Advanced Microeconomics
Lecture Note

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第1章 Kuhn-Tucker Conditions

1.1 Introduction

In this section, we give Kuhn-Tucker necessary and sufficient conditions for a solution to a constrained maximization problem. Our conditions are more general than the standard ones in that they accommodate the case of multiple objective functions, so that the constrained maximization problem is a generalization of the problem of finding a Pareto-efficient allocation. Our proof is more elementary than the standard ones, in that they rely on the Minkowski-Farkas Lemma, which can be proved by an induction argument, but not on the separating hyperplane theorem, whose proof involves a topological argument.

1.2 Minkowski-Farkas Lemma

This materials owe much to Sections 2.1, 2.2, and 2.3 of Gale. Let J and L be two positive integers.

1.2.1 Lemma (Minkowski and Farkas) *Let $A \in \boldsymbol{R}^{J\times L}$ and $b \in \boldsymbol{R}^L$ (column vector). Then one and only one of the following two possibilities holds:*

1. *There exists a $z \in \boldsymbol{R}^J_+$ (column vector) such that $b^\top = z^\top A$.*
2. *There exists an $x \in \boldsymbol{R}^L$ (column vector) such that $Ax \in \boldsymbol{R}^J_+$ and $b^\top x < 0$.*

For each $a \in \boldsymbol{R}^L$, define $a^\perp = \{x \in \boldsymbol{R}^L | a \cdot x = 0\}$.

Let $a \in \boldsymbol{R}^L$ and $b \in \boldsymbol{R}^L$ and suppose that $a \cdot b \neq 0$. Then, for every $x \in \boldsymbol{R}^L$, there are a unique $v \in a^\perp$ and a unique $\lambda \in \boldsymbol{R}$ such that $x = v + \lambda b$. We say that v is the projection of x onto $a^\perp$ along b.

Exercise 1.2.1 Prove that $v = x - \dfrac{a \cdot x}{a \cdot b} b$.

Proof of Lemma 1.2.1 Let's first prove by a contradiction argument that the two possibilities do not hold simultaneously. So suppose there were z and x as in the two possibilities. Multiply x from the right to $b^\top = z^\top A$, then we obtain $b^\top x = z^\top Ax$. But, by Possibility 2 and $z \in \boldsymbol{R}R^J_+$, we have $b^\top x < 0$ and $z^\top Ax \geq 0$. Hence the equality could not be met. This is a contradiction. Hence the two possibilities do not hold simultaneously.

It is now sufficient to show that if Possibility 1 does not hold, then Possibility 2 must necessarily hold. We shall do so by an induction argument on J.

The case of $J = 1$ is left as an exercise.

Let $J \geq 2$ and assume that, for $J-1$, if Possibility 1 does not hold, then Possibility 2 holds. Write

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_{J-1} \\ a_J \end{bmatrix},$$

where $a_1, \ldots, a_{J-1}, a_J$ are row vectors in $\boldsymbol{R}^L$. Also write

$$A' = \begin{bmatrix} a_1 \\ \vdots \\ a_{J-1} \end{bmatrix} \in \boldsymbol{R}^{(J-1)\times L}.$$

Since Possibility 1 is not met by A, it is not met by A' either. That is, there is no $z' \in \boldsymbol{R}_+^{J-1}$ such that $b^\top = z'^\top A'$. Hence, by the induction hypothesis, there exists an $x' \in \boldsymbol{R}^L$ (column vector) such that $A'x' \in \boldsymbol{R}_+^{J-1}$ and $b^\top x' < 0$.

If $a_J \cdot x' \geq 0$, then x' is just as desired in Possibility 2 and the proof is completed. So assume that $a_J \cdot x' < 0$. Denote the projections of $a_1, \ldots, a_{J-1}, b$ onto $x'^\perp$ along a_J by $\widehat{a}_1, \ldots, \widehat{a}_{J-1}, \widehat{b}$, respectively. Define

$$\widehat{A} = \begin{bmatrix} \widehat{a}_1 \\ \vdots \\ \widehat{a}_{J-1} \end{bmatrix} \in \boldsymbol{R}^{(J-1)\times L}.$$

We shall now prove by a contradiction argument that there is no $w \in \boldsymbol{R}_+^{J-1}$ (column vector) such that $\widehat{b} = w^\top \widehat{A}$. So suppose there were such a

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_{J-1} \end{bmatrix},$$

then

$$\begin{aligned} \widehat{b}^\top &= \sum_{j=1}^{J-1} w_j \widehat{a}_j \\ &= \sum_{j=1}^{J-1} w_j \left(a_j - \frac{x' \cdot a_j}{x' \cdot a_J} a_J \right) \\ &= \sum_{j=1}^{J-1} w_j a_j - \left(\sum_{j=1}^{J-1} w_j \frac{x' \cdot a_j}{x' \cdot a_J} \right) a_J. \end{aligned}$$

Thus

$$b^\top = \sum_{j=1}^{J-1} w_j a_j + \left(\frac{x' \cdot b}{x' \cdot a_J} - \sum_{j=1}^{J-1} w_j \frac{x' \cdot a_j}{x' \cdot a_J} \right) a_J.$$

Since

$$\frac{x' \cdot b}{x' \cdot a_J} - \sum_{j=1}^{J-1} w_j \frac{x' \cdot a_j}{x' \cdot a_J} \geq 0,$$

this contradicts our initial hypothesis that Possibility 1 does not hold for A. Hence there is no $w \in \boldsymbol{R}_+^{J-1}$ such that $\widehat{b} = w^\top \widehat{A}$.

By our induction hypothesis, therefore, there exists an $\widehat{x} \in \boldsymbol{R}^L$ (column vector) such that $\widehat{A}\widehat{x} \in \mathbb{R}_+^{J-1}$ and $\widehat{b}^\top \widehat{x} < 0$. Denote by x the projection of $\widehat{x}$ onto $\widehat{a}_J^\perp$ along x'. We shall now prove that x is just as desired in Possibility 2 for A. Indeed, $a_J \cdot x = 0$ because $x \in a_J^\perp$. For each $j \leq J-1$, we have

$$a_j \cdot x = \widehat{a}_j \cdot x = \widehat{a}_j \cdot \widehat{x} \geq 0,$$

where the first equality follows from $\widehat{a}_j - a_j = \lambda a_J$ for some λ and $x \in \widehat{a}_J^\perp$; and the second equality follows from $\widehat{a}_j \in x'^\perp$ and $x - \widehat{x} = \lambda x'$ for some λ. We can similarly show that $b \cdot x < 0$. This completes the proof. ///

1.3 Separating Hyperplane Theorem

1.3.1 Definition A subset C of $\boldsymbol{R}^L$ is *convex* if $\lambda c + (1-\lambda)c' \in C$ for every $c \in C$, every $c' \in C$, and every $\lambda \in [0,1]$.

A subset C is *closed* if, roughly speaking, the boundary of C is completely included in C itself.

1.3.2 Theorem (Separating Hyperplane Theorem) *Let C be a closed, convex subset of $\boldsymbol{R}^L$, and $b \in \boldsymbol{R}^L \backslash C$. Then there exist $x \in \boldsymbol{R}^L$ and $d \in \mathbb{R}$ such that*

$$x \cdot c \geq d > x \cdot d$$

for every $c \in C$.

Exercise 1.3.1 Prove the Minkowski-Farkas Lemma by applying the Separating Hyperplane Theorem by following the steps below:

1. Prove that the cone spanned by the row vectors of A,

$$\left\{ \sum_{j=1}^{J} z_j a_j \in \boldsymbol{R}^N \mid z_1 \geq 0, \ldots, z_J \geq 0 \right\},$$

   is convex.

2. Denote the above cone by C. You can use the fact that C is closed without proof. Apply the Separating Hyperplane Theorem to C and b to show that there exist an $x \in \boldsymbol{R}^N$ such that $x \cdot c \geq 0$ for every $c \in C$ and $x \cdot b < 0$. (Hint: By the separating hyperplane theorem, there exist an $x \in \boldsymbol{R}^N$ and a $d \in \boldsymbol{R}$ such that $x \cdot c \geq d > x \cdot b$ for every $c \in C$. Show then that, for such x and d, we must necessarily have $d \leq 0$ and $x \cdot c \geq 0$ for every $c \in C$.)

3. Show that the result in the second step implies the Minkowski-Farkas Lemma.

1.4 Strict Supportability for a Pointed Cone

1.4.1 Lemma *Let $A \in \boldsymbol{R}^{J \times L}$. Then one and only one of the following two possibilities holds:*

1. *There exists a $z \in \boldsymbol{R}^J_+ \setminus \{0\}$ (column vector) such that $z^\top A = 0$.*
2. *There exists an $x \in \boldsymbol{R}^L$ (column vector) such that $Ax \in \boldsymbol{R}^J_{++}$.*

Proof of Lemma 1.4.1 It is an exercise to show that Possibilities 1 and 2 do not hold simultaneously.

Define

$$e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \boldsymbol{R}^J$$

and define $\widetilde{A} = \begin{bmatrix} \widetilde{A} & e \end{bmatrix} \in \boldsymbol{R}^{J \times (L+1)}$. It is then an easy exercise to show that Possibility 1 is equivalent to saying that there exists a $z \in \boldsymbol{R}^J_+$ (column vector) such that

$$z^\top \widetilde{A} = (\underbrace{0, \dots, 0}_{L}, 1).$$

Thus, according to the Minkowski-Farkas Lemma, if this condition is not met, then there exists an $\widetilde{x} \in \boldsymbol{R}^{L+1}$ (column vector) such that $\widetilde{A}\widetilde{x} \in \boldsymbol{R}^J_+$ and $(0, \dots, 0, 1)\widetilde{x} < 0$. Write

$$\widetilde{x} = \begin{bmatrix} x \\ x_{L+1} \end{bmatrix},$$

where $x \in \boldsymbol{R}^L$ (column vector). Then $\widetilde{A}\widetilde{x} = Ax + x_{L+1}e$ and $(0, \dots, 0, 1)\widetilde{x} = x_{L+1}$. Thus $x_{L+1} < 0$ and $Ax \in \boldsymbol{R}^J_{++}$. Hence Possibility 2 holds. ///

1.5 Constrained Maximization Problem with Multiple Objective Functions

The following formulation of a constrained maximization problem with multiple objective functions is due to Smale.

Let N, M, and L be positive integers. A *constrained maximization problem* with N objective functions is defined by a collection of $(X, f_1, \dots, f_N, g_1, \dots, g_M)$, where X is a subset of $\boldsymbol{R}^L$ and the f_n $(n = 1, \dots, N)$ and the g_m $(m = 1, \dots, M)$ are real-valued functions defined on X. We call X to be the domain, the f_n to be the objective functions, and the g_m to be the constraint functions. This problem can somewhat heuristically be written as:

$$\begin{array}{cc} \max\limits_{x \in X} & (f_1(x), \dots, f_N(x)) \\ \text{subject to} & g_1(x) \geq 0, \\ & \vdots \\ & g_M(x) \geq 0. \end{array}$$

We say that an $x^* \in X$ is a *solution* if $g_m(x^*) \geq 0$ for every m and there is no $x \in X$ such that $g_m(x) \geq 0$ for every m, $f_n(x) \geq f_n(x^*)$ for every n, and $f_n(x) > f_n(x^*)$ for some n.

In the rest of this lecture note, we assume that X is open and the f_n and the g_m are continuously differentiable.

1.6 Kuhn-Tucker Necessary Condition

1.6.1 Theorem (Kuhn-Tucker Necessary Condition) *If $x^* \in X$ is a solution to the constrained maximization problem, then there exists a vector of $N+M$ non-negative numbers, $(\mu_1, \ldots, \mu_N, \lambda_1, \ldots, \lambda_M) \in \boldsymbol{R}_+^{N+M}$, such that:*

1. *At least one of the $N + M$ non-negative numbers is strictly positive;*
2. *For every m, if $g_m(x^*) > 0$, then $\lambda_m = 0$; and*
3. $\displaystyle\sum_{n=1}^{N} \mu_n \nabla f_n(x^*) + \sum_{m=1}^{M} \lambda_m \nabla g_m(x^*) = 0.$

Proof of Theorem 1.6.1 (Sketch)

1. By re-numbering the constraint functions if necessary, we can assume that the first K constraint functions are the binding ones, where $K \leq M$. Then show by a contradiction argument that there is no $v \in \boldsymbol{R}^L$ such that $\nabla f_n(x^*) \cdot v > 0$ for every n and $\nabla g_m(x^*) \cdot v > 0$ for every $m \leq K$.

2. By Lemma 1.4.1, this implies that there exists a vector of $N+K$ non-negative numbers, $(\mu_1, \ldots, \mu_N, \lambda_1, \ldots, \lambda_K) \in \boldsymbol{R}_+^{N+K}$, such that at least one of the $N + K$ non-negative numbers is strictly positive and that
$$\sum_{n=1}^{N} \mu_n \nabla f_n(x^*) + \sum_{m=1}^{K} \lambda_m \nabla g_m(x^*) = 0.$$

3. Finally, let $\lambda_m = 0$ for every $m > K$, then the vector $(\mu_1, \ldots, \mu_N, \lambda_1, \ldots, \lambda_M)$ of $N+M$ numbers has the three properties in the Kuhn-Tucker necessary condition.

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1.7 Kuhn-Tucker Sufficient Condition

1.7.1 Definition Let X be convex and $h : X \to \boldsymbol{R}$. We say that h is *quasi-concave* if the following condition is satisfied: For every $x \in X$, $y \in X$, and $\alpha \in [0, 1]$, if $h(y) \geq h(x)$, then $h(\alpha x + (1 - \alpha)y) \geq h(x)$.

1.7.2 Lemma *Let X be convex and $h : X \to \boldsymbol{R}$ be continuously differentiable. Then h is quasi-concave if and only if $\nabla h(x) \cdot (y - x) \geq 0$ whenever $x \in X$, $y \in X$, and $h(y) \geq h(x)$.*

1.7.3 Definition Let X be convex and $h : X \to \boldsymbol{R}$ be continuously differentiable. Then h is *pseudo-concave* if $\nabla h(x) \cdot (y - x) > 0$ whenever $x \in X$, $y \in X$, and $h(y) > h(x)$.

1.7.4 Lemma *Let X be convex, $h : X \to \boldsymbol{R}$ be continuously differentiable. If h is pseudo-concave, then it is quasi-concave.*

1.7.5 Lemma *Let X be convex, $h : X \to \boldsymbol{R}$ be continuously differentiable, and suppose that $\nabla h(x) \neq 0$ for every $x \in X$. Then h is quasi-concave if and only if it is pseudo-concave.*

1.7.6 Theorem (Kuhn-Tucker Sufficient Condition) *Suppose that X is convex, the f_n are pseudo-concave, and the g_m are quasi-concave. Suppose that $x^* \in X$ and that there exists a vector of $N + M$ non-negative numbers, $(\mu_1, \dots, \mu_N, \lambda_1, \dots, \lambda_M) \in \boldsymbol{R}_+^{N+M}$, such that:*

1. *$g_m(x^*) \geq 0$ for every m;*
2. *$(\mu_1, \dots, \mu_N) \in \boldsymbol{R}_{++}^N$;*
3. *For every m, if $g_m(x^*) > 0$, then $\lambda_m = 0$; and*
4. *$\displaystyle\sum_{n=1}^{N} \mu_n \nabla f_n(x^*) + \sum_{m=1}^{M} \lambda_m \nabla g_m(x^*) = 0.$*

Then x^* is a solution.

Proof of Theorem 1.7.6 (Sketch)

1. By re-numbering the constraint functions if necessary, we can assume that the first K constraint functions are the binding ones, where $K \leq M$. Show then that if x^* were not a solution, there would exist a $v \in \boldsymbol{R}^L$ such that $\nabla f_n(x^*) \cdot v \geq 0$ for every n, $\nabla f_n(x^*) \cdot v > 0$ for some n, and $\nabla g_m(x^*) \cdot v \geq 0$ for every $m \leq K$.
2. Using Conditions 2 and 3 in the Kuhn-Tucker sufficient condition, show then that
$$\sum_{n=1}^{N} \mu_n \nabla f_n(x^*) \cdot v + \sum_{m=1}^{M} \lambda_m \nabla g_m(x^*) \cdot v > 0.$$
3. Since the left hand side of the above inequality equals
$$\left(\sum_{n=1}^{N} \mu_n \nabla f_n(x^*) + \sum_{m=1}^{M} \lambda_m \nabla g_m(x^*) \right) \cdot v,$$
this is a contradiction to Condition 4 in the Kuhn-Tucker sufficient condition.

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Exercise 1.7.1 Give examples to show that each of Conditions 2 and 3 in the Kuhn-Tucker Sufficient Condition is indispensable.

Exercise 1.7.2 Suppose that there are two goods. The price of each good is \$1. Consider a consumer with a utility function $u : \boldsymbol{R}^2_+ \to \boldsymbol{R}$ defined by $u(x_1, x_2) = x_1^{1/2} e^{x_2}$. His income is w, where $w > 0$. We shall consider a standard utility maximization problem under the budget constraint. Note that the domain of the utility function u is $\boldsymbol{R}^2_+$, which is not an open subset of $\boldsymbol{R}^2$, and u is not differentiable at every $x \in \boldsymbol{R}^2_+$ with $x_1 = 0$. In the chapter, we have maintained the assumptions that the domain of the objective function (and the constraint functions) is an open subset of $\boldsymbol{R}^L$ and that the objective and constraint functions are continuously differentiable. In this problem, we will show how to introduce additional constraint functions and modify the domain of the objective function to satisfy these maintained assumptions.

1. Prove that if $x \in \boldsymbol{R}^2_+$ is a solution to the original utility maximization problem, then $x_1 > 0$. This implies that we can restrict the domain to the subset $\{x \in \boldsymbol{R}^2 \mid x_1 > 0 \text{ and } x_2 \geq 0\}$ without altering solutions.

Note that the function u can be extended to the subset $X = \{x \in \boldsymbol{R}^2 | \, x_1 > 0\}$ because the function e^{x_2} can be defined for any values of x_2. Note also that X is an open subset of $\boldsymbol{R}^2$. Define $g_1 : X \to \boldsymbol{R}$ by $g_1(x) = w - x_1 - x_2$. This is simply the budget constraint.

2. Define a constraint function $g_2 : X \to \boldsymbol{R}$ such that the original utility maximization problem is equivalent to the following constrained maximization problem:

$$\begin{array}{ll} \max\limits_{x \in X} & u(x) \\ \text{s.t.} & g_1(x) \geq 0, \\ & g_2(x) \geq 0. \end{array}$$

   (Note that x is chosen from the new domain X.)

3. Apply the Kuhn-Tucker sufficient conditions to find a solution for different values of w.

1.8 Envelope Theorem

Let K, M, and L be positive integers. Let X be an open subset of $\boldsymbol{R}^L$ and P be an open subset of $\boldsymbol{R}^K$. Also let f and g_m $(m = 1, \ldots, M)$ be twice continuously differentiable real-valued functions defined on $X \times P$. For each $p \in P$, we consider the following constrained maximization problem with a single objective function

$$\begin{array}{rc} \max\limits_{x \in X} & f(x, p) \\ \text{subject to} & g_1(x, p) \geq 0, \\ & \vdots \\ & g_M(x, p) \geq 0. \end{array}$$

By varying p over P, we can consider a class of constrained maximization problems. The set P is called the *parameter space* of this class of constrained maximization problems.

In the rest of this lecture note, we assume that for every $p \in P$ there exists a unique solution to the constrained maximization problem with parameter p. Denote the solution by $a(p) \in X$; this defines a mapping $a : P \to X$, often called the *policy function.* Define $b : P \to \boldsymbol{R}$ by $b(p) = f(a(p), p)$. This is called the *value function.*

1.8.1 Continuous Differentiability of the Policy Function

In the statement of the following proposition, we regard gradient vectors as row vectors.

1.8.1 Proposition *Let $(x^*, p^*) \in X \times P$ and suppose that x^* satisfies the Kuhn-Tucker sufficient condition with strictly positive multipliers $(1, \lambda_1, \dots, \lambda_M) \in \boldsymbol{R}^{1+M}_{++}$ for parameter p^*. Suppose also that the $(L+M)\times(L+M)$ matrix*

$$\begin{bmatrix} \nabla_x^2 f(x^*, p^*) + \sum_{m=1}^{M} \lambda_m \nabla_x^2 g_m(x^*, p^*) & \nabla_x g_1(x^*, p^*)^\top & \cdots & \nabla_x g_M(x^*, p^*)^\top \\ \nabla_x g_1(x^*, p^*) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \nabla_x g_M(x^*, p^*) & 0 & \cdots & 0 \end{bmatrix}$$

is invertible. Then there exists an open subset Q of P containing p^* such that the restrictions of a and b on Q are continuously differentiable.

The proof is a direct application of the implicit function theorem.

1.8.2 Envelope Theorem

1.8.2 Theorem (Envelope Theorem) *Let $(x^*, p^*) \in X \times P$ and suppose that x^* satisfies the Kuhn-Tucker sufficient condition with strictly positive multipliers $(1, \lambda_1, \dots, \lambda_M) \in \boldsymbol{R}^{1+M}_{++}$ for parameter p^*. Suppose also that a and b are continuously differentiable. Then*

$$\nabla b(p^*) = \nabla_p f(x^*, p^*) + \sum_{m=1}^{M} \lambda_m \nabla_p g_m(x^*, p^*).$$

Proof of Theorem 1.8.2 (Sketch)

1. By $g_m(a(p), p) = 0$ for every $p \in P$ and m, show that
$$\nabla_x g_m(x^*, p^*) \nabla a(p^*) + \nabla_p g_m(x^*, p^*) = 0.$$
2. Multiply λ_m to both sides of the above equality and apply the Kuhn-Tucker sufficient conditions to show that
$$-\nabla_x f(x^*, p^*) \nabla a(p^*) + \sum_{m=1}^{M} \lambda_m \nabla_p g_m(x^*, p^*) = 0.$$
3. By $b(p) = f(a(p), p)$ for every p, show that
$$\nabla b(p^*) = \nabla_x f(x^*, p^*) \nabla a(p^*) + \nabla_p f(x^*, p^*)$$
4. Combine the above two equalities to complete the proof.

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関連図書

[1] David Gale, *Theory of Linear Economic Models*, the University of Chicago Press

[2] Stephen Smale, Global analysis and economics, in *Handbook of Mathematical Economics*, vol. 1, edited by Kenneth Arrow and M. Intrilligator, North Holland.

第2章 Preference and Choice in an Abstract Setting

2.1 Introduction

In this chapter we explore basic properties of preference and choice in an abstract setting, which includes, but is not restricted to, the consumer's and producer's choice problems to be considered in the subsequent analysis. The materials of this section owe much to Chapter 1 of MWG. The strong axiom of revealed preference is dealt with in the setting of consumption choice in Section J of Chapter 3.

Let X be a nonempty set of *alternatives*. There is an agent (such as a consumer, a producer, an investor, and a social planner) who has some preferences (possibly embodying moral judgments and probability assessments) over these alternatives, and chooses over sets of alternatives (subsets of X).

2.2 Binary Relations

A *binary relation* (or, more simply, *relation*) R *on* X can be considered simply as a mathematical object such that, for every $x \in X$ and $y \in X$, whether the relation "xRy" holds or not has been unambiguously defined. Formally, R can be identified with a (unique) subset of $X \times X$ by regarding "xRy" as equivalent to "$(x, y) \in R$". Whenever it is untrue that xRy, we write $xR^{c}y$. Then, as subsets of $X \times X$, R^{c} is the complement of R.

2.2.1 Definition A binary relation R on X is:

1. *reflexive* if for every $x \in X$, xRx.
2. *irreflexive* if R^{c} is reflexive.
3. *complete* if for every $x \in X$ and every $y \in X$, xRy or yRx.
4. *symmetric* if for every $x \in X$ and every $y \in X$, yRx whenever xRy.
5. *asymmetric* if for every $x \in X$ and every $y \in X$, $yR^{c}x$ whenever xRy.
6. *acyclic* if there is no pair of $N \in \boldsymbol{N}$ and $(x_1, x_2, \dots, x_N) \in X^N$ such that $x_n R x_{n+1}$ for every $n \leq N - 1$ and $x_N R x_1$.
7. *transitive* if for every $x \in X$, every $y \in X$, and every $z \in X$, xRz whenever xRy and yRz.
8. *negatively transitive* if R^{c} is transitive.

Exercise 2.2.1 Prove that:

1. every complete relation is reflexive.
2. no reflexive relation is asymmetric.
3. every acyclic relation is asymmetric.
4. every asymmetric relation is irreflexive.
5. every asymmetric and negatively transitive relation is transitive.
6. every irreflexive and transitive relation is acyclic.

Exercise 2.2.2 Let R be a binary relation on X. Prove that:

1. R is negatively transitive if and only if for every $x \in X$ and $y \in X$, there exists a $z \in X$ such that xRz or zRy whenever xRy.
2. R is transitive if and only if it is negatively transitive, provided that R is complete.
3. R is asymmetric if and only if R^{c} is complete.
4. R is asymmetric and negatively transitive if and only if R^{c} is complete and transitive.

In economics, the interpretation of R depends, among other things, on whether R is reflexive or irreflexive. If R is reflexive, "xRy" is interpreted as "alternative x is at least as preferable as alternative y". In this case, we often use symbol $\succsim$ in place of R. If R is irreflexive, "xRy" is interpreted as "alternative x is more preferable than alternative y". In this case, we often use symbol $\succ$ in place of R.

Exercise 2.2.3
1. Let $X = \boldsymbol{R}^2$ and define a binary relation $\succsim$ by letting, for every $x \in X$ and every $y \in X$, $x \succsim y$ if and only if $x_1 \geq y_1$ and $x_2 \geq y_2$. Prove that $\succsim$ is transitive, and give an example to show that it is not complete.
2. Let $X = \boldsymbol{R}^3$ and define a binary relation $\succsim$ by letting, for every $x \in X$ and every $y \in X$, $x \succsim y$ if and only if $x_n \geq y_n$ for at least two $n \in \{1, 2, 3\}$. Prove that $\succsim$ is complete, and give an example to show that it is not transitive.

Let R be a binary relation on X. Define another binary relation R^{s} by letting $xR^{\mathrm{s}}y$ whenever xRy and $yR^{\mathrm{c}}x$. Then R^{s} is the *strict* or *asymmetric* part of R. Define yet another binary relation R^{i} by letting $xR^{\mathrm{i}}y$ whenever xRy and yRx. Then R^{i} is the *indifference* or *symmetric* part of R. These definitions make sense for any binary relation R, but are useful when R is reflexive. In economics, when a reflexive relation $\succsim$ is given, $\succsim^{\mathrm{s}}$ is often denoted by $\succ$ and $\succsim^{\mathrm{i}}$ is denoted by $\sim$.

Exercise 2.2.4 Let R be a binary relation on X. Prove that:

1. R^{i} is symmetric. It is transitive if R is transitive.
2. R^{s} is asymmetric. It is transitive if R is transitive.

Exercise 2.2.5 Is it true that for every binary relation R, R is transitive if and only if R^{s} and R^{i} are transitive? If so, prove it. If not, present a counterexample and provide additional conditions on R under which this equivalence is guaranteed.

Exercise 2.2.6 For a binary relation R on X, we define another binary relation R^{t} by letting, for every $x \in X$ and every $y \in X$, $xR^{\mathrm{t}}y$ if and only if yRx. Prove that:

1. if R is complete, then $R = ((R^{\mathrm{s}})^{\mathrm{c}})^{\mathrm{t}}$.
2. if R is asymmetric, then $R = ((R^{\mathrm{c}})^{\mathrm{t}})^{\mathrm{s}}$.

2.2.2 Definition Let R be a binary relation on X. A function $u : X \to \boldsymbol{R}$ is a *utility function representing* R if the following condition is met: for every $x \in X$ and every $y \in X$, $u(x) \geq u(y)$ if and only if xRy. If there is a utility function representing R, R is *representable.*

2.2.3 Proposition *If a binary relation is represented by a utility function, then it is complete and transitive.*

Since every complete relation is reflexive, Proposition 2.2.3 implies that when a binary relation is represented by a utility function in the sense of Definition 2.2.2, then it is to be interpreted as an at-least-as-preferable-as relation. A less standard definition of representation is the following:

2.2.4 Definition Let R be a binary relation on X. A function $u : X \to \boldsymbol{R}$ is a *utility function strictly representing* R if the following condition is met: for every $x \in X$ and every $y \in X$, $u(x) > u(y)$ if and only if xRy. If there is a utility function representing R, R is *strictly representable*

2.2.5 Proposition *If a binary relation is strictly represented by a utility function, then it is asymmetric and transitive.*

Since every asymmetric relation is irreflexive, Proposition 2.2.5 implies that when a binary relation is strictly represented by a utility function in the sense of Definition 2.2.4, then it is to be interpreted as a more-preferable-than relation.

Exercise 2.2.7 Prove that:

1. if R is represented by a utility function u, then R^{s} is strictly represented by u.
2. if R is strictly represented by a utility function u, then $(R^{\mathrm{c}})^{\mathrm{t}}$ is represented by u.

Exercise 2.2.8 Use a mathematical induction argument on the the number of the elements of X to prove that if X is finite (that is, it consists of finitely many elements), then every rational preference relation is represented by a utility function.

Exercise 2.2.9 (Difficult) Prove that if a binary relation R on X is strictly represented by a utility function, then there exists an at most countable subset Z of X such that for every $x \in X \setminus Z$ and every $y \in X \setminus Z$, if xRy, then there exists a $z \in Z$ such that xRz and zRy. Would the same conclusion hold if R were represented by a utility function?

2.3 Choice Rules

A *choice structure on* X is a pair of a set of non-empty subsets of X, denoted by $\mathscr{B}$, and a mapping of $\mathscr{B}$ into a set of non-empty subsets of X, denoted by C, that satisfies $C(B) \subseteq B$ for every $B \in \mathscr{B}$.

2.3.1 Definition Let $(\mathscr{B}, C)$ be a choice structure on X. A binary relation R on X is the *revealed at-least-as-preferable-as relation of* $(\mathscr{B}, C)$ if the following condition is met: for every $x \in X$ and every $y \in X$, xRy if and only if there exists a $B \in \mathscr{B}$ such that $\{x, y\} \subseteq B$ and $x \in C(B)$.

2.3.2 Definition Let $(\mathscr{B}, C)$ be a choice structure on X and R be its revealed at-least-as-preferable-as relation. Consider the following condition for each integer $N \geq 2$:

> If $x_n \in X$ for every $n \leq N$ and $x_n R x_{n+1}$ for every $n \leq N - 1$, then $x_1 \in C(B)$ whenever $\{x_1, x_N\} \subseteq B \in \mathscr{B}$ and $x_N \in C(B)$.

Then:

Weak Axiom of Revealed Preference $(\mathscr{B}, C)$ satisfies the *weak axiom of revealed preference* if the above condition is met for $N = 2$.

Strong Axiom of Revealed Preference $(\mathscr{B}, C)$ satisfies the *strong axiom of revealed preference* if the above condition is met for every $N \geq 2$.

Exercise 2.3.1 Let $(\mathscr{B}, C)$ be a choice structure and R be its revealed at-least-as-preferable-as relation. Let Q be the transitive closure of R, that is, xQy if and only if there are an $N \in \boldsymbol{N}$ and an $(x_1, x_2, \ldots, x_N) \in X^N$ such that $x_1 = x$, $x_N = y$, and $x_{n-1} R x_n$ for every $n \leq N$. Prove that:

1. $(\mathscr{B}, C)$ satisfies the weak axiom of revealed preference if and only if for every $x \in X$ and every $y \in X$, $C(B) \cap \{x, y\} \in \{\varnothing, \{x, y\}\}$ whenever $xR^{\mathrm{i}}y$ and $\{x, y\} \subseteq B$.
2. $(\mathscr{B}, C)$ satisfies the strong axiom of revealed preference if and only if it satisfies the weak axiom of revealed preference and $R^{\mathrm{s}} \subseteq Q^{\mathrm{s}}$ (that is, for every $x \in X$ and every $y \in X$, if $xR^{\mathrm{s}}y$, then $xQ^{\mathrm{s}}y$).

Exercise 2.3.2 Let $X = \{x, y, z\}$. Define a choice structure $(\mathscr{B}, C)$ on X by $\mathscr{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$, $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$. Show that $(\mathscr{B}, C)$ satisfies the weak axiom but not the strong axiom of revealed preference.

Exercise 2.3.3 Let $X = \{x, y, z\}$. Define a choice structure $(\mathscr{B}, C)$ on X satisfy $\mathscr{B} = \{\{x, y\}, \{y, z\}, \{z, x\}, \{x, y, z\}\}$, $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$. Show that (regardless of what $C(\{x, y, z\})$ is) $(\mathscr{B}, C)$ does not satisfy the weak axiom of revealed preference.

2.4 Relationship between Preference Relations and Choice Rules

Let $\mathscr{B}$ be a set of non-empty subsets of X.

2.4.1 Definition Let R be a binary relation and assume that for every $B \in \mathscr{B}$, there exists an $x \in B$ such that xRy for every $y \in B$. Define a mapping C of $\mathscr{B}$ into the set of the non-empty subsets of X by $C(B) = \{x \in B \mid xRy \text{ for every } y \in B\}$. Then $(\mathscr{B}, C)$ is the *choice structure of* R.

Definition 2.4.1 makes sense only if R is reflexive (because, otherwise, the assumption that for every $B \in \mathscr{B}$, there exists an $x \in B$ such that xRy for every $y \in B$ may well fail). For an irreflexive R, the following definition of thee choice structure makes sense.

2.4.2 Definition Let R be a binary relation and assume that for every $B \in \mathscr{B}$, there exists an $x \in B$ such that $yR^{c}x$ for every $y \in B$. Define a mapping C of $\mathscr{B}$ into the set of the non-empty subsets of X by $C(B) = \{x \in B \mid yR^{c}x \text{ for every } y \in B\}$. Then $(\mathscr{B}, C)$ is the *strict choice structure of* R.

Exercise 2.4.1 Prove that the choice structure of every complete and transitive relation satisfies the strong axiom of revealed preference.

The following theorem claims that the converse is also true, although we shall not prove it here:

2.4.3 Theorem (Richter) *For every choice structure $(\mathscr{B}, C)$ satisfying the strong axiom of revealed preference, there exists a complete and transitive relation of which the choice structure coincides with $(\mathscr{B}, C)$.*

As can be seen from Exercise 2.3.2, the weak axiom is, in general, not sufficient for the conclusion of Richter's Theorem, that is, the existence of an underlying complete and transitive preference relation. However, it can be shown that if $\mathscr{B}$ contains all subsets of X of up to three elements, then the weak axiom is indeed sufficient for the existence of an underlying complete and transitive relation. Moreover, such a binary relation is uniquely determined.

Exercise 2.4.2 Assume that $\mathscr{B}$ contains all subsets of X of two elements and let $\succsim$ be a complete and transitive relation. Let $(\mathscr{B}, C)$ be the choice structure of $\succsim$ and R be the revealed preference relation of $(\mathscr{B}, C)$. Prove that $\succsim = R$.

2.5 Afriat's Approach

In this section, we briefly review an alternative approach, originally due to S. Afriat, on the possibility of recovering an underlying preference relation from observed choices. In Afriat's approach, the choice structure $(\mathscr{B}, C)$ is such that $C(B)$ is a singleton for every $B \in \mathscr{B}$, and the question is whether there is a complete and transitive relation $\succsim$ on X such that $C(B) \succsim y$ for ever $y \in B$ and every $B \in \mathscr{B}$. The interpretation is that alternative $C(B)$ is chosen when the alternatives in B are available.

The difference between this and the preceding approaches is that, in this approach, we have less information, in the sense that although we know that $C(B)$ is a most preferable choice in B, we do not know whether there is any other equally preferable choice in B. This difference can be made clearer by comparing the notion of an underlying preference relation. In the preceding approach, a preference relation underlies the choice structure $(\mathscr{B}, C)$ if $C(B) = \{x \in B \mid x \succsim y \text{ for every } y \in B\}$, while, in Afriat's apporach, a preference relation underlies the choice structure $(\mathscr{B}, C)$ if $C(B) \in \{x \in B \mid x \succsim y \text{ for every } y \in B\}$.

This notion of an underlying preference may well be superior to the preceding one for the purpose of empirical studies, as we often have a data set of, say, household consumption behavior, but we do not have any data that indicate whether households had equally desirable consumption choices within their budgets. However, since the notion of an underlying relation in Afriat's approach is weaker than that in the preceding apporach, it is easier to guarantee its existence in Afriat's approach. Indeed, the total indifference relation (the relation $\succsim$ such that $x \succsim y$ for every $x \in X$ and every $y \in X$) underlies every choice structure $(\mathscr{B}, C)$ in Ariat's sense. For this reason, in Afriat's approach, it is common to ask whether a given choice structure $(\mathscr{B}, C)$ has an underlying relation within some restricted class of complete and transitive relation defined on X.

関連図書

[1] S. N. Afriat, The construction of utility functions from expenditure data, *International Economic Review* 8 (1967) 67–77.

[2] John S. Chipman and others, *Preferences, Utility and Demand*, Harcourt Brace Jovanovic.

[3] Peter C. Fishburn, *Utility Theory for Decision Making*, John Wiley and Sons.

[4] David M. Kreps, *Notes on the Theory of Choice*, Westview Press.

第3章 Consumer Theory

3.1 Introduction

In this lecture note we present the classical theory of a consumer. The materials are taken from Chapters 2 and 3 of MWG. Debreu gives a succinct account of some of the concepts introduced below. Kreps has a somewhat different, and yet very nice, treatment.

3.2 Commodities and Commodity Space

A *commodity* is defined in terms of its physical characteristics, as well as the time, location, and even contingency when uncertainty is present, at which it is available for consumption or production.

Throughout this lecture course, we denote by L the number of types of commodities. The *commodity space* is the set of all combinations of quantities of the L commodities (called *commodity bundles*) that are regarded as physically feasible. It is therefore a subset of $\boldsymbol{R}^L$, and indeed often taken as $\boldsymbol{R}^L$ itself. It can however be $\boldsymbol{Z}^L$, where $\boldsymbol{Z}$ is the set of all integers, if all commodities are indivisible. For simplicity, we take the commodity space to be $\boldsymbol{R}^L$.

3.3 Consumption Set

The *consumption set* of a consumer is the set of all commodity bundles with which the consumer can survive. We denote it by X. It is on this set where the consumer's preference relation is to be defined. It is a subset of the commodity space and often taken to be $\boldsymbol{R}^L_+$. It can however be $\boldsymbol{Z}^L_+$ or other subsets of $\boldsymbol{R}^L_+$. For simplicity, we take the consumption set to be $\boldsymbol{R}^L_+$. A commodity bundle in the consumption set is a *consumption bundle*.

3.4 Preference and Utility

The *preference relation* of a consumer is nothing but a binary relation defined in Section 2.2 of Chapter 2. We make use of other properties, which relies on the Euclidean structure of X.

3.4.1 Definition Let $\succsim$ be a preference relation (binary relation) on X, with its strict part $\succ$ and indifference part $\sim$. Let x and y be arbitrary consumption bundles and α be an arbitrary number in $[0, 1]$.

Monotonicity $\succsim$ is *monotone* if $y \succ x$ whenever $y - x \in \boldsymbol{R}^L_{++}$.

Strong Monotonicity $\succsim$ is *strongly monotone* if $y \succ x$ whenever $y - x \in \boldsymbol{R}^L_+$ and $y - x \neq 0$.

Local Non-Satiation $\succsim$ is *locally non-satiated* if for every $\varepsilon > 0$ there exists a $z \in X$ such that $\|z - x\| \leq \varepsilon$ and $z \succ x$.

Convexity $\succsim$ is *convex* if $\alpha x + (1 - \alpha)y \succsim z$ whenever $z \in X$, $x \succsim z$, and $y \succsim z$.

Strict Convexity $\succsim$ is *strictly convex* if $\alpha x + (1 - \alpha)y \succ z$ whenever $z \in X$, $x \succsim z$, $y \succsim z$, $x \neq y$, and $\alpha \in (0, 1)$.

Continuity $\succsim$ is *continuous* if $x \succsim y$ whenever $(x^n)_{n=1}^{\infty}$ and $(y^n)_{n=1}^{\infty}$ are sequences in X such that $x^n \succsim y^n$ for every n, and $x^n \to x$ and $y^n \to y$ as $n \to \infty$.

3.4.2 Proposition *Every strongly monotone preference relation on $X = \boldsymbol{R}_+^L$ is monotone. Every monotone preference relation on $X = \boldsymbol{R}_+^L$ is locally non-satiated.*

Exercise 3.4.1 Show that there is a monotone preference relation on $X = \boldsymbol{Z}_+^L$ that is not locally non-satiated. Provide appropriate extensions, to the case of a general consumption set X, of the above definitions of monotonicity and strong monotonicity with respect to which Proposition 3.4.2 is true.

Exercise 3.4.2 Prove that every preference relation that can be represented by a continuous utility function is continuous, complete, and transitive.

3.4.3 Proposition *Every continuous, complete, and transitive preference relation can be represented by a continuous utility function.*

Exercise 3.4.3 Let R be the strict part of the lexicographic ordering. Prove that there is no at most countable subset Z of $X = \boldsymbol{R}_+^2$ for which the property of Proposition 2.2.9 holds.

3.5 Prices, Wealth, and Budget Sets

We assume that there is a complete set of markets, so that there is a price, denoted by p_ℓ, for each commodity ℓ ($\ell = 1, \ldots, L$). Denote by p the L-dimensional column vector consisting of the p_ℓ. This is a price vector.

A *wealth level* of a consumer is presented by a real (and often non-negative or strictly positive) number w. Under a price vector p and a wealth level w, the consumer's *budget set* is $\{x \in X \mid p \cdot x \leq w\}$.

Simple as it may look, this formulation embodies some important assumptions in economics. First, the market is complete, so that all commodities that are relevant to preference relations and utility functions are given prices. Second, there is only one inequality defining the constraint, so that a reduction in expenditure for any one commodity can be used to increase the expenditure for any other commodity. Third, there is no rationing, in the sense that there is no (upper or lower) bound on the amounts of commodities that can be bought. Fourth, the price level for each commodity is unaffected by any change in the choice of the commodity bundle x, so that the consumer is a price-taker, whose influence on the market outcome is negligible relative to the size of the markets.

3.6 Utility Maximization Problem

Let u be a continuous utility function representing a continuous rational preference relation $\succsim$. The continuity assumption guarantees that in the subsequent analysis, all the functions are continuous and correspondences are upper semi-continuous on some appropriate domain (where, for example, the wealth level is strictly positive).

The *utility maximization problem* under a price vector p and a wealth level w is

$$\begin{aligned} \max_{x \in X} \quad & u(x), \\ \text{subject to} \quad & p \cdot x \leq w. \end{aligned}$$

3.6.1 Proposition *If $p \in \boldsymbol{R}^L_{++}$ and $w \geq 0$, then there exists at least one solution to the utility maximization problem. If, in addition, $\succsim$ is strictly convex, then there exists exactly one solution.*

We assume in the rest of this lecture note that there exists at least one solution and denote by $x(p,w)$ the set of all solutions. Then $(p,w) \mapsto x(p,w)$ is the *Walrasian demand correspondence*. It is a function if $\succsim$ is strictly convex.

3.6.2 Proposition *The Walrasian demand correspondence x has the following properties:*

Homogeneity *For every $\alpha > 0$, $x(\alpha p, \alpha w) = x(p,w)$.*

Walras' law *If $\succsim$ is locally non-satiated, then $p \cdot \bar{x} = w$ for every $\bar{x} \in x(p,w)$.*

Strong Axiom of Revealed Preference *x satisfies the strong axiom of revealed preference (Definition 2.3.2).*

Exercise 3.6.1 Give an example of a strictly convex and locally non-satiated (but not necessarily monotone) preference relation on $X = \boldsymbol{R}^2_+$ such that the demand function x is *not* continuous at $p = (0,1)$ and $w = 0$.

The value function of the utility maximization problem is the *indirect utility function* and denoted by v. Then $v(p,w) = u(\bar{x})$ for every $\bar{x} \in x(p,w)$.

3.6.3 Proposition *The indirect utility function v has the following properties:*

Homogeneity *For every $\alpha > 0$, $v(\alpha p, \alpha w) = v(p,w)$.*

Monotonicity *v is non-decreasing in w and non-increasing in every p_ℓ. If $\succsim$ is locally non-satiated, then v is strictly increasing in w.*

Quasi-Convexity *v is quasi-convex, that is, $v(\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w') \leq v(q,b)$ whenever $v(p,w) \leq v(q,b)$, $v(p',w') \leq v(q,b)$, and $\alpha \in [0,1]$.*

3.7 Expenditure Minimization Problem

The *expenditure minimization problem* under a price vector p and a utility level $\bar{u}$ is

$$\begin{aligned} \min_{x \in X} \quad & p \cdot x, \\ \text{subject to} \quad & u(x) \geq \bar{u}. \end{aligned}$$

3.7.1 Proposition *If $p \in \boldsymbol{R}^L_{++}$ and there exists an $\bar{x} \in X$ such that $u(\bar{x}) \geq \bar{u}$, then there exists at least one solution to the expenditure minimization problem. If, in addition, $\succsim$ is strictly convex, then there exists exactly one solution.*

We assume in the rest of this lecture note that there exists at least one solution and denote by $h(p, \bar{u})$ the set of all solutions. Then $(p, \bar{u}) \mapsto h(p, \bar{u})$ is the *Hicksian demand correspondence*. It is a function if $\succsim$ is strictly convex.

3.7.2 Remark The expenditure function e is differentiable with respect to p at $(p, \bar{u})$ if and only if $h(p, \bar{u})$ is a singleton.

3.7.3 Proposition *The Hicksian demand correspondence h has the following properties:*

Homogeneity *For every $\alpha > 0$, $h(\alpha p, \bar{u}) = h(p, \bar{u})$.*

No Excess Utility *If $\succsim$ is continuous and monotone, $p \in \boldsymbol{R}^L_{++}$, and $\bar{u} \geq u(0)$, then $u(\bar{x}) = \bar{u}$ for every $\bar{x} \in h(p, \bar{u})$.*

Compensated Law of Demand *For any two price vectors p and p', and any $\bar{x} \in h(p, \bar{u})$ and $\bar{x}' \in h(p', \bar{u})$, $(p' - p) \cdot (\bar{x}' - \bar{x}) \leq 0$.*

Exercise 3.7.1 Prove the Compensated Law of Demand.

The value function of the expenditure minimization problem is the *expenditure function* and denoted by e. Then $e(p, \bar{u}) = p \cdot \bar{x}$ for every $\bar{x} \in h(p, \bar{u})$.

3.7.4 Proposition *The expenditure function e has the following properties:*

Homogeneity *For every $\alpha > 0$, $e(\alpha p, \bar{u}) = \alpha e(p, \bar{u})$.*

Monotonicity *e is non-decreasing in $\bar{u}$ and every p_ℓ. If $\succsim$ is monotone, then e is strictly increasing in $\bar{u} \geq u(0)$.*

Concavity *e is concave in p, that is, $e(\alpha p + (1-\alpha)p', \bar{u}) \geq \alpha e(p, \bar{u}) + (1-\alpha)e(p', \bar{u})$ for every $\alpha \in [0, 1]$.*

Exercise 3.7.2 Show that the no excess utility property of Proposition 3.7.3 and the strict increasingness in $\bar{u}$ of Proposition 3.7.4 need not be satisfied by a locally non-satiated, but not monotone, preference relation.

3.8 Characterization of Walrasian and Hicksian Demands

3.8.1 Proposition *For every price vector p and utility level $\bar{u}$,*

$$h(p, \bar{u}) = \nabla_p e(p, \bar{u}). \tag{3.1}$$

Exercise 3.8.1 For a price vector $\bar{p}$ and a utility level $\bar{u}$, consider the following maximization problem:

$$\begin{array}{cc} \max\limits_{p} & e(p, \bar{u}), \\ \text{subject to} & p \cdot h(\bar{p}, \bar{u}) \leq e(\bar{p}, \bar{u}). \end{array}$$

Show that $\bar{p}$ is a solution to this maximization problem, and also that the Kuhn-Tucker condition for the solution implies equality (3.1). (*Hint*: Use the homogeneity of e in p to show that the multiplier in the Kuhn-Tucker condition equals one.)

3.8.2 Proposition *For every price vector p and utility level $\bar{u}$, $D_p h(p, \bar{u})$ is symmetric, negative semi-definite, and satisfies $D_p h(p, \bar{u})p = 0$.*

3.8.3 Proposition (Roy's Identity) *For every price vector p and wealth level w,*

$$x(p, w) = -\frac{1}{\nabla_w v(p, w)} \nabla_p v(p, w).$$

Exercise 3.8.2 For a price vector $\bar{p}$ and a wealth level $\bar{w}$, consider the following minimization problem:

$$\begin{array}{cc} \min\limits_{(p,w)} & v(p, w), \\ \text{subject to} & p \cdot x(\bar{p}, \bar{w}) \leq w. \end{array}$$

Show that $(\bar{p}, \bar{w})$ is a solution to this minimization problem, and also that the Kuhn-Tucker condition for the solution implies Roy's Identity.

3.9 Duality

3.9.1 Proposition *For every price vector $p \in \boldsymbol{R}^L_{++}$ and wealth level $w > 0$,*

$$h(p, v(p, w)) = x(p, w)$$

and

$$e(p, v(p, w)) = w.$$

For every price vector $p \in \boldsymbol{R}^L_{++}$ and utility level $\bar{u} \geq u(0)$,

$$x(p, e(p, \bar{u})) = h(p, \bar{u})$$

and

$$v(p, e(p, \bar{u})) = \bar{u}.$$

Exercise 3.9.1 Give examples to show that if $\succsim$ were not locally non-satiated, then it may not be true that $h(p, v(p, w)) = x(p, w)$; and that if $p \notin \boldsymbol{R}^L_{++}$ (but still $p \in \boldsymbol{R}^L_{+}$ and $p \neq 0$) and $w = 0$, then it may not be true that $x(p, e(p, \bar{u})) = h(p, \bar{u})$.

3.9.2 Proposition (Slutsky Equation) *For every price vector $\bar{p} \in \boldsymbol{R}^L_{++}$ and wealth level $\bar{w} > 0$,*

$$D_p x(\bar{p}, \bar{w}) = D_p h(\bar{p}, v(\bar{p}, \bar{w})) - D_w x(\bar{p}, \bar{w}) x(\bar{p}, \bar{w})^\top.$$

Hence $D_p x(\bar{p}, \bar{w}) + D_w x(\bar{p}, \bar{w}) x(\bar{p}, \bar{w})^\top$ is symmetric and negative semi-definite.

3.10 Taxonomy of Properties

No Change in Constraints or Objectives x is homogeneous of degree zero in (p, w); v is homogeneous of degree zero in (p, w); h is homogeneous of degree zero in p; and e is homogeneous of degree one in p.

Monotone Changes in Constraints or Objectives v is non-increasing in p and non-decreasing in w; and e is non-decreasing in p and $\bar{u}$.

Arbitrary Changes in Constraints or Objectives x satisfies the strong axiom of revealed preference; v is quasi-convex in (p, w); h satisfies the Compensated Law of Demand; and e is concave in p.

Exercise 3.10.1 Use Proposition 3.9.1 to derive the monotonicity and concavity of e from the monotonicity and quasi-convexity of v; and also derive the monotonicity and quasi-convexity of v from the monotonicity and concavity of e.

3.11 Welfare Measures of Price Changes

Suppose that the current price vector p^0 now changes to p^1, while the wealth level w is unchanged. The change is preferable if and only if $v(p^1, w) - v(p^0, w) > 0$.

Now take another price vector $\bar{p}$. If e is strictly increasing in $\bar{u}$, then the above inequality is equivalent to

$$e(\bar{p}, v(p^1, w)) - e(\bar{p}, v(p^0, w)) > 0. \tag{3.2}$$

The function $(p, w) \mapsto e(\bar{p}, v(p, w))$ is the *money metric (indirect) utility function (under $\bar{p}$).*

Exercise 3.11.1 Prove that the money metric indirect utility function under any $\bar{p}$ does not depend on the choice of a utility function to represent the preference relation, that is, if u and $\hat{u}$ represent the same preference relation, and v, e, $\hat{v}$, and $\hat{e}$ are the indirect utility functions and the expenditure functions derived from u and $\hat{u}$, then $e(\bar{p}, v(p, w)) = \hat{e}(\bar{p}, \hat{v}(p, w))$ for every (p, w).

If we take $\bar{p} = p^0$, then the left hand side of (3.2) is

$$e(p^0, v(p^1, w)) - e(p^0, v(p^0, w)) = e(p^0, \bar{u}^1) - w,$$

where $\bar{u}^1 = v(p^1, w)$. This is the *equivalent variation of the change from p^0 to p^1 with wealth level w*, and denoted by $EV(p^0, p^1, w)$. On the other hand, if we take $\bar{p} = p^1$, then the left hand side of (3.2) is

$$e(p^1, v(p^1, w)) - e(p^1, v(p^0, w)) = w - e(p^1, \bar{u}^0),$$

where $\bar{u}^0 = v(p^0, w)$. This is the *compensated variation of the change from p^0 to p^1 with wealth level w*, and denoted by $CV(p^0, p^1, w)$.

3.11.1 Proposition *For every p^0, p^1, and w, if $p^0_\ell = p^1_\ell$ for every $\ell \geq 2$, then*

$$EV(p^0, p^1, w) = \int_{p^1_1}^{p^0_1} h_1(p_1, p_{-1}, \bar{u}^1)\ dp_1,$$
$$CV(p^0, p^1, w) = \int_{p^1_1}^{p^0_1} h_1(p_1, p_{-1}, \bar{u}^0)\ dp_1,$$

where $p_{-1} = \left(p^0_2, \ldots, p^0_L\right) = \left(p^1_2, \ldots, p^1_L\right)$, $\bar{u}^0 = v(p^0, w)$, and $\bar{u}^1 = v(p^1, w)$.

Exercise 3.11.2 Prove Proposition 3.11.1.

Another commonly used measure of price changes is what MWG called the *area variation measure*:

$$AV(p^0, p^1, w) = \int_{p^1_1}^{p^0_1} x_1(p_1, p_{-1}, w)\ dp_1.$$

In general, this measure cannot be interpreted as a change in utility levels, but if the preference relation is quasi-linear with respect to a commodity other than the first one, and if the consumption levels of the numeraire are always strictly positive for the relevant range of prices, then

$$AV(p^0, p^1, w) = EV(p^0, p^1, w) = CV(p^0, p^1, w).$$

Exercise 3.11.3 Let p^0, p^1, p^2, p^3 be four price vectors and w be a wealth level.

1. Prove that

$$EV(p^0, p^1, w) = EV(p^0, p^2, w) + EV(p^2, p^1, w) = EV(p^0, p^3, w) + EV(p^3, p^1, w),$$
$$CV(p^0, p^1, w) = CV(p^0, p^2, w) + CV(p^2, p^1, w) = CV(p^0, p^3, w) + CV(p^3, p^1, w).$$

2. Assume that $p^0_\ell = p^1_\ell = p^2_\ell = p^3_\ell$ for every $\ell \geq 3$, $p^0_1 = p^3_1 > p^1_1 = p^2_1$, and $p^0_2 = p^2_2 > p^1_2 = p^3_2$. Give an example in which

$$AV(p^0, p^2, w) + AV(p^2, p^1, w) \neq AV(p^0, p^3, w) + AV(p^3, p^1, w).$$

関連図書

[1] Gerard Debreu, “Theory of Value”, John Wiley, and Sons.

第4章 Producer Theory

4.1 Introduction

In this lecture note we start the classical theory of a producer. We spell out some basic definitions for productions sets and formulate the profit maximization and cost minimization.

4.2 Productions Set

To describe production activities in the commodity space $\boldsymbol{R}^L$, we use the convention that, in a vector $\boldsymbol{R}^L$, the amounts of outputs are measured in positive quantities and the amounts of inputs are measured in negative quantities. A *production set* is the set of all vectors of $\boldsymbol{R}^L$ that are technologically feasible.

Production sets do not presume the availability of inputs. For example, with $L = 2$, the fact that $(-100, 100) \in Y$ merely means that if 100 units of good 1 are used as inputs, then 100 units of good 2 can be produced. It does not imply that there are 100 units of good 1 available for production.

The following assumptions are often used for a production set Y.

Non-emptiness $Y \neq \varnothing$.

Closedness Y is a closed subset of $\boldsymbol{R}^L$.

No-Free Lunch $Y \cap \boldsymbol{R}^L_+ \subseteq \{0\}$.

Possibility of Inaction $0 \in Y$.

Free Disposal $Y - \boldsymbol{R}^L_+ \subseteq Y$.

Irreversibility $Y \cap (-Y) \subseteq \{0\}$.

Non-Increasing Returns to Scale $\alpha\bar{y} \in Y$ for every $\bar{y} \in Y$ and every $\alpha \in [0, 1]$.

Non-Decreasing Returns to Scale $\alpha\bar{y} \in Y$ for every $\bar{y} \in Y$ and every $\alpha \geq 1$.

Constant Returns to Scale $\alpha\bar{y} \in Y$ for every $\bar{y} \in Y$ and every $\alpha \geq 0$.

Additivity $\bar{y} + \bar{y}' \in Y$ for every $\bar{y} \in Y$ and every $\bar{y}' \in Y$.

Convexity Y is a convex subset of $\boldsymbol{R}^L$.

Convex Cone Y is a convex cone in $\boldsymbol{R}^L$, that is, $\alpha\bar{y} + \alpha'\bar{y}' \in Y$ for every $\bar{y} \in Y$, every $\bar{y}' \in Y$, every $\alpha \geq 0$, and every $\alpha' \geq 0$.

Polyhedral Cone Y is a polyhedral cone in $\boldsymbol{R}^L$, that is, there exists a finite subset $\{\bar{y}^1, \ldots, \bar{y}^M\}$ of $\boldsymbol{R}^L$ such that for every $\bar{y} \in \boldsymbol{R}^L$, $\bar{y} \in Y$ if and only if there exists an $(\alpha_1, \ldots, \alpha_M) \in \boldsymbol{R}^M_+$ such that $\bar{y} = \sum_{m=1}^{M} \alpha_m \bar{y}^m$.

Exercise 4.2.1 Prove that:

1. If Y is convex and satisfies the possibility of inaction, then it exhibits non-increasing returns to scale. The converse does not hold.

2. Y is a convex cone if and only if it is convex and additive and satisfies the possibility of inaction.

3. Y is a convex cone if and only if it is additive and exhibits constant returns to scale.

4. If Y is a polyhedral cone, then it is a convex cone. The converse holds only if $L = 2$.

4.3 Transformation Function and Production Function

4.3.1 Definition Let Y be a production set.

1. If a function $F : \boldsymbol{R}^L \to \boldsymbol{R}$ satisfies $Y = \{\bar{y} \in \boldsymbol{R}^L \mid F(\bar{y}) \leq 0\}$, then F is a *transformation function of* Y.

2. Suppose that Y satisfies $\bar{y}_l \leq 0$ for every $\bar{y} \in Y$ and every $\ell < L$. If a function $f : \boldsymbol{R}^{L-1}_+ \to \boldsymbol{R}_+$ satisfies $Y = \{\bar{y} \in \boldsymbol{R}^L \mid \bar{y}_L \leq f(-\bar{y}_1, \ldots, -\bar{y}_{L-1})\}$, then f is a *production function of* Y.

4.4 Profit Maximization Problem

The *profit maximization problem* under a price vector p is

$$\max_{\bar{y} \in Y} p \cdot \bar{y},$$

The set of solutions to this maximization problem is denoted by $y(p)$. This defines a correspondence y, the *supply correspondence of* Y, on a set of price vectors p for which the problem has at least one solution.

Just as in the case of a consumer's budget constraint, this formulation of the profit maximization. First, the markets are complete, so that all inputs and outputs are given prices. Second, the profit is defined as a single inner product, which means that any increment in revenue can, however far distant in the future it will be realized, be counted towards profit, even when the associate cost must be paid immediately. Third, there is no rationing, so that the firm can buy inputs and sell outputs as much as it wishes. Fourth, the price levels are not affected by the any change in the input-output combination y, so that the firm is a price taker in the input and output markets.

4.4.1 Proposition *The supply correspondence y has the following properties:*

Homogeneity *$y(\alpha p) = y(p)$ for every price vector p and every $\alpha > 0$.*

Production Efficiency *$y(p) \subseteq \partial Y$ for every non-zero price vector p.*

Law of Supply *$(p' - p) \cdot (\bar{y}' - \bar{y}) \leq 0$ for any two price vectors p and p', every $\bar{y} \in y(p)$, and every $\bar{y}' \in y(p')$.*

The value function of the profit maximization problem is the *profit function of Y* and denoted by π. Then $\pi(p) = p \cdot \bar{y}$ for every $\bar{y} \in y(p)$.

4.4.2 Proposition *The profit function π has the following properties:*

Homogeneity *$\pi(\alpha p) = \alpha \pi(p)$ for every price vector p and every $\alpha > 0$.*

Convexity *π is a convex function, that is, $\pi(\alpha p + (1-\alpha)p') \leq \alpha \pi(p) + (1-\alpha)\pi(p')$ for every $\alpha \in [0,1]$ and any two price vectors p and p'.*

4.4.3 Remark For every p, π is differentiable at p if and only if $y(p)$ is a singleton.

4.4.4 Lemma (Hotelling) *If π is differentiable at p, then $y(p) = \{\nabla \pi(p)\}$. If y is a continuously differentiable function around p, then $Dy(p)$ is symmetric and positive semi-definite, and satisfies $Dy(p)p = 0$.*

4.5 Cost Minimization Problem

Suppose that Y satisfies $\bar{y}_\ell \leq 0$ for every $\bar{y} \in Y$ and every $\ell < L$. Let $f : \boldsymbol{R}_+^{L-1} \to R_+$ be the production function of Y. The *cost minimization problem* under a vector $w \in \boldsymbol{R}^{L-1}$ of input prices and an output level q is

$$\begin{aligned} \min_{\bar{z} \in \boldsymbol{R}_+^{L-1}} \quad & w \cdot \bar{z}, \\ \text{subject to} \quad & f(\bar{z}) \geq q. \end{aligned}$$

The set of solutions to this minimization problem is denoted by $z(w,q)$. This defines a correspondence z, the *conditional factor demand correspondence of Y*, on a set of price vectors w and output levels q for which the problem has at least one solution.

Let $Y = \{\bar{y} \in \boldsymbol{R}^L \mid \bar{y}_L \leq f(-\bar{y}_1, \ldots, -\bar{y}_{L-1})\}$. If $\bar{y} \in y(p)$, then $(-\bar{y}_1, \ldots, -\bar{y}_{L-1}) \in c(p_1, \ldots, p_{L-1}, y_L)$. That is, profit maximization implies cost minimization. The converse, however, does not hold, that is, even if $\bar{z} \in z(w,q)$, there need not be any p_L such that $(-\bar{z}, q) \in y(w, p_L)$. Although cost minimization is a weaker condition than profit maximization, an advantage of the notion of cost minimization over that of profit maximization is that there may be a cost-minimizing input-output combination even when there is no profit-maximizing one.

Exercise 4.5.1 Prove that if Y exhibits non-decreasing returns to scale and if $y(p) \neq \varnothing$, then $\pi(p) = 0$.

4.5.1 Proposition *The conditional factor demand correspondence z has the following properties:*

Homogeneity *$z(\alpha w, q) = z(w,q)$ for every factor price vector w and every $\alpha > 0$.*

No Excees Output *If f is continuous and $f(0) = 0$, and if $w \in \boldsymbol{R}^{L-1}_{++}$ and $q > 0$, then $f(\bar{z}) = q$ for every $\bar{z} \in z(w, q)$.*

Law of Conditional Factor Demand $(w' - \bar{w}) \cdot (\bar{z}' - \bar{z}) \leq 0$ *for any two input price vectors w and w', every $\bar{z} \in z(w, q)$, and every $\bar{z}' \in z(w', q)$.*

The value function of the cost minimization problem is the *cost function of Y* and denoted by c. Then $c(w, q) = w \cdot \bar{z}$ for every $\bar{z} \in z(q, w)$.

4.5.2 Proposition *The cost function c has the following properties:*

Homogeneity $c(\alpha w, q) = \alpha c(w, q)$ *for every price vector p and every $\alpha > 0$.*

Monotonicity *c is non-decreasing in q and every w_ℓ.*

Concavity *c is a concave function of w, that is, $c(\alpha w + (1-\alpha)w', q) \geq \alpha c(w, q) + (1-\alpha)c(w', q)$ for every $\alpha \in [0, 1]$ and any two input price vectors q and q'.*

4.5.3 Remark For every (w, q), π is differentiable with respect to w at (w, q) if and only if $z(w, q)$ is a singleton.

4.5.4 Lemma (Shephard) *If c is differentiable with respect to w at (w, q), then $z(w, q) = \{\nabla_w c(w, q)\}$. If z is a continuoulsy differentiable function of w around (w, q), then $D_w z(w, c)$ is symmetric and positive semi-definite, and satisfies $D_w z(w, q) w = 0$.*

4.6 When Profit Maximization is Justfied

Imagine that a firm has a production set Y and a consumer has an indirect utility function v, which is strictly increasing in wealth, and owns a share $\theta > 0$ in the profit of the firm. If a production plan $\bar{y} \in Y$ is chosen, $\bar{p}$ is the price vector, and his wealth from other than the shareholding in the firm, then he attains the utility level $u(\bar{p}, \bar{w} + \theta \bar{p} \cdot \bar{y})$.

4.6.1 Proposition *If $\bar{p}$ and $\bar{w}$ does not depend on the choice of a production plan $\bar{y} \in Y$, then the following two maximization problems have the same solutions:*

$$\max_{\bar{y} \in Y} \bar{p} \cdot \bar{y},$$

and

$$\max_{\bar{y} \in Y} v(\bar{p}, \bar{w} + \theta \bar{p} \cdot \bar{y}).$$

4.7 Price Normalization

Suppose now that the price vector $\bar{p}$ depends on the choice of a production plan $\bar{y} \in Y$, and the wealth $\bar{w}$ depends on the price vector $\bar{p}$. Denote the dependence by $p(y)$ and $w(\bar{p})$. When a production plan $\bar{y} \in Y$ is chosen, the consumer attains the utility level

$$u(p(\bar{y}), w(p(\bar{y})) + \theta p(\bar{y}) \cdot \bar{y}).$$

Hence the production plans that are most desirable for the consumer are the solutions to the maximization problem:

$$\max_{\bar{y}\in Y} v(p(\bar{y}), w(p(\bar{y})) + \theta p(\bar{y}) \cdot \bar{y}). \tag{4.1}$$

If v can be written as

$$v(\bar{p}, \bar{w}) = \frac{\bar{w}}{\beta_1 \bar{p}_1 + \cdots + \beta_L \bar{p}_L}$$

and $w(\bar{p}) = 0$ for every price vector $\bar{p}$, then

$$\begin{aligned} & v(p(\bar{y}), w(p(\bar{y})) + \theta p(\bar{y}) \cdot \bar{y}) \\ &= \theta \frac{p(\bar{y}) \cdot \bar{y}}{\beta_1 p_1(\bar{y}) + \cdots \beta_L p_L(\bar{y})} \\ &= \theta \left(\frac{1}{\beta_1 p_1(\bar{y}) + \cdots \beta_L p_L(\bar{y})} p(\bar{y}) \right) \cdot \bar{y}. \end{aligned}$$

Thus the consumer would like the firm to maximize the profit with respect to the price vectors for which the consumption vector $(\beta_1, \ldots, \beta_L)$ is the numeraire. The solution to the maximization problem (4.1) typically depends on the choice of $(\beta_1, \ldots, \beta_L)$.

4.8 Conflict of Interest among Shareholders

If there are two consumers who have shares in the same firm but have different coefficients $(\beta_1, \ldots, \beta_L)$, then there may be a conflict of interest, in that they may like the firm to choose different production plans. For example, suppose that $L = 2$, the first commodity is the input, the second commodity is the output, and Y is given by the production function f. Suppose moreover that the first consumer's indirect utility function equal w/p_1 and the second consumer's indirect utility function equal w/p_2. Then the first consumer would like the firm to choose the input level that solves

$$\max_{z_1 \geq 0} \frac{p_2(-z_1, f(z_1))}{p_1(-z_1, f(z_1))} f(z_1) - z_1,$$

while the second consumer would like the firm to choose the input level that solves

$$\max_{z_1 \geq 0} f(z_1) - \frac{p_1(-z_1, f(z_1))}{p_2(-z_1, f(z_1))} z_1.$$

The solutions are typically different.

第5章　Decision Making under Uncertainty

5.1　Introduction

In this lecture note, we give an introduction to a consumer's utility and choice under uncertainty, covering topics such as the independence axiom; expected utility; absolute and relative risk aversion; and stochastic dominance. The materials of this section can be found Section 6.B of MWG.

5.2　さまざまなくじの定義

5.2.1 Definition (単純くじ) $C = \{1, \ldots, N\}$ を帰結 (consequence) の集合とする．このとき，C 上の任意の確率分布，すなわち各 $n = 1, \ldots, N$ に対して $p_n \geq 0$ が成り立ち，かつ $\sum_n p_n = 1$ を満たす N 次元ベクトル $(p_1, \ldots, p_N)$ を**単純くじ** (simple lottery または単にくじ lottery) と呼ぶ．また，単純くじの集合を $\mathscr{L}$ で表す．

5.2.2 Definition (複合くじ) $\mathscr{L}$ の任意の有限部分集合上の確率分布を**複合くじ**（compound lottery，または逐次くじ sequential lottery，二段階くじ two-stage lottery）と呼ぶ．例えば，K を正の整数，$L_1, L_2, \ldots, L_K$ を K 本の単純くじ，$(\alpha_1, \alpha_2, \ldots, \alpha_K)$ を，各 $k = 1, \ldots, K$ に対して $\alpha_k \geq 0$ が成り立ち，かつ $\sum_k \alpha_k = 1$ を満たす K 次元ベクトルとする．このとき，任意の $k \leq K$ に対し，L_k に確率 α_k を付与する確率分布は複合くじである．この複合くじを

$$\begin{pmatrix} L_1 & L_2 & \cdots & L_K \\ \alpha_1 & \alpha_2 & \cdots & \alpha_K \end{pmatrix} \tag{5.1}$$

と書く．

5.2.3 Definition (単純化くじ) 複合くじ（5.1）において，

$$L_k = \left(p_1^k, p_2^k, \ldots, p_N^k\right)$$

と書くと，$\left(\sum_k \alpha_k p_1^k, \ldots, \sum_k \alpha_k p_N^k\right)$ は単純くじである．これを $\alpha_1 L_1 + \cdots + \alpha_K L_K$ と書き，複合くじ（5.1）の**単純化くじ** (reduced lottery) と呼ぶ．

Exercise 5.2.1 $\Omega = \{1, \ldots, M\}$ を有限確率測度空間とし，$1, \ldots, M$ が実現する確率をそれぞれ $\pi_1, \ldots, \pi_M$ とする．$X : \Omega \to \boldsymbol{R}$，$Y : \Omega \to \boldsymbol{R}$ とし，$C = X(\Omega) \cup Y(\Omega) \cup ((1/2)X + (1/2)Y)(\Omega) = \{c_1, c_2, \ldots, c_N\}$ とする．L_X を，関数 X が C 上に導入するくじ，すなわち，

$$L_X = \left(\sum_{m \in X^{-1}(c_1)} \pi_m, \ldots, \sum_{m \in X^{-1}(c_N)} \pi_m \right)$$

をする．同様に関数 Y が C 上に導入するくじ

$$L_Y = \left(\sum_{m \in Y^{-1}(c_1)} \pi_m, \dots, \sum_{m \in Y^{-1}(c_N)} \pi_m \right)$$

を L_Y とする．$L_{\frac{1}{2}X+\frac{1}{2}Y}$ も同様に定義する．このとき，一般には

$$\frac{1}{2}L_X + \frac{1}{2}L_Y = L_{\frac{1}{2}X+\frac{1}{2}Y}$$

は成立しないことを例を挙げて示せ.（$\Omega = \{1, 2\}, \pi_1 = \pi_2 = 1/2$ の場合を考えれば十分である.）

5.3 独立性公理

以下の分析では，$\succsim$ は $\mathscr{L}$ 上の完備性，推移性を満たす選好関係とする．また，$\mathscr{L}$ は $\boldsymbol{R}^N$ の部分集合なので，ユークリッドの位相が与えられているものとし，$\succsim$ はこの位相について連続性を満たすとする．

5.3.1 独立性公理の定義

5.3.1 Definition (独立性公理) 任意の $L \in \mathscr{L}$，$L' \in \mathscr{L}$，$L'' \in \mathscr{L}$，$\alpha \in [0, 1]$ に対して，$L \succsim L'$ と $\alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$ が同値であるならば，$\succsim$ は**独立性公理** (Independence Axiom) を満たすと言う．

単純くじの集合 $\mathscr{L}$ の上で定義された選好関係 $\succsim$ に独立性公理を課すのは，以下のように正当化される．まず，公理と同様に，$L \in \mathscr{L}, L' \in \mathscr{L}, L'' \in \mathscr{L}$，$\alpha \in [0, 1]$ とする．次に，複合くじ 1 を $\begin{pmatrix} L & L'' \\ \alpha & 1-\alpha \end{pmatrix}$，複合くじ 2 を $\begin{pmatrix} L' & L'' \\ \alpha & 1-\alpha \end{pmatrix}$ とする．どちらの複合くじでも，確率 α の事象が実現するか否かで L'' を与えるか否か決まるので，確率 α で起こるある事象 E において複合くじ 1 は単純くじ L を与え，複合くじ 2 は単純くじ L' を与えると仮定しよう．この場合，もし E が起こらなかったなら，どちらの複合くじも L'' を与えるので，両者の違いは E が起こったときに与える単純くじの違いのみである．E が起こったとき，複合くじ 1 は L を，複合くじ 2 は L' を与える．それゆえ，複合くじの間では選好関係 $\succsim$ は定義されないものの，$L \succsim L'$ なので，複合くじ 1 は複合くじ 2 と少なくとも同程度には好ましいと考えることができよう．複合くじ 1 の単純化くじは $\alpha L + (1-\alpha)L''$，複合くじ 2 の単純化くじは $\alpha L' + (1-\alpha)L''$ であるので，$\alpha L + (1-\alpha)L''$ は $\alpha L' + (1-\alpha)L''$ と少なくとも同程度には好ましいと考えるのが妥当である．

以上の議論は，以下の前提に基づいて組み立てられている．

1. 単純くじの集合 $\mathscr{L}$ 上で定義された選好関係 $\succsim$ は，C に属する各帰結が与えられる確率にのみ依存し，どのような事象において与えられるかには全く依存しない．
2. 複合くじの間の選好関係は，それらの単純化くじの間の選好関係 $\succsim$ によって決められる．
3. 特に，複合くじの好ましさ（効用水準）を評価するにあたり，第 1 段階で選ばれる単純くじに関わる不確実性と，第 1 段階で選ばれた単純くじがどの帰結を与えるかという第 2 段階での不確実性は，全く同様に考慮される．
4. 複合くじの好ましさを評価するにあたり，第 1 段階での異なる事象において得られる単純くじの間には，補完的な関係は存在しない．

以下では，これらの考え方が妥当であるかを，例を挙げて検討していこう．

まず，上記 4 で述べられた，補完性の欠如については，次のような需要理論の例と対比してみると分かりやすいだろう．2 財を右足用の靴，左足用の靴とする．効用関数 $u(x)=\min\{x_1,x_2\}$ が表す選好関係を $\succsim$ とする．$x=(4,4)$，$y=(10,2)$，$z=(2,10)$ とおく．x では靴は 4 足，y では 2 足できるので，$x\succsim y$ が成立すると考えられる．このとき，2 次元ベクトルの凸結合として $\frac{1}{2}x+\frac{1}{2}z$ と $\frac{1}{2}y+\frac{1}{2}z$ を定義すると，$\frac{1}{2}x+\frac{1}{2}z\succsim\frac{1}{2}y+\frac{1}{2}z$ は成立するであろうか？ 2 つの凸結合をそれぞれ計算すると，

$$\frac{1}{2}x+\frac{1}{2}z=\frac{1}{2}(4,4)+\frac{1}{2}(2,10)=\frac{1}{2}(6,14)=(3,7)$$
$$\frac{1}{2}y+\frac{1}{2}z=\frac{1}{2}(10,2)+\frac{1}{2}(2,10)=\frac{1}{2}(12,12)=(6,6)$$

となり，前者からは 3 足，後者からは 6 足の靴が得られる．したがって，$\frac{1}{2}x+\frac{1}{2}z\succsim\frac{1}{2}y+\frac{1}{2}z$ は成立せず，独立性公理が満たされない．ここで独立性公理が満たされない原因は，当然のことではあるが，右足用の靴と左足用の靴は同時に消費され，2 財の間に補完的関係が存在する点にある．

5.3.2　独立性公理の反例

5.3.2 Example (エルスバーグのパラドックス (Ellsberg Paradox)) 次のような 2 つのつぼを考える．どちらにもボールが 100 個入っているが，つぼ 1 には赤 50 個，白 50 個のボールが入ってることがわかっているが，つぼ 2 には赤と白がそれぞれいくつ入っているかはわからないとする．そこで意思決定者は，つぼ 2 には赤 x 個，白 $(100-x)$ 個のボールが入っていると期待しているとしよう．

ここで，次のような実験をする．まず，赤が出たときのみ 1 万円獲得できるとした場合，多くの人はつぼ 1 を好む（つぼ 1 $\succ$ つぼ 2）という結果が出る．このとき人々は，

$$\frac{50}{100}>\frac{x}{100}\Longrightarrow x<50$$

と考えているはずである．

次に，白が出たときのみ 1 万円を獲得できるとした場合，このときも，多くの人はつぼ 1 を好む（つぼ 1 $\succ$ つぼ 2）という結果が出る．つまり人々は，

$$\frac{50}{100}>\frac{100-x}{100}\Longrightarrow x>50$$

と考えているはずである．しかし，これは先ほどの結果と同時には起こりえない．

実験では，このような矛盾する結果が観察される．これは，人々が選択をする際に，必ずしも確率を念頭に置いていないことを示しており，そのとき，独立性公理は成立しない．

5.3.3 Example (帰納主義 (Consequentialism)) 父と 2 人の娘アリスとバーバラが無人島にいるとする．娘 2 人が病気になってしまったが，薬は 1 人分しかない．このとき，帰結の集合は

$$C=\{A,B\}$$

である．ここで A はアリスに投薬し，B はバーバラに投薬することを表す．くじの集合の中には，確率 1 でアリスが助かるくじと確率 1 でバーバラが助かるくじがあるが，両者は父にとって無差別である．つまり

$$(1,0)\ \sim\ (0,1)$$

が成立しているとする．しかし，ここで，父は運を天にまかせ，どちらの娘に薬を与えるかをコインで決めるほうを好むとする．つまり，

$$\left(\frac{1}{2}, \frac{1}{2}\right) \succ (1, 0)$$
$$\left(\frac{1}{2}, \frac{1}{2}\right) \succ (0, 1)$$

が成立している．

$$\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(1, 0) + \frac{1}{2}(0, 1)$$
$$(0, 1) = \frac{1}{2}(0, 1) + \frac{1}{2}(0, 1)$$

が成立する．$(1, 0) \sim (0, 1)$ より，独立性公理が満たされるならば $\left(\frac{1}{2}, \frac{1}{2}\right) \sim (0, 1)$ が成立するはずであるが，実際は $\left(\frac{1}{2}, \frac{1}{2}\right) \succ (1, 0)$ である．つまり独立性公理が満たされない．この原因は，くじを生む手続きが選好には全く影響しない点にある．

5.3.4 Example (情報開示のスピード (Speed of information revelation)) 以下のような 2 つの複合くじを考えよう．くじ 1 は，実現する単純くじに不確実性がなく，確率 $\frac{1}{2}$ で必ず 1 万円が獲得できる単純くじが実現し，確率 $\frac{1}{2}$ で必ず 2 万円が獲得できる単純くじが実現するようなくじである．

くじ 2 は，実現する単純くじの分布に不確実性がなく，確率 $\frac{1}{2}$ で 1 万円，確率 $\frac{1}{2}$ で 2 万円が獲得できる単純くじが，確率 1 で実現するようなくじである．

2 つの複合くじから得られる単純化くじは全く同じであり，独立性公理が成立するならば，2 つの複合くじは無差別である．しかし，くじ 1 はくじ 2 よりも早い段階で，獲得できる賞金の額が明らかになるため，意思決定者によっては，くじ 1 をくじ 2 よりも選好する可能性がある．このようなときは独立性公理は成立しない．情報開示のスピードに効用が依存するケースの分析に対しては，帰納的 (recursive) 効用関数というものが考えられている．

状態依存効用関数は，独立性公理を満たさない．

5.3.5 Example (状態依存効用関数 (State-dependent utility)) 効用関数が状態に依存する場合は，独立性公理を満たさない可能性がある．このことを確認するために次の 2 つの例を見ていこう．

はじめの例は，かさやアイスクリームに対する効用は天気という状態に依存するために，独立性公理が満たされない状況を示すものである．明日の天気の確率は

$$\begin{cases} \text{晴れ} & 50\% \\ \text{雨} & 50\% \end{cases}$$

で与えられているものとする．帰結の集合は $C = \{\,\text{かさ}, \text{アイスクリーム}\,\}$ とし，次のような状態依存くじ，L_1 と L_2 を考える．

$$L_1 = \begin{cases} \text{アイス} & (\text{晴れのとき}) \\ \text{かさ} & (\text{雨のとき}) \end{cases}, \quad L_2 = \begin{cases} \text{かさ} & (\text{晴れのとき}) \\ \text{アイス} & (\text{雨のとき}) \end{cases}$$

通常は，$L_1 \succ L_2$ が成立するが，2 つの確率変数が導入する単純くじは同じで，ともに

$$\begin{pmatrix} \text{かさ} & \text{アイス} \\ 0.5 & 0.5 \end{pmatrix}$$

である．もし二項関係 $\succsim$ が独立性公理を満たすなら，$L_1 \sim L_2$ が成立する．しかしここでは $L_1 \succ L_2$ が成立しており，独立性公理を満たしていない．ここで $\succsim$ が独立性公理を満たさないのは，かさやアイスに対する効用が状態依存的であるためである．

次に，くじの賞金に対する効用が雇用状態によって変わるために独立性公理が満たされない例を見ていこう．労働者は来期に 50% の確率で解雇されるとし，2 つのくじを

$$L_1 = \begin{cases} 20\text{ 万円} & (\text{解雇されたとき}) \\ 0 & (\text{解雇されないとき}) \end{cases}, \quad L_2 = \begin{cases} 0 & (\text{解雇されたとき}) \\ 20\text{ 万円} & (\text{解雇されないとき}) \end{cases}$$

とする．いずれのくじからも確率 50% で賞金 20 万円が得られ，確率 50% で何も得られないにもかかわらず，通常は $L_1 \succ L_2$ が成立し，独立性公理が満たされない．ここで独立性公理が満たされないのは，くじ以外から得られる所得が雇用状態によって異なるためであり，そのためくじから得られる賞金に対する効用もまた雇用状態によって異なるからである．

もし雇用されたときに受け取る賃金を w 万円とすると，L_1 と L_2 はそれぞれ

$$\begin{cases} 20\text{ 万円} & (\text{解雇されたとき}) \\ w\text{ 万円} & (\text{解雇されないとき}) \end{cases}, \quad \begin{cases} 0 & (\text{解雇されたとき}) \\ (20+w)\text{ 万円} & (\text{解雇されないとき}) \end{cases}$$

という総所得を与える．何も持っていないときに獲得できる 20 万円は，w 万円持っているときに獲得できる 20 万円よりも効用を大きく増加させる．つまり，限界効用の観点からも L_1 が L_2 より望ましい．収入の一部のみを表すくじについては，独立性公理は成立しないが，このように総所得を表すくじについては，独立性公理が満たされる可能性がある．

5.3.6 Example (心理的な負の補完性) 帰結の集合が

$$C = \{\,\text{ベネチアへの旅行}, \text{ベネチアに関する映画を見る}, \text{家にいる}\,\}$$

で与えられるとする．通常は，$(1,0,0) \succ (0,1,0) \succ (0,0,1)$ が成立する．Mark Machina (1987, Journal of Economic Perspectives) は，

$$(0.99, 0, 0.01) \succ (0.99, 0.01, 0)$$

という選好を持つ人がいる可能性を紹介し，その場合は独立性公理が満たされないことを指摘した．

単純くじ $(0.99, 0, 0.01)$ は，

$$\begin{pmatrix} (1,0,0) & (0,0,1) \\ 0.99 & 0.01 \end{pmatrix}$$

という複合くじの単純化くじ $(0.99)(1,0,0) + (0.01)(0,0,1)$ に等しい．同様に，$(0.99, 0.01, 0)$ は，

$$\begin{pmatrix} (1,0,0) & (0,1,0) \\ 0.99 & 0.01 \end{pmatrix}$$

という複合くじの単純化くじ $(0.99)(1,0,0) + (0.01)(0,1,0)$ に等しい．

2 つの複合くじを比較すると，確率 0.99 で単純くじ (1,0,0) が実現する点では共通しており，確率 0.01 でそれぞれ $(0,1,0)$ と $(0,0,1)$ が実現する点で異なっている．すでに述べたように，通常は $(0,1,0) \succ (0,0,1)$ が成立し，もし独立性公理が満たされるならば，$(0.99)(1,0,0)+(0.01)(0,1,0) \succ (0.99)(1,0,0)+(0.01)(0,0,1)$ すなわち $(0.99, 0.01, 0) \succ (0.99, 0, 0.01)$ が成立するはずである．したがって，Machina が紹介した選好は独立性公理を満たしていない．

それでは，Machina が紹介したような選好を持つのはどのような人なのであろうか．Machina による説明では，その人は，通常は家にいることよりもベネチアに関する映画を見ることのほうが好ましいと考えているが，ベネチアへ旅行できなくなったという状況の中では，ベネチアに関する映画を見ることに苦痛を感じ，家にいるほうが望ましいと考えているのである．つまり C の要素の間には物理的な補完性は存在しないが，心の中は負の補完性があり，$(1,0,0)$ が実現するかどうかが，$(0,1,0)$ と $(0,0,1)$ の選好に影響を与えているのである．

5.3.7 Example (アレーのパラドックスの数値を変えた例) 賞品の集合が

$$C = \{110\text{ 万円}, 100\text{ 万円}, 0\text{ 円}\}$$

で与えられる場合に，次の 4 つのくじについて考えてみよう．

$$\begin{cases} L_1 = (0,1,0) \\ L_1' = (0.1, 0, 0.9) \end{cases}$$
$$\begin{cases} L_2 = (0, 0.01, 0.99) \\ L_2' = (0.001, 0, 0.999) \end{cases}$$

多くの人の選好関係 $\succsim$ は $L_1 \succ L_1'$ と $L_2' \succ L_2$ を満たすが，このような選好関係は独立性公理を満たさない．実際，$L_3 = (0,0,1)$ とすると，

$$L_2 = 0.01L_1 + 0.99L_3,$$
$$L_2' = 0.01L_1' + 0.99L_3.$$

よって，$L_1 \succ L_1'$ と $L_2 \succ L_2'$ がともに成立するならば，独立性公理は満たされない．

L_1 と L_1' を比較すると，L_1 は確実に 100 万円与えるのに対して L_1' が与える賞金は 110 万円のときもあれば 0 円のときもある．つまり L_1 は L_3 よりもリスクが小さい．他方，L_2 と L_2' に対応する複合くじは，いずれも確率 0.99 で L_3 を与え，この場合は賞金はゼロである．残りの確率 0.01 で，それぞれ L_1 と L_1' を与える．

$L_1 \succ L_1'$ と $L_2 \succ L_2'$ がともに成立するということは，確率 0.99 で賞金がゼロになることがわかると，人々はより大きなリスクをとることを厭わなくなるということである．このとき，独立性公理は満たされない．

5.4 期待効用定理

以上のような多くの反例があるにもかかわらず，実際には $\succsim$ が独立性公理を満たすと仮定することは多い．特に，$\succsim$ が独立性公理を満たすとき，次に述べるような期待効用定理 (Expected Utility Theorem) が成立し，$\succsim$ が期待効用の形で表現できるという利便性は大きい．

5.4.1 Theorem (期待効用定理 (Expected Utility Theorem)) $\succsim$ を $\mathscr{L}$ 上の 2 項関係とする．$\succsim$ が完備性，推移性，連続性，独立性公理を満たすことと，ある $(u_1, \ldots, u_N) \in \boldsymbol{R}^N$ が存

在して，任意の $L=(p_1,\dots,p_N)\in\mathscr{L}$ および $L'=(p'_1,\dots,p'_N)\in\mathscr{L}$ に対して，

$$L\succsim L'\Longleftrightarrow\sum_{n=1}^{N}p_nu_n\geq\sum_{n=1}^{N}p'_nu_n$$

が成立することは同値である．

この定理で，$\sum_{n=1}^{N}p_nu_n$ は，L が定める効用水準の期待値であるから，U は期待効用関数と呼ばれ，その存在を保証するこの定理は期待効用定理と呼ばれる．

ベクトル $(u_1,\dots,u_N)\in\boldsymbol{R}^N$ を所与としたとき，関数 $U:\mathscr{L}\to\boldsymbol{R}$ を任意の $L=(p_1,\dots,p_N)\in\mathscr{L}$ に関して

$$U(L)=\sum_{n=1}^{N}p_nu_n$$

と定義すると，

$$L\succsim L'\Longleftrightarrow U(L)\geq U(L')$$

が成立するので，$\succsim$ を表す効用関数はくじに関して線形である．線形性を要請しなければ，$\succsim$ を表す連続な効用関数の存在は，$\succsim$ の完備性，推移性，連続性のみで保証される．したがって，独立性公理が効用関数に課す条件とは，線形性に他ならないことを期待効用定理は主張する．

$\succsim$ の完備性，推移性，連続性，独立性公理が線形性を満たす U の存在を保証することの証明のステップは以下の通りである．

(i) $N<\infty$ かつ $\succsim$ が連続性を満たすことより，任意の L について $\bar{L}\succsim L$ を満たす $\bar{L}$ が存在する．また，任意の L について $L\succsim\underline{L}$ を満たす $\underline{L}$ が存在する．$\bar{L}$ と $\underline{L}$ は，それぞれ最も好ましいくじと最も嫌われるくじであると言える．

(ii) 任意の L に対して，

$$L\sim\alpha\bar{L}+(1-\alpha)\underline{L}$$

となるような唯一の $\alpha\in[0,1]$ が存在する．この α を $U(L)$ と書くことにする．

(iii) $\succsim$ が独立性公理を満たすことから，任意の $L\in\mathscr{L}$，$L'\in\mathscr{L}$ および $\alpha\in[0,1]$ について

$$U(\alpha L+(1-\alpha)L')=\alpha U(L)+(1-\alpha)U(L')$$

が成立する．

(iv) 任意の $n=1,\dots,N$ に対して，

$$u_n=U\left((0,\dots,0,1,0,\dots,0)\right)$$

と定義する．左辺の $(0,\dots,0,1,0,\dots,0)$ は，n 番目の要素だけが 1 で他はすべて 0 であるようなベクトルであり，n 番目の商品を確率 1 で獲得できるようなくじを表している．このとき，任意の L に対して，

$$U(L)=\sum_{n=1}^{N}p_nu_n$$

が成立する．

Exercise 5.4.1 以上の証明のステップのギャップを埋めよ．

Exercise 5.4.2 U が線形性を満たすなら，$\succsim$ は完備性，推移性，連続性，独立性公理を満たすことを証明せよ．

5.5　リスク回避度とその比較

5.5.1　確実同値額とリスク回避度の比較

$C \in \{\boldsymbol{R}, \boldsymbol{R}_+, \boldsymbol{R}_{++}\}$ とする．連続で厳密に増加関数であるようなすべての u からなる集合を $\mathscr{U}^0$，また，C に含まれるコンパクトな台を持つすべてのボレル確率測度からなる集合を $\mathscr{P}^*$ とする．

5.5.1 Proposition (確実同値額 (certainty equivalent)) 任意の $u \in \mathscr{U}^0$ と任意の $P \in \mathscr{P}$ について，

$$u(x) = \int_C u(z)dP(z)$$

を満たすような $x \in C$ がただ *1* つだけ存在する．このとき x は，P の u に対する確実同値額と呼ばれる．

Proof of Proposition 5.5.1

$$\bar{c} = \max \operatorname{supp} P \in C$$

$$\underline{c} = \min \operatorname{supp} P \in C$$

とする．このとき，

$$u(\underline{c}) \leq \int_C u(z)dP(z) \leq u(\bar{c})$$

が成立する．u が連続だから，中間値の定理より $u(x) = \int_C u(z)dP(z)$ を満たすような $x \in [\underline{c}, \bar{c}]$ が存在する．また u は厳密な増加関数であるから，そのような x は C 上でも一意に定まる．///

確実同値額は，$x = c(P, u) = u^{-1}\left(\int_C u(z)dP(z)\right)$ と書くことができる．

5.5.2 Definition $u_1 \in \mathscr{U}^0$，$u_2 \in \mathscr{U}^0$ とする．任意の $P \in \mathscr{P}^*$ について，$c(P, u_1) \leq c(P, u_1)$ が成立するならば，u_1 は u_2 と少なくとも同程度にリスク回避的である (u_1 is at least as risk aversre as u_2) と言う．

5.5.3 Remark u_1 と u_2 のうちの一方が，もう一方よりも必ず少なくとも同程度にリスク回避的であるとは限らない．u_1 が u_2 より少なくとも同程度にリスク回避的ではなく，かつ u_2 が u_1 と少なくとも同程度にリスク回避的でないような状況がありうる．次の例では，そのような状況が成立している．

5.5.4 Example $C = \{\boldsymbol{R}_+\}$ とし，2 つのくじ P_1 と P_2 を

$$P_1(\{1\}) = P_1(\{3\}) = \frac{1}{2}$$

$$P_2(\{3\}) = P_2(\{5\}) = \frac{1}{2}$$

と定義する．また，効用関数 u_1 と u_2 を

$$u_1(x) = \begin{cases} x & (0 \leq x < 2 \text{ のとき}) \\ \frac{1}{2}x + 1 & (2 \leq x \text{ のとき}) \end{cases}$$

$$u_2(x) = \begin{cases} x & (0 \leq x < 4 \text{のとき}) \\ \frac{1}{2}x + 2 & (4 \leq x \text{のとき}) \end{cases}$$

と定義する．このとき，くじ P_1 が与える期待効用は，それぞれ

$$\int_C u_1(z)dP_1(z) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{5}{2} = \frac{7}{4}$$

$$\int_C u_2(z)dP_1(z) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3 = 2$$

であり，したがって確実同値額はそれぞれ $c(P_1, u_1) = \frac{7}{4}$，$c(P_1, u_2) = 2$ であり，$c(P_1, u_1) < c(P_1, u_2)$ が成立する．同様にして，くじ P_2 が与える確実同値額を求めると，$c(P_2, u_1) = 4$，$c(P_2, u_2) = \frac{15}{4}$ であり，$c(P_2, u_1) > c(P_2, u_2)$ が成立する．

u_1 が u_2 と少なくとも同程度にリスク回避的であるためには，任意の $P \in \mathscr{P}$ について，$c(P, u_1) \leq c(P, u_1)$ が成立する必要がある．しかしこの例では，$P = P_1$ のときは $c(P, u_1) < c(P, u_2)$ が成立し，$P = P_2$ のときは $c(P, u_1) > c(P, u_2)$ が成立している．したがって，u_1 と u_2 のいずれについても，もう一方と少なくとも同程度にリスク回避的であるとは言えない．

もし u が恒等関数であるならば，

$$\int_C u(z)dP(z) = \int_C zdP(z) = P \text{の平均} = c(P, u)$$

が成立する．

5.5.5 Remark $u(x) = \alpha x + \beta$ の形をとる任意の効用関数 u に対して，$c(P, u)$ は P の平均である．

5.5.6 Definition $u \in \mathscr{U}^0$ とする．もし u が id と少なくとも同程度にリスク回避的であるならば，単に，u はリスク回避的であると言う．つまり任意の $P \in \mathscr{P}$ について，$c(P, u) \leq (P$ の平均$)$ が成立するならば，u はリスク回避的 (risk-averse) であると言う．

また任意の $P \in \mathscr{P}$ について，$c(P, u) \geq P$ の平均 が成立するならば，u はリスク愛好的 (risk-laving) であると言い，$c(P, u) = P$ の平均 が成立するならば，u はリスク中立的 (risk-neutral) であると言う．

5.5.7 Proposition u_1 が u_2 と少なくとも同程度にリスク回避的であることは，以下のことと同値である．任意の $x \in C$ と任意の $P \in \mathscr{P}$ について，$u_1(x) \leq \int_C u_1(z)dP(z)$ ならば $u_2(x) \leq \int_C u_2(z)dP(z)$ が成立する．

Exercise 5.5.1 14.7 命題を証明せよ．

関数 $U_1 : \mathscr{P} \to \boldsymbol{R}$ と $U_2 : \mathscr{P} \to \boldsymbol{R}$ を次のように定義する．

$$U_1(P) = \int_C u_1(z)dP(z)$$

$$U_2(P) = \int_C u_2(z)dP(z)$$

また δ_x は $\delta_x(\{x\}) = 1$ を満たす確率測度とする．このとき，u_1 が u_2 と少なくとも同程度にリスク回避的であるならば，$U_1(\delta_x) \leq U_1(P)$ のときは必ず $U_2(\delta_x) \leq U_2(P)$ が成立する．ここで，$U_1(\delta_x) \leq U_1(P)$ は $P \succsim_1 \delta_x$ を，$U_2(\delta_x) \leq U_2(P)$ は $P \succsim_2 \delta_x$ をそれぞれ意味している．これは，$\succsim_1$ と $\succsim_2$ が期待効用関数から導出されなかった場合にも意味を持つ表現になる．

5.5.2 確実同値額を使わないリスク回避度の比較方法

確実同値額を使ってリスク回避度を比較する場合，例えば u_1 が u_2 と少なくとも同程度にリスク回避的であると言うためには，すべての $P \in \mathscr{P}^*$ について $c(P, u_1) \leq c(P, u_1)$ が成立することを確かめなければならず，とても大変である．以下では，確実同値額を使わずにリスク回避度を比較するひとつの方法を見ていくことにしよう．

まず，$u_1 \in \mathscr{U}^0$ かつ $u_2 \in \mathscr{U}^0$ とし，関数 $\varphi : u_2(C) \to u_1(C)$ を $\varphi = u_1 \circ u_2^{-1}$，つまり $\varphi(z) = u_1(u_2^{-1}(z))$ と定義する．また，$P \in \mathscr{P}^*$ とする．ここで，$Q = P \circ u_2^{-1}$ と定義する，つまり，$u_2(C)$ の任意のボレル部分集合 A について，

$$Q(A) = P\left(u_2^{-1}(A)\right)$$

が成立すると定義しよう．

Exercise 5.5.2 Q は $u_2(C)$ 上のボレル確率測度であることを証明せよ．

5.5.8 Definition (変数変換の法則 (Change-of-Variable Formula)) $f : u_2(C) \to \boldsymbol{R}$ が，Q に関して積分可能であるとする．このとき $f \circ u_2 : C \to \boldsymbol{R}$ は P に関して積分可能であり，

$$\int_{u_2(C)} f(z)dQ(z) = \int_C (f \circ u_2)(x)dP(x)$$

が成立する．ここで $(f \circ u_2)(x) = f(u_2(x))$ である．

変数返還の法則の証明は，Patrick Billingsley,“Probability and Measure ,” 3rd ed. (Wiley-Interscience, 1995) を参考にせよ．

5.5.9 Proposition もし φ が凹関数ならば，

$$\int_{u_2(C)} \varphi(z)dQ(z) \leq \varphi\left(\int_{u_2(C)} zdQ(x)\right) \tag{5.2}$$

が成立する．また，もし φ が凸関数ならば，

$$\int_{u_2(C)} \varphi(z)dQ(z) \geq \varphi\left(\int_{u_2(C)} zdQ(x)\right) \tag{5.3}$$

が成立する．

まずは左辺を変数変換の公式を使って変形すると，

$$\begin{aligned}\int_{u_2(C)} \varphi(z)dQ(z) &= \int_C (\varphi \circ u_2)(x)dP(x) \\ &= \int_C u_1(x)dP(x).\end{aligned}$$

次に，恒等関数 $id(z) = z$ と変数変換の公式を使って右辺を変形すると，

$$\begin{aligned}\varphi\left(\int_{u_2(C)} zdQ(x)\right) &= \varphi\left(\int_{u_2(C)} id(z)dQ(x)\right) \\ &= \varphi\left(\int_C (id \circ u_2)(x)dP(x)\right) \\ &= \varphi\left(\int_C u_2(x)dP(x)\right)\end{aligned}$$

したがって，(5.2) は，

$$\int_C u_1(x)dP(x) \leq \varphi\left(\int_C u_2(x)dP(x)\right)$$

と同値である．$\varphi = u_1 \circ u_2^{-1}$ より，これは

$$u_1^{-1}\left(\int_C u_1(x)dP(x)\right) \leq u_2^{-1}\left(\int_C u_2(x)dP(x)\right)$$

と同値である．ここで，確実同値額の定義より，$u_1^{-1}\left(\int_C u_1(x)dP(x)\right) = c(P, u_1)$，$u_2^{-1}\left(\int_C u_2(x)dP(x)\right) = c(P, u_2)$ と書くことができるから，結局，(5.2) は $c(P, u_1) \leq c(P, u_2)$ と同値である．同様に (5.3) は $c(P, u_1) \geq c(P, u_2)$ と同値である．したがって命題はつぎのように書き換えることができる．

5.5.10 Proposition もし φ が凹関数ならば，u_1 は u_2 と少なくとも同程度にリスク回避的である．また φ が凸関数ならば，u_2 は u_1 と少なくとも同程度にリスク回避的である．

実は逆の関係も成立する．したがって，φ が凹関数であることと，u_1 が u_2 と少なくとも同程度にリスク回避的であることは同値であり，また φ が凸関数であることと，u_2 が u_1 と少なくとも同程度にリスク回避的であることは同値である．

逆の関係が成立することを示すには，u_1 が u_2 と少なくとも同程度にリスク回避的であるならば，任意の $z \in u_2(C)$，$z' \in u_2(C)$，および任意の $\alpha \in [0, 1]$ について，

$$\varphi(\alpha z + (1-\alpha)z') \geq \alpha\varphi(z) + (1-\alpha)\varphi(z')$$

が成立することを示せばいい．今，$x = u_2^{-1}(z)$，$x' = u_2^{-1}(z')$ と定義し，確率測度 P を $P(\{x\}) = \alpha$，$P(\{x'\}) = 1-\alpha$ と定義すると，

$$\begin{aligned} c(P, u_1) &= u_1^{-1}\left(\int_C u_1(x)dP(x)\right) \\ &= u_1^{-1}\left(\alpha u_1(x) + (1-\alpha)u_1(x')\right) \\ &= u_1^{-1}\left(\alpha\varphi(z) + (1-\alpha)\varphi(z')\right) \end{aligned}$$

が得られ，同様に

$$c(P, u_2) = u_2^{-1}\left(\alpha z + (1-\alpha)z'\right)$$

が得られる．ここで，u_1 が u_2 と少なくとも同程度にリスク回避的であるならば $c(P, u_1) \leq c(P, u_2)$ が成立するので，

$$u_1^{-1}\left(\alpha\varphi(z) + (1-\alpha)\varphi(z')\right) \leq u_2^{-1}\left(\alpha z + (1-\alpha)z'\right).$$

$\varphi = u_1 \circ u_2^{-1}$ より，

$$\alpha\varphi(z) + (1-\alpha)\varphi(z') \leq \varphi\left(\alpha z + (1-\alpha)z'\right)$$

したがって φ が凹関数であることが示せた．

効用関数 u がリスク回避的であることは，u が恒等関数 id と少なくとも同程度にリスク回避的であることと同値である．このとき，命題より φ は凸関数である．ところで $\varphi = u \circ id^{-1} = u$ であるから，効用関数 u がリスク回避的であることは，u が凸関数であることと同値であると言える．よって，次の命題が得られる．

5.5.11 Proposition 効用関数 u がリスク回避的であるとき，u は凸関数である．

5.5.3 絶対的リスク回避度

$\mathscr{U}^2$ を 2 回連続微分可能で，1 回微分が厳密に正であるような効用関数の集合とし，$u_1 \in \mathscr{U}^2$，$u_2 \in \mathscr{U}^2$ とする．このとき，$\varphi = u_1 \circ u_2^{-1}$ も 2 回連続微分可能である．

命題より，u_1 は u_2 と少なくとも同程度にリスク回避的であることは φ が凸関数，つまり $\varphi'' \le 0$ が成立することと同値である．ところで，φ は u_1 と u_2 を使って定義したのだから，$\varphi'' \le 0$ という条件を u_1 と u_2 を使って表すことはできるはずである．実は $\varphi'' \le 0$ は，任意の $x \in C$ について

$$-\frac{u_1''(x)}{u_1'(x)} \ge -\frac{u_2''(x)}{u_2'(x)}$$

が成立することと同値である．このことを以下で確認しよう．まず，$\varphi'(z)$ と $\varphi''(z)$ は次のように u_1 と u_2 を使って表すことができる．

$$\begin{aligned}\varphi'(z) &= (u_1 \circ u_2^{-1})'(z)\\ &= u_1'(u_2^{-1}(z))(u_2^{-1})'(z)\\ &= u_1'(u_2^{-1}(z))\frac{1}{u_2'(u_2^{-1}(z))}\\ &= \frac{u_1'(u_2^{-1}(z))}{u_2'(u_2^{-1}(z))}.\end{aligned}$$

両辺を微分して整理すれば，

$$\varphi''(z) = \frac{u_1'(u_2^{-1}(z))}{\left(u_2'(u_2^{-1}(z))\right)^2}\left(\frac{u_1''(u_2^{-1}(z))}{u_1'(u_2^{-1}(z))} - \frac{u_2''(u_2^{-1}(z))}{u_2'(u_2^{-1}(z))}\right)$$

ここで $\dfrac{u_1'(u_2^{-1}(z))}{\left(u_2'(u_2^{-1}(z))\right)^2} \ge 0$ より，$\varphi'' \le 0$ のとき，またそのときに限り，任意の $z \in u_2(C)$ について

$$\frac{u_1''(u_2^{-1}(z))}{u_1'(u_2^{-1}(z))} \le \frac{u_2''(u_2^{-1}(z))}{u_2'(u_2^{-1}(z))}$$

が成立する．この関係は $u_2^{-1}(z) = x$ と置き換えれば，任意の $x \in C$ について

$$-\frac{u_1''(x)}{u_1'(x)} \ge -\frac{u_2''(x)}{u_2'(x)}$$

が成立することと同値である．

5.5.12 Definition (アロー・プラットの絶対的リスク回避度) $u \in \mathscr{U}^2$，$x \in C$ とする．このとき

$$r(x,u) = -\frac{u''(x)}{u'(x)}$$

は，u の x におけるアロー・プラットの絶対的リスク回避度 (Arrow-Pratt measure of absolute risk aversion of u at x) と呼ばれる．

先の分析により，以下の命題が成立する．

5.5.13 Proposition u_1 は u_2 と少なくとも同程度にリスク回避的であることは，任意の $x \in C$ について

$$r(x,u_1) \ge r(x,u_2)$$

が成立することと同値である．

5.5.14 Definition $u \in \mathscr{U}^2$, $x \in C$, $x > 0$ とする．このとき

$$-\frac{u''(x)x}{u'(x)}$$

は，u の x における相対的リスク回避度 (Arrow-Pratt measure of relative risk aversion of u at x) と呼ばれる．

ところでアロー・プラットの絶対的リスク回避度は,

$$r(x,u) = -\frac{d}{dx}\log u'(x)$$

と書くことができる．右辺の $\frac{d}{dx}\log u'(x)$ は，$u'(x)$ が何パーセント上昇しているかを表している．マイナスの符号を考慮すると，$r(x,u)$ は，x が 1 単位増えたときに，限界効用が何パーセント減ったかを表している．

一方，相対的リスク回避度は,

$$-\frac{u''(x)x}{u'(x)} = \frac{du'(x)}{dx}\Big/\frac{u'(x)}{x}$$

と変形すると，右辺は $u'(x)$ の弾力性を表していることがわかる．つまり相対的リスク回避度は，x が 1 パーセント増えたときに，限界効用が何パーセント減ったかを表している．

また，アロー・プラットの絶対的リスク回避度の逆数

$$t(x,u) = -\frac{u'(x)}{u''(x)}$$

は，リスク許容度 (risk tolarance) と呼ばれる．

5.5.15 Definition The utility function u over money *exhibits decreasing (constant, increasing) relative risk aversion* if $r_R(x,u)$ is a decreasing (constant, increasing) function of x.

5.5.16 Proposition *For each monetary value $x > 0$ and cumulative distribution function F on $\boldsymbol{R}_+$, define*

$$c_x^R(F,u) = u^{-1}\left(\int u(zx)dF(z)\right).$$

Then u exhibits decreasing (constant, increasing) absolute risk aversion if and only if

$$\frac{x}{c_x^R(F,u)}$$

is a decreasing (constant, increasing) function of x.

5.6 さまざまな効用関数

5.6.1 Example (絶対的リスク回避度一定 (CARA) の効用関数) 効用関数 u の絶対的リスク回避度が一定ならば (CARA)，任意の $x \in \boldsymbol{R}$ について，$r(x,u) = \alpha$ となるような $\alpha > 0$ が存在する．つまり,

$$-\frac{u''(x)}{u'(x)} = \alpha$$

$$-\frac{d}{dx}\log u'(x) = \alpha$$

が得られ，辺々を積分すると，

$$\log u'(x) = -\alpha x + K_0$$

が得られる．ここで K_0 は積分定数である．両辺の指数をとると，

$$\begin{aligned} u'(x) &= \exp(-\alpha x + K_0) \\ &= \exp(K_0)\exp(-\alpha x) \\ &= K_1 \exp(-\alpha x) \end{aligned}$$

が得られる．ここで $K_1 = \exp(K_0) > 0$ と定義している．両辺を積分すると，K_2 を積分定数として，

$$u(x) = -\frac{1}{\alpha} K_1 \exp(-\alpha x) + K_2$$

が得られる．よって CARA 効用関数は，

$$u(x) = -\exp(-\alpha x)$$

または

$$u(x) = -\frac{1}{\alpha}\exp(-\alpha x)$$

と表現することができる．後者の表現は限界効用が $u'(x) = \exp(-\alpha x)$ となるために，この表現が用いられることもある．

5.6.2 Example (相対的リスク回避度一定 (CRRA) の効用関数) 効用関数 u が，相対的リスク回避度一定 (CRRA) ならば，任意の $x \in \boldsymbol{R}_{++}$ について，

$$-\frac{u''(x)x}{u'(x)} = \gamma$$

となるような $\gamma > 0$ が存在する．これは，

$$-\frac{u''(x)}{u'(x)} = \frac{\gamma}{x}$$

と書きかえることができる．このとき，右辺は双曲線を表しているので，CRRA の効用関数は，絶対的リスク回避度が双曲型 (Hyperbolic ARA) の効用関数の特殊ケースであると言える．上の表現を書き換えると，

$$-\frac{d}{dx}\log u'(x) = \frac{\gamma}{x}$$

が得られ，辺々を積分すると，

$$-\log u'(x) = \gamma \log x + K_0$$

が得られる．ここで K_0 は積分定数である．両辺の指数をとると，

$$\begin{aligned} u'(x) &= \exp(-\gamma \log x - K_0) \\ &= K_1 x^{-\gamma} \end{aligned}$$

が得られる．ここで $K_1 = \exp(-K_0) > 0$ と定義している．両辺を積分すると，

$$u(x) = \begin{cases} K_1 \dfrac{1}{1-\gamma} x^{1-\gamma} + K_2 & (\gamma \neq 1) \\ K_1 \log x + K_2 & (\gamma = 1) \end{cases}$$

が得られる．K_1，K_2 は任意だから，

$$u(x) = \begin{cases} \dfrac{1}{1-\gamma} x^{1-\gamma} & (\gamma \neq 1) \\ \log x & (\gamma = 1) \end{cases}$$

と書ける．別の表現としては，

$$u(x) = \begin{cases} x^{1-\gamma} & (\gamma < 1) \\ \log x & (\gamma = 1) \\ -\dfrac{1}{x^{1-\gamma}} & (\gamma > 1) \end{cases}$$

または，

$$u(x) = \frac{x^{1-\gamma} - 1}{1-\gamma}$$

と書くこともできる．

最後の表現は，$\gamma = 1$ のときは分母が 0 になり定義されないが，$\gamma \to 1$ のとき，$u(x) = \log x$ となる．また，$u'(x) = x^{-\gamma}$ だから，$u(x) = \dfrac{x^{1-\gamma} - 1}{1-\gamma}$ は γ の大きさにかかわらず，$u(1) = 0$ かつ $u'(1) = 1$ である．

5.6.3 Example (アフィンリスク許容度または線形リスク許容度 (linear risk tolarance) の効用関数)
任意の $\kappa \in \boldsymbol{R}$ と任意の $\eta \in \boldsymbol{R}$ について，$t(x, u) = \kappa x + \eta$ であるような $x \in C$ が存在するならば，効用関数 u は，アフィンリスク許容度，または，線形リスク許容度 (linear risk tolarance) を持つ．

すぐに確かめることができるように，CARA は $\kappa = 0$ かつ $\eta = \dfrac{1}{\alpha}$ のケースであり，CRRA は $\kappa = \dfrac{1}{\gamma}$ かつ $\eta = 0$ のケースである．また，$t(x, u) = \kappa x + \eta$ ならば $r(x.u) = \dfrac{1}{\kappa x + \eta}$ であるから，絶対的リスク回避度が双曲型 (Hyperbolic ARA) の効用関数の一つのケースでもある．

$\kappa > 0$ のとき，t の定義より $\kappa x + \eta > 0$ だから，$x > -\dfrac{\eta}{\kappa}$ である．よって，帰結の集合は $C = \left(-\dfrac{\eta}{\kappa}, \infty\right)$ である．まず，$\eta < 0$ ならば，$-\dfrac{\eta}{\kappa} > 0$ となる．$-\dfrac{\eta}{\kappa}$ は，最小消費水準 (minimum subsistence level) と呼ばれる．x が $\left(-\dfrac{\eta}{\kappa}, \infty\right)$ に属するとき，相対的リスク回避度は x の減少関数になる．また，$\eta > 0$ ならば，$-\dfrac{\eta}{\kappa} < 0$ となる．x が $\left(-\dfrac{\eta}{\kappa}, 0\right)$ に属するとき，相対的リスク回避度は x の増加関数になる．$\eta = 0$ ならば，相対的リスク回避度は一定となる．

Exercise 5.6.1 相対的リスク回避度に関する上述の主張を証明せよ．

$\kappa < 0$ のとき，t の定義より $\kappa x + \eta > 0$ だから，$x < -\dfrac{\eta}{\kappa}$ である．よって，帰結の集合は $C = \left(-\infty, -\dfrac{\eta}{\kappa}\right)$ である．2 次の効用関数 (quadratic utility function) は，このケースに該当する．実際，$\kappa = -1$ とすると，$r(x.u) = \dfrac{1}{\eta - x}$ より，

$$\frac{d}{dx} \log u'(x) = \frac{d}{dx} \log(\eta - x)$$

両辺の積分をとると，

$$\log u'(x) = \log(\eta - x) + K_0$$

が得られる．K_0 は積分定数である．両辺の指数をとると，

$$u'(x) = K_1(\eta - x)$$

が得られ，限界効用が線形であることが確認できる．ここで，$K_1 = \exp(K_0) > 0$ である．さらに両辺の積分をとると，K_2 を積分定数として，

$$u(x) = -\frac{K_1}{2}(\eta - x)^2 + K_2$$

が得られる．積分定数は任意だから，より一般的には，

$$u(x) = -(\eta - x)^2$$

または

$$u(x) = -\frac{1}{2}(\eta - x)^2$$

と書かれる．

X は賞金を表現する確率変数とし，$X : \Omega \to C$ とする．このとき期待効用水準は，次のように表現することができる．

$$\begin{aligned}
&\int_C u(x)d(P \circ X^{-1})(x)\\
=&\int_\Omega u(X(\omega))dP(\omega)\\
=&E(u(X))\\
=&E(-X^2 + 2\eta X - \eta^2)\\
=&-E(X^2) + 2\eta E(X) - \eta^2\\
=&-(E(X))^2 - \mathrm{Var}(X) + 2\eta E(X) - \eta^2
\end{aligned}$$

よって，期待効用水準が，X の平均と分散によって表現できることが確かめられる．

5.7　単純なポートフォリオ決定問題

帰結の集合を $C \in \{\boldsymbol{R}, \boldsymbol{R}_{++}\}$ とし，効用関数を $u \in \mathscr{U}^2$ とする．また，安全資産と危険資産の 2 種類の資産があるとする．安全資産は債権，危険資産は株式などと考えることができる．投資家の初期資産を $w \in C$ とする．

安全資産に 1 単位投資すると，1 期後に収益 $1 + r$ 単位が得られるとする．r は利子率で，ここでは $r = 0$ とする．一方，危険資産に 1 単位投資すると，1 期後に収益 z 単位が得られるとする．z は確率変数で，z が定める $\boldsymbol{R}$ 上の確率測度は，コンパクトな台 (support) を持つとする．

w のうち a を安全資産に投資し，b を危険資産に投資するとすると，1 期後には $az + b$ の収益が得られる．

ポートフォリオを組む段階でのプレイヤー期待効用は，

$$E\left(u(az + b)\right)$$

であり，彼が解く問題は，

$$\begin{aligned}
\max_{a,b} \quad & E\left(u(az + b)\right)\\
\text{subject to} \quad & a + b \leq w
\end{aligned}$$

である．$a<0$ や $b<0$ であっても構わない．ただし，$az+b\in C$ でなければならない．$C=\boldsymbol{R}_{++}$ ならば $aX+b>0$ であるが，$C=\boldsymbol{R}_{+}$ のときは $az+b\geq 0$ なので，正の確率で $az+b=0$ が成立する可能性がある．このとき，次の 1 階条件は実は成立しないので，以下の議論では，確率 1 で $az+b>0$ が成立すると仮定しよう．

$b=w-a$ を目的関数に代入すると，

$$E\left(u(az+(w-a))\right)=E\left(u(a(z-1)+w)\right)$$

であるから，上の式を a について最大化すればよい．

もし $a=a^*$ が最適であるならば，

$$\frac{d}{da}E\left(u(a^*(z-1)+w)\right)=0$$

を満たす．z が定める $\boldsymbol{R}$ 上の確率測度のサポートコンパクトなので，微分の期待値の順序を入れ替えることができる．よって

$$E\left(u'(a^*(z-1)+w)(z-1)\right)=0 \tag{5.4}$$

が成立する．$u''(\cdot)<0$ かつ $(z-1)^2>0$ なので，

$$E\left(u''(a^*(z-1)+w)(z-1)^2\right) \tag{5.5}$$

は負である．これより，$E\left(u'(a^*(z-1)+w)(z-1)\right)$ は a の減少関数であり，また (5.4) より，$a=a^*$ のときに 0 となることがわかる．

5.7.1 Remark $E(z)>1$ ならば $a^*>0$，$E(z)=1$ ならば $a^*=0$，$E(z)<1$ ならば $a^*<0$ が成立する．また，それぞれ逆も成立する．

5.7.2 Remark 効用関数が u_1，u_2 のときの，最適な危険資産保有量をそれぞれ a_1^*，a_2^* とする．このとき，u_1 が u_2 と少なくとも同程度にリスク回避的であるならば，$a_1^*\leq a_2^*$ が成立する．

5.7.3 Definition The utility function u over money *exhibits decreasing (constant, increasing) absolute risk aversion* if $r_A(x,u)$ is a decreasing (constant, increasing) function of x.

5.7.4 Proposition *For each monetary value x and cumulative distribution function F, define*

$$c_x^A(F,u)=u^{-1}\left(\int u(x+z)dF(z)\right).$$

Then u exhibits decreasing (constant, increasing) absolute risk aversion if and only if

$$x-c_x^A(F,u)$$

is a decreasing (constant, increasing) function of x.

5.7.5 Definition The *Arrow-Pratt measure of relative risk aversion* of u at $x>0$, denoted $r_R(x,u)$, is defined by

$$r_R(x,u)=-\frac{xu''(x)}{u'(x)}.$$

5.7.1 Comparison of Risks

5.7.6 Proposition *Let F and G be two cumulative distribution functions, then the following two conditions are equivalent.*

1. $F(x) \leq G(x)$ *for every* x.
2. *For every non-decreasing utility function u over money, we have* $\int u(x)dF(x) \geq \int u(x)dG(x)$.

When one (and hence both) of them holds, we say that F is first-order stochastically dominates G.

5.7.7 Proposition *Let F and G be two cumulative distribution functions such that* $F(\underline{x}) = G(\underline{x}) = 0$ *for some finite* $\underline{x}$, $F(\overline{x}) = G(\overline{x}) = 1$ *for another finite* $\overline{x}$, *and* $\int xdF(x) = \int xdG(x)$. *Then the following two conditions are equivalent.*

1. $\int_{\underline{x}}^{x} F(z)dz \leq \int_{\underline{x}}^{x} G(z)dz$ *for every* x.
2. *For every concave utility function u over money, we have* $\int u(x)dF(x) \geq \int u(x)dG(x)$.

When one (and hence both) of them holds, we say that F is second-order stochastically dominates G.

Both the first- and second-order stochastic dominance admit equivalent conditions in terms of random variables. In particular, the equivalent condition for the second-order stochastic dominance involves *mean-preserving spreads*, which are often used in applications.

関連図書

[1] avid M. Kreps, *Notes on the Theory of Choice* , Westview Press.

第6章 General Equilibrium Theory

6.1 Introduction to General Equilibrium Theory

Gerard Debreu's "Theory of Value" (Wiley and Sons) is an excellent introduction to general equilibrium theory.

6.1.1 Purpose and Motivation

- Gather consumers and producers in a unifying framework and analyze how the "price mechanism" will lead to an "equilibrium".
- Emphasize the analysis of the interaction between markets for different commodities.

6.1.2 Methodology

- Start from the description of the fundamentals (such as endowments, preference relations, and production possibilities) of the economy.
- Assume the price-taking behavior.
- Take a somewhat "abstract" and "mathematical" approach to the analysis of equilibria.

6.2 Economy, Efficiency, and Equilibrium

The following materials can also be found in 16.C of MWG.

6.2.1 An Economy

We assume that there are L commodities. We study an economy consists of I *consumers* and J *firms*. Each consumer $i = 1, \ldots, I$, is characterized by his *consumption set* $X_i \subset \boldsymbol{R}^L$ and a *preference relation* $\succsim_i$ defined on X_i. Each firm $j = 1, \ldots, J$ is characterized by its *production set* Y_j. The *endowments* in the economy for the L commodities is denoted by $\overline{\omega} \in \boldsymbol{R}^L$.

A *feasible allocation* of this economy is a vector $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ in $X_1 \times \ldots X_I \times Y_1 \cdots \times Y_J$ such that

$$\sum_{i=1}^{I} x_i = \overline{\omega} + \sum_{j=1}^{J} y_j.$$

What we shall call a "Walrasian allocation" is a feasible allocation of this economy led to by the "price (or competitive) mechanism". Other trading mechanisms would lead to other feasible allocations.

6.2.2 Pareto Efficiency

We shall judge the desirability of an allocation, and hence of the price and other mechanisms that lead to them, with respect to the following criterion on efficiency.

6.2.1 Definition A feasible allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is *Pareto efficient* if there is no feasible allocation $(x'_1, \ldots, x'_I, y'_1, \ldots, y'_J)$ such that $x'_i \succsim_i x_i$ for every i and $x'_i \succ_i x_i$ for some i.

6.2.3 A Private Ownership Economy

A private ownership economy is nothing but an economy with a list of specifications regarding who owns what. More specifically, a *private ownership economy* is defined, in addition to the economy as defined in Section 6.2.1, by the consumers' endowments $\omega_i \in \boldsymbol{R}^L$ for $i = 1, \ldots, I$ and shareholdings $\theta_{ij} \geq 0$ in the J firms for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. We assume that $\sum_{i=1}^{I} \omega_i = \overline{\omega}$ and $\sum_{i=1}^{I} \theta_{ij} = 1$ for every j. This list of specifications of the private ownership determines to whom various profits and revenues are paid.

6.2.4 Walrasian Equilibrium

We assume that there is one price for each good. A *price vector* thus belongs to $\boldsymbol{R}^L$.

6.2.2 Definition A feasible allocation $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ and a price vector p constitute a *Walrasian equilibrium* if:

1. For every j and every $y_j \in Y_j$, we have $p \cdot y_j \leq p \cdot y_j^*$.
2. For every i, $p \cdot x_i^* \leq p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j^*$ and, for every $x_i \in X_i$, if $p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j^*$, then $x_i^* \succsim_i x_i$.

An Walrasian equilibrium allocation $(x_1^*, \ldots . x_I^*, y_1^*, \ldots y_J^*)$ is a feasible allocation led to by the price "mechanism", though the definition does not give any concrete idea on what the mechanism is like. The price vector p determines the "exchange rate" between any two commodities. If the "value" of a commodity is defined to be what it can buy in terms of other commodities, the value of a commodity is nothing but its price. General equilibrium theory can then be said to be a theory of value with no explicit trading mechanism.

Another notion of an equilibrium, with no ownership specification, is useful when discussing efficiency properties of Walrasian equilibria.

6.2.3 Definition A feasible allocation $(x_1^*, \ldots . x_I^*, y_1^*, \ldots y_J^*)$ and a price vector p constitute a *price equilibrium with transfers* if:

1. For every j and every $y_j \in Y_j$, we have $p \cdot y_j \leq p \cdot y_j^*$.
2. For every i and every $x_i \in X_i$, if $p \cdot x_i \leq p \cdot x_i^*$, then $x_i^* \succsim_i x_i$.

It is easy to check that this definition is equivalent to Definition 16.B.4 of MWG. The above definition is closer to the definition of an equilibrium in Chapter 6 of Debreu's "Theory of Value".

Exercise 6.2.1 Prove that feasible allocation $(x_1^*, \dots . x_I^*, y_1^*, \dots y_J^*)$ and a price vector p constitute a Walrasian equilibrium if and only if they constitute a price equilibrium with transfers and $p \cdot x_i^* \leq p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j^*$ for every i.

6.3 Two Fundamental Theorems of Welfare Economics

The materials of this sections can also be found Section 16.D of MWG.

6.3.1 First Fundamental Theorem of Welfare Economics

6.3.1 Definition The pair $(X_i, \succsim_i)$ of the consumption set X_i and the preference relation $\succsim_i$ is *locally non-satiated* if, for every $x_i \in X_i$ and every $\varepsilon > 0$, there exists an $x_i' \in X_i$ such that $\|x_i' - x_i\| < \varepsilon$ and $x_i' \succ_i x_i$. We may also say, more simply, that the preference relation $\succsim_i$ is locally non-satiated.

6.3.2 Theorem (First Fundamental Theorem of Welfare Economics) *Suppose that the preference relations are locally non-satiated. If a feasible allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and some price vector constitute a price equilibrium with transfers, then $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ is Pareto efficient.*

6.3.2 Second Fundamental Theorem of Welfare Economics

The following definition is a weaker notion of an equilibrium, termed "quasi-equilibrium". Although it plays only a technical role in the second welfare theorem, it is worth presenting because it will appear again in the existence problem of a Walrasian equilibrium.

6.3.3 Definition A feasible allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and a price vector p constitute a *price quasi-equilibrium with transfers* if:

1. For every j and every $y_j \in Y_j$, we have $p \cdot y_j \leq p \cdot y_j^*$.

2. For every i and every $x_i \in X_i$, if $p \cdot x_i < p \cdot x_i^*$, then $x_i^* \succsim_i x_i$.

A price equilibrium with transfers is a price quasi-equilibrium with transfers. In general, the converse does not hold. Roughly speaking, however, if every consumer i can "survive" under p with an wealth smaller than $p \cdot x_i^*$, then the quasi-equilibrium is also an equilibrium. Note that if $p \cdot x_i^*$ is equal to the minimum wealth necessary for survival, then Condition 2 in Definition 6.3.3 is trivially met.

6.3.4 Theorem (Second Fundamental Theorem of Welfare Economics) *Suppose that the preference relations are convex and locally non-satiated and that Y_j is convex for every j. If a feasible allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ is Pareto efficient, then there exists a price vector p such that $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and p constitute a price quasi-equilibrium with transfers.*

6.4 Examples of Private Ownership Economies

6.4.1 An Edgeworth Box Economy

The materials in this section can also be found in Section 15.B of MWG.

An *Edgeworth Box economy* is an economy such that $L = 2$, $I = 2$, $X_1 = X_2 = \boldsymbol{R}^2_+$, $Y_1 = \cdots = Y_J = \{0\}$, and $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2) \in \boldsymbol{R}^2_{++}$. This is an *exchange economy* with two consumers and two goods. Since the feasibility condition for an allocation is reduced to $x_2 = \overline{\omega} - x_1$, the set of feasible allocations can be identified with a rectangle with length $\overline{\omega}_1$ and height $\overline{\omega}_2$. This economy is a simplest possible framework in which we can see how the prices co-ordinate different consumers' demands to arrive at a feasible allocation. Exercises 15.B.1 and 15.B.2 of MWG are routine but recommended.

6.4.2 A Robinson Crusoe Economy

The materials in this section can be found in Section 15.C of MWG.

A *Robison Crusoe economy* is an economy such that $L = 2$, $I = 1$, $J = 1$, $X_1 = \boldsymbol{R}^2_+$, and $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2) \in \boldsymbol{R}^2_+$. It is often considered that one of the two commodities is an input, whose endowment is positive, and the other is an output, whose endowment is zero. If we let the first commodity is the input and the second the output, then $\overline{\omega}$ lies on the positive part of the horizontal axis and Y_1 is included in the left half space of $\boldsymbol{R}^2$. The feasibility condition is reduced to $x_1 \in Y_1 + \{\overline{\omega}\}$. This economy is a simplest possible framework in which we can describe how the production and consumption decision are made separately and yet the price mechanism leads to a feasible allocation.

6.5 Excess Demand Function

The materials of this section can also be found Section 17.B of MWG.

In the rest of this lecture note, we consider an exchange economy ($Y_1 = \cdots = Y_J = \{0\}$) and assume that $X_i = \boldsymbol{R}^L_+$ or $X_i = \boldsymbol{R}^L_{++}$ for every i, and $\overline{\omega} = \sum_{i=1}^I \omega_i \in \boldsymbol{R}^L_{++}$. We also assume that the preference relations $\succsim_i$ are continuous, strictly convex, and strongly monotone.

The *excess demand function* of this exchange economy is a mapping $z : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ defined by

$$z(p) = \sum_{i=1}^{I} \left(x_i(p, p \cdot \omega_i) - \omega_i \right),$$

where each x_i is the demand function of consumer i. A price vector p is a Walrasian equilibrium price vector if and only if $z(p) = 0$.

6.5.1 Proposition *The excess demand function z has the following properties.*

Continuity *z is continuous.*

Homogeneity *z is homogeneous of degree zero.*

Walras' Law *$p \cdot z(p) = 0$ for every $p \in \boldsymbol{R}^L_{++}$.*

Boundedness from Below *z is bounded from below.*

Boundary Behavior *If a sequence* $p^1, p^2, \ldots$ *in* $\boldsymbol{R}^L_{++}$ *converges to a price vector* $p = (p_1, p_2, \ldots, p_L) \in \boldsymbol{R}^L_+$ *with* $p_\ell > 0$ *for some* ℓ *and* $p_\ell = 0$ *for some other* ℓ, *then*

$$\max\{z_1(p^n), \ldots, z_L(p^n)\} \to \infty$$

as $n \to \infty$, *where* $z_\ell(p^n)$ *is the* ℓ*-th coordinate of* $z(p^n)$.

6.6 Existence of a Walrasian Equilibrium

The materials of this sections can also be found Section 17.C of MWG.

6.6.1 Theorem *Under the assumptions stated in Section 6.5, there exists a Walrasian equilibrium.*

This is a consequence of the fixed point theorem, but the intermediate value theorem is sufficient for the case of $L = 2$, which admits many insightful graphical presentations.

6.7 Sonnenschein-Mantel-Debreu Theorem

The materials of this sections can also be found Section 17.E of MWG.

The SMD theorem asserts that if there are as many consumers as commodities, then the continuity, homogeneity, and Walras' law in Propositions 6.5.1 exhaust all the implications of consumers' utility maximization behavior on the excess demand function of an exchange economy over any compact subset of $\boldsymbol{R}^L_{++}$.

To see why the number of consumers matters, let's assume that the x_i are continuously differentiable. Then

$$Dz_i(p) = S_i(p, p \cdot \omega_i) - D_w x_i(p, p \cdot \omega_i) z_i(p, p \cdot \omega_i)^\top \in \boldsymbol{R}^{L \times L},$$

where $S_i(p, p \cdot \omega_i) \in \boldsymbol{R}^{L \times L}$ is the Slutsky substitution matrix and $z_i(p, p \cdot \omega_i) \in \boldsymbol{R}^L$ (a column vector) is the excess demand of consumer i. (Note that this notation is different from that of Proposition 6.5.1.) Then $Dz_i(p)$ is negative semi-definite on the linear subspace

$$\left\{v \in \boldsymbol{R}^L \mid p \cdot v = z_i(p, p \cdot \omega_i) \cdot v = 0\right\}.$$

Hence $Dz(p) = \sum_{i=1}^I Dz_i(p)$ is negative semi-definite on the linear subspace

$$\begin{aligned} & \bigcap_{i=1}^I \left\{v \in \boldsymbol{R}^L \mid p \cdot v = z_i(p, p \cdot \omega_i) \cdot v = 0\right\} \\ = \; & \left\{v \in \boldsymbol{R}^L \mid p \cdot v = z_1(p, p \cdot \omega_i) \cdot v = \cdots = z_I(p, p \cdot \omega_i) \cdot v = 0\right\}. \end{aligned}$$

If p is an equilibrium price vector, then the dimension of this linear subspace may be $L - I$ but not higher. Hence, if there are fewer consumers than commodities, then there is at least one direction of price variations along which $Dz(p)$ is negative semi-definite.

6.7.1 Theorem (Sonnenschein-Mantel-Debreu) *Let* $z : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ *be an arbitrary function that satisfies the continuity, homogeneity, and Walras law of Proposition 6.5.1. Let* C *be a compact subset of* $\boldsymbol{R}^L_{++}$. *Then there is an exchange economy consisting of* L *consumers whose excess demand function coincides with* z *on* C.

6.8 Generic Determinacy of Walrasian Equilibria

6.8.1 Regular Equilibrium

The materials of this and next sections can also be found Section 17.D of MWG.

In the rest of this lecture note, we assume that the consumers' demand functions are continuously differentiable.

6.8.1 Definition A Walrasian equilibrium price vector p is *regular* if $\operatorname{rank} Dz(p) = L-1$. An exchange economy is *regular* if all of its Walrasian equilibrium price vectors are regular.

Exercise 6.8.1 Show that the regularity is equivalent to each one of the following two conditions:

1. The column space of $Dz(p)$ is equal to the hyperplane normal to p going through the origin.
2. Define $\widehat{z} : \boldsymbol{R}^{L-1}_{++} \to \boldsymbol{R}^{L-1}$ by
$$\widehat{z}\left(\widehat{p}\right) = \left(z_1\left(\widehat{p}, 1\right), \dots, z_{L-1}\left(\widehat{p}, 1\right)\right),$$
for every $\widehat{p} \in \boldsymbol{R}^{L-1}_{++}$, where $z_\ell\left(\widehat{p}, 1\right)$ is the ℓ-th coordinate of $z\left(\widehat{p}, 1\right)$. Then the $(L-1) \times (L-1)$ matrix $D\widehat{z}\left(p_1/p_L, \dots, p_{L-1}/p_L\right)$ is invertible.

6.8.2 Proposition *For every regular Walrasian equilibrium price vector p there exists an $\varepsilon > 0$ such that if a price vector p' is not proportional to p and satisfies $\|p' - p\| < \varepsilon$, then p' is not a Walrasian equilibrium price vector.*

6.8.2 Genericity Analysis

We now consider a class of exchange economies parameterized by $q \in Q$, where Q is an open subset of $\boldsymbol{R}^S$ and S is a positive integer. Denote by $z(\cdot, q) : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ the excess demand function of the exchange economy of parameter $q \in Q$. This defines the *parameterized excess demand function* $z : \boldsymbol{R}^L_{++} \times Q \to \boldsymbol{R}^L$. Note that the domain of z has been expanded to include the *parameter space* Q.

6.8.3 Example We assume that the preference relations $\succsim_i$ $(i = 1, \dots, I)$ of all consumers and the endowments ω_i $(i = 2, \dots, I)$ of all consumers but the first one are prespecified. The economy is parameterized by the (strictly positive) endowments $\omega_1 \in \boldsymbol{R}^L_{++}$ of the first consumer. The parameter space Q is thus equal to $\boldsymbol{R}^L_{++}$.

6.8.4 Example We assume that the preference relations $\succsim_i$ $(i = 2, \dots, I)$ of all consumers but the first one and the endowments ω_i $(i = 1, \dots, I)$ of all consumers are prespecified. The preference relation $\succsim_1$ of the first consumer is represented by the Cobb-Douglas utility function $u_1(x_1) = x_{11}^a x_{21}^{1-a}$, where $x_1 = (x_{11}, x_{21})$ and $a \in (0, 1)$. The parameter space Q is thus equal to the open unit interval $(0, 1)$.

6.8.5 Definition The parametrization by Q is *regular* if the following condition is satisfied: the parameterized excess demand function $z : \boldsymbol{R}^L_{++} \times Q \to \boldsymbol{R}^L$ is continuously differentiable and, for every $(p, q) \in \boldsymbol{R}^L_{++} \times Q$, if p is a Walrasian equilibrium price vector of parameter q then $\text{rank} Dz(p, q) = L - 1$.

Since $Dz(p, q) = [D_p z(p, q)\ D_q z(p, q)]$, the regularity of the parameter space Q is a weaker requirement than the regularity of the exchange economy with every parameter $q \in Q$. It can be shown that Examples 6.8.3 and 6.8.4 are both regular parameterizations.

Given a parameter space Q, we say that a property holds for almost every exchange economy in Q if there exists an open and full-measure subset Q' of Q such that the property holds for every exchange economy in Q'.

6.8.6 Theorem *If the parametrization by Q is regular, then almost every economy in Q is regular.*

6.8.3 Comparative Statics Analysis

Given a parameter space Q and a parameterized excess demand function $z : \boldsymbol{R}^L_{++} \times Q \to \boldsymbol{R}^L$, define $\widehat{z} : \boldsymbol{R}^{L-1}_{++} \times Q \to \boldsymbol{R}^{L-1}$ by

$$\widehat{z}\left(\widehat{p}, q\right) = \left(z_1\left(\left(\widehat{p}, 1\right), q\right), \ldots, z_{L-1}\left(\left(\widehat{p}, 1\right), q\right)\right),$$

for every $(\widehat{p}, q) \in \boldsymbol{R}^{L-1}_{++} \times Q$. If $(\widehat{p}^*, 1)$ is a regular Walrasian equilibrium price vector of q^*, then $\text{rank} D_{\widehat{p}} \widehat{z}\left(\widehat{p}^*, q^*\right) = L - 1$ and hence the implicit function theorem implies that there exist an open subset V of $\boldsymbol{R}^{L-1}_{++}$, an open subset Q' of Q, and a continuously differentiable mapping $p : V \to Q'$ such that $(\widehat{p}^*, q^*) \in V \times Q'$ and, for every $(\widehat{p}, q) \in V \times Q'$, $(\widehat{p}, 1)$ is a regular Walrasian equilibrium price vector of q if and only if $p(q) = \widehat{p}$. The implicit function theorem also implies that

$$Dp(q^*) = -D_{\widehat{p}} \widehat{z}\left(\widehat{p}^*, q^*\right)^{-1} D_q \widehat{z}\left(\widehat{p}^*, q^*\right).$$