if yRx. Prove that:

1. if R is complete, then $R = ((R^{\mathrm{s}})^{\mathrm{c}})^{\mathrm{t}}$.

2. if R is asymmetric, then $R = ((R^{\mathrm{c}})^{\mathrm{t}})^{\mathrm{s}}$.

2.2.2 Definition Let R be a binary relation on X. A function $u : X \to \boldsymbol{R}$ is a *utility function representing* R if the following condition is met: for every $x \in X$ and every $y \in X$, $u(x) \geq u(y)$ if and only if xRy. If there is a utility function representing R, R is *representable.*

2.2.3 Proposition *If a binary relation is represented by a utility function, then it is complete and transitive.*

Since every complete relation is reflexive, Proposition 2.2.3 implies that when a binary relation is represented by a utility function in the sense of Definition 2.2.2, then it is to be interpreted as an at-least-as-preferable-as relation. A less standard definition of representation is the following:

2.2.4 Definition Let R be a binary relation on X. A function $u : X \to \boldsymbol{R}$ is a *utility function strictly representing* R if the following condition is met: for every $x \in X$ and every $y \in X$, $u(x) > u(y)$ if and only if xRy. If there is a utility function representing R, R is *strictly representable*

2.2.5 Proposition *If a binary relation is strictly represented by a utility function, then it is asymmetric and transitive.*

Since every asymmetric relation is irreflexive, Proposition 2.2.5 implies that when a binary relation is strictly represented by a utility function in the sense of Definition 2.2.4, then it is to be interpreted as a more-preferable-than relation.

Exercise 2.2.7 Prove that:

1. if R is represented by a utility function u, then R^{s} is strictly represented by u.

2. if R is strictly represented by a utility function u, then $(R^{\mathrm{c}})^{\mathrm{t}}$ is represented by u.

Exercise 2.2.8 Use a mathematical induction argument on the the number of the elements of X to prove that if X is finite (that is, it consists of finitely many elements), then every rational preference relation is represented by a utility function.

Exercise 2.2.9 (Difficult) Prove that if a binary relation R on X is strictly represented by a utility function, then there exists an at most countable subset Z of X such that for every $x \in X \setminus Z$ and every $y \in X \setminus Z$, if xRy, then there exists a $z \in Z$ such that xRz and zRy. Would the same conclusion hold if R were represented by a utility function?

2.3 Choice Rules

A *choice structure on* X is a pair of a set of non-empty subsets of X, denoted by $\mathscr{B}$, and a mapping of $\mathscr{B}$ into a set of non-empty subsets of X, denoted by C, that satisfies $C(B) \subseteq B$ for every $B \in \mathscr{B}$.

2.3.1 Definition Let $(\mathscr{B}, C)$ be a choice structure on X. A binary relation R on X is the *revealed at-least-as-preferable-as relation of* $(\mathscr{B}, C)$ if the following condition is met: for every $x \in X$ and every $y \in X$, xRy if and only if there exists a $B \in \mathscr{B}$ such that $\{x, y\} \subseteq B$ and $x \in C(B)$.

2.3.2 Definition Let $(\mathscr{B}, C)$ be a choice structure on X and R be its revealed at-least-as-preferable-as relation. Consider the following condition for each integer $N \geq 2$:

> If $x_n \in X$ for every $n \leq N$ and $x_n R x_{n+1}$ for every $n \leq N-1$, then $x_1 \in C(B)$ whenever $\{x_1, x_N\} \subseteq B \in \mathscr{B}$ and $x_N \in C(B)$.

Then:

Weak Axiom of Revealed Preference $(\mathscr{B}, C)$ satisfies the *weak axiom of revealed preference* if the above condition is met for $N = 2$.

Strong Axiom of Revealed Preference $(\mathscr{B}, C)$ satisfies the *strong axiom of revealed preference* if the above condition is met for every $N \geq 2$.

Exercise 2.3.1 Let $(\mathscr{B}, C)$ be a choice structure and R be its revealed at-least-as-preferable-as relation. Let Q be the transitive closure of R, that is, xQy if and only if there are an $N \in \boldsymbol{N}$ and an $(x_1, x_2, \ldots, x_N) \in X^N$ such that $x_1 = x$, $x_N = y$, and $x_{n-1} R x_n$ for every $n \leq N$. Prove that:

1. $(\mathscr{B}, C)$ satisfies the weak axiom of revealed preference if and only if for every $x \in X$ and every $y \in X$, $C(B) \cap \{x, y\} \in \{\varnothing, \{x, y\}\}$ whenever $xR^{\mathrm{i}}y$ and $\{x, y\} \subseteq B$.

2. $(\mathscr{B}, C)$ satisfies the strong axiom of revealed preference if and only if it satisfies the weak axiom of revealed preference and $R^{\mathrm{s}} \subseteq Q^{\mathrm{s}}$ (that is, for every $x \in X$ and every $y \in X$, if $xR^{\mathrm{s}}y$, then $xQ^{\mathrm{s}}y$).

Exercise 2.3.2 Let $X = \{x, y, z\}$. Define a choice structure $(\mathscr{B}, C)$ on X by $\mathscr{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$, $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$. Show that $(\mathscr{B}, C)$ satisfies the weak axiom but not the strong axiom of revealed preference.

Exercise 2.3.3 Let $X = \{x, y, z\}$. Define a choice structure $(\mathscr{B}, C)$ on X satisfy $\mathscr{B} = \{\{x, y\}, \{y, z\}, \{z, x\}, \{x, y, z\}\}$, $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$. Show that (regardless of what $C(\{x, y, z\})$ is) $(\mathscr{B}, C)$ does not satisfy the weak axiom of revealed preference.

2.4 Relationship between Preference Relations and Choice Rules

Let $\mathscr{B}$ be a set of non-empty subsets of X.

2.4.1 Definition Let R be a binary relation and assume that for every $B \in \mathscr{B}$, there exists an $x \in B$ such that xRy for every $y \in B$. Define a mapping C of $\mathscr{B}$ into the set of the non-empty subsets of X by $C(B) = \{x \in B \mid xRy \text{ for every } y \in B\}$. Then $(\mathscr{B}, C)$ is the *choice structure of* R.

Definition 2.4.1 makes sense only if R is reflexive (because, otherwise, the assumption that for every $B \in \mathscr{B}$, there exists an $x \in B$ such that xRy for every $y \in B$ may well fail). For an irreflexive R, the following definition of thee choice structure makes sense.

2.4.2 Definition Let R be a binary relation and assume that for every $B \in \mathscr{B}$, there exists an $x \in B$ such that $yR^{c}x$ for every $y \in B$. Define a mapping C of $\mathscr{B}$ into the set of the non-empty subsets of X by $C(B) = \{x \in B \mid yR^{c}x \text{ for every } y \in B\}$. Then $(\mathscr{B}, C)$ is the *strict choice structure of* R.

Exercise 2.4.1 Prove that the choice structure of every complete and transitive relation satisfies the strong axiom of revealed preference.

The following theorem claims that the converse is also true, although we shall not prove it here:

2.4.3 Theorem (Richter) *For every choice structure $(\mathscr{B}, C)$ satisfying the strong axiom of revealed preference, there exists a complete and transitive relation of which the choice structure coincides with $(\mathscr{B}, C)$.*

As can be seen from Exercise 2.3.2, the weak axiom is, in general, not sufficient for the conclusion of Richter's Theorem, that is, the existence of an underlying complete and transitive preference relation. However, it can be shown that if $\mathscr{B}$ contains all subsets of X of up to three elements, then the weak axiom is indeed sufficient for the existence of an underlying complete and transitive relation. Moreover, such a binary relation is uniquely determined.

Exercise 2.4.2 Assume that $\mathscr{B}$ contains all subsets of X of two elements and let $\succsim$ be a complete and transitive relation. Let $(\mathscr{B}, C)$ be the choice structure of $\succsim$ and R be the revealed preference relation of $(\mathscr{B}, C)$. Prove that $\succsim = R$.

2.5 Afriat's Approach

In this section, we briefly review an alternative approach, originally due to S. Afriat, on the possibility of recovering an underlying preference relation from observed choices. In Afriat's approach, the choice structure $(\mathscr{B}, C)$ is such that $C(B)$ is a singleton for every $B \in \mathscr{B}$, and the question is whether there is a complete and transitive relation $\succsim$ on X such that $C(B) \succsim y$ for ever $y \in B$ and every $B \in \mathscr{B}$. The interpretation is that alternative $C(B)$ is chosen when the alternatives in B are available.

The difference between this and the preceding approaches is that, in this approach, we have less information, in the sense that although we know that $C(B)$ is a most preferable choice in B, we do not know whether there is any other equally preferable choice in B. This difference can be made clearer by comparing the notion of an underlying preference relation. In the preceding approach, a preference relation underlies the choice structure $(\mathscr{B}, C)$ if $C(B) = \{x \in B \mid x \succsim y \text{ for every } y \in B\}$, while, in Afriat's apporach, a preference relation underlies the choice structure $(\mathscr{B}, C)$ if $C(B) \in \{x \in B \mid x \succsim y \text{ for every } y \in B\}$.

This notion of an underlying preference may well be superior to the preceding one for the purpose of empirical studies, as we often have a data set of, say, household consumption behavior, but we do not have any data that indicate whether households had equally desirable consumption choices within their budgets. However, since the notion of an underlying relation in Afriat's approach is weaker than that in the preceding apporach, it is easier to guarantee its existence in Afriat's approach. Indeed, the total indifference relation (the relation $\succsim$ such that $x \succsim y$ for every $x \in X$ and every $y \in X$) underlies every choice structure $(\mathscr{B}, C)$ in Ariat's sense. For this reason, in Afriat's approach, it is common to ask whether a given choice structure $(\mathscr{B}, C)$ has an underlying relation within some restricted class of complete and transitive relation defined on X.

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Chapter 3

Consumer Theory

3.1 Introduction

In this lecture note we present the classical theory of a consumer. The materials are taken from Chapters 2 and 3 of MWG. Debreu gives a succinct account of some of the concepts introduced below. Kreps has a somewhat different, and yet very nice, treatment.

3.2 Commodities and Commodity Space

A *commodity* is defined in terms of its physical characteristics, as well as the time, location, and even contingency when uncertainty is present, at which it is available for consumption or production.

Throughout this lecture course, we denote by L the number of types of commodities. The *commodity space* is the set of all combinations of quantities of the L commodities (called *commodity bundles*) that are regarded as physically feasible. It is therefore a subset of $\boldsymbol{R}^L$, and indeed often taken as $\boldsymbol{R}^L$ itself. It can however be $\boldsymbol{Z}^L$, where $\boldsymbol{Z}$ is the set of all integers, if all commodities are indivisible. For simplicity, we take the commodity space to be $\boldsymbol{R}^L$.

3.3 Consumption Set

The *consumption set* of a consumer is the set of all commodity bundles with which the consumer can survive. We denote it by X. It is on this set where the consumer's preference relation is to be defined. It is a subset of the commodity space and often taken to be $\boldsymbol{R}^L_+$. It can however be $\boldsymbol{Z}^L_+$ or other subsets of $\boldsymbol{R}^L_+$. For simplicity, we take the consumption set to be $\boldsymbol{R}^L_+$. A commodity bundle in the consumption set is a *consumption bundle*.

3.4 Preference and Utility

The *preference relation* of a consumer is nothing but a binary relation defined in Section 2.2 of Chapter 2. We make use of other properties, which relies on the Euclidean structure of X.

3.4.1 Definition Let $\succsim$ be a preference relation (binary relation) on X, with its strict part $\succ$ and indifference part $\sim$. Let x and y be arbitrary consumption bundles and α be an arbitrary number in $[0, 1]$.

Monotonicity $\succsim$ is *monotone* if $y \succ x$ whenever $y - x \in \boldsymbol{R}^L_{++}$.

Strong Monotonicity $\succsim$ is *strongly monotone* if $y \succ x$ whenever $y - x \in \boldsymbol{R}^L_+$ and $y - x \neq 0$.

Local Non-Satiation $\succsim$ is *locally non-satiated* if for every $\varepsilon > 0$ there exists a $z \in X$ such that $\|z - x\| \leq \varepsilon$ and $z \succ x$.

Convexity $\succsim$ is *convex* if $\alpha x + (1-\alpha)y \succsim z$ whenever $z \in X$, $x \succsim z$, and $y \succsim z$.

Strict Convexity $\succsim$ is *strictly convex* if $\alpha x + (1-\alpha)y \succ z$ whenever $z \in X$, $x \succsim z$, $y \succsim z$, $x \neq y$, and $\alpha \in (0,1)$.

Continuity $\succsim$ is *continuous* if $x \succsim y$ whenever $(x^n)_{n=1}^{\infty}$ and $(y^n)_{n=1}^{\infty}$ are sequences in X such that $x^n \succsim y^n$ for every n, and $x^n \to x$ and $y^n \to y$ as $n \to \infty$.

3.4.2 Proposition *Every strongly monotone preference relation on $X = \boldsymbol{R}_+^L$ is monotone. Every monotone preference relation on $X = \boldsymbol{R}_+^L$ is locally non-satiated.*

Exercise 3.4.1 Show that there is a monotone preference relation on $X = \boldsymbol{Z}_+^L$ that is not locally non-satiated. Provide appropriate extensions, to the case of a general consumption set X, of the above definitions of monotonicity and strong monotonicity with respect to which Proposition 3.4.2 is true.

Exercise 3.4.2 Prove that every preference relation that can be represented by a continuous utility function is continuous, complete, and transitive.

3.4.3 Proposition *Every continuous, complete, and transitive preference relation can be represented by a continuous utility function.*

Exercise 3.4.3 Let R be the strict part of the lexicographic ordering. Prove that there is no at most countable subset Z of $X = \boldsymbol{R}_+^2$ for which the property of Proposition 2.2.9 holds.

3.5 Prices, Wealth, and Budget Sets

We assume that there is a complete set of markets, so that there is a price, denoted by p_ℓ, for each commodity ℓ ($\ell = 1, \dots, L$). Denote by p the L-dimensional column vector consisting of the p_ℓ. This is a price vector.

A *wealth level* of a consumer is presented by a real (and often non-negative or strictly positive) number w. Under a price vector p and a wealth level w, the consumer's *budget set* is $\{x \in X \mid p \cdot x \leq w\}$.

Simple as it may look, this formulation embodies some important assumptions in economics. First, the market is complete, so that all commodities that are relevant to preference relations and utility functions are given prices. Second, there is only one inequality defining the constraint, so that a reduction in expenditure for any one commodity can be used to increase the expenditure for any other commodity. Third, there is no rationing, in the sense that there is no (upper or lower) bound on the amounts of commodities that can be bought. Fourth, the price level for each commodity is unaffected by any change in the choice of the commodity bundle x, so that the consumer is a price-taker, whose influence on the market outcome is negligible relative to the size of the markets.

3.6 Utility Maximization Problem

Let u be a continuous utility function representing a continuous rational preference relation $\succsim$. The continuity assumption guarantees that in the subsequent analysis, all the functions are continuous and correspondences are upper semi-continuous on some appropriate domain (where, for example, the wealth level is strictly positive).

The *utility maximization problem* under a price vector p and a wealth level w is

$$\begin{array}{rl} \max\limits_{x \in X} & u(x), \\ \text{subject to} & p \cdot x \leq w. \end{array}$$

3.6.1 Proposition *If $p \in \boldsymbol{R}^L_{++}$ and $w \geq 0$, then there exists at least one solution to the utility maximization problem. If, in addition, $\succsim$ is strictly convex, then there exists exactly one solution.*

We assume in the rest of this lecture note that there exists at least one solution and denote by $x(p, w)$ the set of all solutions. Then $(p, w) \mapsto x(p, w)$ is the *Walrasian demand correspondence.* It is a function if $\succsim$ is strictly convex.

3.6.2 Proposition *The Walrasian demand correspondence x has the following properties:*

Homogeneity *For every $\alpha > 0$, $x(\alpha p, \alpha w) = x(p, w)$.*

Walras' law *If $\succsim$ is locally non-satiated, then $p \cdot \bar{x} = w$ for every $\bar{x} \in x(p, w)$.*

Strong Axiom of Revealed Preference *x satisfies the strong axiom of revealed preference (Definition 2.3.2).*

Exercise 3.6.1 Give an example of a strictly convex and locally non-satiated (but not necessarily monotone) preference relation on $X = \boldsymbol{R}^2_+$ such that the demand function x is *not* continuous at $p = (0, 1)$ and $w = 0$.

The value function of the utility maximization problem is the *indirect utility function* and denoted by v. Then $v(p, w) = u(\bar{x})$ for every $\bar{x} \in x(p, w)$.

3.6.3 Proposition *The indirect utility function v has the following properties:*

Homogeneity *For every $\alpha > 0$, $v(\alpha p, \alpha w) = v(p, w)$.*

Monotonicity *v is non-decreasing in w and non-increasing in every p_ℓ. If $\succsim$ is locally non-satiated, then v is strictly increasing in w.*

Quasi-Convexity *v is quasi-convex, that is, $v(\alpha p + (1 - \alpha)p', \alpha w + (1 - \alpha)w') \leq v(q, b)$ whenever $v(p, w) \leq v(q, b)$, $v(p', w') \leq v(q, b)$, and $\alpha \in [0, 1]$.*

3.7 Expenditure Minimization Problem

The *expenditure minimization problem* under a price vector p and a utility level $\bar{u}$ is

$$\begin{array}{rl} \min\limits_{x \in X} & p \cdot x, \\ \text{subject to} & u(x) \geq \bar{u}. \end{array}$$

3.7.1 Proposition *If $p \in \boldsymbol{R}^L_{++}$ and there exists an $\bar{x} \in X$ such that $u(\bar{x}) \geq \bar{u}$, then there exists at least one solution to the utility maximization problem. If, in addition, $\succsim$ is strictly convex, then there exists exactly one solution.*

We assume in the rest of this lecture note that there exists at least one solution and denote by $h(p, \bar{u})$ the set of all solutions. Then $(p, \bar{u}) \mapsto h(p, \bar{u})$ is the *Hicksian demand correspondence.* It is a function if $\succsim$ is strictly convex.

3.7.2 Proposition *The Hicksian demand function h has the following properties:*

Homogeneity *For every* $\alpha > 0$, $h(\alpha p, \bar{u}) = h(p, \bar{u})$.

No Excess Utility *If* $\succsim$ *is continuous and monotone,* $p \in \boldsymbol{R}^L_{++}$, *and* $\bar{u} \geq u(0)$, *then* $u(\bar{x}) = \bar{u}$ *for every* $\bar{x} \in h(p, \bar{u})$.

Compensated Law of Demand *For any two price vectors* p *and* p', $(p - p') \cdot (h(p, \bar{u}) - h(p', \bar{u})) \leq 0$.

The value function of the expenditure minimization problem is the *expenditure function* and denoted by e. Then $e(p, \bar{u}) = p \cdot \bar{x}$ for every $\bar{x} \in h(p, \bar{u})$.

3.7.3 Proposition *The expenditure function* e *has the following properties:*

Homogeneity *For every* $\alpha > 0$, $e(\alpha p, \bar{u}) = \alpha e(p, \bar{u})$.

Monotonicity e *is non-decreasing in* $\bar{u}$ *and every* p_ℓ. *If* $\succsim$ *is monotone, then* e *is strictly increasing in* $\bar{u} \geq u(0)$.

Concavity e *is concave in* p, *that is,* $e(\alpha p + (1-\alpha)p', \bar{u}) \geq \alpha e(p, \bar{u}) + (1-\alpha)e(p', \bar{u})$ *for every* $\alpha \in [0, 1]$.

Exercise 3.7.1 Show that the no excess utility property of Proposition 3.7.2 and the strict increasingness in $\bar{u}$ of Proposition 3.7.3 need not be satisfied by a locally non-satiated, but not monotone, preference relation.

3.8 Characterization of Walrasian and Hicksian Demands

3.8.1 Proposition *For every price vector* p *and utility level* $\bar{u}$,

$$h(p, \bar{u}) = \nabla_p e(p, \bar{u}). \tag{3.1}$$

Exercise 3.8.1 For a price vector $\bar{p}$ and a utility level $\bar{u}$, consider the following maximization problem:

$$\begin{array}{cc} \max\limits_{p} & e(p, \bar{u}), \\ \text{subject to} & p \cdot h(\bar{p}, \bar{u}) \leq e(\bar{p}, \bar{u}). \end{array}$$

Show that $\bar{p}$ is a solution to this maximization problem, and also that the Kuhn-Tucker condition for the solution implies equality (3.1). (*Hint*: Use the homogeneity of e in p to show that the multiplier in the Kuhn-Tucker condition equals one.)

3.8.2 Proposition *For every price vector* p *and utility level* $\bar{u}$, $D_p h(p, \bar{u})$ *is symmetric, negative semi-definite, and satisfies* $D_p h(p, \bar{u}) p = 0$.

3.8.3 Proposition (Roy's Identity) *For every price vector* p *and wealth level* w,

$$x(p, w) = -\frac{1}{\nabla_w v(p, w)} \nabla_p v(p, w).$$

Exercise 3.8.2 For a price vector $\bar{p}$ and a wealth level $\bar{w}$, consider the following minimization problem:

$$\begin{array}{cc} \min\limits_{(p,w)} & v(p, w), \\ \text{subject to} & p \cdot x(\bar{p}, \bar{w}) \leq w. \end{array}$$

Show that $(\bar{p}, \bar{w})$ is a solution to this minimization problem, and also that the Kuhn-Tucker condition for the solution implies Roy's Identity.

3.9 Duality

3.9.1 Proposition *For every price vector $p \in R^L_{++}$ and wealth level $w > 0$,*

$$h(p, v(p, w)) = x(p, w)$$

and

$$e(p, v(p, w)) = w.$$

For every price vector $p \in R^L_{++}$ and utility level $\bar{u} \geq u(0)$,

$$x(p, e(p, \bar{u})) = h(p, \bar{u})$$

and

$$v(p, e(p, \bar{u})) = \bar{u}.$$

Exercise 3.9.1 Give examples to show that if $\succsim$ were not locally non-satiated, then $h(p, v(p, w)) = x(p, w)$ may not hold; and that if $p \notin R^L_{++}$ and $w = 0$, then $x(p, e(p, \bar{u})) = h(p, \bar{u})$ may not hold.

3.9.2 Proposition (Slutsky Equation) *For every price vector $\bar{p} \in R^L_{++}$ and wealth level $\bar{w} > 0$,*

$$D_p x(\bar{p}, \bar{w}) = D_p h(\bar{p}, v(\bar{p}, \bar{w})) - D_w x(\bar{p}, \bar{w}) x(\bar{p}, \bar{w})^\top.$$

Hence $D_p x(\bar{p}, \bar{w}) + D_w x(\bar{p}, \bar{w}) x(\bar{p}, \bar{w})^\top$ is symmetric and negative semi-definite.

3.10 Taxonomy of Properties

No Change in Constraints or Objectives x is homogeneous of degree zero in (p, w); v is homogeneous of degree zero in (p, w); h is homogeneous of degree zero in p; and e is homogeneous of degree one in p.

Monotone Changes in Constraints or Objectives v is non-increasing in p and non-decreasing in w; and e is non-decreasing in p and $\bar{u}$.

Arbitrary Changes in Constraints or Objectives x satisfies the strong axiom of revealed preference; v is quasi-convex in (p, w); h satisfies the Compensated Law of Demand; and e is concave in p.

Exercise 3.10.1 Use Proposition 3.9.1 to derive the monotonicity and concavity of e from the monotonicity and quasi-convexity of v; and also derive the monotonicity and quasi-convexity of v from the monotonicity and concavity of e.

3.11 Welfare Measures of Price Changes

Suppose that the current price vector p^0 now changes to p^1, while the wealth level w is unchanged. The change is preferable if and only if $v(p^1, w) - v(p^0, w) > 0$.

Now take another price vector $\bar{p}$. If e is strictly increasing in $\bar{u}$, then the above inequality is equivalent to

$$e(\bar{p}, v(p^1, w)) - e(\bar{p}, v(p^0, w)) > 0. \tag{3.2}$$

The function $p \mapsto e(\bar{p}, v(p, w))$ is the *money metric (indirect) utility function (under $\bar{p}$)*. If we take $\bar{p} = p^0$, then the left hand side of (3.2) is

$$e(p^0, v(p^1, w)) - e(p^0, v(p^0, w)) = e(p^0, \bar{u}^1) - w,$$

where $\bar{u}^1 = v(p^1, w)$. This is the *equivalent variation of the change from* p^0 *to* p^1 *with wealth level* w, and denoted by $EV(p^0, p^1, w)$. On the other hand, if we take $\bar{p} = p^1$, then the left hand side of (3.2) is

$$e(p^1, v(p^1, w)) - e(p^1, v(p^0, w)) = w - e(p^1, \bar{u}^0),$$

where $\bar{u}^0 = v(p^0, w)$. This is the *compensated variation of the change from* p^0 *to* p^1 *with wealth level* w, and denoted by $CV(p^0, p^1, w)$.

3.11.1 Proposition *For every* p^0, p^1, *and* w, *if* $p^0_\ell = p^1_\ell$ *for every* $\ell \geq 2$, *then*

$$EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, \bar{u}^1)\ dp_1,$$

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, \bar{u}^0)\ dp_1,$$

where $p_{-1} = \left(p^0_2, \ldots, p^0_L\right) = \left(p^1_2, \ldots, p^1_L\right)$, $\bar{u}^0 = v(p^0, w)$, *and* $\bar{u}^1 = v(p^1, w)$.

Another commonly used measure of price changes is what MWG called the *area variation measure*:

$$AV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} x_1(p_1, p_{-1}, w)\ dp_1.$$

In general, this measure cannot be interpreted as a change in utility levels, but if the preference relation is quasi-linear with respect to a commodity other than the first one, and if the consumption levels of the numeraire are always strictly positive for the relevant range of prices, then

$$AV(p^0, p^1, w) = EV(p^0, p^1, w) = CV(p^0, p^1, w).$$

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Chapter 4

Producer Theory

4.1 Introduction

In this lecture note we start the classical theory of a producer. We spell out some basic definitions for productions sets and formulate the profit maximization and cost minimization.

4.2 Productions Set

To describe production activities in the commodity space R^L, we use the convention that, in a vector R^L, the amounts of outputs are measured in positive quantities and the amounts of inputs are measured in negative quantities. A *production set* is the set of all vectors of R^L that are technologically feasible.

The following assumptions are often used for a production set Y.

Nonemptiness $Y \neq \varnothing$.

Closedness Y is a closed subset of R^L.

No-Free Lunch $Y \cap R^L_+ \subseteq \{0\}$.

Possibility of Inaction $0 \in Y$.

Free Disposal $Y - R^L_+ \subseteq Y$.

Irreversibility $Y \cap (-Y) \subseteq \{0\}$.

Non-Increasing Returns to Scale $\alpha\bar{y} \in Y$ for every $\bar{y} \in Y$ and every $\alpha \in [0, 1]$.

Non-Dncreasing Returns to Scale $\alpha\bar{y} \in Y$ for every $\bar{y} \in Y$ and every $\alpha \geq 1$.

Constant Returns to Scale $\alpha\bar{y} \in Y$ for every $\bar{y} \in Y$ and every $\alpha \geq 0$.

Additivity $\bar{y} + \bar{y}' \in Y$ for every $\bar{y} \in Y$ and every $\bar{y}' \in Y$.

Convexity Y is a convex subset of R^L.

Convex Cone Y is a convex cone in R^L, that is, $\alpha\bar{y} + \alpha'\bar{y}' \in Y$ for every $\bar{y} \in Y$, every $\bar{y}' \in Y$, every $\alpha \geq 0$, and every $\alpha' \geq 0$.

Polyhedral Cone Y is a polyhedral cone in R^L, that is, there exists a finite subset$\{a_1, \ldots, a_M\}$ of R^L such that for every $\bar{y} \in Y$ there exists an$(\alpha_1, \ldots, \alpha_M) \in R^M_+$ such that

$$\bar{y} = \sum_{m=1}^{M} \alpha_m a_m.$$

4.3 Transformation Function and Production Function

4.3.1 Definition Let Y be a production set.

1. If a function $F : R^L \to R$ satisfies $Y = \{\bar{y} \in R^L \mid F(\bar{y}) \leq 0\}$, then F is a *transformation function of* Y.
2. Suppose that Y satisfies $\bar{y}_l \leq 0$ for every $\bar{y} \in Y$ and every $\ell < L$. If a function $f : R_+^{L-1} \to R_+$ satisfies $Y = \{\bar{y} \in R^L \mid \bar{y}_L \leq f(-\bar{y}_1, \ldots, -\bar{y}_{L-1}\}$, then f is a *production function of* Y.

4.4 Profit Maximization Problem

The *profit maximization problem* under a price vector p is

$$\max_{\bar{y} \in Y} p \cdot \bar{y},$$

The set of solutions to this maximization problem is denoted by $y(p)$. This defines a correspondence y, the *supply correspondence of* Y, on a set of price vectors p for which the problem has at least one solution.

4.4.1 Proposition *The supply correspondence y has the following properties:*

Homogeneity $y(\alpha p) = y(p)$ *for every price vector p and every $\alpha > 0$.*

Law of Supply $(p' - p) \cdot (\bar{y}' - \bar{y}) \leq 0$ *for any two price vectors p and p', every $\bar{y} \in y(p)$, and every $\bar{y}' \in y(p')$.*

The value function of the profit maximization problem is the *profit function of* Y and denoted by π. Then $\pi(p) = p \cdot \bar{y}$ for every $\bar{y} \in y(p)$.

4.4.2 Proposition *The profit function π has the following properties:*

Homogeneity $\pi(\alpha p) = \alpha\pi(p)$ *for every price vector p and every $\alpha > 0$.*

Convexity *π is a convex function, that is, $\pi(\alpha p + (1-\alpha)p') \leq \alpha\pi(p) + (1-\alpha)\pi(p')$ for every $\alpha \in [0, 1]$ and any two price vectors p and p'.*

4.4.3 Lemma (Hotelling) *If π is differentiable at a price vector p, then $y(p) = \{\nabla\pi(p)\}$. If π is twice differentiable at a price vector p, then y is a function differentiable at p, and $Dy(p)$ is symmetric and positive semi-definite, and satisfies $Dy(p)p = 0$.*

4.5 Cost Minimization Problem

Suppose that Y satisfies $\bar{y}_\ell \leq 0$ for every $\bar{y} \in Y$ and every $\ell < L$. Let $f : R_+^{L-1} \to R_+$ be the production function of Y. The *cost minimization problem* under a vector $w \in R^{L-1}$ of input prices and an output level q is

$$\begin{array}{cc} \displaystyle\min_{\bar{z} \in R_+^{L-1}} & w \cdot \bar{z}, \\ \text{subject to} & f(\bar{z}) \geq q. \end{array}$$

The set of solutions to this minimization problem is denoted by $z(w, q)$. This defines a correspondence z, the *conditional factor demand correspondence of* Y, on a set of price vectors w and output levels q for which the problem has at least one solution.

4.5.1 Proposition *The conditional factor demand correspondence z has the following properties:*

Homogeneity $z(\alpha w, q) = z(w, q)$ *for every factor price vector w and every $\alpha > 0$.*

Law of Conditional Factor Demand $(w' - \bar{w}) \cdot (\bar{z}' - \bar{z}) \leq 0$ *for any two input price vectors w and w', every $\bar{z} \in z(w, q)$, and every $\bar{z}' \in z(w', q)$.*

The value function of the cost minimization problem is the *cost function of Y* and denoted by c. Then $c(w, q) = w \cdot \bar{z}$ for every $\bar{z} \in z(q, w)$.

4.5.2 Proposition *The cost function c has the following properties:*

Homogeneity $c(\alpha w, q) = \alpha c(w, q)$ *for every price vector p and every $\alpha > 0$.*

Concavity *c is a concave function of w, that is, $c(\alpha w + (1-\alpha)w', q) \geq \alpha c(w, q) + (1-\alpha)c(w', q)$ for every $\alpha \in [0, 1]$ and any two input price vectors q and q'.*

4.5.3 Lemma (Shepard) *If $c(\cdot, q)$ is differentiable at an input price vector w, then $z(w, q) = \{\nabla_w c(w, q)\}$. If $c(\cdot, q)$ is twice differentiable at an input price vector w, then $z(\cdot, q)$ is a function differentiable at w, and $D_w z(w, q)$ is symmetric and negative semi-definite, and satisfies $D_w z(w, q) w = 0$.*

4.6 When Profit Maximization is Justfied

Imagine that a firm has a production set Y and a consumer has an indirect utility function v, which is strictly increasing in wealth, and owns a share $\theta > 0$ in the profit of the firm. If a production plan $\bar{y} \in Y$ is chosen, $\bar{p}$ is the price vector, and his wealth from other than the shareholding in the firm, then he attains the utility level $u(\bar{p}, \bar{w} + \theta \bar{p} \cdot \bar{y})$.

4.6.1 Proposition *If $\bar{p}$ and $\bar{w}$ does not depend on the choice of a production plan $\bar{y} \in Y$, then the following two maximization problems have the same solutions:*

$$\max_{\bar{y} \in Y} \bar{p} \cdot \bar{y},$$

and

$$\max_{\bar{y} \in Y} v(\bar{p}, \bar{w} + \theta \bar{p} \cdot \bar{y}).$$

4.7 Price Normalization

Suppose now that the price vector $\bar{p}$ depends on the choice of a production plan $\bar{y} \in Y$, and the wealth $\bar{w}$ depends on the price vector $\bar{p}$. Denote the dependence by $p(y)$ and $w(\bar{p})$. When a production plan $\bar{y} \in Y$ is chosen, the consumer attains the utility level

$$u(p(\bar{y}), w(p(\bar{y})) + \theta p(\bar{y}) \cdot \bar{y}).$$

Hence the production plans that are most desirable for the consumer are the solutions to the maximization problem:

$$\max_{\bar{y} \in Y} v(p(\bar{y}), w(p(\bar{y})) + \theta p(\bar{y}) \cdot \bar{y}). \tag{4.1}$$

If v can be written as

$$v(\bar{p}, \bar{w}) = \frac{\bar{w}}{\beta_1 \bar{p}_1 + \cdots + \beta_L \bar{p}_L}$$

and $w(\bar{p}) = 0$ for every price vector $\bar{p}$, then

$$\begin{aligned}
&v(p(\bar{y}), w(p(\bar{y})) + \theta p(\bar{y}) \cdot \bar{y}) \\
&= \theta \frac{p(\bar{y}) \cdot \bar{y}}{\beta_1 p_1(\bar{y}) + \cdots \beta_L p_L(\bar{y})} \\
&= \theta \left(\frac{1}{\beta_1 p_1(\bar{y}) + \cdots \beta_L p_L(\bar{y})} p(\bar{y}) \right) \cdot \bar{y}.
\end{aligned}$$

Thus the consumer would like the firm to maximize the profit with respect to the price vectors for which the consumption vector $(\beta_1, \ldots, \beta_L)$ is the numeraire. The solution to the maximization problem (4.1) typically depends on the choice of $(\beta_1, \ldots, \beta_L)$.

4.8 Conflict of Interest among Shareholders

If there are two consumers who have shares in the same firm but have different coefficients $(\beta_1, \ldots, \beta_L)$, then there may be a conflict of interest, in that they may like the firm to choose different production plans. For example, suppose that $L = 2$, the first commodity is the input, the second commodity is the output, and Y is given by the production function f. Suppose moreover that the first consumer's indirect utility function equal w/p_1 and the second consumer's indiredt utility function equal w/p_2. Then the first consumer whold like the firm to choose the input level that solves

$$\max_{z_1 \geq 0} \frac{p_2(-z_1, f(z_1))}{p_1(-z_1, f(z_1))} f(z_1) - z_1,$$

while the second consumer would like the firm to choose the input level that solves

$$\max_{z_1 \geq 0} f(z_1) - \frac{p_1(-z_1, f(z_1))}{p_2(-z_1, f(z_1))} z_1.$$

The solutions are typically different.

Chapter 5

Decision Making under Uncertainty

5.1 Introduction

In this lecture note, we give an introduction to a consumer's utility and choice under uncertainty, covering topics such as the independence axiom; expcted utility; absolute and relative risk aversion; and stochastic dominance. The materials of this section can be found Section 6.B of MWG. Another good textbook is David Kreps's "Notes on the Theory of Choice" (Westview Press).

5.2 Expected Utility

Let C be a set of *outcomes* or *consequences*. Assume that C is finite and index it as $\{1, 2, \ldots, N\}$.

5.2.1 Definition A *simple lottery* is a probability distribution on C, that is, a list $L = (p_1, p_2, \ldots, p_N)$, with $p_n \geq 0$ for every n and $\sum p_n = 1$.

Denote the set of all imple lotteries on C by $\mathscr{L}$.

5.2.2 Definition A *compound lottery* is a probability distribution on a finite subset of $\mathscr{L}$. Namely, a compounded lottery is defined by K simple lotteries $L_k = (p_1^k, p_2^k, \ldots, p_N^k)$, $k = 1, \ldots, K$, and a list $(\alpha_1, \alpha_2, \ldots, \alpha_K)$, with $\alpha_k \geq 0$ for every k and $\sum \alpha_k = 1$, where K is any positive integer. The *reduced lottery* of this compound lottery is the simple lottery

$$\left(\sum_{k=1}^{K} \alpha_k p_1^k, \sum_{k=1}^{K} \alpha_k p_2^k, \ldots, \sum_{k=1}^{K} \alpha_k p_N^k \right).$$

and denoted by

$$\sum_{k=1}^{K} \alpha_k L_k.$$

5.2.3 Definition (Independence Axiom) A preference relation $\succsim$ on $\mathscr{L}$ satisfies the *independence axiom* if, for all simple lotteries L, L', and L'' and for all $\alpha \in [0, 1]$, we have $L \succsim L'$ if and only if $\alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''$.

The independence axiom is justified by the following type of arguments. In the compound lottery $(L, L''; \alpha, 1 - \alpha)$, if you happen not to take L'', then you must be instead taking L. The same is true for the other compound lottery $(L', L''; \alpha, 1 - \alpha)$. Hence if you happen not to take L'' (which occurs with probability α in either simple lottery), then your preference over the two compound lotteries *ought to* be determined by your preference over L and L''. The independence axiom thus follows.

Examples that do not satisfy the independence axiom include:

- Ellsberg Paradox (Example 6.F.1)
- Non-consequentialism
- State-dependent preferences
- Allais Paradox (Example 6.B.2)
- Machina Paradox (Example 6.B.3)
- Sub-problems (Example 6.B.4 and Chapter 13 of Kreps).

5.2.4 Theorem (Expected Utility Theorem) *The following two conditions regarding a preference relation* $\succsim$ *on* $\mathscr{L}$ *are equivalent:*

1. $\succsim$ *is complete and transitive and satisfies the independence and continuity axioms.*
2. *There exists a list* $(u_1, u_2, \dots, u_N)$ *of* N *real numbers such that, for all simple lotteries* $L = (p_1, p_2, \dots, p_N)$ *and* $L' = (p'_1, p'_2, \dots p'_N)$*, we have* $L \succsim L'$ *if and only if*

$$\sum_{n=1}^{N} p_n u_n \geq \sum_{n=1}^{N} p'_n u_n.$$

Moreover, then, such a list of N *numbers is uniquely determined up to affine transformation.*

5.3 Utility Functions over Money

The set C of outcomes is now replaced by the set of possible monetary values or consumption levels, such as $\boldsymbol{R}$, $\boldsymbol{R}_+$, and $\boldsymbol{R}_{++}$. A simple lottery is replaced by a cumulative distribution function $F : \boldsymbol{R} \to [0, 1]$.

With some additional assumptions, it is possible to extend the expected utility theorem to the case where the set of outcomes is $\boldsymbol{R}$, $\boldsymbol{R}_+$, or $\boldsymbol{R}_{++}$. A preference relation satisfying the independence axiom is then replaced by a utility function U over cumulative distribution functions defined in the form

$$U(F) = \int u(x) dF(x),$$

where $u : \boldsymbol{R} \to \boldsymbol{R}$, $u : \boldsymbol{R}_+ \to \boldsymbol{R}$, or $u : \boldsymbol{R}_{++} \to \boldsymbol{R}$ is the utility function over money.

5.3.1 Risk Aversion

Note that if F is the degenerated lottery that yields the amount $y \in \boldsymbol{R}$ with certainty, then $U(F) = u(y)$.

5.3.1 Definition The utility function U over cumulative distribution functions exhibits *risk aversion* if $U(F) \leq u(\int x dF(x))$ for every cumulative distribution function F.

Assume in the sequel that u is continuous and strictly increasing.

5.3.2 Definition The *certainty equivalent* of a cumulative distribution function F given u is the amount y such that $U(F) = u(y)$. Denote $y = c(F, u)$, then

$$c(F, u) = u^{-1}(U(F)) = u^{-1}\left(\int u(x)dF(x)\right).$$

5.3.3 Proposition *The following three conditions are equivalent:*

1. *U exhibits risk aversion.*
2. *u is concave*
3. *$c(F, u) \le \int xdF(x)$ for every cumulative distribution function F.*

There is yet another equivalent condition in terms of the *probability premium*, which we omit here.

5.3.2 Comparison of Risk Aversion

Assume in this subsection that u is twice continuously differentiable and satisfies $u'(x) > 0$ and $u''(x) \le 0$ for every x.

5.3.4 Definition The *Arrow-Pratt measure of absolute risk aversion* of u at x, denoted $r_A(x, u)$, is defined by

$$r_A(x, u) = -\frac{u''(x)}{u'(x)}.$$

We use the same differentiability assumptions for u_1 and u_2 as well.

5.3.5 Proposition *Let u_1 and u_2 be two utility functions defined over money, then the following three conditions are equivalent.*

1. *$r_A(x, u_2) \ge r_A(x, u_1)$ for every x.*
2. *There exists a concave function ψ such that $u_2(x) = \psi(u_1(x))$ for every x.*
3. *$c(F, u_2) \le c(F, u_1)$ for every cumulative distribution function F.*

When one (and hence all) of them holds, we say that u_2 is more risk averse than *u_1.*

5.3.6 Definition The utility function u over money *exhibits decreasing (constant, increasing) absolute risk aversion* if $r_A(x, u)$ is a decreasing (constant, increasing) function of x.

5.3.7 Proposition *For each monetary value x and cumulative distribution function F, define*

$$c_x^A(F, u) = u^{-1}\left(\int u(x + z)dF(z)\right).$$

Then u exhibits decreasing (constant, increasing) absolute risk aversion if and only if

$$x - c_x^A(F, u)$$

is a decreasing (constant, increasing) function of x.

5.3.8 Definition The *Arrow-Pratt measure of relative risk aversion* of u at $x > 0$, denoted $r_R(x, u)$, is defined by

$$r_R(x, u) = -\frac{xu''(x)}{u'(x)}.$$

5.3.9 Definition The utility function u over money *exhibits decreasing (constant, increasing) relative risk aversion* if $r_R(x, u)$ is a decreasing (constant, increasing) function of x.

5.3.10 Proposition *For each monetary value $x > 0$ and cumulative distribution function F on $\boldsymbol{R}_+$, define*

$$c_x^R(F, u) = u^{-1}\left(\int u(zx)dF(z)\right).$$

Then u exhibits decreasing (constant, increasing) absolute risk aversion if and only if

$$\frac{x}{c_x^R(F, u)}$$

is a decreasing (constant, increasing) function of x.

5.3.3 Comparison of Risks

5.3.11 Proposition *Let F and G be two cumulative distribution functions, then the following two conditions are equivalent.*

1. *$F(x) \leq G(x)$ for every x.*
2. *For every non-decreasing utility function u over money, we have $\int u(x)dF(x) \geq \int u(x)dG(x)$.*

When one (and hence both) of them holds, we say that F is first-order stochastically dominates *G.*

5.3.12 Proposition *Let F and G be two cumulative distribution functions such that $F(\underline{x}) = G(\underline{x}) = 0$ for some finite $\underline{x}$, $F(\overline{x}) = G(\overline{x}) = 1$ for another finite $\overline{x}$, and $\int xdF(x) = \int xdG(x)$. Then the following two conditions are equivalent.*

1. *$\int_{\underline{x}}^{x} F(z)dz \leq \int_{\underline{x}}^{x} G(z)dz$ for every x.*
2. *For every concave utility function u over money, we have $\int u(x)dF(x) \geq \int u(x)dG(x)$.*

When one (and hence both) of them holds, we say that F is second-order stochastically dominates *G.*

Both the first- and second-order stochastic dominance admit equivalent conditions in terms of random variables. In particular, the equivalent condition for the second-order stochastic dominance involves *mean-preserving spreads*, which are often used in applications.

Chapter 6

General Equilibrium Theory

6.1 Introduction to General Equilibrium Theory

Gerard Debreu's "Theory of Value" (Wiley and Sons) is an excellent introduction to general equilibrium theory.

6.1.1 Purpose and Motivation

- Gather consumers and producers in a unifying framework and analyze how the "price mechanism" will lead to an "equilibrium".
- Emphasize the analysis of the interaction between markets for different commodities.

6.1.2 Methodology

- Start from the description of the fundamentals (such as endowments, preference relations, and production possibilities) of the economy.
- Assume the price-taking behavior.
- Take a somewhat "abstract" and "mathematical" approach to the analysis of equilibria.

6.2 Economy, Efficiency, and Equilibrium

The following materials can also be found in 16.C of MWG.

6.2.1 An Economy

We assume that there are L commodities. We study an economy consists of I *consumers* and J *firms*. Each consumer $i = 1, \dots, I$, is characterized by his *consumption set* $X_i \subset \boldsymbol{R}^L$ and a *preference relation* $\succsim_i$ defined on X_i. Each firm $j = 1, \dots, J$ is characterized by its *production set* Y_j. The *endowments* in the economy for the L commodities is denoted by $\overline{\omega} \in \boldsymbol{R}^L$.

A *feasible allocation* of this economy is a vector $(x_1, \dots, x_I, y_1, \dots, y_J)$ in $X_1 \times \dots X_I \times Y_1 \cdots \times Y_J$ such that

$$\sum_{i=1}^{I} x_i = \overline{\omega} + \sum_{j=1}^{J} y_j.$$

What we shall call a "Walrasian allocation" is a feasible allocation of this economy led to by the "price (or competitive) mechanism". Other trading mechanisms would lead to other feasible allocations.

6.2.2 Pareto Efficiency

We shall judge the desirability of an allocation, and hence of the price and other mechanisms that lead to them, with respect to the following criterion on efficiency.

6.2.1 Definition A feasible allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is *Pareto efficient* if there is no feasible allocation $(x'_1, \ldots, x'_I, y'_1, \ldots, y'_J)$ such that $x'_i \succsim_i x_i$ for every i and $x'_i \succ_i x_i$ for some i.

6.2.3 A Private Ownership Economy

A private ownership economy is nothing but an economy with a list of specifications regarding who owns what. More specifically, a *private ownership economy* is defined, in addition to the economy as defined in Section 6.2.1, by the consumers' endowments $\omega_i \in \boldsymbol{R}^L$ for $i = 1, \ldots, I$ and shareholdings $\theta_{ij} \geq 0$ in the J firms for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. We assume that $\sum_{i=1}^I \omega_i = \overline{\omega}$ and $\sum_{i=1}^I \theta_{ij} = 1$ for every j. This list of specifications of the private ownership determines to whom various profits and revenues are paid.

6.2.4 Walrasian Equilibrium

We assume that there is one price for each good. A *price vector* thus belongs to $\boldsymbol{R}^L$.

6.2.2 Definition A feasible allocation $(x^*_1, \ldots, x^*_I, y^*_1, \ldots, y^*_J)$ and a price vector p constitute a *Walrasian equilibrium* if:

1. For every j and every $y_j \in Y_j$, we have $p \cdot y_j \leq p \cdot y^*_j$.

2. For every i, $p \cdot x^*_i \leq p \cdot \omega_i + \sum_{j=1}^J \theta_{ij} p \cdot y^*_j$ and, for every $x_i \in X_i$, if $p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^J \theta_{ij} p \cdot y^*_j$, then $x^*_i \succsim_i x_i$.

An Walrasian equilibrium allocation $(x^*_1, \ldots .x^*_I, y^*_1, \ldots y^*_J)$ is a feasible allocation led to by the price "mechanism", though the definition does not give any concrete idea on what the mechanism is like. The price vector p determines the "exchange rate" between any two commodities. If the "value" of a commodity is defined to be what it can buy in terms of other commodities, the value of a commodity is nothing but its price. General equilibrium theory can then be said to be a theory of value with no explicit trading mechanism.

Another notion of an equilibrium, with no ownership specification, is useful when discussing efficiency properties of Walrasian equilibria.

6.2.3 Definition A feasible allocation $(x^*_1, \ldots .x^*_I, y^*_1, \ldots y^*_J)$ and a price vector p constitute a *price equilibrium with transfers* if:

1. For every j and every $y_j \in Y_j$, we have $p \cdot y_j \leq p \cdot y^*_j$.

2. For every i and every $x_i \in X_i$, if $p \cdot x_i \leq p \cdot x^*_i$, then $x^*_i \succsim_i x_i$.

It is easy to check that this definition is equivalent to Definition 16.B.4 of MWG. The above definition is closer to the definition of an equilibrium in Chapter 6 of Debreu's "Theory of Value".

Exercise 6.2.1 Prove that feasible allocation $(x^*_1, \ldots .x^*_I, y^*_1, \ldots y^*_J)$ and a price vector p constitute a Walrasian equilibrium if and only if they constitute a price equilibrium with transfers and $p \cdot x^*_i \leq p \cdot \omega_i + \sum_{j=1}^J \theta_{ij} p \cdot y^*_j$ for every i.

6.3 Two Fundamental Theorems of Welfare Economics

The materials of this sections can also be found Section 16.D of MWG.

6.3.1 First Fundamental Theorem of Welfare Economics

6.3.1 Definition The pair $(X_i, \succsim_i)$ of the consumption set X_i and the preference relation $\succsim_i$ is *locally non-satiated* if, for every $x_i \in X_i$ and every $\varepsilon > 0$, there exists an $x_i' \in X_i$ such that $\|x_i' - x_i\| < \varepsilon$ and $x_i' \succ_i x_i$. We may also say, more simply, that the preference relation $\succsim_i$ is locally non-satiated.

6.3.2 Theorem (First Fundamental Theorem of Welfare Economics) *Suppose that the preference relations are locally non-satiated. If a feasible allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and some price vector constitute a price equilibrium with transfers, then $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ is Pareto efficient.*

6.3.2 Second Fundamental Theorem of Welfare Economics

The following definition is a weaker notion of an equilibrium, termed "quasi-equilibrium". Although it plays only a technical role in the second welfare theorem, it is worth presenting because it will appear again in the existence problem of a Walrasian equilibrium.

6.3.3 Definition A feasible allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and a price vector p constitute a *price quasi-equilibrium with transfers* if:

1. For every j and every $y_j \in Y_j$, we have $p \cdot y_j \leq p \cdot y_j^*$.
2. For every i and every $x_i \in X_i$, if $p \cdot x_i < p \cdot x_i^*$, then $x_i^* \succsim_i x_i$.

A price equilibrium with transfers is a price quasi-equilibrium with transfers. In general, the converse does not hold. Roughly speaking, however, if every consumer i can "survive" under p with an wealth smaller than $p \cdot x_i^*$, then the quasi-equilibrium is also an equilibrium. Note that if $p \cdot x_i^*$ is equal to the minimum wealth necessary for survival, then Condition 2 in Definition 6.3.3 is trivially met.

6.3.4 Theorem (Second Fundamental Theorem of Welfare Economics) *Suppose that the preference relations are convex and locally non-satiated and that Y_j is convex for every j. If a feasible allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ is Pareto efficient, then there exists a price vector p such that $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and p constitute a price quasi-equilibrium with transfers.*

6.4 Examples of Private Ownership Economies

6.4.1 An Edgeworth Box Economy

The materials in this section can also be found in Section 15.B of MWG.

An *Edgeworth Box economy* is an economy such that $L = 2$, $I = 2$, $X_1 = X_2 = \boldsymbol{R}^2_+$, $Y_1 = \cdots = Y_J = \{0\}$, and $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2) \in \boldsymbol{R}^2_{++}$. This is an *exchange economy* with two consumers and two goods. Since the feasibility condition for an allocation is reduced to $x_2 = \overline{\omega} - x_1$, the set of feasible allocations can be identified with a rectangle with length $\overline{\omega}_1$ and height $\overline{\omega}_2$. This economy is a simplest possible framework in which we can see how the prices co-ordinate different consumers' demands to arrive at a feasible allocation. Exercises 15.B.1 and 15.B.2 of MWG are routine but recommended.

6.4.2 A Robinson Crusoe Economy

The materials in this section can be found in Section 15.C of MWG.

A *Robison Crusoe economy* is an economy such that $L = 2$, $I = 1$, $J = 1$, $X_1 = \boldsymbol{R}^2_+$, and $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2) \in \boldsymbol{R}^2_+$. It is often considered that one of the two commodities is an input, whose endowment is positive, and the other is an output, whose endowment is zero. If we let the first commodity is the input and the second the output, then $\overline{\omega}$ lies on the positive part of the horizontal axis and Y_1 is included in the left half space of $\boldsymbol{R}^2$. The feasibility condition is reduced to $x_1 \in Y_1 + \{\overline{\omega}\}$. This economy is a simplest possible framework in which we can describe how the production and consumption decision are made separately and yet the price mechanism leads to a feasible allocation.

6.5 Excess Demand Function

The materials of this section can also be found Section 17.B of MWG.

In the rest of this lecture note, we consider an exchange economy ($Y_1 = \cdots = Y_J = \{0\}$) and assume that $X_i = \boldsymbol{R}^L_+$ or $X_i = \boldsymbol{R}^L_{++}$ for every i, and $\overline{\omega} = \sum_{i=1}^I \omega_i \in \boldsymbol{R}^L_{++}$. We also assume that the preference relations $\succsim_i$ are continuous, strictly convex, and strongly monotone.

The *excess demand function* of this exchange economy is a mapping $z : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ defined by

$$z(p) = \sum_{i=1}^{I} \left(x_i(p, p \cdot \omega_i) - \omega_i\right),$$

where each x_i is the demand function of consumer i. A price vector p is a Walrasian equilibrium price vector if and only if $z(p) = 0$.

6.5.1 Proposition *The excess demand function z has the following properties.*

Continuity *z is continuous.*

Homogeneity *z is homogeneous of degree zero.*

Walras' Law *$p \cdot z(p) = 0$ for every $p \in \boldsymbol{R}^L_{++}$.*

Boundedness from Below *z is bounded from below.*

Boundary Behavior *If a sequence $p^1, p^2, \ldots$ in $\boldsymbol{R}^L_{++}$ converges to a price vector $p = (p_1, p_2, \ldots, p_L) \in \boldsymbol{R}^L_+$ with $p_\ell > 0$ for some ℓ and $p_\ell = 0$ for some other ℓ, then*

$$\max\{z_1(p^n), \ldots, z_L(p^n)\} \to \infty$$

as $n \to \infty$, where $z_\ell(p^n)$ is the ℓ-th coordinate of $z(p^n)$.

6.6 Existence of a Walrasian Equilibrium

The materials of this sections can also be found Section 17.C of MWG.

6.6.1 Theorem *Under the assumptions stated in Section 6.5, there exists a Walrasian equilibrium.*

This is a consequence of the fixed point theorem, but the intermediate value theorem is sufficient for the case of $L = 2$, which admits many insightful graphical presentations.

6.7 Sonnenschein-Mantel-Debreu Theorem

The materials of this sections can also be found Section 17.E of MWG.

The SMD theorem asserts that if there are as many consumers as commodities, then the continuity, homogeneity, and Walras' law in Propositions 6.5.1 exhaust all the implications of consumers' utility maximization behavior on the excess demand function of an exchange economy over any compact subset of $\boldsymbol{R}^L_{++}$.

To see why the number of consumers matters, let's assume that the x_i are continuously differentiable. Then

$$Dz_i(p) = S_i(p, p \cdot \omega_i) - D_w x_i(p, p \cdot \omega_i) z_i(p, p \cdot \omega_i)^\top \in \boldsymbol{R}^{L \times L},$$

where $S_i(p, p \cdot \omega_i) \in \boldsymbol{R}^{L \times L}$ is the Slutsky substitution matrix and $z_i(p, p \cdot \omega_i) \in \boldsymbol{R}^L$ (a column vector) is the excess demand of consumer i. (Note that this notation is different from that of Proposition 6.5.1.) Then $Dz_i(p)$ is negative semi-definite on the linear subspace

$$\left\{ v \in \boldsymbol{R}^L \mid p \cdot v = z_i(p, p \cdot \omega_i) \cdot v = 0 \right\}.$$

Hence $Dz(p) = \sum_{i=1}^I Dz_i(p)$ is negative semi-definite on the linear subspace

$$\begin{aligned} & \bigcap_{i=1}^I \left\{ v \in \boldsymbol{R}^L \mid p \cdot v = z_i(p, p \cdot \omega_i) \cdot v = 0 \right\} \\ = \quad & \left\{ v \in \boldsymbol{R}^L \mid p \cdot v = z_1(p, p \cdot \omega_i) \cdot v = \cdots = z_I(p, p \cdot \omega_i) \cdot v = 0 \right\}. \end{aligned}$$

If p is an equilibrium price vector, then the dimension of this linear subspace may be $L - I$ but not higher. Hence, if there are fewer consumers than commodities, then there is at least one direction of price variations along which $Dz(p)$ is negative semi-definite.

6.7.1 Theorem (Sonnenschein-Mantel-Debreu) *Let $z : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ be an arbitrary function that satisfies the continuity, homogeneity, and Walras law of Proposition 6.5.1. Let C be a compact subset of $\boldsymbol{R}^L_{++}$. Then there is an exchange economy consisting of L consumers whose excess demand function coincides with z on C.*

6.8 Generic Determinacy of Walrasian Equilibria

6.8.1 Regular Equilibrium

The materials of this and next sections can also be found Section 17.D of MWG.

In the rest of this lecture note, we assume that the consumers' demand functions are continuously differentiable.

6.8.1 Definition A Walrasian equilibrium price vector p is *regular* if $\operatorname{rank} Dz(p) = L - 1$. An exchange economy is *regular* if all of its Walrasian equilibrium price vectors are regular.

Exercise 6.8.1 Show that the regularity is equivalent to each one of the following two conditions:

1. The column space of $Dz(p)$ is equal to the hyperplane normal to p going through the origin.
2. Define $\widehat{z} : \boldsymbol{R}^{L-1}_{++} \to \boldsymbol{R}^{L-1}$ by

   $$\widehat{z}(\widehat{p}) = \left(z_1(\widehat{p}, 1), \ldots, z_{L-1}(\widehat{p}, 1) \right),$$

   for every $\widehat{p} \in \boldsymbol{R}^{L-1}_{++}$, where $z_\ell(\widehat{p}, 1)$ is the ℓ-th coordinate of $z(\widehat{p}, 1)$. Then the $(L-1) \times (L-1)$ matrix $D\widehat{z}(p_1/p_L, \ldots, p_{L-1}/p_L)$ is invertible.

6.8.2 Proposition *For every regular Walrasian equilibrium price vector p there exists an $\varepsilon > 0$ such that if a price vector p' is not proportional to p and satisfies $\|p' - p\| < \varepsilon$, then p' is not a Walrasian equilibrium price vector.*

6.8.2 Genericity Analysis

We now consider a class of exchange economies parameterized by $q \in Q$, where Q is an open subset of $\boldsymbol{R}^S$ and S is a positive integer. Denote by $z(\cdot, q) : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ the excess demand function of the exchange economy of parameter $q \in Q$. This defines the *parameterized excess demand function* $z : \boldsymbol{R}^L_{++} \times Q \to \boldsymbol{R}^L$. Note that the domain of z has been expanded to include the *parameter space* Q.

6.8.3 Example We assume that the preference relations $\succsim_i$ $(i = 1, \ldots, I)$ of all consumers and the endowments ω_i $(i = 2, \ldots, I)$ of all consumers but the first one are prespecified. The economy is parameterized by the (strictly positive) endowments $\omega_1 \in \boldsymbol{R}^L_{++}$ of the first consumer. The parameter space Q is thus equal to $\boldsymbol{R}^L_{++}$.

6.8.4 Example We assume that the preference relations $\succsim_i$ $(i = 2, \ldots, I)$ of all consumers but the first one and the endowments ω_i $(i = 1, \ldots, I)$ of all consumers are prespecified. The preference relation $\succsim_1$ of the first consumer is represented by the Cobb-Douglas utility function $u_1(x_1) = x_{11}^a x_{21}^{1-a}$, where $x_1 = (x_{11}, x_{21})$ and $a \in (0, 1)$. The parameter space Q is thus equal to the open unit interval $(0, 1)$.

6.8.5 Definition The parametrization by Q is *regular* if the following condition is satisfied: the parameterized excess demand function $z : \boldsymbol{R}^L_{++} \times Q \to \boldsymbol{R}^L$ is continuously differentiable and, for every $(p, q) \in \boldsymbol{R}^L_{++} \times Q$, if p is a Walrasian equilibrium price vector of parameter q then $\operatorname{rank} Dz(p, q) = L - 1$.

Since $Dz(p, q) = [D_p z(p, q) \; D_q z(p, q)]$, the regularity of the parameter space Q is a weaker requirement than the regularity of the exchange economy with every parameter $q \in Q$. It can be shown that Examples 6.8.3 and 6.8.4 are both regular parameterizations.

Given a parameter space Q, we say that a property holds for almost every exchange economy in Q if there exists an open and full-measure subset Q' of Q such that the property holds for every exchange economy in Q'.

6.8.6 Theorem *If the parametrization by Q is regular, then almost every economy in Q is regular.*

6.8.3 Comparative Statics Analysis

Given a parameter space Q and a parameterized excess demand function $z : \boldsymbol{R}^L_{++} \times Q \to \boldsymbol{R}^L$, define $\widehat{z} : \boldsymbol{R}^{L-1}_{++} \times Q \to \boldsymbol{R}^{L-1}$ by

$$\widehat{z}(\widehat{p}, q) = \left(z_1\left((\widehat{p}, 1), q\right), \ldots, z_{L-1}\left((\widehat{p}, 1), q\right)\right),$$

for every $(\widehat{p}, q) \in \boldsymbol{R}^{L-1}_{++} \times Q$. If $(\widehat{p}^*, 1)$ is a regular Walrasian equilibrium price vector of q^*, then $\operatorname{rank} D_{\widehat{p}} \widehat{z}(\widehat{p}^*, q^*) = L - 1$ and hence the implicit function theorem implies that there exist an open subset V of $\boldsymbol{R}^{L-1}_{++}$, an open subset Q' of Q, and a continuously differentiable mapping $p : V \to Q'$ such that $(\widehat{p}^*, q^*) \in V \times Q'$ and, for every $(\widehat{p}, q) \in V \times Q'$, $(\widehat{p}, 1)$ is a regular Walrasian equilibrium price vector of q if and only if $p(q) = \widehat{p}$. The implicit function theorem also implies that

$$Dp(q^*) = -D_{\widehat{p}} \widehat{z}(\widehat{p}^*, q^*)^{-1} D_q \widehat{z}(\widehat{p}^*, q^*).$$