# Subjective Probability over a Subjective Decision Tree *

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**Abstract**

Since Savage's seminal work, a state space has been assumed as a primitive and as a complete description of the world. However, the decision maker may have subjective states in her mind, which are distinct from the primitive states, but relevant for her decisions. Dekel, Lipman and Rustichini (2001) derive a unique subjective state space from preference.

In a dynamic setting, a state space $S$ and a filtration $\{\mathcal{F}_t\}_{t=0}^T$ over $S$ has been taken as primitives. We derive a triplet $(S, \{\mathcal{F}_t\}_{t=0}^T, \mu)$ from preference, where $S$ is a subjective state space, $\{\mathcal{F}_t\}_{t=0}^T$ is a subjective filtration over $S$, and $\mu$ is a subjective probability over $S$. We also show the uniqueness of the representation.

*Keywords*: preference for flexibility, subjective state space, subjective probability, subjective decision tree.

*JEL classification*: D81

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# 1 Introduction

## 1.1 Motivation and Outline

A state space has been used as the standard tool for modeling uncertainties since Savage [13]. He takes the state space as a primitive, which requires the decision maker (DM) and the analyst (observer) to share the exactly same set of uncertainties. In some situations, however, it would be hard to describe states completely. The DM may have subjective states in her mind, which are distinct from the primitive states, but relevant for her decisions. Hence, a state space should be derived from preference rather than assumed as a primitive. The derivation of *subjective state spaces* is addressed by Kreps [6, 7] and Dekel, Lipman and Rustichini [1] (hereafter DLR).

In a dynamic setting, the standard modeling tool is a pair $(S, \{\mathcal{F}_t\}_{t=0}^T)$, where $S$ is a state space and $\{\mathcal{F}_t\}_{t=0}^T$ is a filtration over $S$. It has been taken as a primitive, which requires the DM and the analyst to share not only the state space, but also how uncertainties are resolving over time. However, the DM may have subjective uncertainties and anticipate they are gradually resolving over time. In this context, we are led to ask whether both $S$ and $\{\mathcal{F}_t\}_{t=0}^T$ can be subjective. The derivation of *subjective decision tree, or subjective filtration over a subjective state space* is the focus of the paper.

The following table summarizes our results and the relation to previous literature:

| | Subjective | | |
|---|---|---|---|
| | $S$ | $\{\mathcal{F}_t\}_{t=0}^T$ | $\mu$ |
| Savage | | | ✓ |
| Kreps, DLR | ✓ | | |
| This Paper | ✓ | ✓ | ✓ |

In Savage [13] and the dynamic counterparts, a state space $S$ and a filtration $\{\mathcal{F}_t\}_{t=0}^T$ over $S$ are primitives. They derive a probability $\mu$ over $S$ as a part of the representation. Kreps [6, 7] and DLR derive $S$ without assuming any objective state space. In this paper, $(S, \{\mathcal{F}_t\}_{t=0}^T)$ is derived from preference. Moreover, unlike DLR, we also provide a subjective probability $\mu$ over $S$. Thus, the triplet $(S, \{\mathcal{F}_t\}_{t=0}^T, \mu)$ can be subjective in this paper.

Kreps and DLR consider preference over menus (or opportunity sets) of alternatives, or menus of lotteries over alternatives. We consider preference over menus of menus of Anscombe-Aumann acts, $h : \Omega \to \Delta(Z)$, where $\Omega$ is a finite objective state space and $\Delta(Z)$ is the set of lotteries over an outcome space $Z$. Thus, there are two changes: (1) menus of menus rather than menus, and (2) acts rather than lotteries. Precisely, our domain is $\mathcal{P}(\mathcal{P}(\mathcal{H}))$, where $\mathcal{P}(X)$ means the set of non-empty subsets of $X$. Though there are some objective states, the DM may have other subjective states and complement the objective state space with them.

We have in mind the following (unmodeled) timing of decisions:

**Period** $0$**:** choose a menu of menus $x_0$.

**Period** $1^-$**:** receive a subjective signal $s_1$.

**Period** $1$**:** choose a menu $x_1$ out of the menu of menus $x_0$.

**Period** $2^-$**:** receive another subjective signal $s_2$.

**Period** $2$**:** choose an act $h$ out of the menu $x_1$.

**Period** $2^+$**:** an objective state is realized and the DM receives the lottery prescribed by $h$.

The DM might expect subjective uncertainties are resolved in the earlier stage. Or, conversely, she may anticipate they are resolved later. Our domain, in principle, can capture such distinctions about resolutions of subjective uncertainties.

We axiomatize preference on $\mathcal{P}(\mathcal{P}(\mathcal{H}))$ having the following representation: there exist a "full" state space $\mathcal{S} \equiv S \times \Omega$, a filtration $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ over $\mathcal{S}$, a probability measure $\mu$ on $\mathcal{S}$, and a mixture linear function $u : \Delta(Z) \to \mathbb{R}$ such that $U_0 : \mathcal{P}(\mathcal{P}(\mathcal{H})) \to \mathbb{R}$ represents preference, where

$$U_t(x_t) \equiv \mathbb{E}_\mu \left[ \sup_{x_{t+1} \in x_t} U_{t+1}(x_{t+1}) \middle| \mathcal{F}_t \right],\ t = 0, 1, 2,$$

and

$$\sup_{x_3 \in x_2} U_3(x_3) \equiv \mathbb{E}_\mu \left[ u(x_2) | \mathcal{F}_3 \right].$$

This is regarded as an axiomatic foundation of subjective probability over a subjective decision tree, $(S, \{\mathcal{F}_t\}_{t=0}^T, \mu)$. Moreover, we show uniqueness of the representation.

An interpretation of the representation is as follows: the DM behaves as if she anticipates the time line described above and is certain about risk preference, but not sure of beliefs about $\Omega$. In other words, subjective uncertainties are only related to beliefs about $\Omega$.

## 1.2 Related Literature

Kreps [6, 7] provides an axiomatic foundation for a subjective state space. DLR show uniqueness of the representation by imposing a richer structure on the domain. Let $\Delta(Z)$ be the set of lotteries over a finite set $Z$ of alternatives, and let $\mathcal{P}(\Delta(Z))$ be the set of non-empty subsets of $\Delta(Z)$. DLR consider $\mathcal{P}(\Delta(Z))$ as the domain. Though they have several different models, we focus on the additive representation with a non-negative measure, that is, $U : \mathcal{P}(\Delta(Z)) \to \mathbb{R}$ defined by

$$U(x) = \int_S \sup_{\beta \in x} u(\beta, s) \mathrm{d}\mu(s),$$

where $S$ is a state space, $\mu$ is a non-negative measure on $S$, and $u(\cdot, s) : \Delta(Z) \to \mathbb{R}$ is a state-dependent mixture linear function. They show that the set of ex-post preferences induced from each $u(\cdot, s)$ is uniquely determined from preference. This set is called the *subjective state space*.

Our domain $\mathcal{P}(\mathcal{P}(\mathcal{H}))$ can be regarded as an extension of the DLR domain $\mathcal{P}(\Delta(Z))$ in two directions. One is the hierarchical part, i.e., considering a menu of menus rather than a menu as a choice object. This direction is important for deriving subjective decision trees. The other direction is to admit objective states. That is, we consider $\mathcal{H}$ rather than $\Delta(Z)$. This extension is important for deriving meaningful subjective probabilities.

Rustichini [12] addresses the multi-period extension of DLR. Let $C$ be the set of consumptions and $C^\infty$ be the set of infinite consumption streams. His domain is $\mathcal{P}(C^\infty)$. He fails to derive a subjective decision tree, because it is essentially a static model in the sense that all subjective uncertainties are resolved in the next period. Modica [8] also considers a menu of menus within the Kreps's framework, i.e., his domain is $\mathcal{P}(\mathcal{P}(Z))$. By the same reason as Kreps [6], however, this framework cannot pin down the representation. Kraus and Sagi [5] take a different approach to address the problem by allowing incompleteness of preference.

We may be tempted to interpret $\mu$ associated with DLR's additive representation as the belief of the DM with regard to subjective uncertainties. However, $\mu$ is not unique because of the state-dependence of the ex-post utility functions. Precisely, there may exist distinct $(S, u(\cdot, s), \mu)$ and $(S', u'(\cdot, s'), \mu')$ representing the same preference but satisfying $\mu \neq \mu'$. Thus, their model fails to derive a subjective probability over a subjective state space.

The special nature of the domain $\mathcal{H}$ helps us to derive meaningful probabilities. Unlike preference on $\Delta(Z)$, a SEU preference on $\mathcal{H}$ has two components: risk preference $u$ and belief $p$ over $\Omega$. In our model, subjective uncertainties are related to beliefs over $\Omega$, and do not affect her risk preference $u$. This state-independence of risk preference is the reason why we can pin down a subjective probability.

The domains consisting of menus with some objective states are not new. Epstein [3] introduces the domain $\mathcal{P}(\mathcal{H})$. He provides non-Bayesian updating models by considering preference over acts from an objective state space $\Omega_1$ into $\mathcal{P}(\mathcal{H})$. Nehring [9] and Ghirardato [4] consider a set-valued Savage act, called opportunity act. Ozdenoren [10] adopts a set-valued AA act as a choice object. The difference between the domains in the above three literatures and $\mathcal{P}(\mathcal{H})$ is the timing of decisions. As has been pointed out, $\mathcal{P}(\mathcal{H})$ implicitly suggests the following: (1) the DM chooses a menu of acts, (2) after receiving a subjective signal, she chooses an act out of the menu, and (3) an objective state is realized and she receives the outcome prescribed by the act. Hence, in this domain, subjective states are realized first, and objective states are resolved later. In the other domains, this order is converse. That is, (1) the DM chooses an opportunity act, (2) an objective state is realized and she receives the menu prescribed by the act. (3) after receiving a subjective signal, she chooses an outcome out of the menu. Hence, the difference is related to in what order subjective and objective uncertainties are going to resolve.

Finally, notice that, on the sub-domain $\mathcal{P}(\mathcal{P}(\Delta(Z))) \subset \mathcal{P}(\mathcal{P}(\mathcal{H}))$, our representation collapses to the following functional form without any subjective states,

$$U(x_0) = \sup_{x_1 \in x_0} \sup_{\beta \in x_1} u(\beta),$$

which does not coincide with the DLR's additive representation. In the sense, our representation is not a generalization of DLR. As shown below, the generalization can be achieved by the similar argument to our main result if one of our main axioms is dropped.

# 2 Model

## 2.1 Domain

Let $\Omega$ be a finite objective state space with $\#\Omega = n$. Let $Z$ be a compact metric outcome space. Let $\Delta(Z)$ be the set of all Borel probability measures over $Z$ and $\mathcal{H}$ the set of all functions, $h : \Omega \to \Delta(Z)$, called Anscombe-Aumann acts (henceforth acts). Notice that $\Delta(Z)$ is a compact metric space under the weak convergence topology and $\mathcal{H}$ is compact metric under the product metric.

Let $\mathcal{P}(\mathcal{H})$ be the set of all non-empty subsets of $\mathcal{H}$. Generic elements are denoted by $x_1$, $y_1$, $\cdots$, and interpreted as menus or opportunity sets of acts. Endow $\mathcal{P}(\mathcal{H})$ with the Hausdorff topology. Details are relegated to Appendix A.

Let $\mathcal{P}(\mathcal{P}(\mathcal{H}))$ be the set of all non-empty subsets of $\mathcal{P}(\mathcal{H})$. Our domain is $\mathcal{D} \equiv \mathcal{P}(\mathcal{P}(\mathcal{H}))$. Generic elements are denoted by $x_0, y_0, \cdots$, and interpreted as menus of menus of acts. The assumptions about the timing of decisions are in Introduction. Endow $\mathcal{D}$ with the Hausdorff topology.

## 2.2 Axioms for Static Choice

Let $\succeq$ be a binary relation on $\mathcal{D}$. The following five axioms on $\succeq$ are formally identical to those of DLR, but are imposed here on $\mathcal{P}(\mathcal{P}(\mathcal{H}))$ rather than on $\mathcal{P}(\Delta(Z))$.

**Axiom 1 (Order):** $\succeq$ is complete and transitive.

**Axiom 2 (Continuity):** For all $x_0 \in \mathcal{D}$, $\{z_0 \in \mathcal{D} | x_0 \succeq z_0\}$ and $\{z_0 \in \mathcal{D} | z_0 \succeq x_0\}$ are closed.

**Axiom 3 (Strong Nondegeneracy):** There exist $l, l' \in \Delta(Z)$ such that $\{\{l\}\} \succ \{\{l'\}\}$.

Define the mixture

$$\lambda x_1 + (1-\lambda)x'_1 \equiv \{\lambda h + (1-\lambda)h' | h \in x_1, h' \in x'_1\},$$

for any $x_1, x'_1 \in \mathcal{P}(\mathcal{H})$ and $\lambda \in [0,1]$, and

$$\lambda x_0 + (1-\lambda)x'_0 \equiv \{\lambda x_1 + (1-\lambda)x'_1 | x_1 \in x_0, x'_1 \in x'_0\},$$

for any $x_0, x'_0 \in \mathcal{D}$ and $\lambda \in [0,1]$. The convex hull of $x_0$ or $x_1$, denoted by $\mathrm{co}(x_0)$ or $\mathrm{co}(x_1)$ respectively, is defined by these mixture operations.

**Axiom 4 (Independence):** For all $x_0, x'_0, x''_0 \in \mathcal{D}$ and for all $\lambda \in (0,1]$,

$$x_0 \succeq x'_0 \;\Leftrightarrow\; \lambda x_0 + (1-\lambda)x''_0 \succeq \lambda x'_0 + (1-\lambda)x''_0.$$

Independence is justified as in DLR. First, consider a mixture space $X$ and hypothetical preference $\succeq^*$ on the set of lotteries over menus of $X$, i.e., $\Delta(\mathcal{P}(X))$. Suppose that $\succeq^*$ satisfies the standard independence axiom, which implies,

$$\delta_x \succeq^* \delta_y \Leftrightarrow \lambda\delta_x + (1-\lambda)\delta_z \succeq^* \lambda\delta_y + (1-\lambda)\delta_z,$$

where $\lambda \in (0,1]$ and $\delta_x, \delta_y$ and $\delta_z$ denote the Dirac measures at $x, y, z \in \mathcal{P}(X)$, respectively. Furthermore, suppose that $\succeq^*$ satisfies timing indifference, i.e.,

$$\lambda\delta_x + (1-\lambda)\delta_z \sim^* \delta_{(\lambda x + (1-\lambda)z)}.$$

This condition is appealing if the DM is sure all ex-post preferences on $X$ are mixture linear, because, for any ex-post preference, she will end up with the same choice out of the above two menus. Consequently, Independence is justified on $\mathcal{P}(X)$. Now, since $\mathcal{H}$ and $\mathcal{P}(\mathcal{H})$ are mixture spaces, Independence on $\mathcal{P}(\mathcal{H})$ or $\mathcal{P}(\mathcal{P}(\mathcal{H}))$ is justified.

The next axiom says that a bigger menu is always weakly preferred.

**Axiom 5 (Monotonicity):** For all $x_0, x'_0 \in \mathcal{D}$, $x_0 \subset x'_0 \Rightarrow x'_0 \succeq x_0$.

This axiom is interpreted as preference for flexibility.

## 2.3 Axioms for Dynamic Choice

The axioms proposed here are related to the multiperiod setup. Thus, they have no counterparts in DLR.

For any $x_0 \in \mathcal{D}$, let

$$\mathrm{co}_1(x_0) \equiv \{\mathrm{co}(x_1) \in \mathcal{P}(\mathcal{H}) | x_1 \in x_0\}.$$

This set consists of the convex hulls of all menus in $x_0$. Notice that $\mathrm{co}_1(x_0)$ and $\mathrm{co}(x_0)$ are distinct objects.

**Axiom 6 (Ex-Post Randomization Indifference):** For all $x_0 \in \mathcal{D}$, $x_0 \sim \mathrm{co}_1(x_0)$.

This axiom expresses the assumption that the DM is sure her preference in period 2 is linear. Indeed, as long as she believes so, a menu $x_1$ and the convex hull $\mathrm{co}(x_1)$ should be indifferent in period 1 no matter what subjective signal she receives then. Hence, she should be indifferent between $x_0$ and $\mathrm{co}_1(x_0)$ in period 0.

Axiom 6 and some static axioms have a relation. Suppose that $\succeq$ satisfies Order, Continuity and Independence. Since the restricted preference on $\mathcal{P}(\mathcal{H})$ also satisfies all these axioms, we know $\{x_1\} \sim \{\text{co}(x_1)\} = \text{co}_1(\{x_1\})$ by DLR[1]. Ex-Post Randomization Indifference requires more than that.

For any $x_0 \in \mathcal{D}$, let

$$\text{cl}_1(x_0) \equiv \{\text{cl}(x_1) \in \mathcal{P}(\mathcal{H}) | x_1 \in x_0\},$$

where $\text{cl}(x_1)$ is the closure of $x_1$. Thus, $\text{cl}_1(x_0)$ is the set of the closures of all menus in $x_0$. Notice that $\text{cl}_1(x_0)$ and $\text{cl}(x_0)$ are distinct objects.

**Axiom 7 (Ex-Post Continuity):** For all $x_0 \in \mathcal{D}$, $x_0 \sim \text{cl}_1(x_0)$.

This axiom reveals that the DM is sure her preference in period 1 is continuous. Indeed, as long as she believes so, a menu $x_1$ and the closure $\text{cl}(x_1)$ should be indifferent in period 1. Hence, she should be indifferent between $x_0$ and $\text{cl}_1(x_0)$ in period 0.

The next axiom says that the DM prefers not to commit to a smaller menu as long as possible.

**Axiom 8 (Aversion to Commitment):** For all $x_0, x_0' \in \mathcal{D}$, $x_0' \cup \{\cup_{x_1 \in x_0} x_1\} \succeq x_0' \cup x_0$.

The axiom implicitly reveals that she is sure her preference in period 1 satisfies Monotonicity, i.e., bigger menu is always preferred. In other words, she anticipates preference for flexibility in period 1.

Define

$$O(h) \equiv \{h' \in \mathcal{H} | \{\{h(\omega)\}\} \succeq \{\{h'(\omega)\}\} \text{ for all } \omega\},$$

for $h \in \mathcal{H}$. Thus, $O(h) \in \mathcal{P}(\mathcal{H})$ is the set of acts dominated by $h$ state by state in terms of the restricted preference on $\Delta(Z)$. For each $x_1 \in \mathcal{P}(\mathcal{H})$, let $O(x_1) \equiv \cup_{h \in x_1} O(h) \in \mathcal{P}(\mathcal{H})$. Formally, the operation $O$ is a mapping from $\mathcal{P}(\mathcal{H})$ into itself. Let

$$O_1(x_0) \equiv \{O(x_1) \in \mathcal{P}(\mathcal{H}) | x_1 \in x_0\}.$$

That is, $O_1(x_0)$ is the image of $x_0$ under the operator $O : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$.

**Axiom 9 (Risk Preference Certainty):** For all $x_0 \in \mathcal{D}$, $x_0 \sim O_1(x_0)$.

Suppose that, though she has subjective uncertainties about her preferences in the future, the DM is sure of her risk preference on $\Delta(Z)$. If so, it should coincide with the restricted preference on $\Delta(Z)$. Hence, she does not care about the dominated acts in terms of the restricted preference on $\Delta(Z)$, which implies $x_1$ and $O(x_1)$ should be indifferent in period 1 no matter what subjective signal she receives then. Consequently, the DM should be indifferent between $x_0$ and $O_1(x_0)$ in period 0.

[1]DLR show that Order, Continuity and Independence on $\mathcal{P}(\Delta(Z))$ imply Indifference to Randomization, i.e., $x_1 \sim \text{co}(x_1)$ for any $x_1 \in \mathcal{P}(\Delta(Z))$.

# 3 Representations

In this section, we provide the main representation theorem.

## 3.1 Additive SEU Representations

First, we examine the "static" domain, that is, $\mathcal{P}(\mathcal{H})$. Consider the functional form $U_1 : \mathcal{P}(\mathcal{H}) \to \mathbb{R}$ defined by,

$$U_1(x_1) \equiv \int_{S_2} \sup_{h \in x_1} U_2(h, s_2) \mathrm{d}\mu_1(s_2)$$

and

$$U_2(h, s_2) \equiv \sum_{\omega \in \Omega} u(h(\omega))\mu_2(\omega, s_2),$$

where $S_2$ is a state space, $\mu_1$ is a finitely additive Borel probability measure over $S_2$, $\mu_2 : S_2 \to \Delta(\Omega)$ is a conditional probability system, and $u : \Delta(Z) \to \mathbb{R}$ is a mixture linear function.

Take any functional form $U_1$ with components $(S_2, \mu_1, \mu_2, u)$ as above. The preference $\succeq_{s_2}$ induced on $\mathcal{H}$ given $s_2 \in S_2$ is

$$h \succeq_{s_2} h' \;\Leftrightarrow\; U_2(h, s_2) \geq U_2(h', s_2).$$

Let $\mathbb{S}_2(S_2, \mu_2, u)$ be the set of all induced preferences, that is,

$$\mathbb{S}_2(S_2, \mu_2, u) \equiv \{\succeq_{s_2} | s_2 \in S_2\}.$$

Given $(S_2, \mu_1, \mu_2, u)$ with finite $S_2$, say that $\succeq_{s_2} \in \mathbb{S}_2(S_2, \mu_2, u)$ is *relevant* if there exist $x_1, y_1 \in \mathcal{P}(\mathcal{H})$ with $x_1 \not\sim y_1$ such that

$$\sup_{h \in x_1} U_2(h, s_2') = \sup_{h \in y_1} U_2(h, s_2')$$

for any $s_2' \in S_2$ with $\succeq_{s_2'} \neq \succeq_{s_2}$.

Now we are ready to define a representation of preference on $\mathcal{P}(\mathcal{H})$. The following is the analogue of the additive EU representation provided by DLR for menus of acts[2]:

**Definition 3.1.** *Preference $\succeq$ on $\mathcal{P}(\mathcal{H})$ admits an additive SEU representation if (i) there exists a functional form with components $(S_2, \mu_1, \mu_2, u)$ representing $\succeq$, (ii) every state $s_2 \in S_2$ is relevant, and (iii) $\succeq_{s_2} \neq \succeq_{s_2'}$ if $s_2 \neq s_2'$.*

An interpretation of this representation is as follows: the DM behaves as if she is certain about risk preference $u$, but not sure of beliefs about $\Omega$. This subjective uncertainty is captured by $\mu_2 : S_2 \to \Delta(\Omega)$.

[2]Details about the additive EU representations are provided in Section 6.2, Definition 6.1.

### 3.2 Second-order Additive SEU Representations

Now we describe a representation of preference on the "dynamic" domain $\mathcal{P}(\mathcal{P}(\mathcal{H}))$. Consider the functional form $U_0 : \mathcal{D} \to \mathbb{R}$ defined by,

$$U_0(x_0) \equiv \int_{S_1} \sup_{x_1 \in x_0} U_1(x_1, s_1) \mathrm{d}\mu_0(s_1),$$

where $S_1$ is a state space, $\mu_0$ is a finitely additive Borel probability measure over $S_1$, and each $U_1(\cdot, s_1) : \mathcal{P}(\mathcal{H}) \to \mathbb{R}$ is a functional form with components $(S_2(s_1), \mu_1(\cdot, s_1), \mu_2(\cdot, s_1), u)$, where $u$ is common across $s_1$.

For each $s_1 \in S_1$, the preference $\succeq_{s_1}$ induced on $\mathcal{P}(\mathcal{H})$ is defined by $U_1(\cdot, s_1)$. Let $\mathbb{S}_1(S_1, U_1)$ be the set of all induced preferences, that is,

$$\mathbb{S}_1(S_1, U_1) \equiv \{\succeq_{s_1} | s_1 \in S_1\}.$$

Relevance of $\succeq_{s_1}$ can be defined as above.

**Definition 3.2.** *Preference $\succeq$ on $\mathcal{D}$ admits a second-order additive SEU representation if (i) there exists a functional form with components $(S_1, U_1, \mu_0, u)$ representing $\succeq$, (ii) every $\succeq_{s_1}$ admits an additive SEU representation, (iii) every $\succeq_{s_1}$ is relevant, and (iv) $\succeq_{s_1} \neq \succeq_{s_1'}$ if $s_1 \neq s_1'$.*

An interpretation is as follows: The DM behaves as if she is certain about risk preference $u$, but not sure of future beliefs about $\Omega$. Thus, subjective uncertainty concerns beliefs about $\Omega$. However, unlike $\mathcal{P}(\mathcal{H})$, where the DM receives a signal once, she may receive subjective signals twice. In other words, subjective uncertainties may not be realized at once, but resolving gradually over time.

Any second-order additive SEU representation $(S_1, U_1, \mu_0, u)$ has the recursive form as in Introduction. Define $S_2 \equiv \cup_{s_1 \in S_1} S_2(s_2)$, $S \equiv S_1 \times S_2$ and $\mathcal{S} \equiv S \times \Omega$. We can regard $\mu_0$ on $S_1$ as the marginal distribution. The conditional probability systems $\mu_1 : S_1 \to \Delta(S_2)$ and $\mu_2 : S_1 \times S_2 \to \Delta(\Omega)$ and the marginal $\mu_0$ jointly determine the Borel probability measure $\mu$ over $\mathcal{S}$. The set $\mathcal{S}$ redefined as the support of $\mu$ is called the *subjective state space*. The *subjective filtration* $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ over $\mathcal{S}$ is defined by

$$\begin{aligned}
\mathcal{F}_0 &\equiv \{\mathcal{S}\}, \\
\mathcal{F}_1 &\equiv \{(\{s_1\} \times S_2 \times \Omega) \cap \mathcal{S} | s_1 \in \mathrm{supp}(\mu_0)\}, \\
\mathcal{F}_2 &\equiv \{(\{(s_1, s_2)\} \times \Omega) \cap \mathcal{S} | s_1 \in \mathrm{supp}(\mu_0), s_2 \in \mathrm{supp}(\mu_1(s_1))\}, \\
\mathcal{F}_3 &\equiv \{\{(s_1, s_2, \omega)\} | s_1 \in \mathrm{supp}(\mu_0), s_2 \in \mathrm{supp}(\mu_1(s_1)), \omega \in \mathrm{supp}(\mu_2(s_1, s_2))\},
\end{aligned}$$

where $\mathrm{supp}(\nu)$ means the support of a measure $\nu$. Then, the second-order additive representation can be rewritten as the recursive form with $(\mathcal{S}, \{\mathcal{F}_t\}_{t=0}^3, \mu)$, that is,

$$U_t(x_t) \equiv \mathbb{E}_\mu \left[ \sup_{x_{t+1} \in x_t} U_{t+1}(x_{t+1}) \middle| \mathcal{F}_t \right], \ t = 0, 1, 2,$$

and

$$\sup_{x_3 \in x_2} U_3(x_3) \equiv \mathbb{E}_\mu \left[ u(x_2) | \mathcal{F}_3 \right].$$

The following is the main theorem:

**Theorem 3.1.** *The following statements are equivalent:*

**(a)** *Preference $\succeq$ on $\mathcal{D}$ satisfies Order, Continuity, Strong Nondegeneracy, Independence, Monotonicity, Ex-Post Randomization Indifference, Ex-Post Continuity, Aversion to Commitment, and Risk Preference Certainty.*

**(b)** *Preference $\succeq$ on $\mathcal{D}$ admits a second-order additive SEU representation.*

# 4 Uniqueness

First consider $\mathcal{P}(\mathcal{H})$ and an additive SEU representation.

**Definition 4.1.** *An additive SEU representation $U_1 : \mathcal{P}(\mathcal{H}) \to \mathbb{R}$ with $(S_2, \mu_1, \mu_2, u)$ is canonical if (i) $S_2 \subset \Delta(\Omega)$, (ii) $\mu_1 \in \Delta(S_2)$, (iii) $\mu_2 : S_2 \to \Delta(\Omega)$ is the identity map, and (iv)*

$$U_1(x_1, \mu_1) \equiv \int_{S_2} \sup_{h \in x_1} U_2(h, p) \mathrm{d}\mu_1(p)$$

*and*

$$U_2(h, p) \equiv \sum_{\omega \in \Omega} u(h(\omega)) p(\omega), \text{ for each } p \in S_2.$$

Given any additive SEU representation $U_1$ with components $(S_2, \mu_1, \mu_2, u)$, there exists also a canonical representation. Let

$$\mathbb{P}(S_2, \mu_2) \equiv \{\mu_2(s_2) \in \Delta(\Omega) | s_2 \in S_2\}.$$

Let $\mu_1^* \in \Delta(\mathbb{P}(S_2, \mu_2))$ be the probability measure induced by $\mu_2 : (S_2, \mu_1) \to \Delta(\Omega)$. Then $(\mathbb{P}(S_2, \mu_2), \mu_1^*, \mathrm{id}, u)$ is canonical, where id means the identity map.

Next consider $\mathcal{D}$ and a second-order additive SEU representation.

**Definition 4.2.** *A second-order additive SEU representation $U_0 : \mathcal{D} \to \mathbb{R}$ with $(S_1, U_1, \mu_0, u)$ is canonical if (i) $S_1 \subset \Delta(\Delta(\Omega))$, (ii) $\mu_0 \in \Delta(S_1)$, (iii) $U_1(\cdot, \mu_1)$ is a canonical SEU representation for each $\mu_1 \in S_1$, and (iv)*

$$U_0(x_0) \equiv \int_{S_1} \sup_{x_1 \in x_0} U_1(x_1, \mu_1) \mathrm{d}\mu_0(\mu_1).$$

Given any second-order additive SEU representation $U_0$ with components $(S_1, U_1, \mu_0, u)$, there exists a canonical representation. For each additive SEU representation $U_1(\cdot, s_1)$ with $(S_2(s_1), \mu_1(\cdot|s_1), \mu_2(\cdot, s_1), u)$, let

$$\mathbb{P}(S_2(s_1), \mu_2(s_1)) \equiv \{\mu_2(s_2, s_1) \in \Delta(\Omega) | s_2 \in S_2(s_1)\}.$$

Let $\mu_1^*(s_1) \in \Delta(\mathbb{P}(S_2(s_1), \mu_2(s_1)))$ be the probability measure induced by $\mu_2(s_1) : (S_2(s_1), \mu_1(s_1)) \to \Delta(\Omega)$. Then, $U_1^* : \mathcal{P}(\mathcal{H}) \to \mathbb{R}$ defined by,

$$U_1^*(x_1, \mu_1) \equiv \int_{\mathbb{P}(S_2(s_1), \mu_2(s_1))} \sup_{h \in x_1} U_2(h, p) \mathrm{d}\mu_1^*(p, s_1),$$

is canonical.

Define

$$\mathbb{P}(S_1, U_1) \equiv \{\mu_1^*(s_1) \in \Delta(\Delta(\Omega)) | s_1 \in S_1\}.$$

Let $\mu_0^* \in \Delta(\mathbb{P}(S_1, U_1))$ be the probability measure induced by $\mu_1 : (S_1, \mu_0) \to \Delta(\Delta(\Omega))$. Then $(\mathbb{P}(S_1, U_1), U_1^*, \mu_0^*, u)$ is canonical.

Given a canonical second-order additive representation $(S_1, U_1, \mu_0, u)$, there exists a canonical subjective decision tree and subjective probability measure. Define $S_2 \equiv \cup_{s_1 \in S_1} S_2(s_1) \subset \Delta(\Omega)$ and

$$\mathcal{S} \equiv S_1 \times S_2 \times \Omega \subset \Delta(\Delta(\Omega)) \times \Delta(\Omega) \times \Omega.$$

The marginal distribution $\mu_0$ on $S_1$, the conditional probability systems id : $S_1 \to \Delta(S_2)$ and id : $S_2 \to \Delta(\Omega)$ jointly determine the Borel probability measure $\mu$ over $\mathcal{S}$, which is called the *canonical subjective probability.* The set $\mathcal{S}$ redefined as the support of $\mu$ is called the *canonical subjective state space.* Finally, the *canonical subjective filtration* $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ over $\mathcal{S}$ is defined as before.

The next theorem ensures that, under suitable conditions, all second-order additive SEU representations of preference have the same associated canonical representation. Thus, the canonical filtered probability space $(\mathcal{S}, \{\mathcal{F}_t\}_{t=0}^3, \mu)$ is uniquely determined from preference.

**Theorem 4.1.** *Suppose that $\succeq$ on $\mathcal{D}$ admits second-order additive SEU representations $(S_1, U_1, \mu_0, u)$ and $(S_1', U_1', \mu_0', u')$. Then:*

**(i)** *$u$ and $u'$ are cardinally equivalent.*

**(ii)** *If the supports of $\mu_0$, $\mu_0'$, $\mu_1(\cdot|s_1)$'s, and $\mu_1'(\cdot|s_1')$'s are finite, $\mu_0^* = {\mu'}_0^*$. Consequently, the associated canonical filtered probability spaces $(\mathcal{S}, \{\mathcal{F}_t\}_{t=0}^3, \mu)$ coincide.*

PROOF: (i) Since $u$ and $u'$ are mixture linear representations of $\succeq|_{\Delta(Z)}$, they are cardinally equivalent by the standard argument.

(ii) Since $u$ and $u'$ are cardinally equivalent, $(S_1', U_1', \mu_0', u)$ also represents $\succeq$. Furthermore, we can assume $u(\Delta(Z)) = [0, 1]$. Let $U_0$ and $U_0'$ be the representations associated with $(S_1, U_1, \mu_0, u)$ and $(S_1', U_1', \mu_0', u)$, respectively.

Let $\mathbb{P}_1 \equiv \mathbb{P}_1(S_1, U_1) \cup \mathbb{P}_1(S'_1, U'_1)$. Then, $\mu_0^*$ and ${\mu'_0}^*$ are regarded as probability measures on $\mathbb{P}_1$.

Suppose $\mu_0^* \neq {\mu'_0}^*$. Then, there is a partition of $\mathbb{P}_1$ consisting of two non-empty subsets $\mathbb{P}_1^1$ and $\mathbb{P}_1^2$ such that

$$\begin{cases} \mu_0^*(\mu_1) > {\mu'_0}^*(\mu_1), \ \forall \mu_1 \in \mathbb{P}_1^1, \\ \mu_0^*(\mu_1) \leq {\mu'_0}^*(\mu_1), \ \forall \mu_1 \in \mathbb{P}_1^2. \end{cases}$$

For any $\mu_1 \in \mathbb{P}_1$, since $U_1(\cdot, \mu_1)$ is relevant and mixture linear on $\mathcal{P}(\mathcal{H})$, there are positive utility levels $\bar{U}_{\mu_1}$ such that the intersection of lower contour sets associated with $\bar{U}_{\mu_1}$ is a non-empty compact convex set of $\mathcal{P}(\mathcal{H})$, and that each lower contour set coincides with a non-trivial part of the boundary of the intersection. Let $x_0$ be this intersection, i.e.,

$$x_0 \equiv \cap_{\mu_1 \in \mathbb{P}_1} \{x_1 \in \mathcal{P}(\mathcal{H}) | U_1(x_1, \mu_1) \leq \bar{U}_{\mu_1}\}.$$

From continuity of $U'_0$, there exists an $\varepsilon_{\mu_1} > 0$ satisfying $U'_0(\tilde{x}_0) > U'_0(x_0)$, where

$$\tilde{x}_0 \equiv \cap_{\mu_1 \in \mathbb{P}_1^1} \{x_1 \in \mathcal{P}(\mathcal{H}) | U_1(x_1, \mu_1) \leq \bar{U}_{\mu_1} - \varepsilon_{\mu_1}\}.$$

Since

$$y_0 \equiv \cap_{\mu_1 \in \mathbb{P}_1} \{x_1 \in \mathcal{P}(\mathcal{H}) | U_1(x_1, \mu_1) \leq \bar{U}_{\mu} - \varepsilon_{\mu_1}\} \subset_{\neq} x_0,$$

$U'_0(x_0) > U'_0(y_0)$. Since $U'_0(\tilde{x}_0) > U'_1(x_0) > U'_1(y_0)$, continuity of $U'_0$ ensures the existence of a compact convex menu of menus $y_0^*$ such that $y_0 \subset y_0^* \subset \tilde{x}_0$ and $U'_0(x_0) = U'_0(y_0^*)$. Since

$$\max_{x_1 \in x_0} U_1(x_1, \mu_1) > \max_{x_1 \in y_0^*} U_1(x_1, \mu_1),$$

for all $\mu_1 \in \mathbb{P}_1$, and

$$\max_{x_1 \in y_0^*} U_1(x_1, \mu_1) \geq \max_{x_1 \in x_0} U_1(x_1, \mu_1),$$

for all $\mu_1 \in \mathbb{P}_2$,

$$\begin{aligned} U_0(x_0) - U'_0(x_0) &= \sum_{\mu \in \mathbb{P}_1} \left( \max_{x_1 \in x_0} U_1(x_1, \mu) \right) (\mu_0^*(\mu) - {\mu'}_0^*(\mu)) \\ &= \sum_{\mu \in \mathbb{P}_1^1} \left( \max_{x_1 \in x_0} U_1(x_1, \mu) \right) (\mu_0^*(\mu) - {\mu'}_0^*(\mu)) \\ &\quad + \sum_{\mu \in \mathbb{P}_1^2} \left( \max_{x_1 \in x_0} U_1(x_1, \mu) \right) (\mu_0^*(\mu) - {\mu'}_0^*(\mu)) \\ &> \sum_{\mu \in \mathbb{P}_1^1} \left( \max_{x_1 \in y_0^*} U_1(x_1, \mu) \right) (\mu_0^*(\mu) - {\mu'}_0^*(\mu)) \\ &\quad + \sum_{\mu \in \mathbb{P}_1^2} \left( \max_{x_1 \in y_0^*} U_1(x_1, \mu) \right) (\mu_0^*(\mu) - {\mu'}_0^*(\mu)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mu \in \mathbb{P}_1} \left( \max_{x_1 \in y_0^*} U_1(x_1, \mu) \right) (\mu_0^*(\mu) - {\mu'}_0^*(\mu)) \\
&= U_0(y_0^*) - U_0'(y_0^*).
\end{aligned}
$$

Hence, $U_0(x_0) > U_0(y_0^*)$. This contradicts that both $U_0$ and $U_0'$ are representations of the same preference $\succeq$. ∎

# 5 Comparative Subjective Decision Trees

In this section, we compare two preferences admitting second-order additive SEU representations and give a behavioral definition capturing distinct attitudes toward resolution of subjective uncertainties.

Consider two preferences $\succeq_1$ and $\succeq_2$ on $\mathcal{D}$ admitting second-order additive SEU representations denoted by $(S_1^i, U_1^i, \mu_0^i, u^i)$, $i = 1, 2$. In order to ensure uniqueness of subjective decision trees, assume that the representation is canonical and that $\mu_0^i$ and $\mu_1^i(\cdot|s_1^i)$'s have the finite supports. (See Theorem 4.1.)

## 5.1 Purely Subjective Decision Trees

For any canonical second-order additive representation $(S_1, U_1, \mu_0, u)$, recall the notation, $S_2 \equiv \cup_{\mu_1 \in S_1} \text{supp}(\mu_1) \subset \Delta(\Omega)$ and

$$
S \equiv S_1 \times S_2 \subset \Delta(\Delta(\Omega)) \times \Delta(\Omega).
$$

Unlike $\mathcal{S}$, $S$ does not include the objective state space $\Omega$. The marginal distribution $\mu_0$ on $S_1$ and the conditional probability system $\text{id} : S_1 \to \Delta(S_2)$ jointly determine the Borel probability measure $\bar{\mu}$ over $S$. The set $S$ redefined as the support of $\bar{\mu}$ is called the *canonical purely subjective state space*. The *purely subjective filtration* $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ over $S$ is defined by

$$
\begin{aligned}
\mathcal{F}_0 &\equiv \{S\}, \\
\mathcal{F}_1 &\equiv \{(\{s_1\} \times S_2) \cap S | s_1 \in \text{supp}(\mu_0)\}, \\
\mathcal{F}_2 &\equiv \{\{(s_1, s_2)\} | s_1 \in \text{supp}(\mu_0), s_2 \in \text{supp}(\mu_1(s_1))\}.
\end{aligned}
$$

From Theorem 4.1, we know that $(S, \{\mathcal{F}_t\}_{t=0}^2, \bar{\mu})$ is uniquely derived from preference.

Consider $\mu_0 \in \Delta(\Delta(\Delta(\Omega)))$ with a finite support $\{\mu_1^1, \cdots, \mu_1^m\} \subset \Delta(\Delta(\Omega))$. Say that $\mu_0$ satisfies the *distinct supports property* if

$$
\text{supp}(\mu_1^i) \cap \text{supp}(\mu_1^j) = \emptyset,
$$

for any $i, j$. Obviously, this property holds generically, that is, the set of $\mu_0$ satisfying the distinct supports property is an open dense subset of $\Delta(\Delta(\Delta(\Omega)))$.

Take $\mu_0$ satisfying the distinct supports property. Notice that, for any canonical representation $(S_1, U_1, \mu_0, u)$, $S$ can be identified with $S_2$. Furthermore,

$\mathcal{F}_0, \mathcal{F}_1$ and $\mathcal{F}_2$ are regarded as partitions of $S_2$. Especially, $\mathcal{F}_0$ is the null-partition and $\mathcal{F}_2$ is the discrete partition. Only $\mathcal{F}_1$ is a non-trivial partition and identified with the set,

$$\{\text{supp}(\mu) | \mu \in S^1\}.$$

## 5.2 Behavioral Definitions and the Implications

Assume that $\mu_0^i$ associated with $\succeq_i$ satisfies the distinct supports property. The following behavioral definition captures distinct attitudes toward resolutions of subjective uncertainties.

**Definition 5.1.** *$\succeq_2$ is more averse to commitment than $\succeq_1$ if, for any $x_1 \in \mathcal{P}(\mathcal{H})$ and $x_0 \in \mathcal{D}$,*

**(i)** *$\{x_1\} \succeq_1 x_0$ implies $\{x_1\} \succeq_2 x_0$.*

**(ii)** *$\{x_1\} \succ_1 x_0$ implies $\{x_1\} \succ_2 x_0$.*

These behavioral differences between two agents can be attributed to the differences between subjective decision trees.

**Theorem 5.1.** *If $\succeq_2$ is more averse to commitment than $\succeq_1$, then:*

**(i)** *$u^1$ and $u^2$ are cardinally equivalent.*

**(ii)** *Both agents have the same state space, that is, $S_2^1 = S_2^2$.*

**(iii)** *$\mathcal{F}_1^1$ is finer than $\mathcal{F}_1^2$, that is, for any $E^1 \in \mathcal{F}_1^1$, there exists $E^2 \in \mathcal{F}_1^2$ such that $E^1 \subset E^2$. (See Figure 1.)*

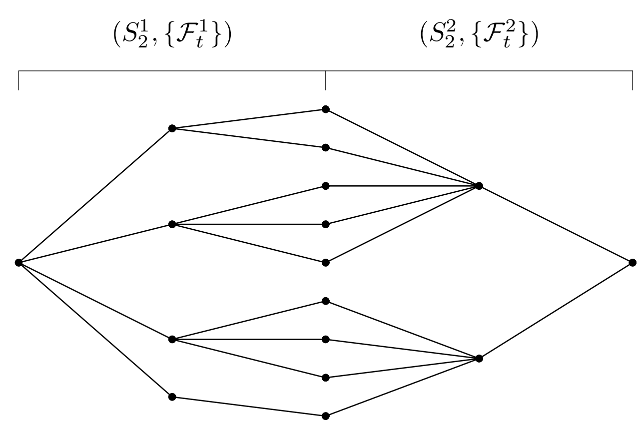


Figure 1: Theorem 5.1

PROOF: (i) First of all, we can show the following:

**Lemma 5.1**: For all $x_1, x_1' \in \mathcal{P}(\mathcal{H})$, $\{x_1\} \succeq_1 \{x_1'\}$ if and only if $\{x_1\} \succeq_2 \{x_1'\}$.

PROOF: From Definition 5.1 (i), $\{x_1\} \succeq_1 \{x_1'\}$ implies $\{x_1\} \succeq_2 \{x_1'\}$. In order to show the converse, suppose that there exists $x_1, x_1' \in \mathcal{P}(\mathcal{H})$ such that $\{x_1'\} \succ \{x_1\}$ and $\{x_1\} \succeq \{x_1'\}$. This immediately contradicts Definition 5.1 (ii). ∎

From Lemma 5.1, for any $l', l \in \Delta(Z)$, $\{\{l'\}\} \succeq_1 \{\{l\}\}$ if and only if $\{\{l'\}\} \succeq_2 \{\{l\}\}$. Hence, $u^1$ and $u^2$ are cardinally equivalent. ∎

(ii) First, from Theorem 5.1 (i), we can assume $u^1 = u^2 = u$.

We will claim the following lemma:

**Lemma 5.2**: If $\{x_1\} \succ_1 \{x_1'\}$ implies $\{x_1\} \succ_2 \{x_1'\}$ for all $x_1, x_1' \in \mathcal{P}(\mathcal{H})$, then $S_2^1 \subset S_2^2$.

PROOF: Suppose $S_2^1 \not\subset S_2^2$. Then, there exists $p^* \in S_2^1 \setminus S_2^2$. For $p \in \Delta(\Omega)$, let

$$U_2(h,p) \equiv \sum_{\omega} u(h(\omega))p(\omega).$$

Since each $U_2(\cdot, p)$ is relevant among $S_2^2 \cup \{p^*\}$ and mixture linear on $\mathcal{H}$, there are positive utility levels $\bar{v}_{p^*}$ and $\bar{v}_p$ such that the intersection of lower contour sets associated with $\bar{v}_{p^*}$ and $\bar{v}_p$'s is a non-empty compact convex subset in $\mathcal{H}$, and that each lower contour set coincides with a non-trivial part of the boundary of the intersection. Let $x_1$ be this intersection and $\hat{x}_1$ be the intersection of those lower contour sets except for $U_2(\cdot, p^*)$. By construction,

$$\max_{h \in x_1} U_2(h,p) = \max_{h \in \hat{x}_1} U_2(h,p),$$

for all $p \in S_2^2$. Hence, the associated representation implies $\{x_1\} \sim_2 \{\hat{x}_1\}$. Also, we have

$$\max_{h \in x_1} U_2(h,p) \leq \max_{h \in \hat{x}_1} U_2(h,p)$$

for all $p \in S_2^1$, and, especially,

$$\max_{h \in x_1} U_2(h,p^*) < \max_{h \in \hat{x}_1} U_2(h,p^*).$$

Since $U_2(\cdot, p^*)$ is relevant, the associated representation implies $\{\hat{x}_1\} \succ_1 \{x_1\}$. This is a contradiction. ∎

From Lemma 5.1, $\{x_1\} \succ_1 \{x_1'\}$ if and only if $\{x_1\} \succ_2 \{x_1'\}$ for all $x_1, x_1' \in \mathcal{P}(\mathcal{H})$. Hence, Lemma 5.2 implies $S_2^1 = S_2^2$. ∎

(iii) Suppose the contrary. Then, there exists $\mu^{1*} \in S_1^1$ such that $\mathrm{supp}(\mu^{1*}) \not\subset \mathrm{supp}(\mu^2)$ for all $\mu^2 \in S_1^2$. Fix any $\mu^2 \in S_1^2$ and choose $p^* \in \mathrm{supp}(\mu^{1*}) \setminus \mathrm{supp}(\mu^2)$.

As Lemma 5.2, define $x_1$ by the intersection of appropriately chosen lower contour sets associated with $U_2(\cdot, p^*)$ and $U_2(\cdot, p)$ for $p \in S_2^2$. If $\#S_1^2 = 1$, we have a contradiction as Lemma 5.2. Hence, assume that $\#S_1^2 \geq 2$. For each $\mu^2 \in S_1^2$, choose each one point out of the boundaries associated with $\mathrm{supp}(\mu^2)$. Let $x_1(\mu^2)$ be the convex hull of the selected points. Clearly, $x_1(\mu^2) \subset_{\neq} x_1$. Let $x_0 \equiv \{x_1(\mu^2) | \mu^2 \in S_1^2\}$. Since, for each $\mu^2 \in S_1^2$,

$$\int_{S_2^2} \max_{h \in x_1} U_2(h, p) \mathrm{d}\mu^2(p) = \max_{x_1' \in x_0} \int_{S_2^2} \max_{h \in x_1'} U_2(h, p) \mathrm{d}\mu^2(p),$$

$\{x_1\} \sim_2 x_0$. On the other hand, since

$$\int_{S_2^1} \max_{h \in x_1} U_2(h, p) \mathrm{d}\mu^1(p) \geq \max_{x_1' \in x_0} \int_{S_2^1} \max_{h \in x_1'} U_2(h, p) \mathrm{d}\mu^1(p),$$

for each $\mu^1 \in S_1^1$ and the strict inequality holds for $\mu^{1*}$, we have $\{x_1\} \succ_1 x_0$. This contradicts the fact that $\succeq_2$ is more averse to commitment than $\succeq_1$. ∎

# 6 Special Cases

## 6.1 Extreme Subjective Decision Trees

We can come up with two extreme cases of subjective decision trees. One extreme is that the DM expects no subjective information arrives in period 1. Precisely, $\#S_1 = 1$. The other extreme is that the DM anticipates all subjective uncertainties are resolved in period 1. In other words, subjective signals in period 1 are fully informative. Formally, $\#S_2(s_1) = 1$ for any $s_1 \in S_1$. Figure 2 illustrates these two cases.

(1) No signal in period 1 (2) Fully informative signals in period 1

Figure 2: Two extreme subjective decision trees

### 6.1.1 No Signal in Period 1

The next axiom says that the DM does not have preference for flexibility in period 0.

**Axiom 5' (Strategic Rationality):** For any $x_0', x_0 \in \mathcal{D}$, $x_0' \succeq x_0$ implies $x_0' \sim x_0' \cup x_0$.

Kreps [6] show that $\succeq$ on $\mathcal{P}(Z)$ with a finite $Z$ satisfies Strategic Rationality if and only if there exists a utility function $u : Z \to \mathbb{R}$ such that $U : \mathcal{P}(Z) \to \mathbb{R}$ defined by,

$$U(x) \equiv \max_{a \in x} u(a),$$

represents $\succeq$. That is, the DM is sure her ex-post preference over $Z$ is $u$. In other words, the DM anticipates no subjective signal arrives. Similarly, Strategic Rationality in our setting is interpreted to mean that the DM expects no information arrives in period 1.

If $\succeq$ satisfies Order, Strategic Rationality implies Monotonicity. Indeed, take any $x_0, x_0'$ with $x_0 \subset x_0'$. From Order, $x_0 \succeq x_0'$ or $x_0' \succeq x_0$. Since $x_0' = x_0 \cup x_0'$, Strategic Rationality implies $x_0' \sim x_0$.

If Monotonicity is replaced with Strategic Rationality in Theorem 3.1, the following is obtained as a corollary:

**Corollary 6.1.** *The following statements are equivalent:*

**(a)** *Preference $\succeq$ on $\mathcal{D}$ satisfies Order, Continuity, Strong Nondegeneracy, Independence, Strategic Rationality, Ex-Post Randomization Indifference, Ex-Post Continuity, Aversion to Commitment, and Risk Preference Certainty.*

**(b)** *The restricted preference $\succeq |_{\mathcal{P}(\mathcal{H})}$ admits an additive SEU representation $U_1 : \mathcal{P}(\mathcal{H}) \to \mathbb{R}$ with $(S_2, \mu_1, \mu_2, u)$, and $U_0 : \mathcal{D} \to \mathbb{R}$ defined by,*

$$U_0(x_0) \equiv \sup_{x_1 \in x_0} U_1(x_1),$$

*represents $\succeq$, where*

$$U_1(x_1) \equiv \int_{S_2} \sup_{h \in x_1} U_2(h, s_2) \mathrm{d}\mu_1(s_2).$$

Proof: $(a \Rightarrow b)$ From Theorem 3.1, there exists a second-order additive SEU representation $(S_1, U_1, \mu_0, u)$. It suffices to show that $\# S_1 = 1$. For simplicity, assume that $S_1$ is finite. Since each $U_1(\cdot, s_1)$'s are relevant and mixture linear, there exist positive utility levels $v_{s_1}$ such that the intersection of lower contour sets associated with $v_{s_1}$'s is a non-empty compact convex subset in $\mathcal{P}(\mathcal{H})$, and that each lower contour set coincides with a non-trivial part of the boundary of the intersection. Let $x_0 \in \mathcal{D}$ be this intersection. Suppose $\# S_1 \neq 1$. Then, there are two states $s_1$ and $s_1'$. Let $x_0(s_1)$ be the intersection of the above lower contour sets except for replacing $v_{s_1}$ with $v_{s_1} + \varepsilon$ for sufficiently small $\varepsilon > 0$. Likewise, $x_0(s_1')$ can be defined. Then, we have

$$x_0(s_1) \cup x_0(s_1') \succ x(s_1) \text{ and } x_0(s_1) \cup x_0(s_1') \succ x(s_1').$$

On the other hand, Order implies, $x_0(s_1) \succeq x_0(s_1')$ or $x_0(s_1') \succeq x_0(s_1)$. Thus, from Strategic Rationality,

$$x_0(s_1) \sim x_0(s_1) \cup x_0(s_1') \text{ or } x_0(s_1') \sim x_0(s_1) \cup x_0(s_1').$$

This is a contradiction. ∎

### 6.1.2 Fully Informative Signals in Period 1

The next axiom says that neither flexibility nor commitment is valueless in period 1.

**Axiom 8' (Neutrality to Commitment):** For all $x_0, x_0' \in \mathcal{D}$, $x_0' \cup \{\cup_{x_1 \in x_0} x_1\} \sim x_0' \cup x_0$.

This axiom reveals that the DM expects all subjective uncertainties are resolved in period 1. In other words, since signals in period 1 is fully informative, there remains no subjective uncertainty in period 2.

Clearly, Neutrality to Commitment implies Aversion to Commitment. If Aversion to Commitment is replaced with Neutrality to Commitment in Theorem 3.1, the following is obtained as a corollary:

**Corollary 6.2.** *The following statements are equivalent:*

**(a)** *Preference $\succeq$ on $\mathcal{D}$ satisfies Order, Continuity, Strong Nondegeneracy, Independence, Monotonicity, Ex-Post Randomization Indifference, Ex-Post Continuity, Neutrality to Commitment, and Risk Preference Certainty.*

**(b)** *Preference $\succeq$ on $\mathcal{D}$ admits a second-order additive SEU representation $U_0 : \mathcal{D} \to \mathbb{R}$ with $(S_1, U_1, \mu_0, u)$ satisfying $\#S_2(s_1) = 1$ for all $s_1 \in S_1$. That is, $U_0 : \mathcal{D} \to \mathbb{R}$ defined by,*

$$U_0(x_0) \equiv \int_{S_1} \sup_{x_1 \in x_0} U_1(x_1, s_1) \mathrm{d}\mu_0(s_1),$$

*represents $\succeq$, where*

$$U_1(x_1, s_1) \equiv \sup_{h \in x_1} U_2(h, p_{s_1}),$$

*for some $p_{s_1} \in \Delta(\Omega)$.*

Proof: ($a \Rightarrow b$) From Theorem 3.1, there exists a second-order additive SEU representation $(S_1, U_1, \mu_0, u)$. Let $S_2 = \cup_{s_1 \in S_1} S_2(s_1)$. From Neutrality to Commitment,

$$\begin{aligned} U_0(x_0) &= U_0(\{\cup_{x_1 \in x_0} x_1\}), \\ &= \int_{S_1} \int_{S_2(s_1)} \sup_{h \in \cup_{x_1 \in x_0} x_1} \left( \sum_{\omega} u(h(\omega)) \mu_2(\omega, s_1, s_2) \right) \mathrm{d}\mu_1(s_2 | s_1) \mathrm{d}\mu_0(s_1), \\ &= \int_{S_2} \sup_{h \in \cup_{x_1 \in x_0} x_1} \left( \sum_{\omega} u(h(\omega)) p(\omega) \right) \mathrm{d}\mu^*(p), \end{aligned}$$

where $\mu^*$ over $\Delta(\Omega)$ is the "mean" probability with respect to $\mu_0$. Redefine $\mu_0$ as $\mu^*$ and $S_1$ by the support of $\mu^*$. For each $p \in S_1$, let

$$U_2(h, p) \equiv \sum_{\omega} u(h(\omega)) p(\omega).$$

Then,

$$\begin{aligned} U_0(x_0) &= \int_{S_1} \sup_{h \in \cup_{x_1 \in x_0} x_1} U_2(h,p) \mathrm{d}\mu_0(p), \\ &= \int_{S_1} \sup_{x_1 \in x_0} \sup_{h \in x_1} U_2(h,p) \mathrm{d}\mu_0(p), \end{aligned}$$

which is the required result. ∎

## 6.2 Special Sub-domains

### 6.2.1 Menus of Acts

One of the special sub-domains is $\mathcal{P}(\mathcal{H})$, that is, menus of acts. If preference on $\mathcal{D}$ admits a second-order additive SEU representation, the restricted preference on $\mathcal{P}(\mathcal{H})$ admits an additive SEU representation with the "mean" probability $\bar{\mu}_1$ with respect to $\mu_0$.

Now, take preference $\succeq$ over $\mathcal{P}(\mathcal{H})$ as a primitive. A question is what axioms on $\succeq$ are necessary and sufficient for an additive SEU representation. The static axioms in Section 2, that is, Order, Continuity, Strong Nondegeneracy, Independence, and Monotonicity can be translated into this setting directly.

The next axiom is essentially same as Risk Preference Certainty on $\mathcal{D}$. The similar interpretation is applicable.

**Axiom 10 (Risk Preference Certainty):** For all $x_1 \in \mathcal{P}(\mathcal{H})$, $x_1 \sim O(x_1)$.

We can show the following:

**Theorem 6.1.** *The following statements are equivalent:*

**(a)** *Preference $\succeq$ on $\mathcal{P}(\mathcal{H})$ satisfies Order, Continuity, Strong Nondegeneracy, Independence, Monotonicity, and Risk Preference Certainty.*

**(b)** *Preference $\succeq$ on $\mathcal{P}(\mathcal{H})$ admits an additive SEU representation.*

Proof: See Appendix.

As Theorem 4.1, the canonical additive SEU representations are uniquely determined from preferences. Hence, if $S_2$ is defined as the support of $\mu_1$, the pair of subjective state space and the subjective probability $(S_2, \mu_1)$ is uniquely derived from preference. However, without the hierarchical domain, we have no subjective filtration.

### 6.2.2 Menus of Menus of Lotteries

Another special sub-domain is $\mathcal{P}(\mathcal{P}(\Delta(Z)))$, that is, the set of menus of menus of lotteries. From now on, assume $Z$ is finite. Then, $\mathcal{P}(\mathcal{P}(\Delta(Z)))$ is the hierarchical extension of the DLR domain $\mathcal{P}(\Delta(Z))$.

**Definition 6.1 (DLR).** *Preference $\succeq$ on $\mathcal{P}(\Delta(Z))$ admits the additive EU representation with a non-negative measure if (i) a functional form $U_1 : \mathcal{P}(\Delta(Z)) \to \mathbb{R}$ defined by*

$$U_1(x_1) \equiv \int_{S_2} \sup_{\beta \in x_1} u(\beta, s_2) \mathrm{d}\mu_1(s_2),$$

*represents preference, where $S_2$ is a state space, $\mu_1$ is a finitely additive non-negative measure on $S_2$, and $u : \Delta(Z) \times S_2 \to \mathbb{R}$ is a state-dependent mixture linear function, (ii) every state $s_2 \in S_2$ is relevant, and (iii) $\succeq_{s_2} \neq \succeq_{s_2'}$ if $s_2 \neq s_2'$.*

DLR show that Order, Continuity, Nondegeneracy[3], Independence and Monotonicity on preference over $\mathcal{P}(\Delta(Z))$ is necessary and sufficient for the additive EU representation with a non-negative measure.

Now, consider preference over $\mathcal{P}(\mathcal{P}(\Delta(Z)))$. The following is the counterpart of Definition 6.1 in the dynamic setting.

**Definition 6.2.** *Preference $\succeq$ on $\mathcal{P}(\mathcal{P}(\Delta(Z)))$ admits a second-order additive EU representation if (i) a functional form $U_0 : \mathcal{P}(\mathcal{P}(\Delta(Z))) \to \mathbb{R}$ defined by*

$$U_0(x_0) \equiv \int_{S_1} \sup_{x_1 \in x_0} U_1(x_1, s_1) \mathrm{d}\mu_0(s_1),$$

*represents preference, where $S_1$ is a state space, $\mu_0$ is a finitely additive non-negative measure on $S_1$, and each $U_1(\cdot, s_1)$ is the additive EU representation with $(S_2(s_1), u(\cdot, \cdot, s_1), \mu(\cdot|s_1))$, (ii) every $\succeq_{s_1}$ is relevant, and (iii) $\succeq_{s_1} \neq \succeq_{s_1'}$ if $s_1 \neq s_1'$.*

Notice that a second-order SEU representation over $\mathcal{P}(\mathcal{P}(\Delta(Z)))$ collapses to the functional without subjecitve state space, that is,

$$U_0(x_0) = \sup_{x_1 \in x_0} \sup_{\beta \in x_1} u(\beta).$$

Hence, subjective uncertainty does not matter on this sub-domain. This is because Risk Preference Certainty is too strong on this sub-domain.

If we drop Risk Preference Certainty from the axioms in Theorem 3.1, the remaining axioms on preference over $\mathcal{P}(\mathcal{P}(\Delta(Z)))$ is necessary and sufficient for a second-order additive EU representation, which might be of interest on its own[4].

**Theorem 6.2.** *The following statements are equivalent:*

[3] Nondegeneracy requires that there exists $x_1, y_1 \in \mathcal{P}(\Delta(Z))$ such that $x_1 \succ y_1$.

[4] Precisely speaking, Strong Nondegeneracy is weakened to Nondegeneracy requiring that there exists $x_0, y_0 \in \mathcal{P}(\mathcal{P}(\Delta(Z)))$ such that $x_0 \succ y_0$

**(a)** *Preference $\succeq$ on $\mathcal{P}(\mathcal{P}(\Delta(Z)))$ satisfies Order, Continuity, Nondegeneracy, Independence, Monotonicity, Ex-Post Randomization Indifference, Ex-Post Continuity, and Aversion to Commitment.*

**(b)** *Preference $\succeq$ on $\mathcal{P}(\mathcal{P}(\Delta(Z)))$ admits a second-order additive EU representation.*

Proof: See Appendix C.

As Section 4, if the canonical representation is properly defined, uniqueness of the representation can be shown. Define $S_2 \equiv \cup_{\mu_1 \in S_1} \mathrm{supp}(\mu_1)$ and $S \equiv S_1 \times S_2$. The marginal distribution $\mu_0$ on $S_1$ and the conditional probability system $\mathrm{id} : S_1 \to \Delta(S_2)$ jointly determine the non-negative measure $\bar{\mu}$ over $S$. The set $S$ redefined as the support of $\bar{\mu}$ is called the *canonical purely subjective state space*. The *purely subjective filtration* $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ over $S$ is defined as before. Then, the pair $(S, \{\mathcal{F}_t\}_{t=0}^2)$ is uniquely derived from preference. However, like DLR, $\bar{\mu}$ is not unique.

# 7 Conclusion

In this paper, we derive a subjective state space with a subjective filtration and a subjective probability $(\mathcal{S}, \{\mathcal{F}_t\}_{t=0}^3, \mu)$ from preference over menus of menus of acts. This result is regarded as an extension of DLR to two direction: a subjective filtration and a meaningful probability.

We introduce some concepts to compare subjective filtrations across agents. For the preferences satisfying a generic property (the distinct supports property), we provide a behavioral definition capturing different subjective filtration over the same subjective state space. Hence, the comparison between different filtrations is meaningful.

# Appendices

### A. Hausdorff Topology

We endow $\mathcal{P}(\mathcal{H})$ with the Hausdorff topology. Precisely, let $d(h, x_1) \equiv \inf_{h' \in x_1} d(h, h')$ and $e(x_1', x_1) \equiv \sup_{h' \in x_1'} d(h', x_1)$. For each $x_1, y_1 \in \mathcal{P}(\mathcal{H})$, define

$$d_h(x_1, y_1) \equiv \max[e(x_1, y_1), e(y_1, x_1)].$$

Then, $d_h$ is a pseudo-metric. That is, $d_h$ satisfies (i) $d_h(x_1, y_1) \geq 0$, (ii) $x_1 = y_1$ implies $d_h(x_1, y_1) = 0$, (iii) $d_h(x_1, y_1) = d_h(y_1, x_1)$, and (iv) $d_h(x_1, z_1) \leq d_h(x_1, y_1) + d_h(y_1, z_1)$. The Hausdorff topology is the topology generated from $\varepsilon$-balls with respect to $d_h$.

Let $\mathcal{D} \equiv \mathcal{P}(\mathcal{P}(\mathcal{H}))$ be the set of all non-empty subsets of $\mathcal{P}(\mathcal{H})$. Generic elements are denoted by $x_0, y_0, \cdots$. Let $D(x_1, x_0) \equiv \inf_{x_1' \in x_0} d_h(x_1, x_1')$ and $E(x_0', x_0) \equiv \sup_{x_1' \in x_0'} D(x_1', x_0)$. For each $x_0, y_0 \in \mathcal{D}$, let

$$d_H(x_0, y_0) \equiv \max[E(x_0, y_0), E(y_0, x_0)].$$

Since $d_H$ is also a pseudo-metric, $\mathcal{D}$ can be endowed with the Hausdorff topology.

## B. Proof of Theorem 3.1

$(a \Rightarrow b)$: (i) We show that there exists a required functional form representing preference. The basic idea of the proof is essentially same as DLR.

For any $x_0 \in \mathcal{P}(\mathcal{P}(\mathcal{H}))$, since $x_0 \sim \mathrm{cl}_1(x_0)$ by Ex-Post Continuity, it suffices to consider the sub-domain $\mathcal{D}_1 \equiv \mathcal{P}(\mathcal{K}(\mathcal{H}))$, where $\mathcal{K}(\mathcal{H})$ is the set of all non-empty compact subsets of $\mathcal{H}$. It is well-known that $\mathcal{K}(\mathcal{H})$ is a compact metric space under the Hausdorff metric $d_h$. Notice that $\mathcal{D}_1$ is convex, i.e., for any $x_0, x_0' \in \mathcal{D}_1$ and $\lambda \in [0,1]$, $\lambda x_0 + (1-\lambda)x_0' \in \mathcal{D}_1$, because $\lambda x_1 + (1-\lambda)x_1' \in \mathcal{K}(\mathcal{H})$ whenever $x_1, x_1' \in \mathcal{K}(\mathcal{H})$. Especially, for any $x_0 \in \mathcal{D}_1$, $\mathrm{co}(x_0)$ is also an element of $\mathcal{D}_1$.

Since Independence implies $x_0 \sim \mathrm{co}(x_0)$, we can restrict our attention to the sub-domain, $\mathcal{D}_2 \equiv \{x_0 \in \mathcal{D}_1 | x_0 = \mathrm{co}(x_0)\}$.

Since $\mathrm{cl}(x_0)$ is convex whenever $x_0$ is convex, it is enough to consider the sub-domain, $\mathcal{D}_3 \equiv \{x_0 \in \mathcal{D}_2 | x_0 = \mathrm{cl}(x_0)\}$, by Continuity. Let $\mathcal{K}(\mathcal{K}(\mathcal{H}))$ be the set of all non-empty compact subsets of $\mathcal{K}(\mathcal{H})$. Since $\mathcal{K}(\mathcal{H})$ is a compact metric space under $d_h$, so is $\mathcal{K}(\mathcal{K}(\mathcal{H}))$ under $d_H$. Since $\mathcal{D}_3 \subset \mathcal{K}(\mathcal{K}(\mathcal{H}))$, $\mathcal{D}_3$ is compact metric and convex.

Order, Continuity and Independence ensures a mixture linear representation $U : \mathcal{D}_3 \to \mathbb{R}$, because $\mathcal{D}_3$ is a mixture space. Let $u : \Delta(Z) \to \mathbb{R}$ be the restriction of $U$ on $\Delta(Z)$, i.e., $u(l) \equiv U(\{\{l\}\})$. Since $\Delta(Z)$ is compact, there exists a maximal element $\bar{l}$ and a minimal element $\underline{l}$ with respect to $u$. Since $u$ is not constant from Strong Nondegeneracy, without loss of generality, we can assume $u(\underline{l}) = 0$ and $u(\bar{l}) = 1$.

Say that a metric $\rho$ on $\Delta(Z)$ is the Prohorv metric if, for any $\mu, \nu \in \Delta(Z)$,

$$\rho(\mu, \nu) \equiv \inf\{\varepsilon > 0 | \mu(E) \leq \nu(U_\varepsilon(E)) + \varepsilon,\ \nu(E) \leq \mu(U_\varepsilon(E)) + \varepsilon,\ \forall E \in \mathcal{B}(Z)\}.$$

It is well-known that the weak convergence topology is metrizable by this metric. Furthermore, the product metric on $\mathcal{H} \equiv \Delta(Z)^n$ is equivalent to the topology generated by $d(h, h') \equiv (\sum_\omega \rho(h(\omega), h'(\omega))^p)^{1/p}$ for some $p \geq 1$. Hence, without loss of generality, we can choose these metrics.

**Lemma B.1**:

**(i)** For any $x_0 \in \mathcal{D}$, $\mathrm{cl}_1(\mathrm{co}_1(x_0)) = \overline{\mathrm{co}}_1(x_0) \equiv \{\overline{\mathrm{co}}(x_1) | x_1 \in x_0\}$.

**(ii)** If $x_0 \in \mathcal{P}(\mathcal{K}(\mathcal{H}))$ is compact, $\mathrm{cl}_1(\mathrm{co}_1(x_0))$ is also compact.

**(iii)** If $x_0 \in \mathcal{P}(\mathcal{K}(\mathcal{H}))$ is convex, $\mathrm{cl}_1(\mathrm{co}_1(x_0))$ is also convex.

**(iv)** The mapping, $\overline{\mathrm{co}}_1 : \mathcal{D}_3 \to \mathcal{D}_3$, is Hausdorff continuous.

PROOF: (i) From Dunford and Schwartz [2, Lemma 4 (ii), p.415], $\mathrm{cl}(\mathrm{co}(x_1)) = \overline{\mathrm{co}}(x_1)$ for any $x_1 \in \mathcal{P}(\mathcal{H})$.

(ii) Consider the closed convex hull operator $\overline{\mathrm{co}}(\cdot) : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$. First of all, since $\mathcal{K}(\mathcal{H})$ is compact, $\overline{\mathrm{co}}(x_1) \in \mathcal{K}(\mathcal{H})$. That is, this operator is well-defined. In order to show the claim, it suffices to show that $\overline{\mathrm{co}}(\cdot) : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ is Hausdorff continuous. Let $d$ be the metric on $Z$. Let $U_\varepsilon(E) \equiv \{z \in Z | d(z, E) < \varepsilon\}$. It can be shown that (1) $\rho(\alpha\mu, \alpha\nu) = \alpha\rho(\mu, \nu)$ for any $\alpha > 0$ and $\mu, \nu \in \mathcal{M}(Z)$, and (2) $\rho(\mu + \nu, \mu' + \nu') \leq \rho(\mu, \nu) + \rho(\mu', \nu')$ for all $\mu, \mu', \nu$ and $\nu'$. Hence, the metric $d$ on $\mathcal{H}$ has the similar properties.

For claiming the continuity of $\overline{\mathrm{co}}(\cdot)$, it suffices to show that $d_h(\overline{\mathrm{co}}(x_1), \overline{\mathrm{co}}(y_1)) \leq d_h(x_1, y_1)$ for all $x_1, y_1 \in K(\mathcal{H})$.

Recall that let $d(h, x'_1) \equiv \min_{h' \in x'_1} d(h, h')$ and $e(x_1, x'_1) \equiv \max_{h \in x_1} d(h, x'_1)$. We will claim that, for any convex $x'_1$, $d(\cdot, x'_1)$ is a convex function. Take any $h_1, h_2 \in \mathcal{H}$, and $\lambda \in [0, 1]$. Let $\bar{h}_i \equiv \mathrm{argmin}_{h' \in x'_1} d(h_i, h')$, $i = 1, 2$. Then,

$$\begin{aligned}
& \lambda d(h_1, x'_1) + (1-\lambda) d(h_2, x'_1), \\
= \quad & d(\lambda h_1, \lambda \bar{h}_1) + d((1-\lambda)h_2, (1-\lambda)\bar{h}_2), \\
\geq \quad & d(\lambda h_1 + (1-\lambda)h_2, \lambda \bar{h}_1 + (1-\lambda)\bar{h}_2), \\
\geq \quad & \min_{h' \in x'_1} d(\lambda h_1 + (1-\lambda)h_2, h'), \\
= \quad & d(\lambda h_1 + (1-\lambda)h_2, x'_1).
\end{aligned}$$

Thus, $d(\cdot, x'_1)$ is convex whenever $x'_1$ is convex.

Turn to the proof of the continuity of $\overline{\mathrm{co}}$. By definition, for any $h \in x_1$,

$$d(h, \overline{\mathrm{co}}(y_1)) \leq e(x_1, \overline{\mathrm{co}}(y_1)).$$

Since $d(\cdot, \overline{\mathrm{co}}(y_1))$ is a convex function by the above claim, this inequality also holds for any $h \in \overline{\mathrm{co}}(x_1)$. Thus,

$$e(\overline{\mathrm{co}}(x_1), \overline{\mathrm{co}}(y_1)) \leq e(x_1, \overline{\mathrm{co}}(y_1)).$$

On the other hand, since $y_1 \subset \overline{\mathrm{co}}(y_1)$,

$$e(x_1, \overline{\mathrm{co}}(y_1)) \leq e(x_1, y_1).$$

Hence,

$$e(\overline{\mathrm{co}}(x_1), \overline{\mathrm{co}}(y_1)) \leq e(x_1, y_1).$$

By the similar argument, this inequality still holds when $x_1$ and $y_1$ is reversed. Hence, $d_h(\overline{\mathrm{co}}(x_1), \overline{\mathrm{co}}(y_1)) \leq d_h(x_1, y_1)$. Consequently, $\overline{\mathrm{co}}(\cdot)$ is Hausdorff continuous.

(iii) From Dunford and Schwartz [2, Lemma 4 (iii) and (iv), p.415], $\overline{\mathrm{co}}(\cdot) : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ is mixture linear, i.e., for any $x_1, y_1 \in \mathcal{K}(\mathcal{H})$ and $\lambda \in [0, 1]$,

$$\overline{\mathrm{co}}(\lambda x_1 + (1-\lambda)y_1) = \lambda \overline{\mathrm{co}}(x_1) + (1-\lambda)\overline{\mathrm{co}}(y_1).$$

Since the mixture linear operator preserves the convexity, $\overline{\text{co}}_1(x_0)$ is convex as long as $x_0 \subset \mathcal{K}(\mathcal{H})$ is convex.

(iv) First of all, this mapping is well-defined by Lemma (ii) and (iii). Let $x_0^n \to x_0$ with $x_0^n, x_0 \in \mathcal{D}_3$. We want to show $\overline{\text{co}}_1(x_0^n) \to \overline{\text{co}}_1(x_0)$. From definition,

$$\begin{aligned} & d_H(\overline{\text{co}}_1(x_0^n), \overline{\text{co}}_1(x_0)) \\ = \; & \max[\max_{x_1 \in x_0^n} \min_{y_1 \in x_0} d_h(\overline{\text{co}}(x_1), \overline{\text{co}}(y_1)), \max_{y_1 \in x_0} \min_{x_1 \in x_0^n} d_h(\overline{\text{co}}(x_1), \overline{\text{co}}(y_1))]. \end{aligned}$$

Since $d_h(\overline{\text{co}}(x_1), \overline{\text{co}}(y_1)) \leq d_h(x_1, y_1)$ for all $x_1, y_1 \in K(\mathcal{H})$ from the argument in Lemma (ii),

$$\begin{aligned} & d_H(\overline{\text{co}}_1(x_0^n), \overline{\text{co}}_1(x_0)) \\ \leq \; & \max[\max_{x_1 \in x_0^n} \min_{y_1 \in x_0} d_h(x_1, y_1), \max_{y_1 \in x_0} \min_{x_1 \in x_0^n} d_h(x_1, y_1)]. \end{aligned}$$

By assumption, this converges to zero. Hence, $\overline{\text{co}}_1(x_0^n) \to \overline{\text{co}}_1(x_0)$. ∎

For $x_0 \in \mathcal{D}_3$, let

$$I(x_1) \equiv \{x'_1 \in \mathcal{K}(\mathcal{H}) | x'_1 \subset x_1\}.$$

Let $I(x_0) \equiv \cup_{x_1 \in x_0} I(x_1)$. So, $I(x_0)$ is the set containing all smaller compact menus of each menu contained in $x_0$.

**Lemma B.2**:

**(i)** If $x_0 \in \mathcal{P}(\mathcal{K}(\mathcal{H}))$ is compact, $I(x_0)$ is also compact.

**(ii)** If $x_0 \in \mathcal{P}(\mathcal{K}(\mathcal{H}))$ is convex, $I(x_0)$ is also convex.

**(iii)** $I : \mathcal{D}_3 \to \mathcal{D}_3$ is Hausdorff continuous.

**(iv)** $x_0 \sim I(x_0)$ for all $x_0 \in \mathcal{D}_3$.

Proof: (i) Since $I(x_0) \subset \mathcal{K}(\mathcal{H})$, it suffices to show that $I(x_0)$ is closed. Let $x_1^n \to x_1$ with $x_1^n \in I(x_0)$. Then there is a sequence $\{y_1^n\}$ in $x_0$ satisfying $y_1^n \supset x_1^n$. Since $x_0$ is compact, without loss of generality we can assume $y_1^n$ converges to some point $y_1 \in x_0$. Suppose there is $h \in x_1 \setminus y_1$. Since $y_1$ is compact, there is an open neighborhood $U(h)$ with $U(h) \cap y_1 = \emptyset$. For all sufficiently large $n$, there is $h^n \in U(h) \cap x_1^n$, because $x_1^n \to x_1$. Since $x_1^n \subset y_1^n$, $h^n \in y_1^n$. This contradicts $y_1^n \to y_1$. Thus, $x_1 \subset y_1$. Hence, $I(x_0)$ is closed.

(ii) Take $x'_1, x_1 \in I(x_0)$. Then there are $y'_1, y_1 \in x_0$ such that $x'_1 \subset y'_1$ and $x_1 \subset y_1$. Since $x_0$ is convex, $\alpha y'_1 + (1-\alpha) y_1 \in x_0$ for any $\alpha \in [0,1]$. Clearly, $\alpha x'_1 + (1-\alpha) x_1 \subset \alpha y'_1 + (1-\alpha) y_1$. Hence, $\alpha x'{}_1 + (1-\alpha) x_1 \in I(x_0)$.

(iii) First of all, the mapping $I(\cdot)$ is well-defined by Lemma (i) and (ii). Let $x_0^n \to x_0$. We have a sequence $\{I(x_0^n)\}_{n=0}^{\infty}$. Since $\mathcal{D}_3$ is a compact metric space, without loss of generality, assume $I(x_0^n) \to y_0$ for some $y_0 \in \mathcal{D}_3$. We want to show $I(x_0) = y_0$.

First, we will claim $y_0 \subset I(x_0)$. Suppose there is $y_1 \in y_0 \setminus I(x_0)$. There is $z_1 \in y_0$ such that $y_1 \subset z_1$. Since $I(x_0^n) \to y_0$, we can find a sequence $z_1^n \to z_1$ with $z_1^n \in I(x_0^n)$. Thus, there is a sequence $x_1^n \in x_0^n$ with $z_1^n \subset x_1^n$. Since the sequence $\{x_1^n\}$ is in $\mathcal{K}(\mathcal{H})$, we can assume $x_1^n \to x_1$ for some $x_1 \in \mathcal{K}(\mathcal{H})$. Since $x_1^n \in x_0^n$, $x_0^n \to x_0$ and $x_1^n \to x_1$, we have $x_1 \in x_0$. This implies $y_1 \in I(x_0)$, because $y_1 \subset z_1 \subset x_1$, which contradicts the assumption.

Next, we will claim $I(x_0) \subset y_0$. Suppose there is $x_1 \in I(x_0) \setminus y_0$. There is $z_1 \in x_0$ such that $x_1 \subset z_1$. Since $x_0^n \to x_0$, there is a sequence $z_1^n \in x_0^n$ with $z_1^n \to z_1$. It suffices to find a sequence $x_1^n \subset z_1^n$ such that $x_1^n$ is compact, and $x_1^n \to x_1$, because $x_1^n \to x_1$, $I(x_0^n) \to y_0$ and $x_1^n \in I(x_0^n)$ imply $x_1 \in y_0$. If $x_1$ is finite, for each $h \in x_1$, take a sequence $\{k_h^n\}$ such that $k_h^n \to h$ and $k_h^n \in z_1^n$. Let $x_1^n \equiv \{k_h^n | h \in x_1\}$. Clearly, $x_1^n \to x_1$ with $x_1^n \subset z_1^n$. Next, consider the case where $x_1$ is a compact subset of $z_1$. From a property of the Hausdorff metric, there is a sequence $y_1^n \to x_1$ such that $y_1^n$ is a finite subset of $x_1$. Take the open $1/n$-neighborhood of $x_1$, denoted by $U(x_1, 1/n)$. Without loss of generality, we can assume $y_1^n \in U(x_1, 1/n)$ for all $n \geq 1$. Since $y_1^n$ is finite, from the above argument we can choose a finite set $x_1^n \subset z_1^n$ with $x_1^n \in U(x_1, 1/n)$. Then, $x_1^n \to x_1$. This completes the proof.

(iv) First, we will claim that the statement holds whenever $x_0$ is finite. By Monotonicity, it suffices to show that $x_0 \succeq I(x_0)$. Denote $x_0 = \{x_1^1, \cdots, x_1^n\}$. By definition, $I(x_0) = \cup_{i=1}^n I(x_1^i)$, where $I(x_1^i) \equiv \{x_1 \in \mathcal{D} | x_1 \subset x_1^i\}$. Since $\cup_{x_1 \in I(x_1^i)} x_1 = x_1^i$ for each $i$, Aversion to Commitment implies

$$\begin{aligned} x_0 &= (x_0 \setminus \{x_1^1\}) \cup \{x_1^1\} \\ &= (x_0 \setminus \{x_1^1\}) \cup \{\cup_{x_1 \in I(x_1^1)} x_1\} \\ &\succeq (x_0 \setminus \{x_1^1\}) \cup I(x_1^1). \end{aligned}$$

By the same argument,

$$\begin{aligned} (x_0 \setminus \{x_1^1\}) \cup I(x_1^1) &= (x_0 \setminus (\cup_{i=1}^2 \{x_1^i\})) \cup \{x_1^2\} \cup I(x_1^1) \\ &\succeq (x_0 \setminus (\cup_{i=1}^2 \{x_1^i\})) \cup (\cup_{i=1}^2 I(x_1^i)). \end{aligned}$$

By repeating this argument finite times, $x_0 \succeq I(x_0)$.

Next, we will claim the general case. Since $x_0$ is compact, we can find a sequence $x_0^n \to x_0$ such that $x_0^n \subset x_0$ and $x_0^n$ is finite for all $n$. From the above argument, $x_0^n \sim I(x_0^n)$. By compactness of $\mathcal{D}_3$, we can assume $\{I(x_0^n)\}$ converges to a point $z_0 \in \mathcal{D}_3$ without loss of generality. If we can show $z_0 = I(x_0)$, the required result follows from Continuity. One direction is easy. Since $x_0^n \subset x_0$, $I(x_0^n) \subset I(x_0)$. So, $z_0 \subset I(x_0)$. In order to show the other direction, suppose that there is $x_1 \in I(x_0) \setminus z_0$. Since $x_1 \in I(x_0)$, there is $y_1 \in x_0$ with $x_1 \subset y_1$. For any open neighborhood $U(y_1)$, there is $y_1^n \in U(y_1) \cap x_0^n$ for all sufficiently large $n$, because $x_0^n \to x_0$. Hence, we can find a sequence $y_1^n \to y_1$ with $y_1^n \in x_0^n$. Since $x_1$ is compact, there exists a sequence $x_1^m \to x_1$ such that $x_1^m \subset x_1$ and $x_1^m$ is finite for all $m$. For each fixed $m$, $x_1^m$ is denoted as $\{h_1^m, \cdots, h_{j_m}^m, \cdots, h_{J_m}^m\}$. Since $h_{j_m}^m \in x_1^m \subset y_1$ and $y_1^n \to y_1$, for any open neighborhood $V(h_{j_m}^m)$, there exists $\beta_{j_m}^n \in V(h_{j_m}^m) \cap y_1^n$ for all sufficiently large $n$. So, there is a sequence

$\{h^n_{j_m}\}$ such that $h^n_{j_m} \in y^n_1$ and $h^n_{j_m} \to h^m_{j_m}$ as $n \to \infty$. Since $x^m_1$ is finite, we can find sequences $\{\{h^n_{j_m}\}_n\}_{j_m}$ such that $h^n_{j_m} \in y^n_1$ and $h^n_{j_m} \to h^m_{j_m}$ as $n \to \infty$ for all $j_m$. Let $x^{m,n}_1 \equiv \{h^n_{j_m} | j_m = 1, \cdots, J_m\}$. Then, $x^{m,n}_1 \subset y^n_1$ and $x^{m,n}_1 \to x^m_1$ as $n \to \infty$. Hence, an appropriately chosen sequence $x^l_1 \subset y^l_1$ converges to $x_1$. On the other hand, since $x_1 \notin z_0$ and $z_0$ is compact, there is an open neighborhood $W(x_1)$ with $W(x_1) \cap z_0 = \emptyset$. This contradicts $I(x^n_0) \to z_0$, because $x^l_1 \to x_1$ with $x^l_1 \in I(x^l_0)$. ∎

**Lemma B.3**:

**(i)** If $x_1 \in \mathcal{K}(\mathcal{H})$, $O(x_1) \in \mathcal{K}(\mathcal{H})$.

**(ii)** If $x_1 \in \mathcal{K}(\mathcal{H})$ is convex, $O(x_1)$ is also convex.

**(iii)** If $x_0 \in \mathcal{P}(\mathcal{K}(\mathcal{H}))$ is compact, $O_1(x_0)$ is also compact.

**(iv)** If $x_0 \in \mathcal{P}(\mathcal{K}(\mathcal{H}))$ is convex, $O_1(x_0)$ is also convex.

**(v)** $O_1 : \mathcal{D}_3 \to \mathcal{D}_3$ is Hausdorff continuous.

Proof: (i) We want to show $O(x)$ is compact. Since $\mathcal{D}$ is a compact metric space, it suffices to show that $O(x)$ is closed. Let $h^n \to h$ with $h^n \in I(x)$. Then there is a sequence $\{k^n\}_{n=0}^{\infty}$ in $x$ satisfying $\{k^n(\omega)\} \succeq \{h^n(\omega)\}$ for all $\omega$. Since $\Delta(Z)$ is compact, for each $\omega$, the sequence $\{k^n(\omega)\}_{n=0}^{\infty}$ has a convergent subsequence $\{k^{n_i}(\omega)\}_{i=0}^{\infty}$ with a limit point $\zeta_\omega \in \Delta(Z)$. Define $k^* \in \mathcal{H}$ by $k^*(\omega) \equiv \zeta_\omega$. From the finiteness of $\Omega$, we can find a subsequence $\{k^m\}_{m=0}^{\infty}$ of $\{k^n\}_{n=0}^{\infty}$ satisfying $k^m \to k^*$. Notice that $k^* \in x$. Since $\{k^m(\omega)\} \succeq \{h^m(\omega)\}$ for all $\omega$, $\{k^*(\omega)\} \succeq \{h(\omega)\}$ because of Cont. Thus, $h \in O(k^*) \subset O(x)$.

(ii) We will claim that $O(x)$ is convex whenever $x$ is convex. Take $h, h' \in O(x)$. Then there are $k, k' \in x$ such that $\{k(\omega)\} \succeq \{h(\omega)\}$ and $\{k'(\omega)\} \succeq \{h'(\omega)\}$ for all $\omega$. Since $x$ is convex, $\lambda k + (1-\lambda)k' \in x$ for any $\lambda \in [0,1]$. From Order and Independence,

$$\lambda\{k(\omega)\} + (1-\lambda)\{k'(\omega)\} \succeq \lambda\{h(\omega)\} + (1-\lambda)\{h'(\omega)\},$$

for all $\omega$, which is equivalent to

$$\{\lambda k(\omega) + (1-\lambda)k'(\omega)\} \succeq \{\lambda h(\omega) + (1-\lambda)h'(\omega)\}.$$

Hence, $\lambda h + (1-\lambda)h' \in O(x)$.

(iii) It suffices to show that $O(\cdot) : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ is continuous with respective to the Hausdorff metric. Let $x^n_1 \to x_1$. We want to show $O(x^n) \to O(x_1)$. Since $\mathcal{K}(\mathcal{H})$ is compact, the sequence $\{O(x^n_1)\}_{n=1}^{\infty}$ has a convergent subsequence $\{O(x^m_1)\}_{m=1}^{\infty}$ with a limit $y_1 \in \mathcal{K}(\mathcal{H})$. Suppose $O(x_1) \neq y_1$. We have two cases; (a) $O(x_1) \not\subset y_1$ or (b) $y_1 \not\subset O(x_1)$. First, let us consider case (a). Suppose there is $h \in O(x_1) \setminus y_1$. Since $y_1$ is a compact subset of $\mathcal{H}$, there is an open neighborhood of $h$, $U(h) \subset \mathcal{H}$, satisfying $U(h) \cap y_1 = \emptyset$. From the definition of Hausdorff metric, $U(h) \cap O(x^n_1) = \emptyset$, for all sufficiently large $n$. On the other hand, since

$h \in O(x_1)$, there exists $\bar{h} \in x_1$ such that $\{\{\bar{h}(\omega)\}\} \succeq \{\{h(\omega)\}\}$ for all $\omega$. Since $x_1^m \to x_1$, we can find a sequence $\{\bar{h}^m\}_{m=1}^{\infty}$ in $\mathcal{H}$ satisfying $\bar{h}^m \in x_1^m$ and $\bar{h}^m \to \bar{h}$ in the sense of the metric on $\mathcal{H}$, equivalently, for all $\omega$, $\bar{h}^m(\omega) \to \bar{h}(\omega)$ in the sense of the metric on $\Delta(Z)$. Now we are going to construct a sequence $\{h^m\}_{m=1}^{\infty}$ in $\mathcal{H}$ with $h^m \in O(x_1^m)$ satisfying $h^m \to h$. Let us fix an arbitrary $\omega$. There are two cases; (1) $\{\{\bar{h}(\omega)\}\} \succ \{\{h(\omega)\}\}$, and (2) $\{\{\bar{h}(\omega)\}\} \sim \{\{h(\omega)\}\}$. If case (1) holds, from Continuity, $\{\{\bar{h}^m(\omega)\}\} \succ \{\{h(\omega)\}\}$ for all sufficiently large $m$. Hence, define $h^m(\omega) \equiv h(\omega)$ for all sufficiently large $m$, and $h^m(\omega)$ can be taken to be an arbitrary point, otherwise. If case (2) holds, define $h^m(\omega) \equiv h(\omega)$ as long as $\bar{h}^m(\omega) \succeq \bar{h}(\omega) \sim h(\omega)$. Otherwise, let $k \geq 1$ be the first natural number satisfying $\bar{h}(\omega) \sim h(\omega) \succ \bar{h}^k(\omega)$. Let $l_\omega(\lambda) \equiv \lambda \bar{h}^k(\omega) + (1-\lambda)h(\omega)$. Continuity ensures that there is $\lambda^m$ such that $l_\omega(\lambda^m) \sim \bar{h}^m(\omega)$. Define $h^m(\omega) \equiv l_\omega(\lambda^m)$. From case (1) and (2), we now have a sequence $\{h^m\}_{m=1}^{\infty}$ such that $h^m \to h$ and $\{\{\bar{h}^m(\omega)\}\} \succeq \{\{h^m(\omega)\}\}$ for all $\omega$, i.e., $h^m \in O(x_1^m)$. This contradicts $U(h) \cap O(x_1^m) = \emptyset$, for all sufficiently large $m$.

Next consider case (b). Take $h \in y_1 \setminus O(x_1)$. Since $O(x_1)$ is compact as long as $x_1$ is compact, there exists an open neighborhood of $h$, $U(h)$, such that $U(h) \cap O(x_1) = \emptyset$. Then, we can find a sequence $h^n \in O(x_1^n)$ with $h^n \to h$ in the sense of the metric on $\mathcal{H}$. By definition of $O(x_1^n)$, there is $\bar{h}^n \in x_1^n$ such that $\{\{\bar{h}^n(\omega)\}\} \succeq \{\{h^n(\omega)\}\}$ for all $\omega$. Since $\mathcal{H}$ is compact, we can assume $\{\bar{h}^n\}$ converges to a limit $h^* \in \mathcal{H}$ without loss of generality. Since $\bar{h}^n \to h^*$ and $x_1^n \to x_1$ with $\bar{h}^n \in x_1^n$, $h^* \in x_1$. From Continuity, $\{\{h^*(\omega)\}\} \succeq \{\{h(\omega)\}\}$ for all $\omega$. Thus, $h \in O(x_1)$. This is a contradiction.

Since we have contradictions in the both cases, it must be the case that $O(x_1) = y_1$. That is, $O$ is Hausdorff-continuous.

(iv) It suffices to show that $O(\cdot) : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ is linear, i.e., for any $x_1, x_1' \in \mathcal{K}(\mathcal{H})$ and $\lambda \in [0,1]$,

$$\lambda O(x_1) + (1-\lambda)O(x_1') = O(\lambda x_1 + (1-\lambda)x_1').$$

Take any $h'' \in \lambda O(x_1) + (1-\lambda)O(x_1')$. Then, there are $h \in O(x_1)$ and $h' \in O(x_1')$ satisfying $h'' = \lambda h + (1-\lambda)h'$. By definition of $O(x_1)$ and $O(x_1')$, there are $\bar{h} \in x_1$ and $\bar{h}' \in x_1'$ such that

$$\{\{\bar{h}(\omega)\}\} \succeq \{\{h(\omega)\}\}, \{\{\bar{h}'(\omega)\}\} \succeq \{\{h'(\omega)\}\},$$

for all $\omega$. Consider $\lambda\bar{h} + (1-\lambda)\bar{h}' \in \lambda x_1 + (1-\lambda)x_1'$. By Independence,

$$\begin{aligned}
& \{\{\lambda\bar{h}(\omega) + (1-\lambda)\bar{h}'(\omega)\}\}, \\
= \;& \lambda\{\{\bar{h}(\omega)\}\} + (1-\lambda)\{\{\bar{h}'(\omega)\}\}, \\
\succeq \;& \lambda\{\{h(\omega)\}\} + (1-\lambda)\{\{h'(\omega)\}\}, \\
= \;& \{\{\lambda h(\omega) + (1-\lambda)h'(\omega)\}\}.
\end{aligned}$$

Thus, $h'' \in O(\lambda x_1 + (1-\lambda)x_1')$.

Conversely, take any $h'' \in O(\lambda x_1 + (1-\lambda)x_1')$. There are $\bar{h} \in x_1$ and $\bar{h}' \in x_1'$ satisfying $\{\{\lambda\bar{h}(\omega) + (1-\lambda)\bar{h}'(\omega)\}\} \succeq \{\{h''(\omega)\}\}$ for all $\omega$. We are going to find $h \in O(x_1)$ and $h' \in O(x_1')$ satisfying $h'' = \lambda h + (1-\lambda)h'$. Let us consider an arbitrarily fixed $\omega$. For a while, assume $\{\{\bar{h}(\omega)\}\} \succeq \{\{\bar{h}'(\omega)\}\}$. By Independence,

$$\{\{\bar{h}(\omega)\}\} \succeq \{\{\lambda\bar{h}(\omega) + (1-\lambda)\bar{h}'(\omega)\}\} \succeq \{\{\bar{h}'(\omega)\}\}.$$

We have the following two cases; (1) $\{\{\bar{h}'(\omega)\}\} \succeq \{\{h''(\omega)\}\}$ and (2) $\{\{\bar{h}(\omega)\}\} \succeq \{\{h''(\omega)\}\} \succ \{\{\bar{h}'(\omega)\}\}$.

If case (1) holds, define $h(\omega) = h'(\omega) \equiv h''(\omega)$. Since $\{\{\bar{h}(\omega)\}\} \succeq \{\{\bar{h}'(\omega)\}\}$,

$$\{\{\bar{h}(\omega)\}\} \succeq \{\{h(\omega)\}\} \text{ and } \{\{\bar{h}'(\omega)\}\} \succeq \{\{h'(\omega)\}\},$$

and $h''(\omega) = \lambda h(\omega) + (1-\lambda)h'(\omega)$ hold.

If case (2) holds, Continuity ensures the existence of $l_\omega \sim \bar{h}'(\omega)$ such that $h''(\omega) = \alpha\bar{h}(\omega) + (1-\alpha)l_\omega$ with $\alpha \in (0,1]$. From Independence, $\lambda \geq \alpha$. Define

$$h(\omega) \equiv \frac{\alpha}{\lambda}\bar{h}(\omega) + \left(1 - \frac{\alpha}{\lambda}\right) l_\omega,$$

and $h'(\omega) \equiv l_\omega$. Then, we have

$$\{\{\bar{h}(\omega)\}\} \succeq \{\{h(\omega)\}\} \text{ and } \{\{\bar{h}'(\omega)\}\} \succeq \{\{h'(\omega)\}\},$$

and $h''(\omega) = \lambda h(\omega) + (1-\lambda)h'(\omega)$. The similar argument works when $\{\{\bar{h}'(\omega)\}\} \succeq \{\{\bar{h}(\omega)\}\}$.

By the construction, it is easy to see that $h \in O(\bar{h}) \subset O(x_1)$, $h' \in O(\bar{h}') \subset O(x_1')$, and $h'' = \lambda h + (1-\lambda)h'$. Therefore, $h'' \in \lambda O(x_1) + (1-\lambda)O(x_1')$.

(v) From Lemma (i), (iii) and (iv), $O_1 : \mathcal{D}_3 \to \mathcal{D}_3$ is well-defined. Let $x_0^n \to x_0$ with $x_0^n, x_0 \in \mathcal{D}_3$. We want to show $O_1(x_0^n) \to O_1(x_0)$. From definition,

$$\begin{aligned} & d_H(O_1(x_0^n), O_1(x_0)) \\ = \quad & \max[\max_{x_1 \in x_0^n} \min_{y_1 \in x_0} d_h(O(x_1), O(y_1)), \max_{y_1 \in x_0} \min_{x_1 \in x_0^n} d_h(O(x_1), O(y_1))]. \end{aligned}$$

We have to show that, for any $\varepsilon > 0$, there exists $N$ such that

$$\max_{x_1 \in x_0^n} \min_{y_1 \in x_0} d_h(O(x_1), O(y_1)) < \varepsilon, \text{ and } \max_{y_1 \in x_0} \min_{x_1 \in x_0^n} d_h(O(x_1), O(y_1)) < \varepsilon,$$

for all $n \geq N$. Suppose that there exists an $\varepsilon > 0$ such that

$$\max_{x_1 \in x_0^n} \min_{y_1 \in x_0} d_h(O(x_1), O(y_1)) \geq \varepsilon$$

for any $n$. There exists $x_1^n \in x_0^n$ satisfying $\min_{y_1 \in x_0} d_h(O(x_1^n), O(y_1)) \geq \varepsilon$. Since $\mathcal{K}(\mathcal{H})$ is compact, without loss of generality, we can assume that the sequence $\{x_1^n\}$ converges to a limit $x_1^*$. Since $x_0^n \to x_0$ and $x_1^n \to x_1^*$ with $x_1^n \in x_0^n$, $x_1^* \in x^0$. From the Hausdorff continuity of $O : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ (See the argument in Lemma (iii).), $O(x_1^n) \to O(x_1^*)$. Since $d_h(O(x_1^n), O(y_1)) \geq \varepsilon$ for all $n$ and $y_1 \in x_0$, we have $d_h(O(x_1^*), O(y_1)) \geq \varepsilon$ for all $y_1 \in x_0$, which contradicts $x_1^* \in x_0$.

Next, suppose that there exists an $\varepsilon > 0$ such that

$$\max_{y_1 \in x_0} \min_{x_1 \in x_0^n} d_h(O(x_1), O(y_1)) \geq \varepsilon$$

for any $n$. There exists $y_1^n \in x_0$ satisfying $\min_{x_1 \in x_0^n} d_h(O(x_1), O(y_1^n)) \geq \varepsilon$ for all $n$. Since $x_0$ is compact, without loss of generality, we can assume that the sequence $\{y_1^n\}$ converges to a limit $y_1^* \in x_0$. Since $x_0^n \to x_0$, we can find a sequence $x_1^n \in x_0^n$ such that $x_1^n \to y_1^*$. By the continuity of $O$, $O(x_1^n) \to O(y_1^*)$ and $O(y_1^n) \to O(y_1^*)$. Since $d_h(O(x_1^n), O(y_1^n)) \geq \varepsilon$ for all $n$, we have $0 = d_h(O(y_1^*), O(y_1^*)) \geq \varepsilon$, which is a contradiction. Therefore, we have the required result. ∎

Thus, from Lemma (iv), Ex-Post Randomization Indifference, and Risk Preference Certainty, we can consider

$$\mathcal{D}_4 \equiv \{x_0 \in \mathcal{D}_3 | x_0 = O_1(\overline{\text{co}}_1(I(\overline{\text{co}}_1(x_0))))\}$$

as the domain. Since the function $O_1(\overline{\text{co}}_1(I(\overline{\text{co}}_1(\cdot))))$ on $\mathcal{D}_3$ into itself is Hausdorff continuous from Lemma, $D_4$ is compact. From Lemma (ii), any $x_1 \in x_0 \in \mathcal{D}_4$ is convex.

Let $S_2 \equiv \Delta(\Omega)$. Since $\#\Omega = n$, $S_2$ is identified with the $(n-1)$-dimensional unit simplex. Let $\mathcal{C}(S_2)$ be the set of real-valued continuous functions on $S_2$ with the sup-norm metric. Let

$$\mathcal{K}^*(\mathcal{H}) \equiv \{x_1 \in \mathcal{K}(\mathcal{H}) | x_1 = O(\overline{\text{co}}(x_1))\}.$$

From Lemma 1-(i) and 2-(i), $\mathcal{K}^*(\mathcal{H})$ is well-defined. From Lemma (ii), any $x_1 \in \mathcal{K}^*(\mathcal{H})$ is convex. Since $\overline{\text{co}} : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ and $O : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ are Hausdorff continuous and mixture linear from the argument in Lemma, $\mathcal{K}^*(\mathcal{H})$ is compact and convex. For each $x_1 \in \mathcal{K}^*(\mathcal{H})$ and $p \in S_2$, let

$$U_2(h, p) \equiv \sum_{\omega \in \Omega} u(h(\omega))p(\omega)$$

and

$$\zeta_{x_1}(p) \equiv \max_{h \in x_1} U_2(h, p).$$

This defines a mixture linear and continuous function $\zeta : \mathcal{K}^*(\mathcal{H}) \to \mathcal{C}(S_2)$.

**Lemma B.4**: $\zeta$ is injective.

Proof: Take any $x_1, x_1' \in \mathcal{K}^*(\mathcal{H})$ with $x_1 \neq x_1'$. Then, there is $\bar{h} \in x_1' \setminus x_1$. Let $\bar{u} \equiv (u(\bar{h}(\omega)))_{\omega \in \Omega} \in \mathbb{R}^n$ and $u(x_1) \equiv \{(u(h(\omega)))_{\omega \in \Omega} \in \mathbb{R}^n | h \in x_1\}$. Since $u$ is continuous and mixture linear, $u(x_1)$ is a compact and convex set in $\mathbb{R}^n$.

Suppose that $(u(x_1) - \bar{u}) \cap \mathbb{R}^n_+ \neq \emptyset$. Since there exists $\hat{h} \in x_1$ such that $u(\hat{h}(\omega)) - u(\bar{h}(\omega)) \geq 0$ for all $\omega$, we have $\bar{h} \in O(x_1)$, which contradicts $O(x_1) =$

$x_1 \not\ni \bar{h}$. Hence, $(u(x_1) - \bar{u}) \cap \mathbb{R}^n_+ = \emptyset$. By the separating hyperplain theorem, there is $\bar{p} \in \Delta^{n-1}$ strictly separating $\bar{u}$ and $u(x_1)$. That is,

$$\sum_{\omega \in \Omega} u(\bar{h}(\omega))\bar{p}(\omega) > \sum_{\omega \in \Omega} u(h(\omega))\bar{p}(\omega),$$

for any $h \in x_1$. Hence,

$$\max_{h \in x_1'} \sum_{\omega \in \Omega} u(h(\omega))\bar{p}(\omega) > \max_{h \in x_1} \sum_{\omega \in \Omega} u(h(\omega))\bar{p}(\omega).$$

This means $\zeta_{x_1'} \neq \zeta_{x_1}$. ∎

Since $S_2$ is a compact metric space, $\Delta(S_2)$ is also a compact metric space under the weak convergence topology. Let $S_1 \subset \Delta(S_2)$ be the set of probability measures on $S_2$ with finite supports. It is well-known that $S_1$ is dense in $\Delta(S_2)$. For each $x_1 \in \mathcal{K}^*(\mathcal{H})$ and $\mu \in S_1$, define

$$U_1(x_1, \mu) \equiv \int_{S_2} \zeta_{x_1}(p)\mathrm{d}\mu(p) = \int_{S_2} \max_{h \in x_1} U_2(h, p)\mathrm{d}\mu(p).$$

Thus, each $\mu$ defines a mixture linear functional on $\mathcal{K}^*(\mathcal{H})$.

Let $\mathcal{C}(S_1)$ be the set of bounded real-valued continuous functions on $S_1$. This space is endowed with the sup norm metric. For each $x_0 \in \mathcal{D}_4$, let

$$\sigma_{x_0}(\mu) \equiv \max_{x_1 \in x_0} U_1(x_1, \mu).$$

Since any $x_1 \in x_0 \in \mathcal{D}_4$ belongs to $\mathcal{K}^*(\mathcal{H})$, $\sigma_{x_0}(\mu)$ is well-defined. This defines a function $\sigma : \mathcal{D}_4 \to \mathcal{C}(S_1)$.

**Lemma B.5**: $\sigma$ is injective.

Proof: Let $x_0 \neq x_0'$. Then, there is $\bar{x}_1 \in x_0' \setminus x_0$. Since $\zeta : \mathcal{K}^*(\mathcal{H}) \to \mathcal{C}(S_2)$ is injective by Lemma, $\zeta_{\bar{x}_1} \in \zeta(x_0') \setminus \zeta(x_0)$, where $\zeta(x_0)$ and $\zeta(x_0')$ are the images of $x_0$ and $x_0'$ under $\zeta$, respectively.

We will claim that there exists a positive linear functional $\Lambda$ on $\mathcal{C}(S_2)$ strictly separating $\zeta_{\bar{x}_1}$ and $\zeta(x_0)$, that is, $\Lambda(\zeta_{\bar{x}_1}) > \Lambda(\zeta_{x_1})$ for all $\zeta_{x_1} \in \zeta(x_0)$. Suppose that there exists $\zeta_{\hat{x}_1} \in \zeta(x_0)$ such that $\zeta_{\hat{x}_1}(p) \geq \zeta_{\bar{x}_1}(p)$ for all $p \in S_2$ and $\zeta_{\hat{x}_1}(\underline{p}) > \zeta_{\bar{x}_1}(\underline{p})$ for some $\underline{p} \in S_2$. Then, $\bar{x}_1 \subset \hat{x}_1$. From the construction of the domain $\mathcal{D}$, $\bar{x}_1 \in x_0$. This contradicts $\bar{x}_1 \in x_0' \setminus x_0$. Therefore, it must be the case that $\zeta(x_0) \cap (\mathcal{C}_+(S_2) + \{\zeta_{\bar{x}_1}\}) = \emptyset$, where $\mathcal{C}_+(S_2) \subset \mathcal{C}(S_2)$ is the set of non-negative continuous functions on $S_2$. Equivalently, $(\zeta(x_0) - \{\zeta_{\bar{x}_1}\}) \cap \mathcal{C}_+(S_2) = \emptyset$. Let $A \equiv \zeta(x_0) - \{\zeta_{\bar{x}_1}\}$ and $M \equiv \mathcal{C}_+(S_2) - A$. We know that $A$ is compact and convex and that $M$ is closed and convex.

It is easy to see that the origin is not in $M$. Indeed, suppose that $0 \in M$. Then there are $f_+ \in \mathcal{C}_+(S_2)$ and $x_1 \in x_0$ such that $0 = f_+ - (\zeta_{x_1} - \zeta_{\bar{x}_1})$. After rearrangement, $\zeta_{x_1} - \zeta_{\bar{x}_1} = f_+$. This contradicts $(\zeta(x_0) - \{\zeta_{\bar{x}_1}\}) \cap \mathcal{C}_+(S_2) = \emptyset$.

Since $0 \notin M$, by the separation hyperplain theorem (See Schaefer [14, p.65, Theorem 9.2 (Second Separation Theorem)]), there exists a linear functional $\Lambda$ on $\mathcal{C}(S_2)$ strictly separating 0 and $M$. That is, $\Lambda(f) > 0$ for all $f \in M$. Since $0 \in \mathcal{C}_+(S_2)$, we know that $\Lambda(\zeta_{\bar{x}_1}) > \Lambda(\zeta_{x_1})$ for all $\zeta_{x_1} \in \zeta(x_0)$.

The remaining task is to show that $\Lambda$ is positive. That is, $\Lambda(f_+) \geq 0$ if $f_+ \in \mathcal{C}_+(S_2)$. Since $\Lambda(f) \geq 0$ for all $f \in M$, $\Lambda(f_+) > \Lambda(\zeta_{x_1} - \zeta_{\bar{x}_1})$ for all $f_+ \in \mathcal{C}_+(S_2)$ and $\zeta_{x_1} \in \zeta(x_0)$. This means that $\Lambda$ is bounded from below on $\mathcal{C}_+(S_2)$. Take a lower bound $\alpha \in \mathbb{R}$. Then $\Lambda(f_+) \geq \alpha$ for all $f_+ \in \mathcal{C}_+(S_2)$. Suppose that $\Lambda$ is not positive. There is $\bar{f}_+ \in \mathcal{C}_+(S_2)$ with $\Lambda(\bar{f}_+) < 0$. Since $\theta\bar{f}_+ \in \mathcal{C}_+(S_2)$ for all $\theta > 0$, $\Lambda(\theta\bar{f}_+) = \theta\Lambda(\bar{f}_+)$ diverges to $-\infty$ as $\theta$ tends to $\infty$. This contradicts $\Lambda(f_+) \geq \alpha$ for all $f_+ \in \mathcal{C}_+(S_2)$.

The Riesz Representation theorem (See Rudin [11, p40, Theorem 2.14]) assures the existence of positive measure $\nu$ on $S_2$ satisfying

$$\Lambda(f) = \int_{S_2} f(p)\mathrm{d}\nu(p) \text{ for all } f \in \mathcal{C}(S_2).$$

Let $\bar{\mu} \in \Delta(S_2)$ be the normalization of $\nu$. Since $\Lambda(\zeta_{\bar{x}_1}) > \Lambda(\zeta_{x_1})$ for all $\zeta_{x_1} \in \zeta(x_0)$, we have

$$\int_{S_2} \zeta_{\bar{x}_1}(p)\mathrm{d}\bar{\mu}(p) > \int_{S_2} \zeta_{x_1}(p)\mathrm{d}\bar{\mu}(p),$$

for all $\zeta_{x_1} \in \zeta(x_0)$. Since $S_1$ is dense in $\Delta(S_2)$, there is $\mu^* \in S_1$ satisfying $U_1(\bar{x}_1, \mu^*) > U_1(x_1, \mu^*)$ for all $x_1 \in x_0$. This implies $\max_{x_1 \in x_0'} U_1(x_1, \mu^*) > \max_{x_1 \in x_0} U_1(x_1, \mu^*)$. Hence, $\sigma : \mathcal{D}_4 \to \mathcal{C}(S_1)$ is injective. ∎

Let $C \subset \mathcal{C}(S_1)$ be the range of $\sigma$. We will see several properties of $C$.

**Lemma B.6**:

**(i)** $C$ is convex.

**(ii)** The zero function, i.e., $f(\mu) = 0$ for any $\mu \in S_1$, is in $C$.

**(iii)** The unit function, i.e., $f(\mu) = 1$ for any $\mu \in S_1$, is in $C$.

**(iv)** The supremum of any two points $f, f' \in C$ is also in $C$. That is, $\max[f(\mu), f'(\mu)]$ is also in $C$.

**(v)** $f \geq 0$ for all $f \in C$.

Proof: (i) At first, we will claim that, for all $x_0', x_0 \in \mathcal{D}_4$,

$$\lambda\sigma_{x_0'} + (1-\lambda)\sigma_{x_0} = \sigma_{O_1(\overline{\mathrm{co}}_1(I(\lambda x_0' + (1-\lambda)x_0)))}.$$

For each $\mu \in S_1$, let $x_1^*$ and $x_1^{**}$ satisfy $U_1(x_1^*, \mu) = \max_{x_1 \in x_0'} U_1(x_1, \mu)$ and $U_1(x_1^{**}, \mu) = \max_{x_1 \in x_0} U_1(x_1, \mu)$. Since $\lambda x_1^* + (1-\lambda)x_1^{**} \in \lambda x_0' + (1-\lambda)x_0$, mixture linearity of $U_1(\cdot,\mu)$ implies,

$$\begin{aligned}
\lambda\sigma_{x_0'}(\mu)+(1-\lambda)\sigma_{x_0}(\mu) &= \lambda U_1(x_1^*,\mu)+(1-\lambda)U_1(x_1^{**},\mu)\\
&= U_1(\lambda x_1^*+(1-\lambda)x_1^{**},\mu)\\
&= \max_{x_1\in O_1(\overline{\mathrm{co}}_1(I(\lambda x_0'+(1-\lambda)x_0)))} U_1(x_1,\mu)\\
&= \sigma_{O_1(\overline{\mathrm{co}}_1(I(\lambda x_0'+(1-\lambda)x_0)))}(\mu).
\end{aligned}$$

Hence, the claim holds.

Take any $f,f'\in C$ and $\lambda\in[0,1]$. There are $x_0',x_0\in\mathcal{D}_4$ satisfying $f'=\sigma_{x_0'}$ and $f=\sigma_{x_0}$. From the above claim,

$$\begin{aligned}
\lambda f'+(1-\lambda)f &= \lambda\sigma_{x_0'}+(1-\lambda)\sigma_{x_0},\\
&= \sigma_{O_1(\overline{\mathrm{co}}_1(I(\lambda x_0'+(1-\lambda)x_0)))}\in\mathcal{D}_4.
\end{aligned}$$

Hence, $C$ is convex.

(ii) Let $x_0\equiv O_1(\{\{\underline{l}\}\})\in\mathcal{D}_4$. Since $u(\underline{l})=0$, $\sigma_{x_0}(\mu)=0$ for all $\mu\in S_1$.

(iii) Let $x_0\equiv O_1(\{\{\bar{l}\}\})=\{\mathcal{H}\}\in\mathcal{D}_4$. Since $u(\bar{l})=1$, $\sigma_{x_0}(\mu)=1$ for all $\mu\in S_1$.

(iv) Take any $f',f\in C$. There exist $x_0',x_0\in\mathcal{D}_4$ such that $f=\sigma_{x_0}$ and $f'=\sigma_{x_0'}$. Let $x_0''\equiv O_1(\overline{\mathrm{co}}_1(I(\overline{\mathrm{co}}_1(x_0\cup x_0'))))\in\mathcal{D}_4$ and $f''\equiv\sigma_{x_0''}\in C$. Then, $f''(\mu)=\max[\sigma_{x_0}(\mu),\sigma_{x_0'}(\mu)]$.

(v) Since each $x_1\in x_0\in\mathcal{D}_4$ contains the constant act $\underline{l}$,

$$\max_{h\in x_1}\left(\sum_{\omega\in\Omega}u(h(\omega))p(\omega)\right)\geq\sum_{\omega\in\Omega}u(\underline{l})p(\omega)=0,$$

for any $p\in S_2$. Hence, $\sigma_{x_0}\geq 0$. ∎

Since $\sigma$ is injective, define $W:C\to\mathbb{R}$ by $W(f)\equiv U(\sigma^{-1}(f))$. Notice that $W(0)=0$ and $W(1)=1$, where 0 and 1 are identified with the zero function and the unit function, respectively. Since $U$ and $\sigma$ is continuous, so is $W$ under the sup norm metric. Furthermore, $W$ is linear in the following sense:

**Lemma B.7**: $W(\alpha f+\beta f')=\alpha W(f)+\beta W(f')$ as long as $f,f',\alpha f+\beta f'\in C$, where $\alpha,\beta\in\mathbb{R}_+$.

Proof: First of all, we will claim that $W:C\to\mathbb{R}$ is mixture linear. Take any $f,f'\in C$ and $\lambda\in[0,1]$. There are $x_0,x_0'\in\mathcal{D}$ such that $f=\sigma_{x_0}$ and $f'=\sigma_{x_0'}$. Since $U$ is mixture linear,

$$\begin{aligned}
W(\lambda f'+(1-\lambda)f) &= W(\lambda\sigma_{x_0'}+(1-\lambda)\sigma_{x_0}),\\
&= U(\sigma^{-1}(\lambda\sigma_{x_0'}+(1-\lambda)\sigma_{x_0})),\\
&= U(\sigma^{-1}(\sigma_{O_1(\overline{\mathrm{co}}_1(I(\lambda x_0'+(1-\lambda)x_0)))})),\\
&= U(O_1(\overline{\mathrm{co}}_1(I(\lambda x_0'+(1-\lambda)x_0)))),\\
&= U(\lambda x_0'+(1-\lambda)x_0),\\
&= \lambda U(x_0')+(1-\lambda)U(x_0),\\
&= \lambda W(f')+(1-\lambda)W(f).
\end{aligned}$$

Next, for any $\alpha \in [0,1]$, $W(\alpha f) = W(\alpha f + (1-\alpha)0) = \alpha W(f) + (1-\alpha)W(0) = \alpha W(f)$, where 0 is the zero function. For any $\alpha > 1$, let $f'' \equiv \alpha f$. Since

$$W\left(\frac{1}{\alpha}f''\right) = \frac{1}{\alpha}W(f''),$$

$\alpha W(f) = W(\alpha f)$. Finally,

$$W(f+f') = 2W\left(\frac{1}{2}f + \frac{1}{2}f'\right) = W(f) + W(f').$$

This completes the proof. ∎

By the same argument as DLR, we will extend $W$ to $\mathcal{C}(S_1)$ by several steps. For any $r \geq 0$, let $rC \equiv \{rf | f \in C\}$. Let $H \equiv \cup_{r \geq 0} rC$. At first, we will extend $W : C \to \mathbb{R}$ to $H$. For any $f \in H \setminus 0$, there is $r > 0$ satisfying $(1/r)f \in C$. Define $W(f) \equiv rW((1/r)f)$.

**Lemma B.8**: $W : H \to \mathbb{R}$ is (i) well-defined, (ii) monotonic, and (iii) linear.

Proof: (i) We want to show that $rW((1/r)f) = r'W((1/r')f)$ even if there is another $r' > 0$ satisfying $(1/r')f \in C$. Without loss of generality, assume $r' > r$. From the linearity of $W$ on $C$,

$$r'W\left(\frac{1}{r'}f\right) = r'W\left(\frac{r}{r'}\frac{1}{r}f\right) = r'\frac{r}{r'}W\left(\frac{1}{r}f\right) = rW\left(\frac{1}{r}f\right).$$

(ii) Take $f', f \in H$ with $f' \geq f$. There is $r > 0$ such that $f' \in rC$. Since $f' \geq f$, $f \in rC$. Hence, $(1/r)f', (1/r)f \in C$ and $(1/r)f' \geq (1/r)f$. Since $W$ is monotonic on $C$, $W((1/r)f') \geq W((1/r)f)$, which implies $W(f') \geq W(f)$.

(iii) It is easy to see that $W$ on $H$ is mixture linear. Take any $f, f' \in H$ and $\lambda \in [0,1]$. Without loss of generality, we can assume there exists $r > 0$ such that $W(f) = rW((1/r)f)$ and $W(f') = rW((1/r)f')$. Since $\lambda f + (1-\lambda)f' \in rC$,

$$\begin{aligned}
W(\lambda f + (1-\lambda)f') &= rW\left(\lambda\frac{f}{r} + (1-\lambda)\frac{f'}{r}\right) \\
&= \lambda rW\left(\frac{f}{r}\right) + (1-\lambda)rW\left(\frac{f'}{r}\right) \\
&= \lambda W(f) + (1-\lambda)W(f').
\end{aligned}$$

By the same argument in Lemma, $W$ is linear. ∎

Let

$$H^* \equiv H - H = \{f_1 - f_2 \in \mathcal{C}(S_1) | f_1, f_2 \in H\}.$$

Since $0 \in H$, $H \subset H^*$. For any $f \in H^*$, there are $f_1, f_2 \in H$ satisfying $f = f_1 - f_2$. Define $W(f) \equiv W(f_1) - W(f_2)$.

**Lemma B.9**: $W : H^* \to \mathbb{R}$ is (i) well-defined, and (ii) linear.

PROOF: (i) Suppose that $f_1, f_2, f_3$ and $f_4$ in $H$ satisfy $f = f_1 - f_2 = f_3 - f_4$. Since $f_1 + f_4 = f_2 + f_3$, $W(f_1) + W(f_4) = W(f_2) + W(f_3)$ by the linearity of $W$ on $H$, which implies $W(f_1) - W(f_2) = W(f_3) - W(f_4)$. Thus, $W$ is well-defined on $H^*$.

(ii) Let $f, f' \in H^*$. There are $f_1, f_2, f'_1, f'_2 \in H$ satisfying $f = f_1 - f_2$ and $f' = f'_1 - f'_2$. Since $\alpha f_1 + \beta f'_1$, $\alpha f_2 + \beta f'_2 \in H$ for any $\alpha, \beta \in \mathbb{R}_+$, linearity of $W$ on $H$ implies,

$$\begin{aligned} W(\alpha f + \beta f') &= W((\alpha f_1 + \beta f'_1) - (\alpha f_2 + \beta f'_2)) \\ &= W(\alpha f_1 + \beta f'_1) - W(\alpha f_2 + \beta f'_2) \\ &= \alpha W(f_1) + \beta W(f'_1) - \alpha W(f_2) - \beta W(f'_2) \\ &= \alpha W(f) + \beta W(f'). \blacksquare \end{aligned}$$

We will see that $H^*$ is dense in $\mathcal{C}(S_1)$.

**Lemma B.10**: $H^*$ is dense in $\mathcal{C}(S_1)$.

PROOF: From the Stone-Weierstrass theorem, it is enough to show that (i) $H^*$ is a vector sublattice, (ii) $H^*$ separates the points of $S_1$, i.e., for any two points $\mu, \mu' \in S_1$, there is $f \in H^*$ such that $f(\mu) > f(\mu')$, and (iii) $H^*$ contains the constant functions equal to one.

By the exactly same argument as Lemma 11 (p.928) in DLR, (i) holds.

In order to show (ii), take $\mu, \mu' \in S_1$ with $\mu \neq \mu'$. Suppose that $U_1(x_1, \mu) = U_1(x_1, \mu')$ for any $x_1 \in \mathcal{P}(\mathcal{H})$. Then, trivially, for any $x_1, y_1 \in \mathcal{P}(\mathcal{H})$, $U_1(x_1, \mu) \geq U_1(y_1, \mu)$ if and only if $U_1(x_1, \mu') \geq U_1(y_1, \mu')$. Thus, these two functional form on $\mathcal{P}(\mathcal{H})$ represent the same preference.

Let $S_2^* \equiv \mathrm{supp}(\mu) \cup \mathrm{supp}(\mu') \subset \Delta(\Omega)$. Since

$$\bigcap_{p \in S_2^*} \left\{ v \in [0,1]^n \,\middle|\, v \cdot p \leq \frac{1}{2} \right\}$$

is non-empty and $u(\Delta(Z)) = [0, 1]$, the menu $x_1 \in \mathcal{P}(\mathcal{H})$ defined by,

$$x_1 \equiv \bigcap_{p \in S_2^*} \left\{ h \in \mathcal{H} \,\middle|\, U_2(h, p) \leq \frac{1}{2} \right\},$$

is a non-empty, compact and convex menu such that each lower contour set coincides with a non-trivial part of the boundary of $x_1$. By the same argument as Theorem 4.1 (ii), we must have $\mu = \mu'$, which is a contradiction. Hence, there exists $\bar{x}_1 \in \mathcal{P}(\mathcal{H})$ such that $U_1(\bar{x}_1, \mu) > U_1(\bar{x}_1, \mu')$. Let $\bar{x}_0 \equiv O_1(\overline{\mathrm{co}}_1(I(\overline{\mathrm{co}}_1(\{\bar{x}_1\}))))$. Then,

$$\sigma_{\bar{x}_0}(\mu) = U_1(\bar{x}_1, \mu) > U_1(\bar{x}_1, \mu') = \sigma_{\bar{x}_0}(\mu').$$

Since $\sigma_{\bar{x}_0} \in C$, (ii) holds.

Finally, (iii) directly follows from Lemma B.6 (iii) and the definition of $H$. ∎

By the same argument as Lemma 12 (p.929) in DLR, it can be shown that there is a constant $K > 0$ such that $W(f) \leq K||f||$ for any $f \in H^*$, because $\mathcal{D}_4$ is compact. By the Hahn-Banach theorem, we can extend $W$ to $\bar{W} : \mathcal{C}(S_1) \to \mathbb{R}$ in a linear, continuous and increasing way. The uniqueness of this extension follows from Lemma B.10. Now we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{D}_4 \arrow[r, "U"] \arrow[d, "\sigma"'] & \mathbb{R} \\
\mathcal{C}(S_1) \arrow[ur, "\bar{W}"'] &
\end{tikzcd}$$

Since $\bar{W}$ is a positive linear functional on $\mathcal{C}(S_1)$, the Riesz representation theorem ensures that there is a unique non-negative measure $\eta$ on $S_1$ satisfying

$$\bar{W}(f) = \int_{S_1} f(\mu)\mathrm{d}\eta(\mu),$$

for all $f \in \mathcal{C}(S_1)$. Especially, $\eta$ can be taken to be a probability measure. Thus, for any $x_0 \in \mathcal{D}_4$,

$$U(x_0) = \bar{W}(\sigma(x_0)) = \int_{S_1} \max_{x_1 \in x_0} \int_{S_2} \max_{h \in x_1} \left( \sum_{\omega} u(h(\omega))p(\omega) \right) \mathrm{d}\mu(p)\mathrm{d}\eta(\mu).$$

This is the required result. ∎

(ii) It is easy to see that, for any finite subset $S_2^* \subset S_2$ and any $p \in S_2^*$, $U_2(\cdot, p)$ is relevant. Hence, (ii) holds obviously. ∎

(iii) Take any finite subset $S_1^* \equiv \{\mu^1, \cdots, \mu^j, \cdots, \mu^J\} \subset S_1$. It suffices to show that any $\mu^j \in S_1^*$ is relevant. Let

$$S_2^* \equiv \cup_{j=1}^{J} \mathrm{supp}(\mu^j) \subset \Delta(\Omega).$$

Let $m \equiv \#S_2^*$. Then, each $\mu^j$ is regarded as an element of $(m-1)$-dimensional unit simplex. We can find $\{a^1, \cdots, a^J\} \subset \mathbb{R}^m$ such that $\mu^j a^j > \mu^j a^k$ for any $j$ and $k \neq j$. Without loss of generality, the absolute value of each coordinate $a_p^j$ of each $a^j$ is sufficiently small.

The menu $x_1 \in \mathcal{P}(\mathcal{H})$ defined by,

$$x_1 \equiv \bigcap_{p \in S_2^*} \left\{ h \in \mathcal{H} \,\middle|\, U_2(h, p) \leq \frac{1}{2} \right\},$$

is a non-empty, compact and convex menu such that each lower contour set coincides with a non-trivial part of the boundary of $x_1$. For each $j$, define

$$x_1^j \equiv \bigcap_{p \in S_2^*} \left\{ h \in \mathcal{H} \,\middle|\, U_2(h,p) \leq \frac{1}{2} + a_p^j \right\} \in \mathcal{P}(\mathcal{H}).$$

Since each $a_p^j$ is sufficiently small, each lower contour set still coincides with a non-trivial part of the boundary of $x_1^j$. Then, $x_1^j$ is a unique maximizer of $U_1(\cdot, \mu^j)$ among $\{x_1^1, \cdots, x_1^J\}$. Indeed,

$$\begin{aligned} U_1(x_1^j, \mu^j) &= \sum_p \max_{h \in x_1^j} U_2(h,p) \mu_p^j \\ &= \sum_p \left( \frac{1}{2} + a_p^j \right) \mu_p^j \\ &= \frac{1}{2} + \sum_p a_p^j \mu_p^j \\ &> \frac{1}{2} + \sum_p a_p^k \mu_p^j \\ &= U_1(x_1^k, \mu^j), \end{aligned}$$

for any $k \neq j$.

Let

$$x_0 \equiv \cap_{\mu^j \in S_1^*} \{x_1 \in \mathcal{P}(\mathcal{H}) | U_1(x_1, \mu^j) \leq U_1(x_1^j, \mu^j)\} \in \mathcal{D}.$$

From the above argument, $x_0$ is a non-empty, compact and convex menu of menus such that each lower contour set coincides with a non-trivial part of the boundary of $x_0$. Now, any $\mu^j \in S_1^*$ is relevant because $x_0$ and $y_0^j$ defined by

$$\begin{aligned} y_0^j \equiv\ & \cap_{k \neq j} \{x_1 \in \mathcal{P}(\mathcal{H}) | U_1(x_1, \mu^k) \leq U_1(x_1^k, \mu^k)\} \\ & \cap \{x_1 \in \mathcal{P}(\mathcal{H}) | U_1(x_1, \mu^j) \leq U_1(x_1^j, \mu^j) + \varepsilon^j\}, \end{aligned}$$

where $\varepsilon^j > 0$ is a sufficiently small number, satisfy the required condition. ▮

(iv) It is enough to show that $U_1(\cdot, \mu)$ and $U_1(\cdot, \mu')$ generate distinct preferences whenever $\mu \neq \mu'$. This is the direct consequence of Theorem 4.1. ▮

## C. Proof of Theorem 6.2

($a \Rightarrow b$) Since the procedure of the proof is almost same as Theorem 3.1, we provide the sketch of the proof.

(i) We show that there exists a required functional form representing preference. From Lemma B.1 and Lemma B.2, we can consider

$$\mathcal{D}_4^- \equiv \{x_0 \in \mathcal{D}_3 | x_0 = \overline{\text{co}}_1(I(\overline{\text{co}}_1(x_0)))\}$$

as the domain. Since the function $\overline{\mathrm{co}}_1(I(\overline{\mathrm{co}}_1(\cdot)))$ on $\mathcal{D}_3$ into itself is Hausdorff continuous from Lemma B.1 and B.2, $D_4^-$ is compact. Notice that any $x_1 \in x_0 \in \mathcal{D}_4^-$ is convex.

Order, Continuity, Nondegeneracy and Independence ensure a non-constant mixture linear representation $U : \mathcal{D}_3 \to \mathbb{R}$, because $\mathcal{D}_3$ is a mixture space. By abuse of notation, the restriction of $U$ on $\mathcal{D}_4^-$ is also denoted by $U$.

Let $S_2 \equiv \{u \in \mathbb{R}^n | \sum_{i=1}^n u_i = 0,\ \sum_{i=1}^n |u_i| = 1\}$. Let $\mathcal{C}(S_2)$ be the set of real-valued continuous functions on $S_2$ with the sup-metric. Let $\mathcal{K}(\Delta(Z))$ be the set of compact subsets of $\Delta(Z)$ and $\mathcal{K}_c(\Delta(Z))$ be the set of compact and convex subsets of $\Delta(Z)$. Notice that $\mathcal{K}_c(\Delta(Z))$ is compact and convex. For each $x_1 \in \mathcal{K}_c(\Delta(Z))$ and $u \in S_2$, let

$$\zeta_{x_1}(u) \equiv \max_{\beta \in x_1} u(\beta).$$

DLR show that $\zeta : \mathcal{K}_c(\Delta(Z)) \to \mathcal{C}(S_2)$ is injective, continuous, and mixture linear.

Let $S_1 \subset \Delta(S_2)$ be the set of probability measures with finite supports. Endow $\Delta(S_2)$ the weak convergence topology. It is well-known that $S_1$ is dense in $\Delta(S_2)$ under this topology. Let $\mathcal{C}(S_1)$ be the set of bounded real-valued continuous functions on $S_1$. For each $x_0 \in \mathcal{D}_4^-$ and $\mu \in S_1$, let

$$U_1(x_1, \mu) \equiv \int_{S_1} \zeta_{x_1}(u) \mathrm{d}\mu(u) = \int_{S_1} \max_{\beta \in x_1} u(\beta) \mathrm{d}\mu(u).$$

For each $x_0 \in \mathcal{D}_4^-$ and $\mu \in S_1$, let

$$\sigma_{x_0}(\mu) \equiv \max_{x_1 \in x_0} U_1(x_1, \mu).$$

As Lemma B.5, we can show that the function $\sigma : \mathcal{D}_4^- \to \mathcal{C}(S_1)$ is injective. Let $C \subset \mathcal{C}(S_1)$ be the range of $\sigma$. Since $\sigma$ is injective, $\sigma : \mathcal{D}_4^- \to C$ is bijective.

**Lemma C.1**:

**(i)** $C$ is convex.

**(ii)** The zero function, that is, $f(\mu) = 0$ for any $\mu \in S_1$, is in $C$.

**(iii)** There exists $c > 0$ such that the constant function equal to $c$, that is, $f(\mu) = c$ for any $\mu \in S_1$, is in $C$.

**(iv)** The supremum of any two points $f, f' \in C$ is also in $C$. That is, $\max[f(\mu), f'(\mu)]$ is also in $C$.

Proof: (i) At first, we will claim that, for all $x_0', x_0 \in \mathcal{D}_4^-$,

$$\lambda \sigma_{x_0'} + (1-\lambda)\sigma_{x_0} = \sigma_{\overline{\mathrm{co}}_1(I(\lambda x_0' + (1-\lambda)x_0))}.$$

For each $\mu \in S_1$, let $x_1^*$ and $x_1^{**}$ satisfy $U_1(x_1^*, \mu) = \max_{x_1 \in x_0'} U_1(x_1, \mu)$ and $U_1(x_1^{**}, \mu) = \max_{x_1 \in x_0} U_1(x_1, \mu)$. Since $\lambda x_1^* + (1-\lambda)x_1^{**} \in \lambda x_0' + (1-\lambda)x_0$, mixture linearity of $U_1(\cdot, \mu)$ implies,

$$\begin{aligned}
\lambda\sigma_{x_0'}(\mu) + (1-\lambda)\sigma_{x_0}(\mu) &= \lambda U_1(x_1^*, \mu) + (1-\lambda)U_1(x_1^{**}, \mu) \\
&= U_1(\lambda x_1^* + (1-\lambda)x_1^{**}, \mu) \\
&= \max_{x_1 \in \overline{\mathrm{co}}_1(I(\lambda x_0' + (1-\lambda)x_0))} U_1(x_1, \mu) \\
&= \sigma_{\overline{\mathrm{co}}_1(I(\lambda x_0' + (1-\lambda)x_0))}(\mu).
\end{aligned}$$

Hence, the claim holds.

Take any $f, f' \in C$ and $\lambda \in [0,1]$. There are $x_0', x_0 \in \mathcal{D}_4^-$ satisfying $f' = \sigma_{x_0'}$ and $f = \sigma_{x_0}$. From the above claim,

$$\begin{aligned}
\lambda f' + (1-\lambda)f &= \lambda\sigma_{x_0'} + (1-\lambda)\sigma_{x_0}, \\
&= \sigma_{\overline{\mathrm{co}}_1(I(\lambda x_0' + (1-\lambda)x_0))} \in \mathcal{D}_4^-.
\end{aligned}$$

Hence, $C$ is convex.

(ii) Let $x_0 \equiv \{\{(1/n, \cdots, 1/n)\}\} \in \mathcal{D}_4^-$. Since $u((1/n, \cdots, 1/n)) = 0$ for all $u \in S_2$, $\sigma_{x_0}(\mu) = 0$ for all $\mu \in S_1$.

(iii) Let $x_1$ be the intersection of $\Delta(B)$ and the closed $c$-ball at $(1/n, \cdots, 1/n)$ for a sufficiently small $c > 0$. Let $x_0 \equiv \{x_1\}$. Since $\max_{\beta \in x_1} u(\beta) = c$ for all $u \in S_2$, $\sigma_{x_0}(\mu) = c$ for all $\mu \in S_1$.

(iv) Take any $f', f \in C$. There exist $x_0', x_0 \in \mathcal{D}_4^-$ such that $f = \sigma_{x_0}$ and $f' = \sigma_{x_0'}$. Let $x_0'' \equiv co_1(I(\mathrm{co}(x_0 \cup x_0'))) \in \mathcal{D}$ and $f'' \equiv \sigma_{x_0''} \in C$. Then, $f''(\mu) = \max[\sigma_{x_0}(\mu), \sigma_{x_0'}(\mu)]$. ∎

Since $\sigma$ is injective, define $W : C \to \mathbb{R}$ by $W(f) \equiv U(\sigma^{-1}(f))$. Notice that $W(0) = 0$ and $W(c) = c$, where 0 and $c$ are identified with the zero function and the constant function equal to $c$, respectively. Since $U$ and $\sigma$ is continuous, so is $W$ under the sup norm metric. As Lemma B.7, we can show $W$ is linear.

We extend $W : C \to \mathbb{R}$ on $\mathcal{C}(S_1)$ by several steps. First, restrict $W$ on $C_+ \equiv \{f \in C | f \geq 0\}$. For any $r \geq 0$, let $rC_+ \equiv \{rf | f \in C_+\}$. Let $H \equiv \cup_{r \geq 0} rC_+$ and

$$H^* \equiv H - H = \{f_1 - f_2 \in \mathcal{C}(S_1) | f_1, f_2 \in H\}.$$

For any $f \in H \setminus 0$, there is $r > 0$ satisfying $(1/r)f \in C_+$. Define $W(f) \equiv rW((1/r)f)$. As Lemma B.8, $W : H \to \mathbb{R}$ is well-defined, monotonic, and linear. For any $f \in H^*$, there are $f_1, f_2 \in H$ satisfying $f = f_1 - f_2$. Define $W(f) \equiv W(f_1) - W(f_2)$. As Lemma B.9, $W : H^* \to \mathbb{R}$ is well-defined and linear.

In order to show that $H^*$ is dense in $\mathcal{C}(S_1)$, we prepare the following Lemma:

**Lemma C.2**: Assume that $U_1(\cdot, \mu)$ and $U_1(\cdot, \mu')$ generate the same preference on $\mathcal{K}_c(\Delta(Z))$, that is, $U_1(x_1, \mu) \geq U_1(y_1, \mu)$ if and only if $U_1(x_1, \mu') \geq U_1(y_1, \mu')$ for any $x_1, y_1 \in \mathcal{K}_c(\Delta(Z))$. Then, $\mu = \mu'$.

PROOF: Suppose $\mu \neq \mu'$. Let $\mathbb{S} \subset S_2$ be the union of the supports of $\mu$ and $\mu'$. Then, there exists the partition of $\mathbb{S}$ consisting of two non-empty subsets $\mathbb{S}_1, \mathbb{S}_2$, such that

$$\begin{cases} \mu(u) > \mu'(u), \ \forall u \in \mathbb{S}_1, \\ \mu(u) \leq \mu'(u), \ \forall u \in \mathbb{S}_2. \end{cases}$$

Since each $u \in \mathbb{S}$ is relevant and mixture linear on $\Delta(Z)$, there exist positive utility levels $\bar{v}_u$ such that the intersection of lower contour sets associated with $\bar{v}_u$ is a non-empty compact convex set of $\Delta(Z)$, and that each lower contour set coincides with a non-trivial part of the boundary of the intersection. Let $x_1$ be this intersection, i.e.,

$$x_1 \equiv \cap_{u \in \mathbb{S}} \{\beta \in \Delta(Z) | u(\beta) \leq \bar{v}_u\}.$$

From continuity of $U_1(\cdot, \mu')$, there exists a small $\varepsilon > 0$ satisfying $U_1(\tilde{x}_1, \mu') > U_1(x_1, \mu')$, where

$$\tilde{x}_1 \equiv \cap_{u \in \mathbb{S}_1} \{\beta \in \Delta(Z) \, | u(\beta) \leq \bar{v}_u - \varepsilon\}.$$

Since

$$y_1 \equiv \cap_{u \in \mathbb{S}} \{\beta \in \Delta(Z) \, | u(\beta) \leq \bar{v}_p - \varepsilon\} \subset_{\neq} x_1,$$

$U_1(x_1, \mu') > U_1(y_1, \mu')$. From continuity of $U_1(\cdot, \mu')$, there is a compact convex menu $y_1^* \subset \tilde{x}_1$ such that $U_1(x_1, \mu') = U_1(y_1^*, \mu')$. Since

$$\max_{\beta \in x_1} u(\beta) > \max_{\beta \in y_1^*} u(\beta),$$

for all $u \in \mathbb{S}_1$, and

$$\max_{\beta \in y_1^*} u(\beta) \geq \max_{\beta \in x_1} u(\beta),$$

for all $u \in \mathbb{S}_2$,

$$\begin{aligned} & U_1(x_1, \mu) - U_1(x_1, \mu') \\ = & \sum_{u \in \mathbb{S}} \left( \max_{\beta \in x_1} u(\beta) \right) (\mu(u) - \mu'(u)) \\ = & \sum_{u \in \mathbb{S}_1} \left( \max_{\beta \in x_1} u(\beta) \right) (\mu(u) - \mu'(u)) + \sum_{u \in \mathbb{S}_2} \left( \max_{\beta \in x_1} u(\beta) \right) (\mu(u) - \mu'(u)) \\ > & \sum_{u \in \mathbb{S}_1} \left( \max_{\beta \in y_1^*} u(\beta) \right) (\mu(u) - \mu'(u)) + \sum_{u \in \mathbb{S}_2} \left( \max_{\beta \in y_1^*} u(\beta) \right) (\mu(u) - \mu'(u)) \\ = & \sum_{u \in \mathbb{S}} \left( \max_{\beta \in y_1^*} u(\beta) \right) (\mu(u) - \mu'(u)) \\ = & U_1(y_1^*, \mu) - U_1(y_1^*, \mu'). \end{aligned}$$

Hence, $U_1(x_1, \mu) > U_1(y_1^*, \mu)$. This contradicts the assumption. ∎

Now, we can show the following:

**Lemma C.3**: $H^*$ is dense in $\mathcal{C}(S_1)$.

PROOF: From the Stone-Weierstrass theorem, it is enough to show that (i) $H^*$ is a vector sublattice, (ii) $H^*$ separates the points of $S_1$, i.e., for any two points $\mu, \mu' \in S_1$, there is $f \in H^*$ such that $f(\mu) > f(\mu')$, and (iii) $H^*$ contains the constant functions equal to one.

By the exactly same argument as Lemma 11 (p.928) in DLR, (i) holds.

In order to show (ii), take $\mu, \mu' \in S_1$ with $\mu \neq \mu'$. Suppose that $U_1(x_1, \mu) = U_1(x_1, \mu')$ for any $x_1 \in \mathcal{K}_c(\Delta(Z))$. Then, trivially, for any $x_1, y_1 \in \mathcal{K}_c(\Delta(Z))$, $U_1(x_1, \mu) \geq U_1(y_1, \mu)$ if and only if $U_1(x_1, \mu') \geq U_1(y_1, \mu')$. Thus, these two functional form on $\mathcal{K}_c(\Delta(Z))$ represent the same preference. From Lemma C.2, $\mu = \mu'$, which is a contradiction. Hence, there exists $\bar{x}_1 \in \mathcal{K}_c(\Delta(Z))$ such that $U_1(\bar{x}_1, \mu) > U_1(\bar{x}_1, \mu')$. Let $\bar{x}_0 \equiv \overline{\text{co}}_1(I(\overline{\text{co}}_1(\{\bar{x}_1\})))$. Then,

$$\sigma_{\bar{x}_0}(\mu) = U_1(\bar{x}_1, \mu) > U_1(\bar{x}_1, \mu') = \sigma_{\bar{x}_0}(\mu').$$

Since $\sigma_{\bar{x}_0} \in C_+$, (ii) holds.

Finally, (iii) directly follows from Lemma C.1 (iii) and the definition of $H$. ∎

By the same argument as Lemma 12 (p.929) in DLR, it can be shown that there is a constant $K > 0$ such that $W(f) \leq K||f||$ for any $f \in H^*$, because $\mathcal{D}_4^-$ is compact. By the Hahn-Banach theorem, we can extend $W$ to $\bar{W} : \mathcal{C}(S_1) \to \mathbb{R}$ in a linear, continuous and increasing way. The uniqueness of this extension follows from Lemma C.3. Since $\bar{W}$ is a positive linear functional on $\mathcal{C}(S_1)$, the Riesz representation theorem ensures that there is a unique non-negative measure $\eta$ on $S_1$ satisfying

$$\bar{W}(f) = \int_{S_1} f(\mu) \mathrm{d}\eta(\mu),$$

for all $f \in \mathcal{C}(S_1)$. Especially, $\eta$ can be taken to be a probability measure. Thus, for any $x_0 \in \mathcal{D}_4^-$,

$$U(x_0) = \bar{W}(\sigma(x_0)) = \int_{S_1} \max_{x_1 \in x_0} \int_{S_2} \max_{\beta \in x_1} u(\beta) \mathrm{d}\mu(u) \mathrm{d}\eta(\mu).$$

This is the required result. ∎

(ii) Take any finite subset $S_1^* \equiv \{\mu^1, \cdots, \mu^j, \cdots, \mu^J\} \subset S_1$. It suffices to show that any $\mu^j \in S_1^*$ is relevant. Let

$$S_2^* \equiv \cup_{j=1}^J \text{supp}(\mu^j) \subset S_2.$$

Let $m \equiv \# S_2^*$. Then, each $\mu^j$ is regarded as an element of $(m-1)$-dimensional unit simplex. We can find $\{a^1, \cdots, a^J\} \subset \mathbb{R}^m$ such that $\mu^j a^j > \mu^j a^k$ for any $j$ and $k \neq j$. Without loss of generality, the absolute value of each coordinate $a_u^j$ of each $a^j$ is sufficiently small.

For a small $c > 0$, since

$$x_1 \equiv \cap_{u \in S_2^*} \{\beta \in \Delta(Z) \,|u(\beta) \leq c\},$$

is a non-empty, compact and convex menu such that each lower contour set coincides with a non-trivial part of the boundary of $x_1$. For each $j$, define

$$x_1^j \equiv \cap_{u \in S_2^*} \left\{\beta \in \Delta(Z) \,\middle|u(\beta) \leq c + a_u^j\right\} \in \mathcal{P}(\Delta(Z)).$$

Since each $a_u^j$ is sufficiently small, each lower contour set still coincides with a non-trivial part of the boundary of $x_1^j$. Then, $x_1^j$ is a unique maximizer of $U_1(\cdot, \mu^j)$ among $\{x_1^1, \cdots, x_1^J\}$. As the proof of relevance in Theorem 3.1, we can find $x_0$ and $y_0$ satisfying the requirement for any $U_1(\cdot, \mu^j)$. ∎

(iii) It is enough to show that $U_1(\cdot, \mu)$ and $U_1(\cdot, \mu')$ generate distinct preferences whenever $\mu \neq \mu'$. This is the direct consequence of Lemma C.2. ∎

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