STRATEGIC CORES AND COST SHARE EQUILIBRIUM IN A PUBLIC GOODS ECONOMY

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ABSTRACT. In this paper, we consider a public goods economy and the corresponding strategic form game, called a voluntary contribution game. We show that the set of the allocations attained at the cost share equilibria is equivalent to the set of the allocations attained by the strategy profiles which induce the payoff vectors which are in the α-core. Moreover, we show that the voluntary contribution game has a nonempty α-core. Hence the existence of the cost share equilibrium is also shown.
Keywords: cost share equilibrium, α-core, voluntary contribution
JEL Classification; C71, D62, H41

1. INTRODUCTION

The cost share equilibrium is an equilibrium concept for a public goods economy, which is introduced by Mas-Colell and Silvestre [10]. At the cost share equilibrium, the amount of public goods which is chosen by each agent in order to maximize her preference under a given cost sharing rule coincides each other among the agents. The cost share equilibrium can be regarded as a kind of generalizations of a Lindahl equilibrium. The cost sharing rule plays the role of price mechanism. Hence the cost share equilibrium is an equilibrium under the decentralized mechanism.

On the other hand, the α-core is a core concept for the strategic form games, which is introduced by Aumann and Peleg [2]. We derive a strategic form game from the public goods economy. Our model is that of a voluntary contribution game, which is a simpler version of the model due to Utsumi and Nakayama [13], and the roots of which can be traced back to the Scarf [12]. In the voluntary contribution game, each player chooses how much private good she contributes to produce each public good from her endowment . Hence the α-core can

Date: June 3, 2004.
This is part of my master's thesis submitted to graduate school of economics, Keio university. I thank my advisor Mikio Nakayama. I am also grateful to Shinsuke Nakamura, Midori Hirokawa and Tomoki Inoue for helpful comments.
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be regarded as a set of the agreements attained among the players in the public goods economy.

In this paper, we prove that the set of the allocations attained at the cost share equilibria is equivalent to the set of the allocations attained by the strategy profiles which induce the payoff vectors which are in the α-core. Such characterizations for the cost share equilibrium and the special cases of the cost share equilibrium have been considered in several papers.

In Foley [4], it was shown that the Lindahl equilibrium is in the core of the public goods economy when the production technology is linear. Mas-Colell [9] showed that the set of the allocations attained at the cost share equilibria is equivalent to the core of the public goods economy. But, the core may be empty. Weber and Wiesmeth [14] and Diamantaras and Gilles [3] have extended the result of Mas-Colell [9]. The nonemptiness of the core has not been assured.

The characterizations from another perspective are also considered. Kaneko [8] showed that the equivalence between the ratio equilibria and the core of the voting game, which is a game of political procedure. This result was extended by Hirokawa [6] to the equivalence between the cost share equilibria and the core of the voting game.

Moreover, we show that the α-NTU coalitional game is balanced. This result induces the existence of the cost share equilibrium. In fact, some special cases of the cost share equilibrium has been proved in several papers. Kaneko [7] verified the existence of the ratio equilibrium which is a special case of the cost share equilibrium with proportional cost sharing rule. Mas-Colell and Silvestre [10] verified the existence of the other two special cases of the cost share equilibrium, called a linear cost share equilibrium and a balanced linear cost share equilibrium. But, these results do not assure the existence of the cost share equilibrium in our model. Hence we showed the existence of the more general cost share equilibrium.

In the next section, we define a public goods economy, and derive a voluntary contribution game from the public goods economy. Section 3 includes the results of this paper. In the final section, we conclude with some remarks.

2. A Public Goods Economy and the Strategic Form Game

We consider a public goods economy consisting of n agents, m public goods, and one private good. Let us denote by $N = \{1, \cdots, n\}$ the set of the agents, by $M = \{1, \cdots, m\}$ the set of the public goods. The private good is either consumed or used as the input to provide the public goods.

The amount of the public goods each agent consumes is denoted by $y \in \mathbb{R}^m_+$ and j-th component of y, y_j, denotes the amount of j-th public good. For each public good $j \in M$, the cost to provide an amount of the public good y_j is determined by the cost function $C_j : \mathbb{R}_+ \to \mathbb{R}_+$. We assume that for any $j \in M$, $C_j(0) = 0$, $C_j(y_j)$ is convex and strictly increasing in y_j. This kind of separability of the cost means that the cost to provide an amount of y_j of public good j is independent of the amount of the other public goods provided. Let us define $\bar{C}(y) = \sum_{j \in M} C_j(y_j)$ the total cost to provide y, and denote $Y = \{y \in \mathbb{R}^m_+ | \bar{C}(y) \leq \sum_{i \in N} \omega_i\}$. Thus, the set Y means the producible amount of the public goods.

Each agent i is characterized by (Q_i, u_i, ω_i), where $Q_i \subset \mathbb{R}^m_+ \times \mathbb{R}_+$ is the consumption set of agent i, $u_i : Q_i \to \mathbb{R}$ is the utility function of agent i, and each agent i is endowed with an amount of the private good ω_i. We assume that $Y \times [0, \omega_i] \subset Q_i$ for all $i \in N$. Also, we assume that each utility function u_i satisfies the assumptions below.

Assumption 1. *For all $i \in N$, the utility function u_i satisfies conditions below.*

- **(a):** *Monotonicity in the amount of the public and private goods, that is, for any $(y, \omega_i - z_i), (y', \omega_i - z'_i) \in Q_i$ with $(y, \omega_i - z_i) \geq (y', \omega_i - z'_i)$, $u_i(y, \omega_i - z_i) \geq u_i(y', \omega_i - z'_i)$.*
- **(b):** *Indispensability of the private good, that is, for any $(y, \omega_i - z_i) \in Q_i$ with $z_i < \omega_i$, $u_i(y, \omega_i - z_i) \geq u_i(y', 0)$ for any $(y', 0) \in Q_i$.*

Remark that for any $(y, 0), (y', 0) \in Q_i$, $u_i(y, 0) = u_i(y', 0)$ for all $i \in N$ by Assumption 1(b) if u_i is continuous.

Here, we define as $E = (N, M, \{Q_i\}_{i \in N}, \{u_i\}_{i \in N}, \{\omega_i\}_{i \in N}, \{C_j\}_{j \in M})$ a *public goods economy.* In the public goods economy, we call a tuple of consumption vectors $((y, \omega_i - z_i)_{i \in N}) \in \prod_{i \in N} Q_i$ an *allocation.* Note that the amount of public goods consumed by each agent is same at any allcation. We say an allocation $((y, \omega_i - z_i)_{i \in N})$ is *feasible* iff $\bar{C}(y) \leq \sum_{i \in N} z_i$ and $z_i \leq \omega_i$ for all $i \in N$. Let us define as A the set of the feasible allocations. We introduce an equilibrium concept defined by Mas-Colell and Silvestre [10] that is called the *cost share equilibrium.*

Definition 1. *(Mas-Colell and Silvestre* [10]*)*

A collection of functions $g = (g_1, \cdots, g_n)$, where for all $i \in N$, $g_i : Y \to [0, \omega_i]$, and $\sum_{i \in N} g_i(y) = \bar{C}(y)$, for any $y \in Y$, is called a cost share system.

Note that for any $y \in Y$, $((y, \omega_i - g_i(y))_{i\in N}) \in A$ if g is a cost share system. Note also that the nonnegativity of the range of the cost share system means that there is no transfers of the private good among the agents at the feasible state attained under the cost share system. Let us define as G the set of cost share systems.

Definition 2. *(Mas-Colell and Silvestre* [10]*)*
A pair of $((y, \omega_i - z_i)_{i\in N}) \in A$ *and* $g \in G$ *is called a cost share equilibrium iff for all* $i \in N$, $z_i = g_i(y)$ *and* $u_i(y, \omega_i - z_i) \geq u_i(y', \omega_i - g_i(y'))$ *for all* $y' \in Y$.

Here, we derive a strategic form game $\Gamma = (N, \{X_i\}_{i\in N}, \{v_i\}_{i\in N})$ which represents the public goods economy, called a *voluntary contribution game*, where $N = \{1, \cdots, n\}$ is the set of players, $X_i = \{x_i \in \mathbb{R}^m_+ | \sum_{j\in M} x_{ij} \leq \omega_i\}$ is the set of strategies for player $i \in N$ and $v_i : \prod_{i\in N} X_i \to \mathbb{R}$ is the payoff function of player i. The payoff function of player i is defined by the composite function of $f_i : \prod_{i\in N} X_i \to Q_i$ and $u_i : Q_i \to \mathbb{R}$, where f_i is the outcome function to agent i which we define

$$f_i(x) = (C_1^{-1}(\sum_{i\in N} x_{i1}), \cdots, C_m^{-1}(\sum_{i\in N} x_{im}), \omega_i - \sum_{j\in M} x_{ij}),$$

and u_i is the utility function defined above.

Note that the convexity and strict increasingness of the cost function imply the cost function is continuous, one-to-one and onto function. Hence there exists the inverse function $C_j^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$. Furthermore, C_j^{-1} is increasing and continuous for all $j \in M$. Note also that $f(x) = (f_1(x), \cdots, f_n(x))$ induces a feasible allocation for all $x \in \prod_{i\in N} X_i$.

Any nonempty subset of N is called a coalition. For any coalition S, we denote $X_S = \prod_{i\in S} X_i$ and $x_S \in X_S$. For the simplicity, let us denote $X = \prod_{i\in N} X_i$ and $x \in X$.

The α-core is a strategic core concept introduced by Aumann and Peleg [2]. We first introduce a correspondence $V_\alpha : 2^N \twoheadrightarrow \mathbb{R}^n_+$ which is said to be an α-NTU characteristic function. A correspondence V_α is said to be an α-NTU characteristic function iff, $\nu \in V_\alpha(S) \subset \mathbb{R}^N$ iff $\exists x_S \in X_S,\ \ \forall x_{N\setminus S} \in X_{N\setminus S},\ \ \forall i \in S,\ \ \nu_i \leq u_i(x_S, x_{N\setminus S})$. The α-core is a core in the game (N, V_α), that is, a payoff vector ν is in the α-core iff there is no $S \subset N$ such that $\exists \nu' \in V_\alpha(S),\ \ \forall i \in S,\ \ \nu'_i > \nu_i$. It is easily checked that the α-core is nonempty if and only if there exists some $x \in X$ such that $v(x) = (v_1(x), \cdots, v_n(x))$ is in the α-core.

For any coalition $S \subset N$, a strategy tuple d_S is the *dominant punishment strategy (against* $N \setminus S$*)* (see Hirai, Masuzawa and Nakayama [5]) iff for any $x_{N\setminus S} \in X_{N\setminus S}$, any $x_S \in X_S$, $v_i(x) \geq v_i(d_S, x_{N\setminus S})$ for all $i \in N \setminus S$.

In the voluntary contribution game, for any $S \subset N$, $d_S = 0$ is a dominant punishment strategy. To show this, fix any $x_S \in X_S$ with $x_S \neq 0$, and any $x_{N\setminus S} \in X_{N\setminus S}$. Then, since each C_j^{-1} is increasing,

$$(C_1^{-1}(\sum_{i\in N} x_{i1}), \cdots, C_m^{-1}(\sum_{i\in N} x_{im})) \geq (C_1^{-1}(\sum_{i\in N\setminus S} x_{i1}), \cdots, C_m^{-1}(\sum_{i\in N\setminus S} x_{im}))$$

for any $x_S \in X_S$. Then, by Assumption 1(a), $v_i(x) \geq v_i(0, x_{N\setminus S})$ for all $i \in N \setminus S$. Thus, in the voluntary contribution game, by the definition of the α-core, we can say that a payoff vector $\nu \in V_\alpha(N)$ is in the α-core iff there exists no coalition $S \subset N$ which has some $x_S \in X_S$ with $v_i(x_S, 0) > \nu_i$ for all $i \in S$.

3. The Results

This section contains the results of this paper.

Theorem 1. *Let E be a public goods economy such that u_i*

1: *satisfies Assumption 1, and*
2: *is continuous,*

for all $i \in N$, and $((y, \omega_i - z_i)_{i\in N})$ denote an allocation in E. Let Γ be a voluntary contribution game derived from E. Then, there exists a cost share system g such that $((y, \omega_i - z_i)_{i\in N}, g)$ is a cost share equilibrium in E if and only if any strategy profile x such that $f(x) = ((y, \omega_i - z_i)_{i\in N})$ induces the payoff vector $v(x)$ which is in the α-core of Γ.

To prove this theorem, we prove two lemmata. The first lemma can be verified in partially a similar way to the proof of Proposition 1(3) of Mas-Colell [9].

Lemma 1. *Let E be a public goods economy which satisfies the conditions of Theorem 1. Let $v(x^*) = (v_1(x^*), \cdots, v_n(x^*))$ be in the α-core of Γ derived from E Then, there is a cost share system g such that $(f(x^*), g)$ is a cost share equilibrium.*

Proof. Let $x^* \in X$ be any strategy profile such that $v(x^*)$ is in the α-core.

Fix any player $i \in N$. Let us denote $(y^*, \omega_i - \sum_{j\in M} x^*_{ij}) = f_i(x^*)$.

Define $Z_i(y) = \{z_i \in [0, \omega_i] | u_i(y, \omega_i - z_i) = u_i(f_i(x^*))\}$ for any $y \in Y$. Note that $Z_i(y)$ is a closed set for any $y \in Y$ by the continuity of u_i.

And, let us define a function $h_i : Y \to \mathbb{R}_+$ such that for any $y \in Y$,

$$h_i(y) = \begin{cases} \min Z_i(y) & \text{if } Z_i(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Note that $h_i(y^*) \leq \sum_{j\in M} x^*_{ij}$.

If $Z_i(y) = \emptyset$ for some $y \in Y$, then, we claim that $u_i(y^*, \omega_i - \sum_{j\in M} x^*_{ij}) > u_i(y, \omega_i)$. Suppose, to the contrary, that $u_i(y^*, \omega_i - \sum_{j\in M} x^*_{ij}) \leq u_i(y, \omega_i)$. Then, by the indispensability of the private good, we obtain

$$u_i(y, \omega_i) \geq u_i(y^*, \omega_i - \sum_{j\in M} x^*_{ij}) \geq u_i(y, 0).$$

Then, by the continuity of u_i, there exists some $z_i \in [0, \omega_i]$ such that $u_i(y, \omega_i - z_i) = u_i(y^*, \omega_i - \sum_{j\in M} x^*_{ij})$. This contradicts the hypothesis that $Z_i(y) = \emptyset$. Hence $u_i(y^*, \omega_i - \sum_{j\in M} x^*_{ij}) > u_i(y, \omega_i)$ if $Z_i(y) = \emptyset$. This implies that

$$u_i(y^*, \omega_i - \sum_{j\in M} x^*_{ij}) \geq u_i(y, \omega_i - h_i(y)) \tag{1}$$

for all $y \in Y$.

Next, we claim that $\sum_{i\in N} h_i(y) \leq \bar{C}(y)$ for any $y \in Y$. Suppose, to the contrary, that there exists some $y' \in Y$ such that $\sum_{i\in N} h_i(y') > \bar{C}(y')$. Let us define $T = \{k \in N | h_k(y') > 0\}$. If $T = \emptyset$, then $\sum_{i\in N} h_i(y') = 0 \leq \bar{C}(y')$. This contradicts that $\sum_{i\in N} h_i(y') > \bar{C}(y')$. Thus, $T \neq \emptyset$. Note that $u_k(y', \omega_k - h_k(y')) = u_k(y^*, \omega_k - \sum_{j\in M} x^*_{kj})$ by the definition of h_k. Define $x'_{ij} = \frac{h_i(y')}{\sum_{k\in T} h_k(y')} C_j(y'_j)$ for all $i \in N$ and $j \in M$. Note that $x'_{ij} = 0$ for all $i \in N \setminus T$ and $j \in M$. Then, since $\sum_{k\in T} x'_{kj} = C_j(y'_j)$ for all $j \in M$, we obtain $f_k(x'_T, 0_{N\setminus T}) = (y', \omega_k - \sum_{j\in M} x'_{kj})$ for all $k \in T$. Moreover, by the hypothesis, $\frac{\bar{C}(y')}{\sum_{k\in T} h_k(y')} \in [0, 1)$. Hence

$$\sum_{j\in M} x'_{kj} = \frac{h_k(y')}{\sum_{k\in T} h_k(y')} \bar{C}(y') < h_k(y')$$

for all $k \in T$.

Let k be any player in T. Then, $u_k(y', \omega_k - \sum_{j\in M} x'_{kj}) \geq u_k(y', \omega_k - h_k(y'))$ by Assumption 1(a). And, if $u_k(y', \omega_k - \sum_{j\in M} x'_{kj}) = u_k(y', \omega_k - h_k(y'))$, this contradicts that $h_k(y') = \min Z_k(y')$ since $\sum_{j\in M} x'_{kj} < h_k(y')$. Hence $u_k(y', \omega_k - \sum_{j\in M} x'_{kj}) > u_k(y', \omega_k - h_k(y'))$. Since $u_k(f_k(x'_T, 0_{N\setminus T})) = u_k(y', \omega_k - \sum_{j\in M} x'_{kj})$ and $u_k(y', \omega_k - h_k(y')) = u_k(y^*, \omega_i - \sum_{j\in M} x^*_{ij}) = u_k(f_k(x^*))$, we obtain that $u_k(f_k(x'_T, 0_{N\setminus T})) >$

$u_k(f_k(x^*))$. This contradicts that x^* gives a payoff vector which is in the α-core. Hence $\sum_{i\in N} h_i(y) \le \bar{C}(y)$.

Next, we define a function $g_i : Y \to \mathbb{R}_+$, such that

$$g_i(y) = \begin{cases} \sum_{j\in M} x^*_{ij} & \text{if } y = y^*, \\ h_i(y) & \text{if } y \neq y^* \text{ and } \bar{C}(y) = \sum_{i\in N} h_i(y), \\ h_i(y) + \epsilon_i & \text{if } y \neq y^* \text{ and } \bar{C}(y) > \sum_{i\in N} h_i(y), \end{cases}$$

where $\epsilon_i = \frac{\bar{C}(y) - \sum_{i\in N} h_i(y)}{\sum_{i\in N}\omega_i - \sum_{i\in N} h_i(y)} \cdot (\omega_i - h_i(y))$. Recall that $\sum_{i\in N}\omega_i \ge \bar{C}(y)$ for any $y \in Y$. Hence $\sum_{i\in N}\omega_i - \sum_{i\in N} h_i(y) > 0$ for any $y \in Y$ such that $\bar{C}(y) > \sum_{i\in N} h_i(y)$.

Let us fix any $y \in Y$. The definition of g_i implies that $\sum_{i\in N} g_i(y) = \bar{C}(y)$, $g_i(y) \ge h_i(y)$, and $g_i(y) \in [0, \omega_i]$. Hence $g = (g_1, \cdots, g_n)$ is a cost share system. This fact, inequality (1) and Assumption 1(a) imply that

$$\begin{aligned} u_i(y^*, \omega_i - \sum_{j\in M} x^*_{ij}) &\ge u_i(y, \omega_i - h_i(y)) \\ &\ge u_i(y, \omega_i - g_i(y)). \end{aligned} \tag{2}$$

By the definition of g_i, $g_i(y^*) = \sum_{j\in M} x^*_{ij}$. This fact and inequality (2) imply that $((y^*, \omega_i - \sum_{j\in M} x^*_{ij})_{i\in N}, g)$ is a cost share equilibrium.

□

Then, we prove the inverse.

Lemma 2. *Assume that E satisfies the conditions of Theorem 1. Let $((y^*, \omega_i - z^*_i)_{i\in N}, g)$ be a cost share equilibrium in E. Let x^* be any strategy profile such that $f_i(x^*) = ((y^*, \omega_i - z^*_i))$ for all $i \in N$. Then, $v(x^*) = (v_1(x^*), \cdots, v_n(x^*))$ is in the α-core of Γ.*

Proof. Let $((y^*, \omega_i - z^*_i)_{i\in N}), g)$ be a cost share equilibrium in E. Let x^* be any strategy profile such that $f_i(x^*) = (y^*, \omega_i - z^*_i)$ for all $i \in N$. Suppose that there exists some $S \subset N$, which has some $x_S \in X_S$ with $u_i(f_i(x_S, 0)) > u_i(y^*, \omega_i - z^*_i)$ for all $i \in S$.

Define $y = (C_1^{-1}(\sum_{i\in S} x_{i1}), \cdots, C_m^{-1}(\sum_{i\in S} x_{im}))$. Thus, $f(x_S, 0_{N\setminus S}) = ((y, \omega_i - \sum_{j\in M} x_{ij})_{i\in S}, (y, \omega_k)_{k\in N\setminus S})$. We claim that $\sum_{j\in M} x_{ij} < g_i(y)$ for all $i \in S$. Suppose, to the contrary, that $\sum_{j\in M} x_{kj} \ge g_k(y)$ for some $k \in S$. By the fact that $((y^*_, \omega_i - z^*_i)_{i\in N}), g)$ is a cost share equilibrium and Assumption 1(a), it follows that

$$u_k(y, \omega_k - \sum_{j\in M} x_{kj}) \le u_k(y, \omega_k - g_k(y)) \le u_k(y^*, \omega_k - z^*_k).$$

This contradicts the hypothesis that $u_k(f_k(x_S, 0_{N\setminus S})) > u_k(y^*, \omega_k - z_k^*)$. Hence $\sum_{j\in M} x_{ij} < g_i(y)$ for all $i \in S$. Thus,

$$\sum_{i\in S}\sum_{j\in M} x_{ij} < \sum_{i\in S} g_i(y) \leq \sum_{i\in N} g_i(y) = \bar{C}(y).$$

But, this contradicts that $y = (C_1^{-1}(\sum_{i\in S} x_{i1}), \cdots, C_m^{-1}(\sum_{i\in S} x_{im}))$. Hence $v(x^*)$ is in the α-core of Γ. □

Proof of Theorem 1.

If part is immediate from Lemma 1. And, only if part is immediate from Lemma 2.

□

The β-core is another strategic core concept also introduced by Aumann and Peleg [2]. It is easily checked that the α-core coincides with the β-core if all coalition $S \subsetneq N$ has a dominant punishment strategy (see also Nakayama [11]). Thus, we may state as follows.

Corollary 1. *Let E be a public goods economy which satisfies the conditions of Theorem 1, and $((y, \omega_i - z_i)_{i\in N})$ denote an allocation in E. Let Γ be a voluntary contribution game derived from E. Then, there exists a cost share system g such that $((y, \omega_i - z_i)_{i\in N}, g)$ is a cost share equilibrium in E if and only if any strategy profile x such that $f(x) = ((y, \omega_i - z_i)_{i\in N})$ induces the payoff vector $v(x)$ which is in the β-core of Γ.*

We must note that Theorem 1 does not assure the nonemptiness of the α-core, or the existence of the cost share equilibrium. It was shown in Scarf [12] that if in a strategic form game, the strategy set of each player is compact and convex, and the payoff function of each player is continuous and quasi-concave, then the α-core of this game is nonempty. Next proposition shows that the voluntary contribution game Γ satifies these conditions above.

Proposition 1. *Let E be a public goods economy such that u_i*

- **(i):** *satisfies Assumption 1(a), and*
- **(ii):** *is continuous and quasi-concave,*

for all $i \in N$. Then, the voluntary contribution game Γ derived from E satifies that

- **(a):** *the set of strategies X_i is compact and convex, and*
- **(b):** *the payoff function v_i is continuous and quasi-concave.*

for all $i \in N$.

Proof. (a)This is obvious from the definition of X_i.

(b)Fix any $i \in N$. From the discussions above, the outcome function f_i of the strategic form game Γ is defined as

$$f_i(x) = (C_1^{-1}(\sum_{i\in N} x_{i1}), \cdots, C_m^{-1}(\sum_{i\in N} x_{im}), \omega_i - \sum_{j\in M} x_{ij}).$$

Since C_j^{-1} is continuous for any $j \in M$, f_i is obviously continuous. Thus, since u_i is continuous, the function $v_i = u_i \circ f_i$ is also continuous.

Next, we show that C_j^{-1} is concave for any $j \in M$. Suppose, to the contrary, that there exists some $z, z' \in \mathbb{R}_+$ and some $\delta \in [0,1]$ such that $C_j^{-1}(\delta z + (1-\delta)z') < \delta C_j^{-1}(z) + (1-\delta)C_j^{-1}(z')$. By the convexity and the strict monotonicity of C_j, we obtain

$$\begin{aligned} C_j(C_j^{-1}(\delta z + (1-\delta)z')) &< C_j(\delta C_j^{-1}(z) + (1-\delta)C_j^{-1}(z')) \\ &\leq \delta C_j(C_j^{-1}(z)) + (1-\delta)C_j(C_j^{-1}(z')). \end{aligned}$$

Then, $\delta z + (1-\delta)z' < \delta z + (1-\delta)z'$. This is a contradiction. Hence C_j^{-1} is concave for any $j \in M$.

Here, we show that v_i is quasi-concave. Let $x, x' \in X$ be any strategy profile. Without loss of generality, we can assume that $u_i(f_i(x)) \leq u_i(f_i(x'))$. Fix any $\theta \in [0,1]$ and denote $x(\theta) = \theta x + (1-\theta)x'$. Then, for any $j \in M$, $\sum_{i\in N} x(\theta)_{ij} = \theta \sum_{i\in N} x_{ij} + (1-\theta)\sum_{i\in N} x'_{ij}$. By the concavity of C_j^{-1},

$$C_j^{-1}(\sum_{i\in N} x(\theta)_{ij}) \geq \theta C_j^{-1}(\sum_{i\in N} x_{ij}) + (1-\theta)C_j^{-1}(\sum_{i\in N} x'_{ij}),$$

for any $j \in M$. Hence $f_i(x(\theta)) \geq \theta f_i(x) + (1-\theta)f_i(x')$. Then, by Assumption 1(a) and the quasi-concavity of u_i,

$$\begin{aligned} v_i(x(\theta)) &= u_i(f_i(x(\theta))) \\ &\geq u_i(\theta f_i(x) + (1-\theta)f_i(x')) \\ &\geq u_i(f_i(x)) \\ &= v_i(x). \end{aligned}$$

Thus, v_i is quasi-concave. □

Hence the α-core of the voluntary contribution game is nonempty.

Next corollary is not quite new. But, in strictly speaking, any of the existence theorem for the cost share equilibrium from Kaneko [7] and Mas-colell and Silvestre [10] does not include our model.

Corollary 2. *In the public goods economy E, a cost share equilibrium exists if the utility function u_i*

1: *satisfies Assumption 1, and*

2: *is continuous and quasi-concave*
for all $i \in N$.

The proof is immediate from Theorem 1 and Proposition 1.

Note that to assure the existence of the cost share equilibrium in E, the quasi-concavity of the utility function of each plyaer must be added to the conditions of Theorem 1.

4. Concluding Remarks

We have proved that the set of the cost share equilibria is equivalent to the strategic cores of the corresponding voluntary contribution game. Mas-Colell [9] showed that the set of the cost share equilibria is equivalent to the core of the economy. But, it was pointed out at there that the core of the economy may be empty. In our model, we also prove that the α-core of the voluntary contribution game is nonempty by applying the theorem of Scarf [12]. Moreover, it can be directly verified that the core of the economy is equivalent to the α-core of the voluntary contribution game.

The voting game is a game of politics, in which the only winning coalitions can make out new proposal. Kaneko [8] and Hirokawa [6] showed that the cost share equilibrium allocation is attained through such the political procedure. The voluntary contributions by agents can be regarded as a more primitive procedure to decide how much amount of private good each agent pays to provide the public goods. We showed that the voluntary contribution can yields the same results as ones attained through such a political procedure.

On the other hand, the *strong Nash equilibrium* is an extended concept of the Nash equilibrium introduced by Aumann [1]. A strategy profile x is a strong Nash equilibrium iff there exists no coalition S such that there exists some $x'_S \in X_S$ with $v_i(x'_S, x_{N\setminus S}) > v_i(x)$ for all $i \in S$. By the definitions of the strong Nash equilibrium and the α-core, it is easily checked that the any payoff vector attained by a strong Nash equilibrium is in the α-core. Thus, any strong Nash equilibrium induces the cost share equilibrium if it exists.

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