Incomplete Preference, Iterated Strict Dominance and Rationalizability *

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Abstract:
In his seminal paper, Epstein (1997) generalized to non-subjective utility model that (correlated) rationalizability is a consequence of common knowledge of rationality in finite normal form games, and showed related results of iterated deletion of strictly dominated strategies and a posteriori equlibrium. In this paper we extend his result to incomplete preference using incomplete type spaces. We also show that existence of maximal element with respect to dominance relation is sufficient for order independence of iteration of strictly dominance in infinite normal form games. This extends a result of Dufwenberg and Stegeman (2002).

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*Preliminary. Comments are welcome.

1. Introduction

Solution concept of rationalizability in noncooperative games was introduced by Perace [18] and Bernheim [4] as the consequence of common knowledge of rationality alone, and its decision theoretic foundation was done by Tan and Werlang [21]. On the other hand, its relation to equilibrium notion, a posteriori equilibrium which is a refinement of subjective correlated equilibrium, was shown by Brandenburger and Dekel [7]. These are generalized by Epstein [10] to non-subjective utility model which includes probabilistically sophisticated preference model, multiple-priors model or monotonic preference model and others.

As in most paper these author assumed that preference is complete, that is, any two acts are comparable: either one is more preferable or indifferent. But in real life completeness of preference is not satisfied often. This is stressed by many authors (see for instance Aumann [1], Bewley [5] and its references). The difficulty is that players cannot make decisions when one is not comparable to the other, and solution concepts relies on this. But in iterated deletion of strict dominated strategies and its counter part of (correlated) rationalizability, completeness of preference is not needed essentially. Rational player does not choose dominated strategies, so she is not needed what to choose but what not choose. So it is natural not to assume completeness of preference in this case.

In this paper we extend the results of Epstein [10] to incomplete preference model using incomplete type space. For our purpose, to define common knowledge, exhaustiveness of uncertainty of all events of type spaces which expresses 'beliefs about beliefs about beliefs. . . ' is not needed. It only needs for events for finite order level. As in Epstein [10] we define "knowledge" in the sense of Savage: an event is known if its complement is null. In the expected utility model this means the event has subjective probability 1, and in the model of Epstein this means a player is indifferent between two acts that are same except its complement. In our incomplete preference model this means a player cannot compare two acts which are same except its complement.

In this case of considering common knowledge, the assumption that preference is complete is not appropriate much more. When we define common knowledge from preferences, 'preferences about preferences about preferences $\cdots$' type hierarchies are needed, and we cannot mostly compare two acts over high level of preferences.

On the other hand, Dufwenberg and Stegeman [9] showed that iterated elimination of strictly dominated strategies is order dependant procedure, and gave a sufficient condition to be order independent: compactness of strategy set and upper-semicontinuity of payoff function is sufficient. We show that in our general setting, their argument on order independence can be applied. We show that the presence of maximal element with respect to dominance relation in maximal reduction is sufficient to be order independent.

We proceed as follows. In section 2, we define admissible preferences correspondence where preferences are not necessarily complete. In section 3, iterated deletion of strict dominance and rationalizability with these preferences are defined for not necessarily finite normal form games. And we and give a sufficient condition that order of iterated deletion of strict dominance does not matter and it coincides with (correlated) rationalizability. In section 4, we give decision theoretic foundation for iterated deletion of strict dominance and (correlated) rationalizability. In section 5, we give an equivalence of a-posteriori equilibrium and (correlated) rationalizability with incomplete preferences. Most proofs are relegated to appendix.

2. Admissible Preferences Correspondence

We define an admissible preferences correspondence where preferences are not necessarily complete as Epstein [10].

Definition 2.1. An act f on S is a mapping from S to a set of consequences $\mathcal{X}$. We denote by $F(S) := \mathcal{X}^S$ the set of acts on S. Let $\mathcal{PO}(S)$ be the set of (a) irreflexive, (b) transitive relations (preference order), $\succ$, on $F(S)$. That is, for $f, f', f'' \in F(S)$, $(a) f \not\succ f, (b) f' \succ f,$ and $f'' \succ f'$ implies $f'' \succ f$.

We say $\succ \in \mathcal{PO}(S)$ knows a subset S' of S if $f' \sim f$ for all $f, f' \in F(S)$ with $f = f'$ on S', where $\sim$ is the symmetric part of $\succ$ as usual. That is, $f' \sim f \overset{def}{\iff} f' \not\succ f$ and $f \not\succ f'$. Note that since $\succ \in \mathcal{PO}(S)$ is not nececcarily complete, its symmetric part $\sim$ is not transitive. Let $\mathcal{PO}(S|S')$ be the set of preferences, $\succ$, that know S'.

Let $\iota : S \hookrightarrow S'$ be an injective mapping for sets S, S'. We define its associated mapping $\iota^* : \mathcal{F}(S) \leftarrow \mathcal{F}(S')$ by restriction, and define its associated mapping $\iota_* : \mathcal{PO}(S) \to \mathcal{PO}(S')$ by $f' \iota_*(\succ) f \overset{def}{\iff} \iota^*(f') \succ \iota^*(f)$ for $\succ \in \mathcal{PO}(S), f, f' \in F(S')$. Let $f_0 \in \mathcal{F}(S' \setminus S)$. We define $(\cdot S f_0) : \mathcal{F}(S) \to \mathcal{F}(S')$ by

$$(\cdot S f_0)(f) = f S f_0 = \begin{cases} f \text{ on } S \\ f_0 \text{ on } S' \setminus S. \end{cases}$$

We define its associated mapping $(\cdot S f_0)^* : \mathcal{PO}(S') \to \mathcal{PO}(S)$ by

$$f' (\cdot S f_0)^*(\succ) f \overset{def}{\iff} (\cdot S f_0) f' \succ (\cdot S f_0) f \text{ for } \succ \in \mathcal{PO}(S'), f, f' \in F(S).$$

Let $\pi = \pi_S : S \times S' \to S$ be projection mapping. We define its associated mapping $\pi^* : F(S \times S') \leftarrow F(S)$ by $\pi^*(f) = f \circ \pi$, and its associated marginal mapping on $S, \pi_* : \mathcal{PO}(S \times S') \to \mathcal{PO}(S)$ by $f' \pi_*(\succ) f \overset{def}{\iff} \pi^*(f') \succ \pi^*(f)$ for $\succ \in \mathcal{PO}(S \times S'), f, f' \in F(S)$.

We consider a game with a set of players N, and let $i \in N$ be a player. Let S be a set which expresses uncertainty a player i faces, and let $\mathcal{P}^*_i(S)$ be a set of admissible preferences of player i with a set S of uncertainty. We say $\mathcal{P}^* = (\mathcal{P}^*_i)_{i \in N}$ an admissible preferences correspondence which is defined on class of sets if it satisfies the following conditios for each player $i \in N$ and each S.

Definition 2.2.

PREF 1. $\emptyset \neq \mathcal{P}_i^*(S) \subseteq \mathcal{PO}(S)$.

PREF 2. For an injective mapping $\iota : S \hookrightarrow S'$,

(a) $\iota_* \mathcal{P}_i^*(S) \subseteq \mathcal{P}_i^*(S')$,

(b) $(\cdot S f_0)^* \mathcal{P}_i^*(S'|\ S) \subseteq \mathcal{P}_i^*(S)$.

PREF 3. $\pi_* \mathcal{P}_i^*(S \times S') \subseteq \mathcal{P}_i^*(S)$, here π_* is marginal on S.

Here the set of admissible preferences that know a subset S' of S is defined by $\mathcal{P}_i^*(S|\ S') := \mathcal{PO}(S|\ S') \cap \mathcal{P}_i^*(S)$.

We allow preference to be incomplete, so it is very natural to assume PREF 1. If we assume completeness, PREF 1 means that player i can compare any two acts on S. $\mathcal{P}_i^*(\cdot)$ includes a model of preference introduced by Epstein and Wang [11] and Epstein [10]. They defined it as a subset of regular preferences $\mathcal{P}(S)$ for compact Hausdorff spaces S. Here $\mathcal{P}(S)$ is the set of utility functions over the set of Borel measurable functions from S to unit intervals satisfying regularity conditions; see Epstein and Wang [11] for details.

Epstein [10, subsection 4.2] gave examples of a model of preference $\mathcal{P}^*(S)$ as follows: (a) standard expected utility, (b) ordinal expected utility where vNM indices are not common knowledge but only preferences over pure strategy outcomes of the game are common knowledge, which was studied by Börgers [6], (c) probabilistic sophistication by Machina and Schmeidler [14], (d) multiple-priors utility by Gilboa and Schmeidler [12], (e) ε-contamination, (f) monotonic utility.

Remark 2.3. (a) In PREF 2, $\iota_* \mathcal{P}_i^*(S) \subseteq \mathcal{P}_i^*(S'|\ S)$.

(b) $(\cdot S f_0)^* \iota_* =$identity. So ι_* is injective, and $(\cdot S f_0)^*$ is surjective. So for a subset S' of S, the set of i's admissible preferences on S that know S is identified with the set of i's admissible preferences on S'. That is, $\mathcal{P}_i^*(S'|\ S) \cong \mathcal{P}_i^*(S)$.

(c) In PREF 3, we have (a) $\pi_* \mathcal{P}_i^*(S \times S') = \mathcal{P}_i^*(S)$, and (b) for $E \in 2^{S \times S'}, \pi_* \mathcal{P}_i^*(S \times S'|\ E) \subseteq \mathcal{P}_i^*(\pi(S \times S')|\ \pi E) = \mathcal{P}_i^*(S|\ \pi E)$, here $\pi : S \times S' \to S$ is projection.

3. Iterated Deletion of Strict Dominance

Dufwenberg and Stegeman [9] gave various examples that order matters in iterated deletion of strictly dominated strategies in infinite normal form games, and gave sufficient condition of order independence: compactness of strategy set and upper semicontinuity of payoff functions suffices to be order independent. In this section, we define dominance for incomplete preference which extends Epstein [10], and gave sufficient condition to be order independent in infinite games. We also define (correlated) rationalizability for incomplete preference and show that the same condition is sufficient that maximal reduction of iterated deletion of this dominance and this rationalizability are the same.

Definition 3.1. Let $G = (N, (A_i, r_i)_{i\in N})$ be a normal form game, here N is a finite set of players, A_i are strategy sets which are not necessarily finite, $r_i : A := \times_{i\in N} A_i \to \mathcal{X}$ are outcome functions and the game is common knowledge. Here $\mathcal{X}$ is a set of outcomes. We say $X = \times_{i\in N} X_i \subseteq A, X_i \subseteq A_i$, a paring of A, and let $P(A)$ be the set of parings of A. We denote $X_{-i} := \times_{j\neq i} X_j$. For an admissible preference on X_{-i}, $\succ_X \in \mathcal{P}_i^*(X_{-i})$, we consider $\succ_X$ as an irrefrexive, transitive relation on A_i by the following:for $a', a \in A_i$,

$$a' \succ_X a \overset{def}{\Longleftrightarrow} r_i(a', \cdot) \succ_X r_i(a, \cdot) \tag{3.1}$$

We say $a \in A_i$ is $\mathcal{P}_i^*$-*dominated* in $X \in P(A)$ if for each $\succ_X \in \mathcal{P}_i^*(X_{-i})$ there exist $a' \in X_i$ such that $a' \succ_X a$. This concept is a generalization of strict dominance.

For $X, X' \in P(A)$, we write $X \to X'$ if $X_i' \subseteq X_i$ for all $i \in N$, and each $a \in X_i \setminus X_i'$ is $\mathcal{P}_i^*$-dominated. We say $(M, (X^n)_{n\geqslant 0})$ is a *reduction* if $X^0 = A$, $X^n \to X^{n+1}$ for each $n \geqslant 0$, and $M = \bigcap_{n\geqslant 0} X^n$. We say $(M, (X^n)_{n\geqslant 0})$ is *maximal* if $M \to M'$ implies $M = M'$. For an example of reduction which is not maximal, see Dufwenberg and Stegeman [9, Example 3] which is a Cournot competition with outside wager.

For a paring $X \in P(A)$, we say i's strategy a_i is *never worse response* against $\succ_X \in \mathcal{P}_i^*(X_{-i})$ if

$$r_i(a', \cdot) \not\succ_X r_i(a, \cdot) \text{ for all } a' \in A_i.$$

Let denote by $NW(\succ_X)$ the set of all never worse responses against $\succ_X \in \mathcal{P}_i^*(X_{-i})$. For $x \in A_i, \succ_X \in \mathcal{P}_i^*(X_{-i})$, we denote by $\max(\cdot \trianglerighteq_X x)$ the strategies of player i which are never worse responses aginst $\succ_X$ and more preferable than or equall to x. That is, let define $y \trianglerighteq_X x$ for $y \in A_i$ if $y \succ_X x$ or $y = x$, and

$$\max(\cdot \trianglerighteq_X x) := \{z^* \in A|\ z^* \trianglerighteq_X x \text{ and } z \not\succ_X z^* \text{ for all } z \in A\}.$$

The following Proposition gives sufficient condition for the order independence of iterated deletion of $\mathcal{P}^*$-dominance which extends Dufwenberg and Stegeman [9, Theorem 1 (a)]. That is, existence of maximal element with respect to $\mathcal{P}^*$-dominance relation which is more preferable or equal to each element of maximal reduction is sufficient for the order independence.

Proposition 3.2. *Assume that for each maximal reduction* $(M, (X^n)_{n\geqslant 0})$*, each* $x \in M_i$*, each* $\succ_M \in \mathcal{P}_i^*(M_{-i}), \max(\cdot \trianglerighteq_M x) \neq \emptyset$*. Then maximal reduction is unique.*

Let $U_i(X)$ be the set of $\mathcal{P}_i^*$-undominated strategies in X of player i, and let $R_i(X)$ be the set of never worse responses against some $\succ_X \in \mathcal{P}_i^*(A_{-i}|\ X_{-i})$ of player i. And put

$U(X) := \bigtimes_{i\in N} U_i(X), R(X) := \bigtimes_{i\in N} R_i(X)$. For a mapping $\lambda : 2^A \to 2^A$, for $B \in 2^A$, we define $\lambda^0(B) = B, \lambda^n(B) = \lambda(\lambda^{n-1}(B))$, and

$$\mathcal{U} = \bigcap_{n\geqslant 0} U^n(A),\ \mathcal{R} = \bigcap_{n\geqslant 0} R^n(A).$$

$\mathcal{U}$ is the set of survivals of iterated deletion by $\mathcal{P}^*$-dominance (by countable process).

Remark 3.3. We see that from PREF 1, R_i is monotone, that is, $X \subseteq Y$ implies $R_i X \subseteq R_i Y$. So R and R^n is monotone for every n, implying $R\mathcal{R} \subseteq \mathcal{R}$.

We define $\mathcal{P}^*$-rationalizability for incomplete preferences.

Definition 3.4. The set of $\mathcal{P}^*$-*rationalizable strategy profiles* is the largest $Z \in P(A)$ such that for each player i, each $a_i \in Z_i$ is never worse response against some $\succ_Z \in \mathcal{P}_i^*(A_{-i}|\, Z_{-i})$. That is, largest Z such that $Z \subseteq RZ$.

The following Proposition shows that existence of maximal element with respect to $\mathcal{P}^*$-dominance relation which is more preferable or equals to each element of maximal reduction is sufficient that maximal reduction by iterated deletion of $\mathcal{P}^*$-dominance and $\mathcal{P}^*$-rationalizable strategy profiles are the same. Consider the following property about $X \in P(A)$:

$$\forall i \in N, \forall \succ_X \in \mathcal{P}_i^*(X_{-i}), \forall x \in A_i, \max(\cdot \succeqq x) \neq \emptyset. \tag{3.2}$$

Proposition 3.5. (a) *Assume (3.2) holds for $X = R^n(A)$ for each $n \geqslant 0$. Then we have $\mathcal{U} = \mathcal{R}$.*

(b) *Assume furthermore that (3.2) holds for $X = \mathcal{R}$. Then we have $U\mathcal{U} = \mathcal{U}$ iff $R\mathcal{R} = \mathcal{R}$.*

(c) *Assume furthermore that for each maximal reduction M, (3.2) holds for $X = M$. Then if $\mathcal{U}$ is a maximal reduction, any maximal reduction and $\mathcal{R}$ are the same $\mathcal{P}^*$-rationalizable strategy profiles.*

We consider an example for finite game of Epstein [10] ($\gamma = 0$ in his example).

	L	R
T	$1-\delta,\ .9$	$2,\ 1$
M	$1,\ 100$	$1,\ 1$
B	$2,\ .9$	$1-\delta,\ 1$

FIGURE 1. Example in Epstein, $\delta \geqslant 0$

Example 3.6. Let φ_i be the mapping from $\mathcal{P}_i^*(A_{-i})$ to the set of irreflexive, transitive relations on A_i defined by (3.1), and let $\bar{\mathcal{P}}_i^*(A_{-i}|\ X_{-i})$ be its image of $\mathcal{P}_i^*(A_{-i}|\ X_{-i})$ under φ_i, here $X_{-i} \subseteq A_{-i}$. Under this notation, let assume the following.

$$\bar{\mathcal{P}}_1^*(L) = \{B \succ M \sim T\},\ \bar{\mathcal{P}}_1^*(R) = \{T \succ M \sim B\},$$
$$\bar{\mathcal{P}}_1^*(LR) = \{T \succ M, B \succ M, B \sim T\} \cup \bar{\mathcal{P}}_1^*(LR|\ L) \cup \bar{\mathcal{P}}_1^*(LR|\ R),$$

$$\bar{\mathcal{P}}_2^*(T) = \bar{\mathcal{P}}_2^*(B) = \{R \succ L\},\ \bar{\mathcal{P}}_2^*(M) = \{L \succ R\},$$
$$\bar{\mathcal{P}}_2^*(TM) = \{L \sim R\} \cup \bar{\mathcal{P}}_2^*(TM|\ T) \cup \bar{\mathcal{P}}_2^*(TM|\ M), \dots,$$
$$\bar{\mathcal{P}}_2^*(TMB) = \{L \sim R\} \cup \bar{\mathcal{P}}_2^*(TMB|\ TM) \cup \bar{\mathcal{P}}_2^*(TMB|\ MB) \cup \dots$$
$$\cup\, \bar{\mathcal{P}}_2^*(TMB|\ T) \cup \bar{\mathcal{P}}_2^*(TMB|\ M) \cup \bar{\mathcal{P}}_2^*(TMB|\ B),$$

and $\bar{\mathcal{P}}_i^*(A_j|\ X_j) = \bar{\mathcal{P}}_i^*(X_j)$ for $(i,j) = (1,2),(2,1)$, all $X_j \subseteq A_j$. Then (T,R) is the survival of iterated deletion by $\mathcal{P}^*$-dominance and $\mathcal{P}^*$-rationalizable profile of strategy. When $\delta \geqslant 1$ this is different from other models in examples of Epstein [10].

4. Decision Theoretic Foundation

In this section, first we define incomplete type spaces for incomplete preferences as usual. Here type spaces are not complete means that they do not express exhaustive uncertainty of all events which expresses 'beliefs about beliefs about beliefs...'. It only expresses uncertainty of events of finite order level, and this is sufficient for our purpose. And next using these type spaces we define (repeated) common knowledge for finite order events and show that as a result of (repeated) common knowledge of rationality, a survival of iterated $\mathcal{P}^*$-dominance and a $\mathcal{P}^*$-rationalizable strategy profile attains with some assumptions. Here rationality means each player does not choose never worse response against his preference.

Definition 4.1. We consider a game and let fix a player i. Let $S_i^0 = S_i$ be a space of uncertainty that i faces of level 0. We define inductively as follows. For $n \geqslant 1, i \in N$,

$$T_i^n = \mathcal{PO}(S_i^{n-1}),$$
$$S_i^n = S_i^{n-1} \times T_{-i}^n = \dots = S_i \times T_{-i}^1 \times \dots \times T_{-i}^n.$$

S_i^{n-1} is player i's state space of uncertainty of level n consisting of other players' preferences up to level $n-1$, and T_i^n is player i's preferences of level n over its state space S_i^{n-1}. Let $\pi_i^n : S_i^{n-1} = S_i \times T_{-i}^1 \times \dots \times T_{-i}^{n-2} \times T_{-i}^{n-1} \to S_i \times T_{-i}^1 \times \dots \times T_{-i}^{n-2} = S_i^{n-2}$ be the projecition mapping, and let π_{i*}^n be its associated marginal mapping:

$$\pi_{i*}^n : T_i^n = \mathcal{PO}(S_i^{n-1}) \to \mathcal{PO}(S_i^{n-2}) = T_i^{n-1}.$$

Let define type space of player i, T_i, to be coherent types in $\bigtimes_{i \in N} T_i^n$. That is,

$$T_i = \underleftarrow{\lim}(T_i^n, \pi_{i*}^n) := \{(t_i^n) \in \bigtimes_{n=1}^{\infty} T_i^n|\ \pi_{i*}^n(t_i^n) = t_i^{n-1} \text{ for } n \geqslant 2\}.$$

And let $T := \times_{i\in N} T_i$ be the set of type spaces. Let $p^{(n)} : \times_{k=0}^{\infty} T_i^k \to \times_{k=0}^{n} T_i^k$ be projection mapping, and for $n \geqslant 1$, let $T_i^{(n)} := p^{(n)} T_i$, $T^{(n)} := \times_{i\in N} T_i^{(n)}$. Put $T_i^{(0)} := S_i$.

Let consider a normal form game $G = (N, (A_i, r_i)_{i\in N})$. In the game G, uncertainty player i faces is other player's strategies, so put $S_i := A_{-i}$. Let $\Omega = A \times T$, the set of states. A subset of Ω is called an event. Put $\Omega^{(n)} = A \times T^{(n)}, \Omega_i^{(n)} = A_{-i} \times T_i^{(n)}$, and $\Omega_{-i}^{(n)} = A \times T_{-i}^{(n)}$. We consider $\Omega^{(n)} \hookrightarrow \Omega$ by $E \mapsto E \times T^{>n}$, here $T^{>n} := \times_{i\in N} \times_{k>n} T_i^k$.

Definition 4.2. Let $\mathcal{E}$ be the set of finite level events, that is, $\mathcal{E} := \bigcup_{n\geqslant 0} 2^{\Omega^{(n)}} \subset 2^{\Omega}$. We define $K_i : \mathcal{E} \to \mathcal{E}$ as follows: for $E = E^{(n)} \times T^{>n} \in 2^{\Omega^{(n)}}$.

$$K_i E := \{(a,t) \in \Omega |\ t_i^{n+1} \in \mathcal{P}_i^*(S_i^n |\ E_{-i}^{(n)})\} \in 2^{\Omega^{(n+1)}},$$
$$KE := \cap_{i\in N} K_i E, \text{ here } E_{-i}^{(n)} = \times_{j\neq i} E_j^{(n)}.$$

Since (t_i^n) are coherent, this is well defined.

We define *repeated common knowledge* (see Morris [16]) by

$$RC(E) := KE \cap K(E \cap KE) \cap K(E \cap K(E \cap KE)) \cap \dots.$$

Formally putting $K^E(F) := K(E \cap F), RC(E) := \bigcap_{n\geqslant 1} (K^E)^n(E)$.

Remark 4.3. Consider the following properties. Let $E, F \in \mathcal{E}$.

(a) (monotonicity) $E \subseteq F$ implies $K_i E \subseteq K_i F$.
(b) (intersection) $K_i(E \cap F) = K_i E \cap K_i F$.

Then K_i satisfies monotonicity, but since preference is not complete, K_i does not satisfy intersection property. If preference is complete, then intersection property is satisfied. So in this case, repeated common knowledge and common knowledge are the same, that is, $RC(E) = \cap_{n\geqslant 1} K^n(E)$.

Definition 4.4. We say player i is $\mathcal{P}^*$-rational at $(a_i, t_i^1) \in A_i \times T_i^1 \hookrightarrow \Omega$ if $t_i^1 \in \mathcal{P}_i^*(A_{-i})$ and a_i is never worse response against t_i^1. Let Q_i be the event that player i is $\mathcal{P}^*$-rational, and let Q be the event that players are $\mathcal{P}^*$-rational. That is,

$$Q_i = \{(a,t) \in \Omega |\ t_i^1 \in \mathcal{P}_i^*(A_{-i}),\ a_i \in NW(t_i^1)\} \in 2^{\Omega^{(1)}},\ Q = \cap_{i\in N} Q_i.$$

For $E \in 2^{\Omega}$, let define $[E]^0 = \{a \in A |\ (a,t) \in E\}$. This is the set of profiles of strategies when the event E occurs.

The following Theorem extends Epstein [10, Theorem 6.3] to incomplete preferences.

Theorem 4.5. *Let $\mathcal{P}^*$ be an admissible preferences correspondence and assume that (3.2) holds for $X = R^n(A)(\forall n \geqslant 0), \mathcal{R}$ and M for each maximal reduction M, and $\mathcal{U}$ is a maximal reduction. Then the strategy profile $a^* \in A$ is $\mathcal{P}^*$-rationalizable iff a^* is a survival of a maximal reduction by iterated $\mathcal{P}^*$-dominance iff each player is $\mathcal{P}^*$-rational and this is* *repeated common knowledge. That is,* $[Q \cap RC(Q)]^0 = M = \mathcal{U} = \mathcal{R} = \mathcal{P}^*$*-rationalizable profiles of strategies, for each maximal reduction* M*.*

5. A Posteriori equilibrium

Brandenburger and Dekel [7] showed equivalence between (correlated) rationalizability and a posteriori equilibrium which is a refinement of subjective correlated equilibrium. And Epstein [10] generalized to a model of preference, $\mathcal{P}^*$. In this section, we show that the same argument apply to incomplete preferences. Let $G = (N, (A_i, r_i)_{i \in N})$ be a normal form game, henceforth we call A_i, player i's action set.

Definition 5.1. Let $\mathcal{P}^*$ be an admissible preferences correspondence. We call a tuple $(\Omega, (H_i, (\succ_i^\omega)_{\omega \in \Omega}, \sigma_i))_{i \in N}$ $\mathcal{P}^*$-a posteriori equilibrium, where (a) Ω is a state space. (b) H_i is an information partition of Ω. (c) $\succ_i^\omega$ is player i's admissible conditional preference on Ω at $\omega \in \Omega$ that knows $H_i(\omega)$. That is, $\succ_i^\omega \in \mathcal{P}_i^*(\Omega | \, H_i(\omega))$. (d) $\sigma_i : (\Omega, H_i) \to A_i$ is player i's measrable strategy mapping such that i's preference on other players' actions associated with i's conditional preference at ω by other players' strategy, σ_{-i}, is admissible. That is, $(\sigma_{-i})_*(\succ_i^\omega) \in \mathcal{P}_i^*(A_{-i})$ for all $\omega \in \Omega$. And it satisfies the following:

$$r_i(a_i, \sigma_{-i}(\cdot)) \not\succ_i^\omega r_i(\sigma_i(\omega), \sigma_{-i}(\cdot)) \text{ for all } a_i \in A_i. \tag{5.3}$$

We have the following Theorem extending Epstein [10, Theorem 5.1]. Proof proceed just as Epstein [10] or Brandenburger and Dekel [7].

Theorem 5.2. *Let* $\mathcal{P}^*$ *be an admissible preferences correspondence and assume that conditions of Theorem 4.5 are satisfied. Then* $a^* \in A$ *is* $\mathcal{P}^*$*-rationalizable iff* a^* *is a survival of a maximal reduction by iterated* $\mathcal{P}^*$*-dominance iff there exists* $\mathcal{P}^*$*-a posteriori equilibrium* $(\Omega, (H_i, (\succ_i^\omega)_{\omega \in \Omega}, \sigma_i)_{i \in N})$ *and* $\omega^* \in \Omega$ *such that* $a^* = \sigma(\omega^*)$*.*

6. Appendix

(Proof of Remark 2.3)

(a) Let $\succ \in \mathcal{P}_i^*(S)$, and let $f, f' \in \mathcal{F}(S')$ be such that $f = f'$ on S. Since $\iota^* f = f_{|S} = f'_{|S} = \iota^* f'$, we hav $f' \iota_*(\sim) f$.

(b) $\iota^*(\cdot S f_0)(f) = \iota^*(f S f_0) = f$. So $\iota^*(\cdot S f_0)$ is identity, implying $(\cdot S f_0)^* \iota_*$ is identity.

(c) (i) For $s' \in S'$, let define $\iota = \iota_{s'} : S \hookrightarrow S \times S'$ by $\iota(s) = (s, s')$. Then for $f \in \mathcal{F}(S)$, we have $\iota^* \pi^*(f)(s) = \pi^*(f)(s, s') = f(s)$. So $\iota^* \pi^*$ =identity, implying $\pi_* \iota_*$:=identity. And we have π_* surjective.

(ii) Let $\succ_i \in \mathcal{P}_i^*(S \times S' | \, E)$, and let $f, f' \in F(S)$ be such that $f = f'$ on πE. Then we have $\pi^* f = \pi^* f'$ on E implying $f' \pi_*(\sim_i) f$. So we have $\pi_*(\succ_i) \in \mathcal{P}_i^*(S | \, \pi E)$.

(Proof of Proposition 3.2) Let $(M, (X^n)_{n\geqslant 0})$ and $(M', (Y^m)_{m\geqslant 0})$ be maximal reductions. We prove that $\forall m, M \subseteq Y^m$ by induction on m. Assume that $M \subseteq Y^{m-1}$. Let $i \in N, x \in M_i, \succ_M \in \mathcal{P}_i^*(M_{-i})$ and let $z^* \in \max(\cdot \trianglerighteq_M x)$. Let $\iota^n : M_{-i} \hookrightarrow X_{-i}^n$ be injection, and put $\succ_{X^n} = \iota_*^n(\succ_M) \in \mathcal{P}_i^*(X_{-i}^n)$. We prove that $\forall z \in A_i, \forall n, z \not\succ_{X^n} z^*$. Assume $\exists z \in A_i, \exists n$ such that $z \succ_{X^n} z^*$. Then we have $z \succ_M z^*$, a contradiction. So we have $z^* \in \bigcap_{n\geqslant 0} X_i^n = M_i$. From maximality of M we have $z^* = x$. So for all $z \in A_i, z \not\succ_M x$. Put $\iota^{m-1} : M_{-i} \hookrightarrow Y_{-i}^{m-1}$ be injection and put $\succ_{Y^{m-1}} := \iota_*^{m-1}(\succ_M) \in \mathcal{P}_i^*(Y_{-i}^{m-1})$. Then we have $z \not\succ_{Y^{m-1}} x$ for all $z \in Y_i^{m-1}$ implying $x \in Y_i^m$. □

Lemma 6.1. *Assume that for every* $\succ_X \in \mathcal{P}_i^*(X_{-i}), x \in A_i$, $\max(\cdot \trianglerighteq_X x) \neq \emptyset$. *Then* $R(X) \subseteq X$ *iff* $U(X) = R(X)$.

(Proof of Lemma 6.1) Only if: $R_i(X) \subseteq U_i(X)$ is obvious. We prove $U_i(X) \subseteq R_i(X)$. Let $x \in U_i(X)$ and let $\succ_X \in \mathcal{P}_i^*(X_{-i})$ be such that $y \not\succ_X x$ for all $y \in X_i$. We prove that for every $y \in A_i$ this holds. (recall Remark 2.3 (b).) Assume that there exists $y' \notin X_i$ such that $y' \succ_X x$ and let $\bar{y} \in \max(\cdot \trianglerighteq_X y') \subseteq R_i(X) \subseteq X_i$. Then we have $\bar{y} \succ_X x$, a contradiction. □

(Proof of Proposition 3.5)

(a) By induction, using Lemma (6.1) for $X = R^{n-1}(A)$ we have $U^n(A) = UU^{n-1}(A) = UR^{n-1}(A) = RR^{n-1}(A)$. This implies $\mathcal{U} = \mathcal{R}$.

(b) Since $R\mathcal{R} \subseteq \mathcal{R}$, this follows from Lemma (6.1) and (a). (Since $U\mathcal{R} = R\mathcal{R}$, we have $R\mathcal{R} = \mathcal{R} \iff U\mathcal{R} = \mathcal{R} \iff U\mathcal{U} = \mathcal{U}$.)

(c) Let M be a maximal reduction. Then from Proposition 3.2, (a) and (b) we have $M = \mathcal{U} = \mathcal{R}$ and $R\mathcal{R} = \mathcal{R}$. Let Z be rationalizable profiles of strategies. Since $Z \subseteq A$ we have $Z \subseteq R^n Z \subseteq R^n(A)$ implying $Z \subseteq \cap_{n\geqslant 0} R^n(A) = \mathcal{R}$. So we have $Z = \mathcal{R}$. □

(Proof of Theorem 4.5) From Proposition 3.5, we only prove $[Q \cap RC(Q)]^0 = \mathcal{R}$. We prove $[Q \cap (K^Q)^{n-1}(Q)]^0 = R^n(A)$ by induction of n. Since $t_i^1 \in T_i^1 = \mathcal{P}_i^*(A_{-i})$ and $[Q]^0 = R(A)$, it is true for $n = 1$. Assume

$$[Q \cap (K^Q)^{n-1}(Q)]^0 = R^n(A). \tag{6.4}$$

We prove it for $n + 1$.

(a) We prove $[Q \cap (K^Q)^n Q]^0 \subseteq R^{n+1}(A)$. Let $(a^*, t) \in Q \cap (K^Q)^n Q = Q \cap KE$, here $E := Q \cap (K^Q)^{n-1} Q \subseteq \Omega^{(n)}$. Put $E^{(k)} = p^{(k)} E$. Then $t_i^{n+1} \in \mathcal{P}_i^*(S_i^{(n)} |\ E_{-i}^{(n)})$. And from Remark 2.3 (c)(ii) we have

$$\begin{aligned} t_i^n = (\pi_{i*}^{n+1})_* t_i^{n+1} &\in (\pi_{i*}^{n+1})_* \mathcal{P}_i^*(S_i^n |\ E_{-i}^{(n)}) \\ &\subseteq \mathcal{P}_i^*(\pi_i^{n+1}(S_i^n) |\ \pi_i^{n+1} E_{-i}^{(n)}) = \mathcal{P}_i^*(S_i^{n-1} |\ E_{-i}^{(n-1)}). \end{aligned}$$

And successively we have $t_i^1 \in \mathcal{P}_i^*(S_i^0|\ E_{-i}^0) = \mathcal{P}_i^*(A_{-i}|\ R^n(A)_{-i})$ by (6.4). Since $(a^*, t) \in Q_i$, we have $a_i^* \in NW(t_i^1)$ implying $a_i^* \in R_i(R^n(A))$. And we get $[Q \cap (K^Q)^n Q]^0 \subseteq R^{n+1}(A)$.

(b) We prove $R^{n+1}(A) \subseteq [Q \cap (K^Q)^n Q]^0$. Let $a^* \in R^{n+1}(A)$. Then for each player i,

$$\exists \succ_i^0 \in \mathcal{P}_i^*(A_{-i|}\ R^n(A)_{-i}) \text{ such that } a_i^* \in NW(\succ_i^0). \tag{6.5}$$

We prove that there exists $\bar{t}$ such that $(a^*, \bar{t}) \in Q_i \cap K_i(Q \cap (K^Q)^{n-1}Q)$. For $a \in R^n(A)$ there exists $t = t(a)$ such that $(a, t) \in Q \cap (K^Q)^{n-1}Q$ by (6.4). We define $t_i^{n+1} \in \mathcal{P}_i^*(S_i^{n-1}|\ E_{-i}^{(n)})$ as follows. Let

$$\sigma : A_{-i} \hookrightarrow S_i^n$$

be $\sigma(a_{-i}) = (a_{-i}, t_{-i}(a_{-i})) \in Q_{-i} \cap (K^Q)^{n-1}(Q)_{-i} = E_{-i}$ for $a_{-i} \in R^n(A)_{-i}$ and arbitrary for $a_{-i} \notin R^n(A)_{-i}$. Then $t_i^{n+1} := \sigma_*(\succ_i^0) \in \mathcal{P}_i^*(S_i^n)$. We prove

$$t_i^{n+1} \in \mathcal{P}_i^*(S_i^n|\ E_{-i}^{(n)}). \tag{6.6}$$

Let $f, f' \in \mathcal{F}(S_i^{n-1})$ be such that $f = f'$ on $E_{-i}^{(n)}$. Since $\sigma^* f = \sigma^* f'$ on $\sigma^{-1}(E_{-i}) = R^n(A)_{-i}$ and $\succ_i^0 \in \mathcal{P}_i^*(A_{-i}|\ R^n(A)_{-i})$, we have $\sigma^* f \sim_i^0 \sigma^* f'$ implying $f \sigma_*(\sim_i^0) f'$. So we have $t_i^{n+1} = \sigma_*(\succ_i^0) \in \mathcal{P}_i^*(S_i^n|\ E_{-i}^{(n)})$. Put $\bar{t} = (\pi_{i*}^{n+1,k}(t^{n+1}))_{k=2,\ldots,n+1}$, here $\pi_{i*}^{n+1,k} = \pi_{i*}^k \circ \ldots \circ \pi_{i*}^n \circ \pi_{i*}^{n+1}$. Since $\pi_i^{n+1,2} \circ \sigma(a_{-i}) = a_{-i}$ we have $t_i^1 = \pi_{i*}^{n+1,2}(t_i^{n+1}) = \pi_{i*}^{n+1,2} \circ \sigma_*(\succ_i^0) = \succ_i^0$. So from (6.5) and (6.6) we have $(a^*, \bar{t}) \in Q_i \cap K_i(E)$. $\square$

(Proof of Theorem 5.2) Only if: Put $\Omega := \mathcal{R}, \sigma_i(a) = a_i, H_i(a) = \{a_{-i}\} \times \mathcal{R}_{-i}$. We define $\succ_i^a \in \mathcal{P}_i^*(\Omega|\ H_i(a))$ as follows. For $a_i \in \mathcal{R}_i$, there exists

$$\succ_i \in \mathcal{P}_i^*(A_{-i}|\ \mathcal{R}_{-i}) \tag{6.7}$$

such that $a_i \in NW(\succ_i)$. Let $\iota_{a_i} : \mathcal{R}_{-i} \hookrightarrow \mathcal{R}$ be such that $\iota_{a_i}(a_{-i}) = (a_i, a_{-i})$ and let $(\cdot \mathcal{R}_{-i} \iota_{a_i}^* r_i) \circ \iota_{a_i} : F(\mathcal{R}) \to F(A_{-i})$ be such that $(\cdot \mathcal{R}_{-i} \iota_{a_i}^* r_i) \circ \iota_{a_i}^*(f) = (\iota_{a_i}^* f \mathcal{R}_{-i} \iota_{a_i}^* r_i) = f(a_i, \cdot) \mathcal{R}_{-i} r_i(a_i, \cdot) \in F(A_{-i})$ for $f \in F(\mathcal{R})$.

Put $\succ_i^a = (\iota_{a_i})_*(\cdot \mathcal{R}_{-i} \iota_{a_i}^* r_i)^*(\succ_i)$. Then $\succ_i^a \in \mathcal{P}_i^*(\mathcal{R}) = \mathcal{P}_i^*(\Omega)$ by PREF 2. We prove $\succ_i^a \in \mathcal{P}_i^*(\Omega|\ H_i(a))$. Let $f, f' \in F(\mathcal{R})$ be such that $f = f'$ on $H_i(a) = \{a_i\} \times \mathcal{R}_{-i}$ and put $g = (\cdot \mathcal{R}_{-i} \iota_{a_i}^* r_i) \iota_{a_i}(f), g' = (\cdot \mathcal{R}_{-i} \iota_{a_i}^* r_i) \iota_{a_i}(f')$. Then $g = g'$ on $\mathcal{R}_{-i}$. So from (6.7) we have $g' \sim_i g'$, implying $f' \sim_i^a f$. So we have that $\succ_i^a \in \mathcal{P}_i^*(\Omega|\ H_i(a))$. Since

$$\begin{aligned} & r_i(a_i', \sigma_{-i}(\cdot)) \not\succ_i^a r_i(\sigma_i(a), \sigma_{-i}(\cdot)) \\ \Longleftrightarrow\ & r_i(a_i', \sigma_{-i}(\cdot)) \mathcal{R}_{-i} r_i(a_i, \cdot) \not\succ_i r_i(a_i, \cdot) \mathcal{R}_{-i} r_i(a_i, \cdot) \\ \Longleftrightarrow\ & r_i(a_i', \cdot) \mathcal{R}_{-i} r_i(a_i, \cdot) \not\succ_i r_i(a_i, \cdot), \end{aligned}$$

from (6.7) we see that (5.3) is satisfied.

If: Put $A_i^* := \{\sigma_i(\omega)|\ \omega \in \Omega\}$. We prove $A^* \subseteq \mathcal{R}$. For this, we show that for $a_i^* \in A_i^*$, there exists $\succ_i \in \mathcal{P}_i^*(A_{-i}|\ A_{-i}^*)$ such that $a_i^* \in NW(\succ_i)$. Let $\omega^* \in \Omega$ be such that $\sigma_i(\omega^*) = a_i^*$. Put $\succ_i := (\sigma_{-i})_*(\succ_i^\omega) \in \mathcal{P}_i^*(A_{-i})$. For $f, f' \in F(A_{-i})$, if $f = f'$ on A_{-i}^* then we have $f \circ \sigma_{-i} = f' \circ \sigma_{-i}$, implying $\succ_i = (\sigma_{-i})_*(\succ_i^\omega) \in \mathcal{P}_i^*(A_{-i}|\ A_{-i}^*)$. And $r_i(a_i, \sigma_{-i}(\cdot)) \not\succ_i^\omega r_i(\sigma_i(\omega), \sigma_{-i}(\cdot)) \iff r_i(a_i, \cdot)\sigma_{-i} \not\succ_i^\omega r_i(a_i^*, \cdot)\sigma_{-i} \iff r_i(a_i, \cdot) \not\succ_i r_i(a_i^*, \cdot)$. So we have $a_i^* \in NW(\succ_i)$. $\square$

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