Modeling A Player's Perspective I: Info-memory Protocols*†

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Abstract

This two-part paper studies a player's perspective on the social (game) situation he is in. In Part I we develop the theory of info-memory protocols from the viewpoint of the objective observer. This will be used for the development of the theory of inductive derivations of social views by an individual player in Part II. An info-memory protocol consists of an information protocol and a memory function for each player. An information protocol is an alternative formulation of an extensive game emphasizing information pieces and actions as primitives. We show that under the notion of an *infomorphism*, the class of information protocols satisfying certain axioms is equivalent to the class of extensive games in Kuhn's sense. A memory function is a separate and explicit description of a player's memory that allows us to capture and distinguish various aspects of bounded memory. We reconsider memory conditions on extensive games using a memory function.

1. Introduction

1.1. General Motivations and a Rough Sketch of our Theory

In this two-part paper, we develop a theory of info-memory protocols and inductive derivations of individual views on a game. Specifically, we discuss the following subjects

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in the two parts:

(I): *Information protocols* and individual *memory functions* from the viewpoint of the objective outside observer;

(II): *Inductive derivations of an individual view* on the game from the subjective viewpoint of an inside player using his memories of experiences.[1]

Each part is closely related to the other, and both should be, more or less, simultaneously developed. However, we cannot avoid piecewise theoretical developments, as in any mathematical theory. Since the piecewise developments may prevent the reader from understanding even each part, we should provide a rough sketch of the entire picture of our research.

The entire situation is conceptually described as Figure 1.1. Each of $\Gamma_0, \Gamma_1, \Gamma_2, ...$ is a *finite* extensive game. We focus on an individual player's perception and behavior in the particular game Γ_0. It is our presumption that each player participating in Γ_0 notices the occurrence of that game every time it starts to be played, and that the individual behavior depends only upon Γ_0. Hence, the situation depicted in Figure 1.1 is reduced into that in Figure 1.2.

$$\begin{array}{ccccccccc}
 & & & \nearrow & & \nearrow & & \cdots & \\
 & & \Gamma_2 & \rightarrow & \Gamma_0 & \rightarrow & \Gamma_0 & \rightarrow & \cdots \\
 & \nearrow & & & & & & & \\
\rightarrow \Gamma_0 & \rightarrow & \Gamma_1 & \rightarrow & \Gamma_2 & \rightarrow & \cdots & & \\
 & \searrow & & \searrow & & & & & \\
 & & \Gamma_0 & \rightarrow & \Gamma_1 & \rightarrow & \Gamma_0 & \rightarrow & \cdots \\
 & & & \searrow & & \searrow & & \ddots &
\end{array}$$

Figure 1.1

$$\rightarrow \; \Gamma_0 \rightarrow \Gamma_0 \; \rightarrow \cdots \rightarrow \; \Gamma_0 \; \rightarrow$$

Figure 1.2

We neither formulate the situation of Figure 1.1 as a large (infinite) extensive game nor take the standard repeated game approach to the situation of Figure 1.2. Both need the presumption that the player has full or partial knowledge about the structure of the situation. The aim of the present research is to consider the origin and/or emergence of a player's knowledge/beliefs about the structure from his memories of past plays, under the presumption that he has no *a priori* knowledge/beliefs on the structure of Γ_0.[2]

[1] The term "induction" is used in the original philosophical sense that one obtains a general law from finite observations. This differs from that of "mathematical induction", which is still a deductive inference.

[2] This might sound similar to some papers in the learning literature of game theory such as fictitious in Brown [2], or the slightly more sophisticated learning processes in Kalai-Lehrer [7]. In those approaches, some behavioral rules determine actions based on past histories, and typically the conver-

We differentiate our theory from the standard game theoretical approach which is stated as:

(*Ex Ante* **Approach**): Each player makes a decision before the actual play of the game. Here, each player is presumed to know, *a priori*, the structure of the game or the candidates (types) of the games. By the structure of the game we mean players, decision nodes, information sets, available actions, preferences, etc.

If the entire situation of Figure 1.1 or Figure 1.2 is regarded as a large one-shot game, then the *ex ante* approach is inevitable. The theory of incomplete information games initiated by Harsanyi [5] is also in the scope of this approach.

In this paper, we take the following alternative approach.

(**Inductive Approach**): Each player has been playing the game Γ_0 repeatedly and acquiring information and forming beliefs about Γ_0. He takes a (partially or fully) optimal behavior using his formed beliefs about the game. The presumption here is that in the beginning, each player is ignorant of the structure of the game.[3]

The basic idea for the inductive (game) approach was given in Kaneko-Matsui [10]. This two part paper makes more thorough investigations of the idea, and widens its scope quite a lot. In fact, the *ex ante* decision making will be included as part of our inductive approach, which will be discussed in part II. For these developments, we take various radical steps, while borrowing extant basic concepts from game theory.

The salient difference between our theory and the standard game theory appears in the treatment of (partial) knowledge/belief on the structure of a game. In particular, the treatment of information is different and basic.

To motivate this difference, we refer to Plato's [19] *"analogy of the cave"*: In the cave, prisoners have been chained from their birth, and their faces have been fixed to see only the wall of the bottom of the cave. From time to time, they have noticed shadows on the wall projected from the outside. From those experienced shadows, each prisoner may have developed his view of the world. Plato continues this analogy to discuss the nature of individual beliefs and knowledge.

Here, a shadow is a piece of information each prisoner receives. The analogy intends to describe a situation where a prisoner has no idea about the possible causes behind each shadow. In fact, he might just accumulate information for some time, without

gence or nonconvergence to some point such as Nash equilibrium is considered. These do not address the question of how a player comes to understand the structure of the game he is playing, which is our concern.

[3] In the recent literature of game theory, there is another approach, called the *evolutionary game approach*. Sometimes, this approach is regarded as an inductive approach. However, in the evolutionary approach, "memory" is typically treated as externally stored in the distribution of strategies (genes). No epistemological ability is assumed in each individual "player (gene)". See the introduction of Kaneko-Matsui [10], and for evolutionary games, see Weibull [22].

thinking about the structure of his world or cause of different events. This is consistent with our behavior in much of our life. We go to work, parties, excercise, eat, sleep and usually do not think about the cause or structure of our social situation. Then, at some point, we might try to understand our world. We will capture this attempt to understand our social situation by the inductive derivation of a view in Part II. Before we get there, however, we need to lay the foundation, which is the subject matter of Part I.

To capture the feature that each shadow is nothing more than a piece of information, we will not treat information recieved as an information set, as in the standard game theory. Instead, we take information to be a symbolic expression, perhaps a sentence, which we call an *information piece*. A shadow in Plato's analogy is just an information piece.

In the theory of extensive games or Harsanyi's [5] incomplete information game approach, partial knowledge/belief is captured by a set of candidates or types (with a probability distribution over the set of candidates). The identification of information with a set is also quite common in the standard mathematics.[4] However, if we adopt the *identification of a sentence with a set* as a general principle of expressing knowledge/belief, then the presumption that a player is cognizant of elements of the set sneaks into the theory. We want to avoid this, and so we treat information simply as pieces. This affects not only the treatment of an individual belief but also the choice of the very foundation of our theory.

Let us continue a sketch of our theory. In the game situation, the activity of each player is divided into:

(a): external interactions with other players;

(b): internal mental activities.

Although we discuss both in this paper, an *information protocol* (sometimes called a protocol) is introduced in Part I to describe the aspect (a). A *memory function* of a player is also introduced in Part I and is the interface between (a) and (b). The inductive derivation of a view on the game by a player falls into (b), and is the subject of Part II. This derivation is based on past individual experiences, which have been accumulated through his memory function.

In our theory, we need to discuss another aspect of the game situation:

(c): experienced past history over repeated plays of Γ_0.

For example, if the same actions in Γ_0 have always been played in the past, the history and experiences give almost no ingredients for induction. In this case, only a poor view

[4]The starting point of mathematical logic is to separate symbolic expressions from its set-theoretical extension (if it ever exists). That is, syntax (proof theory) and semantics (model theory) are separated, and are connected by the completeness theorem (cf. Mendelson [14], Kaneko [8]). This is also the starting point of our study.

is derived. Thus, the domain of experiences would matter in the inductive derivation of Part II. In a separate paper, we will consider the generation of the domain of experiences, and also equilibrium behavior.

1.2. Developments in Part I

We are prepared for a more detailed discussion of our approach of Part I which involves (a) and the interface to (b). We can now discuss the foundations of our theory and compare it to the extant game theory.

The aspect (a) (external interactions) has a *passive* and an *active* part. The passive part is the receipt of information and the active part is the choice of an action. The active part is not problematic, but in the passive part, we assume that information comes to a player as an *information piece*, as already mentioned. In the analogy of the cave, a shadow is an information piece. An information piece can also be understood as a sentence, e.g., a well-formed formula in the sense of mathematical logic. Here, we avoid the identification of a sentence with a set. Technically speaking, an information piece could be an information set in the standard sense, but our intention is that information pieces are symbolic expressions.

Yet another structure is needed to describe (a). It is the causality relation between a history of actions together with information pieces and a new information piece. An information protocol is formulated as a triple consisting of the set of information pieces, available actions, and a *causality relation*. That is, it consists of the following three primitives:

(i) information pieces to be received;

(ii) available actions;

(iii) causal relation from information pieces and actions to a new information piece.

The primary purpose of the introduction of an information protocol is the investigation of the individual subjective description of a game situation. However, in order to compare a player's subjective view to the objective situation, we will also use an information protocol for the objective description. These two purposes are separated by different sets of axioms: a set of *basic* axioms is adopted for the individual subjective description, while additional *non-basic* axioms are added for the objective description.

We should give one comment on our axiomatic method. It is taken from the viewpoint of an *analyst* to investigate the objective situation and subjective descriptions. The axiomatic consideration is the only method of investigation for the analyst. The *objective observer* knows the entire structure of the protocol and serves as a reference point. Thus, the axiomatic consideration is irrelevant for him. It is also irrelevant to a player until he starts to analyze the situation, in which case he might act as an analyst. In Part I, the axiomatic consideration is taken from the viewpoint of an analyst comparing extensive games with information protocols. In Part II, some theorems will be considered from the viewpoint of a player acting as an analyst.

Here, we make the research strategic assumption that the objective game situation is, in effect, given as an extensive game in Kuhn's [12] sense. We are taking the position that Kuhn's extensive game is an appropriate description for the objective view. Nevertheless, since we use the notion of an information protocol, we need to show that a protocol with certain axioms is "equivalent" to an extensive game in Kuhn's sense. Our full set of axioms is chosen for this "equivalence". In Part I, we will show this using the notion of an "infomorphism", which preserves the shape of information and causality.[5]

The reason for not adopting directly an extensive game in Kuhn's sense as the description of the objective situation is for ease of comparison of the subjectively derived protocols with the objective protocol. The "equivalence" result gives a well specified reference point.

As already stated, the interface between (a) and (b) is:

(iv): the memory function $\mathfrak{m}_i$ of player i.

We separate individual memory from information pieces. Individual memory consists of a historical sequence of received information pieces and actions taken. Each information piece has possibly some description of past histories, but we take the position that a memory function plays the role of describing such historical memories.

A memory function connects the memories in the mind of a player with the external world. In our theory, these memories are explicitly formulated as objects in the mind of a player. They are distinguished from the information pieces and objective histories.

An information set in an extensive game involves the two roles of describing information received at an information set, and describing individual memory at that information set. If we forget the second role of memory in an information set, we have the conceptual correspondence between

$$\text{information piece (in our theory)} \longleftrightarrow \text{information set (in an extensive game).}$$

This correspondence will be established in the equivalence results in Part I of Sections 5 and 6. When an information set contains memories, there are other ways to compare extensive games with our theory. Another possibility is to compare a pair of an information protocol and memory functions with an extensive game. We will show how such a comparison can be done in Section 7.1. This comparison involves refining the information protocol using the memory function. We show that certain information about the memory function is lost in this refinement.

Once the individual memories are given, we can talk about the inductive derivation of a personal view from his memories described by his memory function. This is the

[5] "Infomorphism" is composed of "information" and "-morphism".

subject of Part II. A player's accumulation of memorized perceptions is the source for the derivation of his view on the objective protocol (the game Γ_0). Our separate treatment of memory by a memory function faciliates our consideration of the inductive derivation of a personal view.

Part I is written as follows. In Section 2, we give a formulation of an information protocol with four basic axioms and two nonbasic axioms. In Section 3 we formulate individual memories using a memory function and show how to capture various types of imperfect recall. Section 4 is a survey of Kuhn's [12] theory of extensive games. In Sections 5 and 6 we show the "equivalence" between an information protocol (which does not include memory functions) and an extensive game. In Section 7, we compare our formulation of memory with that of extensive games. Section 8 gives conclusions and connections to Part II.

2. An Information Protocol and Axioms for It

An information protocol is a description of a game situation with multiple players. In our analysis, an information protocol is used to describe such a game situation objectively, and it is also used separately to describe a player's subjective view of the situation. The definition of an information protocol is given in terms of primitives that are observable for each player while playing the game.

In Section 2.1, we give a definition of an information protocol. In Section 2.2, we provide axioms for it. The first set of axioms are called basic and we require them of objective and subjective descriptions. The other axioms are called nonbasic. We adopt Kuhn's formulation of an extensive game as an objective description. The full set of axioms imply that an information protocol is "equivalent" to an extensive game, which is the subject of Sections 5 and 6.

2.1. Information Protocol

An *information protocol* is given as a triple $\Pi = (W, A, \prec)$:

(IP1): W is a nonempty set of *information pieces*;

(IP2): A is a nonempty set of *actions*;

(IP3): $\prec$ is a subset of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$.[6]

Here $(W \times A)^0 \times W$ is stipulated to be W. Throughout the paper, we assume $W \cap A = \emptyset$ to avoid unnecessary complications. IP3 means that $\prec$ is the union of a unary relation on $(W \times A)^0 \times W = W$, a binary relation on $(W \times A)^1 \times W$, a trinary relation on

[6]Note that this is the union of $((W \times A)^m \times W$ over the natural numbers $0, 1, ...$ The infinite product is not included.

$(W \times A)^2 \times W$,..., etc. In this paper, we consider only *finite* information protocols, i.e., those W, A and $\prec$ are all finite sets.

The subset $\prec$ of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$ is called a *causality relation*. We call each element $\langle(w_1, a_1), ..., (w_m, a_m), w\rangle \in\prec$ a *feasible sequence*. We often write $[(w_1, a_1), ..., (w_m, a_m)] \prec w$ for $\langle(w_1, a_1), ..., (w_m, a_m), w\rangle \in\prec$. We will use $\langle\xi, w\rangle$ to denote an generic element of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$. When $\langle\xi, w\rangle \in \prec \cap((W \times A)^0 \times W)$, the feasible sequence is just $\langle w\rangle$ and is denoted by $\prec w$.

An information protocol $\Pi = (W, A, \prec)$ is intended to describe objectively or subjectively the finite game situation such as Γ_0 in Figure 1.1. In Π, we interpret $[(w_1, a_1), ..., (w_m, a_m)] \prec w$ as meaning that the information pieces $w_1, ..., w_m$ have successively occurred and the action a_t was taken at each w_t $(t = 1, ..., m)$, and then the information piece w occurs. The sequence $[(w_1, a_1), ..., (w_m, a_m)]$ is *not necessarily* the exhaustive history before w. The concept of an exhaustive history to w will be defined later.

We have various interpretations of an element $w \in W$. As already mentioned in Section 1.2, w is interpreted purely as a symbol like a shadow in the Analogy of the Cave. A more rigorous interpretation of w is a well-formed formula or a finite set of well-formed formulae in some language in the sense of mathematical logic. The third one is that w is an information set in the sense of von Neumann-Morgenstern [23] and Kuhn [12]. However, our intention is to take the first or second interpretation rather than the third, though the third interpretation will be used in the translating an extensive game to an information protocol in Section 5.

Since the introduction of "information pieces" is very basic to our approach and is already a big departure from the standard extensive game, we illustrate "information pieces" by giving a simple formulation of the Analogy of the Cave. This example will be used several times in Part I as well as in Part II.

Example 2.1 (Analogy of the Cave with one Prisoner (1)). The story is simplified as follows: The day starts in the *morning*, the shadow of a *man* passes, and then the shadow of a *cart* passes. Finally, a prisoner, 1, recalls the events of the day in the *evening*. No actions are taken by prisoner 1, but time is *passing* which we regard as a unique action p. Here, $W = \{m(orning), M(an), C(art), e(vening)\}, A = \{p\}$, and we depict $\prec$ as Figure 2.1:

$$m \underset{p}{\rightarrow} M \underset{p}{\rightarrow} C \underset{p}{\rightarrow} e$$

Figure 2.1

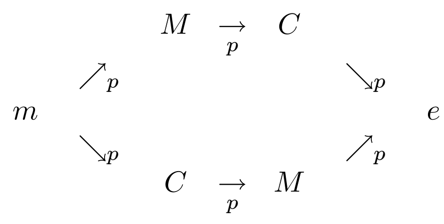

Figure 2.2

The elements in W are symbolic expressions of the shadows of *morning, man, cart* and *evening*. They are merely pieces of information, and the player may not see any causality or connection between the elements. In our theory, he will construct his view of how the pieces are connected from his memory.

Figure 2.1 represents the true objective order in which the pieces appear. In terms of the information protocol describing this situation, $\prec$ has various sequences. Under the axioms to be presented in Section 2.2, $\prec$ has 15 feasible sequences: (1) 4 sequences of length 1 such as $\langle C\rangle$; (2) 6 sequences of length 2 such as $\langle (m,p),C\rangle$; (3) 4 sequences of length 3 such as $\langle (m,p),(C,p),e\rangle$; (4) 1 sequence of length 4, i.e., $\langle (m,p),(M,p),(C,p),e\rangle$. We skip the detailed description of this protocol.

If prisoner 1 tries to recall the events of the day in the evening, but he does not recall the order of M and C, he may construct the subjective information protocol described by Figure 2.2. The precise formulation of his individual memory will be given in Section 3 of Part I. We separate the description of individual memory of a player from the formulation of the objective situation in order to discuss how a player might take the "information pieces" he finds in his memory, and combine them with other information contained in his memory to obtain his subjective protocol. The derivation of his subjective protocol from the "information pieces" in his memory is the topic of Part II.

Let $\Pi = (W, A, \prec)$ be an information protocol. We partition W into:

(Decision Pieces): $W^D = \{w \in W : [(w,a)] \prec u$ for some some $a \in A$ and $u \in W\}$;

(Endpieces): $W^E = W - W^D$.

In the protocol of Figure 2.1, $W^D = \{m, M, C\}$ and $W^E = \{e\}$. These may be regarded as corresponding to information sets and endnodes in an extensive game.

We next introduce the player set $N = \{0, 1, ..., n\}$ to the information protocol Π. Player 0 is called the *chance* player, and players $1, ..., n$ are called *personal* players. A *player assignment* is a function $\pi : W \to 2^N$ satisfying $\pi(w)$ consists of a single player for all $w \in W^D$, and $\pi(w) = \{1, ..., n\}$ for all $w \in W^E$. This means that only one player receives a decision piece[7], but all players receive an endpiece. We will sometimes denote the set of information pieces received by player i by $W_i = \{w \in W : i \in \pi(w)\}$. By the definition of a player assignment, W_i includes W^E for each personal player i.

Each personal player i has a *payoff function* h_i, which is a real-valued function over the set of endpieces W^E. We call (Π, π) an information protocol with a *specification of players*. A triple (Π, π, h) of Π, π and $h = (h_1, ..., h_n)$ is called an information

[7] For the purpose of applications such as the problem of discrimination and prejudices in Kaneko-Matsui [10], it would be more convenient to allow simultaneous moves like Dubey-Kaneko [3]. However, since this paper does not deal with such problems, we assume that only one player moves at a time, to avoid notational complications.

protocol with a *specification of players and payoff functions.* Since this paper is about the construction of a player's perspective, payoff functions will play only a small role.

2.2. Basic and Nonbasic Axioms for an Information Protocol

We give four basic axioms, B1-B4, and two nonbasic axioms, N1-N2, for an information protocol $\Pi = (W, A, \prec)$. Both basic and non-basic axioms will be required of an objective observer. In contrast, only the basic axioms will be required of an inside player. Recall that we consider a finite information protocol $\Pi = (W, A, \prec)$.

2.2.1. Basic Axioms

The first and second axioms require that all information pieces and actions are potentially used in the game.

Axiom B1 (All Pieces Used): $\prec w$ for any $w \in W$.

Axiom B2 (All Actions Used): for any $a \in A$, $[(u, a)] \prec v$ for some $u, v \in W$.

For the next axiom, we need the notion of a subsequence of an sequence in $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$. To this end, we regard each (v_t, a_t) as a component in the sequence, $[(v_1, a_1), ..., (v_m, a_m)] \in \bigcup_{m=1}^{\infty}(W \times A)^m$. We say that $\langle(v_1, a_1), ..., (v_m, a_m), v_{m+1}\rangle$ is a *subsequence* of $\langle(u_1, b_1), ..., (u_k, b_k), u_{k+1}\rangle$ iff $[(v_1, a_1), ..., (v_m, a_m), (v_{m+1}, a)]$ is a subsequence of $[(u_1, b_1), ..., (u_k, b_k), (u_{k+1}, b)]$ for some a and b. For example, $\langle u_{k+1}\rangle$ is a subsequence of $\langle(u_1, b_1), ..., (u_k, b_k), u_{k+1}\rangle$ since $[(u_{k+1}, a)]$ is a subsequence of $[(u_1, b_1), ..., (u_k, b_k), (u_{k+1}, a)]$ for any a. The *supersequence* relation is defined likewise.

The third basic axiom states that all subsequences of feasible sequences are also feasible. This is a requirement for mathematical convenience.

Axiom B3 (Contraction): Let $\langle \xi, v\rangle$ be a feasible sequence, and $\langle \xi', v'\rangle$ be a subsequence of $\langle \xi, v\rangle$. Then $\langle \xi', v'\rangle$ is a feasible sequence.

The last basic axiom is slightly more substantive than the first three. It guarantees that if a player receives a decision piece $w \in W^D$, then the player has some action to take whatever a history to w is.

Axiom B4 (Weak Extension): If $\xi \prec w$ and $w \in W^D$, then there is $a \in A$ and $v \in W$ such that $[\xi, (w, a)] \prec v$.

Any protocol that satisfies Axioms B1 to B4 is called a *basic protocol.*

The information protocol $\Pi = (W, A, \prec)$ described in Figure 2.1 of Example 2.1 satisfies Axioms B1-B4. Axiom B3 requires that any subsequence of the feasible sequence $\langle(m, p), (M, p), (C, p), e\rangle$ is a feasible sequence, and thus $\prec$ has 15 sequences. We can also formulate Figure 2.2 as a protocol satisfying Axioms B1-B4. Axiom B3 is a convenient axiom for subsequent mathematical analyses, but gives too many sequences for illustrations. Presently, we will give a lemma which enables us to focus on some subset of $\prec$.

Before that, however, we remark that an information protocol may be regarded as a concept similar to a graph. One might wonder then, why feasible sequences of lengths more than 2 are needed. The necessity is caused by using one information piece for various "nodes" in a graph. We will give one example to illustrate this necessity. In order to discuss this issue, we introduce more technical terms. In the end of this subsection, we will note another problem in graphical representations of informational protocols.

We say that a feasible sequence $\langle \xi, v \rangle$ is a *maximal feasible sequence* iff there is no proper feasible supersequence $\langle \xi', v' \rangle$ of $\langle \xi, v \rangle$. We say that $\langle w_1 \rangle$ and $\langle (w_1, a_1), ..., (w_k, a_k), w_{k+1} \rangle$ for $k = 1, ..., m$ are *initial fragments* of $\langle \xi, w_{m+1} \rangle = \langle (w_1, a_1), ..., (w_m, a_m), w_{m+1} \rangle$. When $k = m$, $\langle \xi, w_{m+1} \rangle$ itself is an initial fragment of $\langle \xi, w_{m+1} \rangle$. A *position* $\langle \xi, v \rangle$ is an initial fragment of some maximal feasible sequence. Each position can be regarded as an exhaustive history to v. We denote the set of positions by Ξ, and partition Ξ into the set of *end positions* $\Xi^E = \{\langle \xi, w \rangle \in \Xi : w \in W^E\}$ and the set of *decision positions* $\Xi^D = \{\langle \xi, w \rangle \in \Xi : w \in W^D\}$.

Example 2.2 (Absentminded Driver Game (1)). Figure 2.3 is one possible formulation of the absentminded driver game of Piccione-Rubinstein [16]. Specifically, let $W = \{\alpha, \mathbf{0}, \mathbf{1}, \mathbf{2}\}$, $A = \{a, b\}$. This protocol is characterized by the two decision positions $\langle \alpha \rangle$ and $\langle (\alpha, b), \alpha \rangle$ and three ending positions $\langle (\alpha, a), \mathbf{0} \rangle$, $\langle (\alpha, b), (\alpha, a), \mathbf{2} \rangle$ and $\langle \langle (\alpha, b), (\alpha, b), \mathbf{1} \rangle$. At two decision points $\langle \alpha \rangle$ and $\langle (\alpha, b), \alpha \rangle$, player 1 receives the same information piece α. It is the standard interpretation that player 1 does not recall, at the second α, that he already received α. In our theory, this type of forgetfulness can be described separately by a memory function introduced in Section 3.

The purpose of presenting this example here, however, is to explain the necessity of feasible sequences of lengths more than 2. If we restrict our description to feasible sequences of length 2, i.e., the binary part of $\prec$, then we could not distinguish between Figure 2.3 and Figure 2.4. That is, the set of binary relations extracted from these figures are the same. When we use longer feasible sequences, the protocol of Figure 2.4 has the same decision positions as Figure 2.3, but the end positions are different and thus the protocols are distinguished.

$$\begin{array}{ccc} & & \mathbf{1} \\ & & \uparrow_b \\ \mathbf{2} & \longleftarrow_a & \alpha \\ & & \uparrow_b \\ \mathbf{0} & \longleftarrow_a & \alpha \end{array}$$

Figure 2.3

$$\begin{array}{ccc} & & \mathbf{1} \\ & & \uparrow_b \\ \mathbf{0} & \longleftarrow_a & \alpha \\ & & \uparrow_b \\ \mathbf{2} & \longleftarrow_a & \alpha \end{array}$$

Figure 2.4

$$\begin{array}{ccccc} \mathbf{0} & & & & \mathbf{1} \\ & \nwarrow_a & & \nearrow_b & \\ \mathbf{2} & \longleftarrow_a & \alpha & \rightarrow_b & \alpha \end{array}$$

Figure 2.5

As far as only the binary relation is considered, even Figure 2.5 would become a candidate.

We have the following results about positions for basic protocols.

Lemma 2.3. Let $(W, A, \prec)$ be a basic protocol.

(a): If $\langle \xi, w \rangle$ is a feasible sequence, there is a position $\langle \eta, w \rangle$ and η is a supersequence of ξ.

(b): For any $w \in W$, there is a position $\langle \xi, w \rangle$.

(c): $\langle \xi, w \rangle$ is a feasible sequence if and only if $\langle \xi, w \rangle$ is a subsequence of a maximal feasible sequence.

(d): $\langle \xi, w \rangle$ is a maximal feasible sequence if and only if $\langle \xi, w \rangle$ is an end position.

Proof. (a): Let $\langle \xi, w \rangle$ be a feasible sequence. If there is no proper feasible supersequence of it, it is maximal and thus a position. Suppose that there is a proper feasible supersequence $\langle \eta', v \rangle$ of $\langle \xi, w \rangle$. We can take $\langle \eta', v \rangle$ as a maximal feasible sequence since $(W, A, \prec)$ is finite. Then, there is an initial fragment (position) $\langle \eta, w \rangle$ of $\langle \eta', v \rangle$ and η is a supersequence of ξ.

(b): By Axiom B1, we have $\prec w$. Hence, $\langle w \rangle$ is a feasible sequence. By (a), there is a position $\langle \xi, w \rangle$.

(c): By Axiom B3, every subsequence of a maximal feasible sequence is a feasible sequence. Conversely, let $\langle \xi, w \rangle$ be a feasible sequence. Then by (a), there is a position $\langle \eta, w \rangle$ and η is a supersequence of ξ. Since $\langle \eta, w \rangle$ is a position, it is an intial fragment of a maximal feasible sequence $\langle \eta', w' \rangle$ Hence, $\langle \xi, w \rangle$ is a subsequence of the maximal feasible sequence $\langle \eta', w' \rangle$.

(d): Suppose that $\langle \xi, w \rangle$ is a maximal feasible sequence. Then it is a position. If it were a decision position, then by Axiom B4, there would be a feasible sequence $\langle \xi, (w, a), v \rangle$ for some $a \in A$ and $v \in W$. But this is impossible. Hence, $\langle \xi, w \rangle$ is an end position.

Conversely, let $\langle \xi, w \rangle$ be an end position. Then it is an initial fragment of a maximal feasible sequence $\langle \eta, v \rangle$. If $\langle \xi, w \rangle$ is strictly shorter than $\langle \eta, v \rangle$, then there would be an initial fragment $\langle \xi, (w, a), v \rangle$ of $\langle \eta, v \rangle$. Then, $[(w, a)] \prec v$ by Axiom B3, which means $w \in W^D$ contradicting that $\langle \xi, w \rangle$ is an end position. Hence, $\langle \xi, w \rangle$ is a maximal feasible sequence. ■

In particular, Lemma 2.3.(c) and (d) suggest that we can simplify our description of a basic protocol by just listing the set of end positions, rather than the entire list of feasible sequences. *A fortiori*, the set of positions is enough as well. We will use this simplification throughout the paper.

The reader will notice that Figure 2.3 and Figure 2.4 can be represented as basic protocols, but Figure 2.5 cannot. The sequence $\langle (\alpha, b), \alpha \rangle$ would be maximal there, but not an end position violating part (d) of Lemma 2.3 (indeed, Axiom B4 is violated).

Before moving on to the non-basic axioms, we give three more examples of basic protocols to illustrate that basic protocols can be quite different from extensive games.

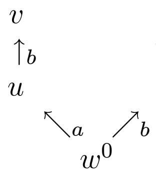

Figure 2.6

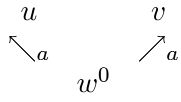

Figure 2.7

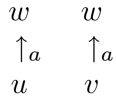

Figure 2.8

The protocol of Figure 2.6 has the feature that the actions available to the player who receives the information piece u depend on the action taken before u. If a was taken, then only b is available, but if b was taken, then only a is available. If u is interpreted as an information set in an extensive game, then it would typically be required that the actions available there be independent of the history before u.

In the protocol of Figure 2.7, the same action a taken at w^0 leads to two different information pieces. This may be caused by a hidden player (maybe, nature) between w^0 and u, v. The protocol of Figure 2.8 has two root pieces, but this still satisfies the basic axioms.

	a	b
a	$5,5$	$1,6$
b	$6,1$	$3,3$

Figure 2.9: Prisoner's Dilemma

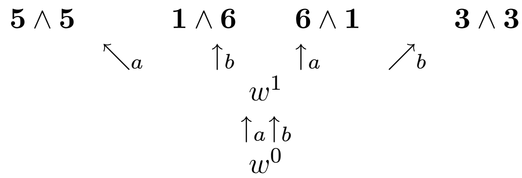

Figure 2.10

Although we will very often use graphical representations, the concept of an information protocol is not truly a geometric one. Let us see this fact. Figure 2.9 is the Prisoner's Dilemma, which is represented by the information protocol of Figure 2.10. As an information protocol, this figure has a certain ambiguity: We may read this figure as meaning that the end piece $\mathbf{1} \wedge \mathbf{6}$ is reached in two ways: $(w^0, a), (w^1, b)$ and $(w^0, b), (w^1, b)$, though we intend to allow only $(w^0, a), (w^1, b)$. To avoid this ambiguity, we may use the graphical representation of Figure 2.11, which is the standard extensive game representation of the Prisoner's Dilemma.

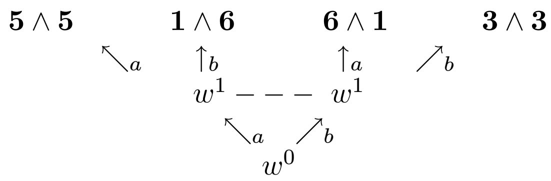

Figure 2.11

2.2.2. Non-basic Axioms

To describe the objective view, we require two further axioms. The first one is a stronger form of Axiom B4 which will be explained after the two axioms are presented.

Axiom N1 (History Independent Extension): If $\langle \xi, u\rangle$ is a position and $[(u,a)] \prec v$, then there is a $w \in W$ such that $\langle \xi, (u,a), v\rangle$is a position.

The next axiom states that a position is an exhaustive history determining the present information piece. For this axiom, we read $\langle \xi, u\rangle$ and $\langle \eta, v\rangle$ so that if $\langle \xi, u\rangle = \langle u\rangle$ and $\langle \eta, v\rangle = \langle v\rangle$, then ξ and η are the same, i.e., $\xi = \eta$.

Axiom N2 (Determination): Let $\langle \xi, u\rangle$ and $\langle \eta, v\rangle$ be positions. If $\xi = \eta$, then $u = v$.

When a basic protocol Π also satisfies Axioms N1 and N2, we call it a *full protocol.* It will be shown in Section 7 that a full information protocol is "equivalent" to an extensive game.

Now, let us consider some implications of Axioms N1-N2. First, Axiom N1 adds to Axiom B4 in two respects: (1) the set of available actions at an information piece $w \in W$ is independent of the history to w; and (2) a position can be extended to a longer position if the last w in the position is a decision piece. The information protocol of Figure 2.6 is eliminated by Axiom N1 by the first additional aspect. We could find an example violating N1 with respect to the second aspect as well, but we just mention it here.

Under Axiom B3, Axiom N1 is really an extension of Axiom B4.

Lemma 2.4. Let $(W, A, \prec)$ be an information protocol satisfying Axiom B3. Axiom N1 implies Axiom B4.

Proof. Let $\xi \prec w$ and $[(w,a)] \prec v$ for some $a \in A$ and $v \in W$. By Lemma 2.3(a), there is a position $\langle \eta, w\rangle$ such that η is a supersequence of ξ. By Axiom N1, there is a $u \in W$ such that $\langle \eta, (w,a), u\rangle$ is a position. Applying Axiom B3 to $\langle \eta, (w,a), u\rangle$, we have the desired subsequence $\langle \xi, (w,a), u\rangle$, i.e., $[\xi, (w,a)] \prec u$. ■

Axiom N2 eliminates the protocols of Figures 2.2, 2.5, 2.7 and 2.8, that are all basic protocols. It would be natural to require Axiom N2 if we give a full description of the world from the objective point of view. On the other hand, a player's perspective might not satisfy this axiom when he has limited knowledge of the causality relation. So as to satisfy Axiom N2, a player may invent new players to describe the lack of determination in his information protocol. In Part II we discuss these types of issues.

Axiom N2 guarantees the existence of a distinguished information piece $w^0 \in W$ which appears at the start of every position. We call this piece the *root piece*, which corresponds to the root information set in the tree of an extensive game.

Lemma 2.5 (Root Piece). Let $(W, A, \prec)$ be a basic protocol satisfying Axiom N2. There is a distinguished piece $w^0 \in W$ called the root and:

(a): $\langle u \rangle$ is a position if an only if $u = w^0$.

(b): If $\langle (u_1, a_1), ..., (u_m, a_m), u_{m+1} \rangle$ is a position, then $u_1 = w^0$.

Proof. Let $\langle u \rangle$ and $\langle v \rangle$ be any positions. The existence of at least one such position is guaranteed for a basic protocol. Indeed, Lemma 2.3.(b) states the existence of a position, and the initial fragment of length 1 of this position is also a position. By Axiom N2 (with the stipulation of the null ξ), we have $u = v$. We denote u by w_0. By this, we have (a). Consider (b). Let $\langle (u_1, a_1), ..., (u_m, a_m), u_{m+1} \rangle$ be a position. Then $\langle u_1 \rangle$ is also a position. Hence, $u_1 = w^0$ by (a). ■

For the sake of reference, we mention the following axiom. It requires that the same information piece is not received more than once in any single play of the protocol. This is related to the basic requirement by Kuhn [12] that every path from the root to an endnode crosses an information set at most once.

Irreflexivity Axiom: For any $v \in W$ and any $a \in A$, not $[(v, a)] \prec v$.

We do not include this axiom in our formulation. The absentminded driver game of Example 2.3 violates the Irreflexivity Axiom.

3. Memory Functions

There are two procedures to describe individual memory in the present setting. One is to include individual memory in each information piece, and the other is to introduce a separate concept to express individual memory. In this section, we adopt the second procedure. Our procedure provides a direct and precise description of individual memory. This enables us to conduct the research in Part II on the inductive derivations of individual views about the game situation.

3.1. Definition and Examples

Memory has a lot of aspects such as forgetfulness, partial and/or false perceptions, memory of past memories, etc. We would like our definition of memory to encompass those different aspects. First, we need to prepare some new concepts.

We may distinguish between an actual information piece received by a player and his perception of it. To allow for this possibility and some others, we extend W and A to W^* and A^*, which are sets of perceived pieces including all elements of W and A, respectively. If the players accurately perceive and record all elements observed, then we could restrict $W^* = W$ and $A^* = A$.

A memory of a player at a point of time consists of a finite number of (memory) threads of past perceived information pieces, available action sets, and actions taken. Formally, a *(memory) thread* is a finite sequence

$$\mu = \langle (v_1, B_1, b_1), ..., (v_m, B_m, b_m), (v_{m+1}, B_{m+1}) \rangle \tag{3.1}$$

with $v_t \in W^*$, $b_t \in B_t \subseteq A^*$ for all $t = 1, ..., m$ and $v_{m+1} \in W^*, B_{m+1} \subseteq A^*$. Each component (v_t, B_t, b_t) or (v_{m+1}, B_{m+1}) in a thread μ is called a *(memory) knot*. A thread differs from a position since the perceived available action sets, B_t's, are included. We denote the ending pair (v_{m+1}, B_{m+1}) of μ by $\varepsilon(\mu) = (\varepsilon_1(\mu), \varepsilon_2(\mu))$ which we call the *tail* of thread μ. This is meant to describe the player's current perception of his current information piece and current available action set. Also, we denote the set of positions for player i by Ξ_i, i.e., $\Xi_i = \{\langle \xi, w \rangle \in \Xi : i \in \pi(w)\}$.

Definition 3.1. A function $\mathfrak{m}_i$ with the domain Ξ_i is an *individual memory function* of player i iff for each position $\langle \xi, w \rangle \in \Xi_i$, $\mathfrak{m}_i\langle \xi, w \rangle$ is a finite non-empty set of threads with the conditions[8] that for all $\mu, \eta \in \mathfrak{m}_i\langle \xi, w \rangle$,

$$\varepsilon(\mu) = \varepsilon(\eta) \tag{3.2}$$

$$\varepsilon_2(\mu) \neq \emptyset \Longleftrightarrow w \in W^D. \tag{3.3}$$

We are now in a position to define the basic mathematical structure for our theory.

Definition 3.2. An *info-memory protocol* is a triple $(\Pi, \pi, \mathfrak{m})$ of an information protocol Π with a player player assignment π and a profile $\mathfrak{m} = (\mathfrak{m}_1, ..., \mathfrak{m}_n)$ of memory functions.

If we add a profile of payoff functions to $(\Pi, \pi, \mathfrak{m})$, we would obtain a complete objective description of the game situation. We will use the structure of an info-memory protocol to describe the objective description and the subjective description. The objective description will be compared to an extensive game later in Part I. The subjective description will be described fully and discussed in Part II.

Now we give some comments on the concept of a memory function, thread, knot, and the relationship between the value of a memory function $\mathfrak{m}_i\langle \xi, w \rangle$ and the actual

[8]It would be more natural to restrict the domain Ξ_i to a smaller subset of actually "experienced" positions. This will be discussed in a future paper.

position $\langle \xi, w \rangle$. A memory at a position is a set of threads and each thread is a sequence of knots. We can think of knots as the basic building blocks for a memory.

A knot (v_t, B_t, b_t) or (v_{m+1}, B_{m+1}) includes the perceived set of available actions, while an objective set of available actions is determined by the information protocol itself. A player i may perceive only a small part of objective available actions. This affects his subjective consideration of the objective situation. This will play an important role in Part II.

We can, however, use the following notation to transform a position into a complete and correct memory thread. First, define, for any $w \in W$, the set of *objectively available actions* at the information piece w,

$$A_w = \{a \in A : [(w, a)] \prec u \text{ for some } u \in W\}. \tag{3.4}$$

For a position $\langle \xi, w \rangle = \langle (w_1, a_1), ..., (w_m, a_m), w \rangle \in \Xi$, we define the *correct and complete memory thread*,

$$\langle \xi, w \rangle^* = \langle (w_1, A_{w_1}, a_1), ..., (w_m, A_{w_m}, a_m), (w, A_w) \rangle. \tag{3.5}$$

In general, however, we allow that a perceived information piece in the player's memory may differ from the actual piece. Each thread μ in $\mathfrak{m}_i\langle \xi, w \rangle$ may differ entirely from $\langle \xi, w \rangle^*$ in addition to the fact that μ contains the description of perceived set of available actions. Also, the length of μ may be very different from that of $\langle \xi, w \rangle$.

Furthermore, multiplicity of threads in $\mathfrak{m}_i\langle \xi, w \rangle$ is used to describe ambiguity about the past. In particular, if he forgets historical orders of occurrences of past knots, $\mathfrak{m}_i\langle \xi, w \rangle$ contains multiple memory threads. The lengths of the threads may even differ, if for example, he forgets the number of times something happened in the past.

We impose conditions (3.2) and (3.3) on a memory function $\mathfrak{m}_i$ as a minimum requirement. Condition (3.2) means that the current perception of the current information piece and available actions is unique, i.e., he does not consider multiple possible perceptions of his current information recieved. Condition (3.3) means that player i perceives the game situation as ending precisely when it ends.

To show what kind of memory structures Definition 3.1 captures, we give a variety of examples of memory functions. First, we return to the Analogy of the Cave given in Figure 2.1.

Example 3.3 (Analogy of the Cave with one Prisoner (2)). This protocol has four positions: $\Xi_1 = \{\langle m \rangle, \langle (m, p), M \rangle, \langle (m, p), (M, p), C \rangle, \langle (m, p), (M, p), (C, p), e \rangle\}$. Now, suppose that prisoner 1 perceives only the current information piece (and available action set) in the day time, and that he tries to recall the events of the day in the evening. If he does recall the past events with their order in the evening, then his memory function is described as

$$\mathfrak{m}_i\langle \xi, w\rangle = \begin{cases} \{\langle (w,\{p\})\rangle \text{ if } w = m, M, C \\ \\ \{\langle \xi, w\rangle^*\} \text{ if } w = e. \end{cases} \tag{3.6}$$

Note that $\langle \xi, w\rangle^* = \langle (m,\{p\},p),(M,\{p\},p),(C,\{p\},p),(e,\emptyset)\rangle$ for $w = e$. At each position $\langle \xi, w\rangle$ with $w = m, M, C$, his memory function gives only the current information piece and available action set, and at $\langle \xi, w\rangle$ with $w = e$, his memory function gives the complete correct history.

Next, suppose that in the evening, prisoner 1 can recall the events of the day, but not their order. In this case, the memory function is modified with the replacement of the second line of (3.6) by

$$\mathfrak{m}_i\langle \xi, w\rangle = \{\langle \xi, w\rangle^*, \langle \xi', w\rangle^*\} \text{ if } w = e, \tag{3.7}$$

where $\langle \xi', w\rangle^* = \langle (m,\{p\},p),(C,\{p\},p),(M,\{p\},p),(e,\emptyset)\rangle\}$. This means that for prisoner 1, there are two possible orderings of the events. He cannot discriminate one from the other. As mentioned earlier, if prisoner 1 tries to reconstruct a protocol based on his memory, he may find the information protocol of Figure 2.2 which is incorrect, but consistent with his memory.

Example 3.4 (Absentminded Driver Game (2)). Let us consider the absentminded driver game of Figure 2.3 of Example 2.2. In this protocol, a simple memory function is given by:

$$\mathfrak{m}_1^M\langle \xi, w\rangle = \begin{cases} \langle (w,\{a,b\})\rangle & \text{if } w = \alpha \\ \\ \langle (w,\emptyset)\rangle & \text{if } w = \mathbf{2},\mathbf{1},\mathbf{0}. \end{cases} \tag{3.8}$$

In this case, player 1 has no past memory and only perceives the current piece and available actions. This memory function will be discussed more generally below. In this specific example, this memory function describes a player who at the position $\langle (\alpha, b), \alpha\rangle$ has forgotten that he received α previously.

In the standard game theory, a player is presumed to know the structure of the game, which means in this context that he is cognizant of his forgetfulness. The next memory function turns out to be better suited to this interpretation while capturing a similar type of forgetfulness. It is given by,

$$\mathfrak{m}_1^C\langle \xi, w\rangle = \{\langle \zeta, v\rangle^* : \langle \zeta, v\rangle \in \Xi \text{ and } v = w\}. \tag{3.9}$$

Recall that $\langle \zeta, v\rangle^*$ is defined by (3.5). His memory at $\langle \alpha\rangle$ is the same as his memory at $\langle (\alpha, b), \alpha\rangle$, and consists of two threads: $\langle (\alpha,\{a,b\})\rangle$ and $\langle (\alpha,\{a,b\},b),(\alpha,\{a,b\})\rangle$. This multiplicity of threads describes that the player cannot distinguish the two positions. In this sense, this memory function, like its predecessor, describes a player who at $\langle(\alpha, b), \alpha\rangle$ has forgotten that he received α previously. Nevertheless, these memory functions are extremely different. Both will be discussed in the Section in 3.2. In Part II, we will see that the construction of a subjective protocol based on these two memory functions leads to quite different results.

3.2. Some Classes of Memory Functions

The memory functions we have looked at thus far are defined for a specific information protocol. Here, we give five general classes of memory functions.

Markov Memory Function: In this function, any observations immediately lapse from memory. We define the *Markov* memory function $\mathfrak{m}_i^M$ as:

$$\mathfrak{m}_i^M\langle\xi, w\rangle = \{\langle(w, A_w)\rangle\} \text{ for } \langle\xi, w\rangle \in \Xi_i. \tag{3.10}$$

Here, player i has the actual current piece w and actual action set A_w in his memory, but he does not recall the past at all. Once he has moved, all is forgotten. A particular case is given in (3.8).

Perfect Information Function and (exact) Perfect Recall Function: The opposite extreme to the Markov memory is perfect information. We define the $\mathfrak{m}$-*perfect information* function $\mathfrak{m}_i^{PI}$ by:

$$\mathfrak{m}_i^{PI}\langle\xi, w\rangle = \{\langle\xi, w\rangle^*\} \text{ for } \langle\xi, w\rangle \in \Xi_i. \tag{3.11}$$

Recall that $\langle\xi, w\rangle^*$ is defined by (3.5) as the correct and complete memory thread. This function describes the idea that player i both observes and recalls accurately all realized triples (v_t, A_{v_t}, a_t) of himself as well as of the others up to w.

A variant of $\mathfrak{m}_i^{PI}$ is the (*exact*) $\mathfrak{m}$-*perfect recall* function $\mathfrak{m}_i^{PR}$, which is defined by restricting $\langle\xi, w\rangle^*$ to his own positions. That is, let $\langle\xi, w\rangle_i^*$ is the subthread of $\langle(w_1, A_{w_1}, a_1), ..., (w_m, A_{w_m}, a_m), (w, A_w)\rangle$ consisting only his part $w_t \in W_i$. Then, we define $\mathfrak{m}_i^{PR}$ as follows:

$$\mathfrak{m}_i^{PR}\langle\xi, w\rangle = \{\langle\xi, w\rangle_i^*\} \text{ for } \langle\xi, w\rangle \in \Xi_i. \tag{3.12}$$

Thus, he recalls exactly what he has received and has done including the accurate perceptions of his action sets.

Classical Memory Function: We define the *classical memory function* $\mathfrak{m}_i^C$ by:

$$\mathfrak{m}_i^C\langle\xi, w\rangle = \{\langle\eta, v\rangle^* : \langle\eta, v\rangle \in \Xi \text{ and } v = w\} \text{ for } \langle\xi, w\rangle \in \Xi_i. \tag{3.13}$$

This function differs from the perfect information memory function since it gives all threads $\langle \eta, w\rangle^*$ ending with w, not just the true one $\langle \xi, w\rangle^*$. The memory function given by (3.9) in Example 3.4 is a special case. This memory function is based on the classical interpretation that player i is cognizant of the entire structure of the game. That is, he can infer all the histories consistent with the current piece w since he knows the entire structure of the game. This interpretation will be discussed further in Section 7.2.

Up to now, all the memory functions could be formulated with $W^* = W$ and $A^* = A$. Suppose now that player i recalls only his moves but not their details. To facilitate this forgetfulness, we introduce the piece θ to W^* and A^*, i.e., $W^* = W \cup \{\theta\}$ and $A^* = A \cup \{\theta\}$. When $\mathfrak{m}_i\langle \xi, w\rangle$ includes the triple $(\theta, \{\theta\}, \theta)$, this is interpreted as a memory of one move but no details about the piece, available action set and action taken. Alternatively, the triple $(v_t, \{\theta\}, \theta)$ with $v_t \in W_i$ means that player i has a memory of his information piece v_t, but not the available actions or the action he took.

Orwell Memory Function: We define the *Orwell* memory function $\mathfrak{m}_i^O$: for $\langle \xi, w\rangle \in \Xi_i$,

$$\mathfrak{m}_i^O\langle \xi, w\rangle = \{\langle(\theta, \{\theta\}, \theta), ..., (\theta, \{\theta\}, \theta), (w, A_w)\rangle\}, \tag{3.14}$$

where the length of the sequence preceding (w, A_w) is the number of times he had received information pieces before w. This memory function is interpreted as meaning that player i makes a mark each time he receives an information piece and can count those past marks at any of his positions, but forgets the details of his moves.[9]

4. Extensive Games and Individual Memory

In this section, we review Kuhn's [12] formulation of extensive games.[10] We will adopt this as a full representation of an objective situation. In Sections 5 and 6, we will show the "equivalence" between an information protocol with all the axioms and an extensive game.

4.1. Extensive Games

We define an *extensive game* $\Gamma = ((X \cup Z, x^0, <), \mathfrak{I}, \{C_I\}_{I \in \mathfrak{I}})$ *without a specification of players and payoff functions* as follows:

[9] We take the name "Orwell" from George Orwell's novel *Animal Farm* ([17]). This book described the process of pigs dominating and exploiting other animals by taking advantage of their inability to recall details.

[10] There are various formulations of extensive games such as in Selten [21], Dubey-Kaneko [3], Osborne-Rubinstein [18] and Ritzberger [20]. Those are essentially the same formulations, except for that given in Dubey-Kaneko [3] where a simultaneous move form is given. The formulation here is also regarded as the same as Kuhn's formulation.

(K1): $(X \cup Z, x^0, <)$ is a finite tree with the *root* x^0, *decision nodes* X and *endnodes* Z;

(K11): $X \cup Z$ is a finite set of nodes with $x^0 \in X$, and $<$ is a partial ordering over $X \cup Z$ with the smallest element x^0;

(K12): $\{x \in X \cup Z : x < y\}$ is totally ordered with $<$ for any $y \in X \cup Z$;[11]

(K13): every node in X has at least one successor, and every node in Z has no successors;[12]

(K2): $\mathfrak{I} = \mathfrak{I}_X \cup \mathfrak{I}_Z$, where $\mathfrak{I}_X$ is a partition of X and $\mathfrak{I}_Z$ is a partition of Z;

(K3): C_I is a finite set of *available choices* for each $I \in \mathfrak{I}$;

(K31): for any $x \in I \in \mathfrak{I}_X$, there is a bijection φ_x from C_I to the set of immediate successors of x;

(K32): $C_I = \emptyset$ for any $I \in \mathfrak{I}_Z$.

Each element I in $\mathfrak{I}$ defined by K2 is called an *information set*. It is additional in K2 to Kuhn's formulation that Z is partitioned into the information sets $\mathfrak{I}_Z$. This is included to capture the notion of end-pieces in an information protocol. Condition K31 states that each $x \in I \in \mathfrak{I}_X$ has the same set of available actions C_I, and condition K32 states that each node $x \in I \in \mathfrak{I}_Z$ has no available actions.

In an extensive game, the binary relation $<$ over nodes is given. Now, we extend it to information sets. For any $x, y \in X \cup Z$ with $x \in I \in \mathfrak{I}_X$ and $c \in C_I$, we define $x <_c y$ iff $\varphi_x(c) = y$ or $\varphi_x(c) < y$. This means that choice c at node x leads to node y. We define $I <_c y$ iff $x <_c y$ for some $x \in I$, and $I <_c I'$ ($I < I'$) iff $I <_c y$ for some y (and c).

As mentioned at the end of Section 2.2, Kuhn [12] restricted attention to irreflexive games, i.e., those games where $I \not< I$ for all I in $\mathfrak{I}$. We do not include this restriction in the definition of an extensive game.

In the same way as in Section 2.1, we can define an extensive game *with* a specification of players and payoff functions. Using the player set $N = \{0, 1, ..., n\}$, we partition $\mathfrak{I}_X$ into $(\mathfrak{I}_{0X}, \mathfrak{I}_{1X}, ..., \mathfrak{I}_{nX})$, where $\mathfrak{I}_{iX}$ $(i = 1, ..., n)$ is the *information partition* of (personal) player i and $\mathfrak{I}_{0X}$ is the information partition of the chance player. We let $\mathfrak{I}_i = \mathfrak{I}_{iX} \cup \mathfrak{I}_Z$ for $i \in N$. When the players are specified, the game is denoted by $\Gamma_N = ((X \cup Z, x^0, <), \mathfrak{I}_0, \mathfrak{I}_1, ..., \mathfrak{I}_n, \{C_I\}_{I \in \mathfrak{I}})$.

[11] The binary relation $<$ is called a *partial ordering* on $X \cup Z$ iff it satisfies (i)(irreflexivity): $x \not< x$; and (ii)(transitivity): $x < y$ and $y < z$ imply $x < z$. It is a *total* ordering iff it is a partial ordering and satisfies (iii)(totality): $x < y$, $x = y$ or $y < x$ for all $x, y \in X \cup Z$. A node x is called the *smallest* element iff $x < y$ or $x = y$ for all $y \in X \cup Z$.

[12] We say that y is a *successor* of x iff $x < y$, and that y is an *immediate* successor of x iff $x < y$ but not $x < y' < y$ for any y'.

Payoff function h_i for each personal player i is defined over $\mathfrak{I}_{iZ}$ as in Section 2.1. A pair $(\Gamma_N, , h) = (\Gamma_N, h_1, ..., h_n)$ is called an *extensive game with a specification of players and payoff functions.*

4.2. Memory Conditions in an Extensive Game

Information sets are used to express individual memory as well as information received. Here, memory itself is not defined; instead, a memory condition such as "perfect recall" is defined as a condition on the information partition of the player. In this section, we review the conditions of perfect information and perfect recall in an extensive game with a specification of players $\Gamma_N = ((X \cup Z, x^0, <), \mathfrak{I}_0, \mathfrak{I}_1, ..., \mathfrak{I}_n, \{C_I\}_{I \in \mathfrak{I}})$.

We recall the definition that player i has $\mathfrak{ip}$-*perfect information* iff every information set $I_i \in \mathfrak{I}_i$ consists of one node. It is standard to interpret this condition as meaning that player i has observed all the players' moves and recalls those observations perfectly.

Next we consider the condition called perfect recall in the literature of game theory. We say that player i has $\mathfrak{ip}$-*perfect recall* in Γ iff for any $x, y \in I_i \in \mathfrak{I}_i, I_i' \in \mathfrak{I}_i$ and $c \in C_{I_i'}$,

$$I_i' <_c x \text{ implies } I_i' <_c y. \tag{4.1}$$

It is interpreted as meaning that player i recalls perfectly (1) what he has learned and (2) what he has done.[13] The following lemma is a simple observation.

Lemma 4.1. Player i has $\mathfrak{ip}$-perfect recall if and only if for any $x, y \in I_i \in \mathfrak{I}_i$, x and y have exactly the same history of his own information sets and actions in the sense that for any $I_i' \in \mathfrak{I}_i$ and $c \in C_{I_i'}, I_i' <_c x \Longleftrightarrow I_i' <_c y$.

5. Comparing Information Protocols to Extensive Games (1)

In this and the next section, we make a comparison between an extensive game in Kuhn's [12] sense and an information protocol but, not an entire info-memory protocol. First, we discuss what should be compared, and particularly, why memory functions are not included in this comparison.

5.1. Comparisons between the two Theories

We took a quite relativistic attitude toward the use of an info-memory protocol in the sense that we will adopt a basic protocol requiring only Axioms B1-B4 for a personal

[13] The perfect recall condition can be decomposed into (1) and (2). Condition (1) is formalized by Okada [15]. Bonanno [1] formalized (2) and gave a careful consideration of the equivalence between perfect recall and (1), (2). More discussions on the $\mathfrak{ip}$-perfect recall condition are given in Kaneko-Kline [9] and Kline [11].

view, while we require all the axioms for an objective description. Here, we would like to fix the meaning of an objective description. A subjective description will be discussed in Part II.

First, it is our basic assumption that an extensive game in Kuhn's [12] sense is adequate for a full description of the objective game situation. Then, our theory of info-memory protocols should capture an extensive game in a certain manner. We will show that our theory of information protocols with all the axioms is "equivalent" to the theory of extensive games in Kuhn's sense.[14] Since these two theories are described by different languages, we need to introduce some concept to make a comparison. The concept of an "infomorphism" to be introduced in Section 6.1 enables us to make such a comparison.

Nevertheless, since our treatment of individual memory in terms of a memory function is not parallel to that in an extensive game, we should elaborate on the relationship between these two treatments. This is relevant to the issue of why an *information* protocol and not the entire *info-memory* protocol is compared with an extensive game.

In our theory, we have two possible procedures to describe individual memories:

(a): individual memory is described by a separate concept from an information protocol;

(b): individual memory is only described in each information piece.

We have adopted procedure (a) in Section 3. It is our research strategy that we separate information pieces from individual memory as much as possible. Nevertheless, it is possible to describe the same content by procedure (b), which we will discuss presently. This fact makes the comparisons between the two theories more complicated.

Even in (a), there are various alternative assumptions on how player i can understand each memory set $\mathfrak{m}_i\langle\xi, w\rangle$. Suppose that player i has some *reading device* to read the set $\mathfrak{m}_i\langle\xi, w\rangle$. Consider the following two possible reading devices:

(1): the reading device can read each memory thread in $\mathfrak{m}_i\langle\xi, w\rangle$;

(2): the reading device can only distinguish $\mathfrak{m}_i\langle\xi, w\rangle$ from $\mathfrak{m}_i\langle\xi', w'\rangle$ set-theoretically, i.e., the device may read $\mathfrak{m}_i\langle\xi, w\rangle$ in the sense of (1) but can only tell whether the sets of memory threads are identical or different.

The devices for (1) and (2) may be very different. For example, consider a perfect information information protocol $(W, A, \prec)$, i.e., one satisfying

$$\text{for any positions } \langle\xi, w\rangle, \langle\eta, v\rangle,\ w = v \ \Rightarrow \xi = \eta. \tag{5.1}$$

From the viewpoint of (1), the Markov and perfect information memory functions $\mathfrak{m}_i^M$ and $\mathfrak{m}_i^{PI}$ are different since $\mathfrak{m}_i^M$ gives memories only of length 1 but $\mathfrak{m}_i^{PI}$ gives longer

[14] Note that here the theory of extensive games means purely what Section 4 describes, but does not include "equilibrium theory" such as Harsanyi's incomplete information game.

memories. From the viewpoint of (2), they are effectively the same in the sense that that for any two positions $\langle \xi, w \rangle, \langle \eta, v \rangle$, $\mathfrak{m}_i^M \langle \xi, w \rangle = \mathfrak{m}_i^M \langle \eta, v \rangle$ if and only if $\mathfrak{m}_i^{PI} \langle \xi, w \rangle = \mathfrak{m}_i^{PI} \langle \eta, v \rangle$.

In our approach we associate the device in the sense of (1) with the memory function of each player which is described as (a1).

When a memory is described only inside a piece w as in procedure (b), the device might read the description fully – (1), or it might read it only up to the set theoretical extension – (2). These cases are denoted by (b1) and (b2). Below, we will explain that (a1) can actually be captured by some method of (b1).

Now, let us see that (b2) corresponds to the treatment of individual memory in the theory of extensive games as described in Section 4.2. In an extensive game, possibly different memories are described within information sets – (b) and are distinguished only set-theoretically – (2). The latter is caused by a set-theoretical description of an information partition. This means that (b2) is effectively assumed in the extensive game theory.

The conclusion of the above paragraph seems to imply that an extensive game should be compared with an info-memory protocol with the device in the sense of (2). However, as far as only (2) is assumed, an info-memory protocol and an information protocol are neither conceptually nor technically different. The main point of the introduction of a memory function is the association with the device in the sense of (1) and so we should like to keep this aspect. This leads us to compare only the information protocol part of an info-memory protocol with an extensive game.

Finally, we argue that (b1) could include (a1). Let us start with a given info-memory protocol $(\Pi, \pi, \mathfrak{m})$. We can construct an information protocol (Π', π') so that it includes the description of $(\Pi, \pi, \mathfrak{m})$. Here we give only a rough idea of this translation. Each information piece w in an information protocol (Π, π) is replaced by the set

$$\{w\} \cup \mathfrak{m}_i \langle \xi, w \rangle, \text{ where } \langle \xi, w \rangle \in \Xi_i, i \in N,$$

and is now regarded as an information piece in (Π', π'). Thus, each piece $\{w\} \cup \mathfrak{m}_i \langle \xi, w \rangle$ includes the information about w and $\mathfrak{m}_i \langle \xi, w \rangle$. In this newly defined information protocol (Π', π'), we would not need to define memory functions if the device in the sense of (1) is assumed. In the same manner, we can argue that (b2) includes (a2).

In sum, with respect to the descriptive power, (a) and (b) together with the corresponding reading devices are the same. The crucial difference comes from the difference between (1) and (2). Nevertheless, the separate treatment of memory by a memory function will facilitate our developments in Part II.[15]

[15] The reader may notice that the choice of (a1) or (b1) ((a2) or (b2)) is, technically speaking, a matter of convenience. But we would like to keep the research principle that conceptually different problems should have different descriptions.

5.2. Extensive Games as Information Protocols

In the following comparisons, we do not include the specifications of players and payoff functions, since the inclusion of them would be easy but make the comparisons cumbersome. Therefore, an information protocol $\Pi = (W, A, \prec)$ is compared with an extensive game $\Gamma = ((X \cup Z, x^0, <), \mathfrak{I}, \{C_I\}_{I \in \mathfrak{I}})$.

Here, we show that an extensive game in the sense of Section 4 can be regarded as an information protocol. Since the relation $<$ in Γ is local, we cannot directly compare it with the causality relation $\prec$ in $\Pi = (W, A, \prec)$. Therefore, first, we extend the local relation $<$ to the global relation $<^*$ over $\bigcup_{m=0}^{\infty}(\mathfrak{I} \times \bigcup_{I \in \mathfrak{I}} C_I)^m$. We define $<^*$ by

(L1): $\langle I \rangle \in <^*$ for all $I \in \mathfrak{I}$;

(L2): for any $\langle (I_1, c_1), ..., (I_k, c_k), I_{k+1} \rangle \in \bigcup_{m=1}^{\infty}((\mathfrak{I} \times \bigcup_{I \in \mathfrak{I}} C_I)^m \times \mathfrak{I})$,
$\langle (I_1, c_1), ..., (I_k, c_k), I_{k+1} \rangle \in <^* \Longleftrightarrow y_t <_{c_t} y_{t+1}$ for some $y_1 \in I_t, ..., y_{k+1} \in I_{k+1}$.

Now, $(\mathfrak{I}, \bigcup_{I \in \mathfrak{I}} C_I, <^*)$ is an information protocol. We say that $(\mathfrak{I}, \bigcup_{I \in \mathfrak{I}} C_I, <^*)$ is the information protocol *induced* from Γ, which we denote by $\Pi(\Gamma)$. Then, we have the following theorem.

Theorem 5.1. Let $\Gamma = ((X \cup Z, x^0, <), \mathfrak{I}, \{C_I\}_{I \in \mathfrak{I}})$ be an extensive game satisfying K1-K3. Then the induced information protocol $\Pi(\Gamma) = (\mathfrak{I}, \bigcup_{I \in \mathfrak{I}} C_I, <^*)$ is a full protocol, i.e., it satisfies Axioms B1-B3 and N1-N2.

Proof. The verifications of B1-B3 are straightforward. Consider Axiom N2. A position $\langle (I_1, c_1), ..., (I_k, c_k), I_{k+1} \rangle$ is an exhaustive history to I_{k+1}. More specifically, $c_1, ..., c_k$ determine the unique exhaustive path $y_1 = x^0 \in I_1, y_2 \in I_2, ..., y_k \in I_k$ and $y_{k+1} \in I_{k+1}$ with $y_t <_{c_t} y_{t+1}$ for $t = 1, ..., k$. Hence, for any $\langle (J_1, d_1), ..., (J_k, d_k), J_{k+1} \rangle$, if $[(I_1, c_1), ..., (I_k, c_k)] = [(J_1, d_1), ..., (J_k, d_k)]$, then $I_{k+1} = J_{k+1}$ is the information set containing y_{k+1}. We can prove N1 similarly. ■

We denote the set of all extensive games with K1-K3 by $\mathcal{G}$, and the set of all induced information protocols by $\mathcal{G}^*$. Also, we denote the set of all information protocols satisfying Axioms B1-B3 and N1-N2 by $\mathcal{P}$. The above theorem is expressed as

$$\mathcal{G}^* \subseteq \mathcal{P}. \tag{5.2}$$

We would like to prove that the three sets $\mathcal{G}, \mathcal{G}^*$ and $\mathcal{P}$ can be regarded as equivalent. This equivalence will discussed in the next section.

6. Comparing Information Protocols to Extensive Games (2)

In Section 5, we have seen that the set of extensive games can be regarded as a set of full protocols, i.e., (5.2). In this section, We will show the converse. This comparison is not so simple as that in Section 5.2, since an extensive game is described by a richer language than an information protocol. For this comparison, we introduce the notion of an infomorphism between two information protocols, which means that two protocols are identical with respect to the information pieces, available actions and causalities. Then, we argue that an information protocol is regarded as an extensive game.

6.1. Infomorphism

Let $\Pi = (W, A, \prec)$ and $\Pi' = (W', A', \prec')$ be two information protocols. We say that ψ is an *infomorphism* from Π to Π' iff

(I1): ψ is a bijection from $W \cup A$ to $W' \cup A'$ with $u \in W \Longleftrightarrow \psi(u) \in W'$;

(I2)(causality-preserving): for any $\langle \xi, u \rangle \in \bigcup_{m=0}^{\infty} (W \times A)^m \times W$,

$\langle \xi, u \rangle \in \prec \Longleftrightarrow \psi \langle \xi, u \rangle \in \prec'$,

where $\psi\langle \xi, u \rangle = \langle (\psi(v_1), \psi(a_1)), ..., (\psi(v_m), \psi(a_m)), \psi(u) \rangle$ for $\xi = [(v_1, a_1), ..., (v_m, a_m)]$.

Condition I1 states simply that ψ is a bijection from W to W' as well as from A to A'. Condition I2 states that the causality relation $\prec$ in Π is faithfully preserved in Π' with ψ.

When there is an infomorphism from Π to Π', Π is said to be *infomorphic* to Π', which is denoted as $\Pi \cong \Pi'$.

Then, this relation is an equivalence relation.

Lemma 6.1^o. The relation $\cong$ is irreflexive, symmetric and transitive, i.e., $\Pi \cong \Pi$, $\Pi \cong \Pi'$ implies $\Pi' \cong \Pi$, and $\Pi \cong \Pi'$ & $\Pi \cong \Pi''$ imply $\Pi \cong \Pi''$.

Proof. We show only that $\cong$ is symmetric. Let ψ be an infomorphism from Π to Π'. Then we show that ψ^{-1} is an infomorphism from Π' to Π. Condition I1 for ψ^{-1} follows I1 for ψ. Consider I2 for ψ^{-1}. Take an arbitrary $\langle \xi', u' \rangle \in \bigcup_{m=0}^{\infty} (W' \times A')^m \times W'$. Then we have $\psi^{-1}\langle \xi', u' \rangle \in \bigcup_{m=0}^{\infty} (W \times A)^m \times W$. Hence, by I2 for ψ, $\psi^{-1}\langle \xi', u' \rangle \in \prec$ $\Longleftrightarrow \psi \circ \psi^{-1}\langle \xi', u' \rangle = \langle \xi', u' \rangle \in \prec'$. ■

We can compare Π with an extensive game Γ. We write $\Pi \cong \Gamma$ iff $\Pi \cong \Pi(\Gamma)$, where $\Pi(\Gamma)$ is the induced information protocol of Γ defined in Section 5.2. In this case, we stipulate $\Gamma \cong \Pi$. For two extensive games Γ, Γ', we write $\Gamma \cong \Gamma'$ iff $\Pi(\Gamma) \cong \Pi(\Gamma')$. Then we extend Lemma 6.1^o.

Lemma 6.1. The relation $\cong$ is an equivalence relation over $\mathcal{P} \cup \mathcal{G}$.

Proof. Irreflexivity and transitivity are straightforward. Symmetry can be verified in the same manner as in the proof of Lemma 6.1^o noting the stipulation that $\Pi \cong \Gamma$ implies $\Gamma \cong \Pi$ ■

It follows from Theorem 5.1 and the definition of $\cong$ that

$$\text{for any } \Gamma \in \mathcal{G},\ \Pi(\Gamma) \in \mathcal{P} \text{ and } \Pi(\Gamma) \cong \Gamma. \tag{6.1}$$

The converse of this is the main step for the equivalence between $\mathcal{G}$ and $\mathcal{P}$, which can now be stated. The proof of Theorem 6.2 will be given in Section 6.3.

Theorem 6.2 (Equivalence up to infomorphisms). For any $\Pi \in \mathcal{P}$, there is a $\Gamma(\Pi) \in \mathcal{G}$ such that $\Gamma(\Pi) \cong \Pi$.

An information protocol $\Gamma(\Pi)$ asserted in Theorem 6.2 is not uniquely determined, but a concrete construction of $\Gamma(\Pi)$ will be given in Section 6.2.

Looking at (6.1) and Theorem 6.2, we can see that the two sets $\mathcal{P}$ and $\mathcal{G}$ are equivalent up to information, actions and causalities. First, using (6.1), Theorem 6.2 and Lemma 6.1 repeatedly, we have

$$\Gamma \cong \Pi(\Gamma) \cong \Gamma(\Pi(\Gamma)) \text{ imply } \Gamma \cong \Gamma(\Pi(\Gamma))$$

$$\Pi \cong \Gamma(\Pi) \cong \Pi(\Gamma(\Pi)) \text{ imply } \Pi \cong \Pi(\Gamma(\Pi)).$$

The first line starts with an extensive game $\Gamma \in \mathcal{G}$, then we obtain the equivalent induced protocol $\Pi(\Gamma)$ by (6.1) and then another extensive game $\Gamma(\Pi(\Gamma))$ by Theorem 6.2. The second line starts with an information protocol $\Pi \in \mathcal{P}$, and we have $\Gamma(\Pi)$ and $\Pi(\Gamma(\Pi))$ by Theorem 6.2 and (6.1). In both cases, we have $\Gamma \cong \Gamma(\Pi(\Gamma))$ and $\Pi \cong \Pi(\Gamma(\Pi))$ by Lemma 6.1. In particular, the first line suggests that $\mathcal{G}^*$ is regarded as equivalent to $\mathcal{G}$.[16]

These results have the implication that as far as we ignore the differences in the underlying structures, the class $\mathcal{P}$ of information protocols satisfying Axioms B1-B3 and N1-N2 could be regarded as the same as the class $\mathcal{G}$ of extensive games defined by K1-K3. As emphasized several times already, however, the primitives of these theories differ. If these primitives matter, so do the theories.

6.2. Derived Extensive Games

In this section, we construct an extensive game $\Gamma(\Pi)$ of Theorem 6.2 for any given information protocol $\Pi = (W, A, \prec) \in \mathcal{P}$. We postpone, to the next subsection, the verification of the fact that $\Gamma(\Pi)$ is an extensive game and is infomorphic to Π.

Let $\Pi = (W, A, \prec) \in \mathcal{P}$. Recall that Ξ is the set of all positions. This Ξ is a finite tree with the naturally defined ordering $<$. Then, we introduce the information sets $\mathfrak{I}$ and the sets of actions $\{C_I\}_{I \in \mathfrak{I}}$.

[16] We can prove the following claim that if $\Gamma \cong \Gamma'$, then there is an isomorphism φ from $X \cup Z$ to $X' \cup Z'$ in the sense that φ fully preserves the structure determined by relation $<$ on nodes, information sets and actions in Γ'.

Now, we define the *derived game structure* $\Gamma(\Pi) = ((X \cup Z, x^0, <), \mathfrak{I}, \{C_I\}_{I \in \mathfrak{I}})$ from $\Pi = (W, A, \prec)$ with its root w^0 as follows:

(T1)(Nodes): $X \cup Z = \Xi$, and $\langle \xi, v \rangle \in Z \Longleftrightarrow \langle \xi, v \rangle$ is an end position in Π;

(T2)(Root): $x^0 = \langle w^0 \rangle$;

(T3)(Causality Relation): for any two nodes $\langle \xi, v \rangle, \langle \eta, u \rangle \in X \cup Z$,

$$\langle \xi, v \rangle <_a \langle \eta, u \rangle \Longleftrightarrow [\xi, (v, a)] \text{ is an initial fragment of } \eta; \tag{6.2}$$

$$\langle \xi, v \rangle < \langle \eta, u \rangle \Longleftrightarrow \langle \xi, v \rangle <_a \langle \eta, u \rangle \text{ for some } a \in A. \tag{6.3}$$

(T4) (Information Sets): $\mathfrak{I} = \{\{\langle \xi, u \rangle : \langle \xi, u \rangle \in \Xi \text{ and } u = v\} : v \in W\}$;

(T5)(Choice Sets): $C_{I_v} = \{a \in A : [(v, a)] \prec u \text{ for some } u \in W\}$ for each $I_v = \{\langle \xi, u \rangle : \langle \xi, u \rangle \in \Xi \text{ and } u = v\} \in \mathfrak{I}$.

It will be shown in Section 6.3 that $\Gamma(\Pi)$ satisfies K1-K3, and that Π is infomorphic to $\Gamma(\Pi)$. Thus, the derived extensive game $\Gamma(\Pi)$ is an extensive game asserted in Theorem 6.2.

Furthermore, we have the following facts.

Lemma 6.3. Let $\Gamma(\Pi)$ and $\Gamma(\Pi')$ be the extensive games defined by T1-T5 from Π and Π'. Then $\Pi \cong \Pi'$ implies $\Gamma(\Pi) \cong \Gamma(\Pi')$.

Proof. Let $\Pi \cong \Pi'$. Then $\Gamma(\Pi) \cong \Pi \cong \Pi' \cong \Gamma(\Pi')$. Thus, by Lemma 6.1, $\Gamma(\Pi) \cong \Gamma(\Pi')$. ■

Here, we give one small example to illustrate the derived extensive game. This example will be used in Section 7 to show the difference between the perfect information in our sense and $\mathfrak{ip}$-perfect information.

Example 6.4. Consider the information protocol Π of Figure 6.1. This protocol has two maximal feasible sequences: $\langle (w^0, a), w^1 \rangle$ and $\langle (w^0, b), (w^2, a), w^2 \rangle$. Figure 6.2 is the derived extensive game $\Gamma(\Pi)$, where $\{x^0, x^1, x^2, x^3\}$ is the set of positions, e.g., $x^1 = \langle (w^0, a), w^1 \rangle$ and $x^3 = \langle (w^0, b), (w^2, a), w^1 \rangle$. In $\Gamma(\Pi)$, $I_{w^1} = \{x^1, x^3\}$ forms an information set. Then the induced protocol $\Pi(\Gamma(\Pi))$ from $\Gamma(\Pi)$ is described by Figure 6.3. It is easy to see $\Pi \cong \Pi(\Gamma(\Pi))$.

$$\begin{array}{ccc} w^1 & \longleftarrow_a & w^2 \\ \uparrow_a & \nearrow_b & \\ w^0 & & \end{array} \quad \cong \quad \begin{array}{ccc} x^1 & -- & x^3 \\ \uparrow_a & & \uparrow_a \\ x^0 & \rightarrow_b & x^2 \end{array} \quad \cong \quad \begin{array}{ccc} \{x^1, x^2\} & \longleftarrow_a & \{x^2\} \\ \uparrow_a & \nearrow_b & \\ \{x^0\} & & \end{array}$$

Figure 6.1. Figure 6.2 Figure 6.3

The above constructions from Π to $\Gamma(\Pi)$ and from $\Gamma(\Pi)$ to $\Pi(\Gamma(\Pi))$ are not completely reversible, but as discussed in Section 6.1, these are reversible up to infomorphisms.

So far, we have ignored a player assignment π in deriving the extensive game $\Gamma(\Pi)$ from an information protocol. However, once this derivation is done, it is easy to take a player assignment π into account in this derivation. Let $\Gamma(\Pi) = ((X \cup Z, x^0, <), \mathfrak{I}, \{C_I\}_{I\in\mathfrak{I}})$ be the derived extensive game from Π. Then, we define

$$\mathfrak{I}_i = \{\{\langle\xi, v\rangle : \langle\xi, v\rangle \in \Xi\} : v \in W \text{ and } \pi(v) \ni i\} \tag{6.4}$$

for $i = 0, 1, ..., n$. Then, $(\mathfrak{I}_0, \mathfrak{I}_1, ..., \mathfrak{I}_n)$ is a partition of $\mathfrak{I}$. The *derived game* $((X \cup Z, x^0, <), \mathfrak{I}_0, \mathfrak{I}_1, ..., \mathfrak{I}_n, \{C_I\}_{I\in\mathfrak{I}})$ with the *specification of players* is will be denoted by $\Gamma(\Pi, \pi)$.

6.3. Proof of Theorem 6.2

Suppose that an information protocol $\Pi = (W, A, \prec)$ with its root w^0 satisfying Axioms B1–B3 and N1-N2 is given. We show that $\Gamma(\Pi) = ((\Xi, x^0, <), \mathfrak{I}, \{C_I\}_{I\in\mathfrak{I}}) = ((X \cup Z, x^0, <), \mathfrak{I}, \{C_I\}_{I\in\mathfrak{I}}))$ defined in Section 6.2 satisfies K1-K3, and that $\Gamma(\Pi) \cong \Pi$.

We first prove that $(\Xi, x^0, <) = (X \cup Z, x^0, <)$ satisfies K1.

Lemma 6.5. $(\Xi, x^0, <) = (X \cup Z, x^0, <)$ is a finite tree with the root x^0.

Proof. First, we show that $(X \cup Z, <)$ is a partially ordered set.

Transitivity: Let $\langle\xi, w\rangle <_a \langle\xi', u\rangle$ and $\langle\xi', u\rangle <_b \langle\xi'', v\rangle$. Then $[\xi, (w, a)]$ is an initial fragment of ξ', and $[\xi', (u, b)]$ is an initial fragment of ξ''. Then ξ' is an initial fragment of ξ''. Thus, $[\xi, (w, a)]$ itself is an initial fragment of ξ''.

Irreflexivity: Since $\langle\xi, w\rangle$ cannot be an initial fragment of ξ, we have $\langle\xi, w\rangle \not< \langle\xi, w\rangle$.

Now, let us see that $x^0 = \langle w^0\rangle$ is the root of $(X \cup Z, <)$. By Lemma 2.5.(a), $x^0 = \langle w^0\rangle$ is a position. Lemma 2.5.(b) states that every position starts with w^0. Therefore, $\langle w^0\rangle < \langle\xi, w\rangle$ for any $\langle\xi, w\rangle \in \Xi$.

It remains to show that for any $\langle\xi, w\rangle \in \Xi$, the set $\{\langle\xi', v\rangle \in X \cup Z : \langle\xi', v\rangle < \langle\xi, w\rangle\}$ is total. Let $\langle\xi', v\rangle <_a \langle\xi, w\rangle$ and $\langle\xi'', u\rangle <_b \langle\xi, w\rangle$. This means $[\xi', (v, a)]$ is an initial fragment of ξ, and also $[\xi'', (u, b)]$ is an initial fragment of ξ. Then one of the following three cases holds: (1) $[\xi', (v, a)] = [\xi'', (u, b)]$, (2) $[\xi', (v, a)]$ is an initial fragment of ξ'', (3) $[\xi'', (u, b)]$ is an initial fragment of ξ'. These three cases correspond, respectively, to : (1′) $\langle\xi', v\rangle = \langle\xi'', u\rangle$, (2′) $\langle\xi', v\rangle <_a \langle\xi'', u\rangle$, (3′) $\langle\xi'', u\rangle <_b \langle\xi', v\rangle$. Thus, $\{\langle\xi', v\rangle \in X \cup Z : \langle\xi', v\rangle < \langle\xi, w\rangle\}$ is total. ■

Now, let us see K2. Recall $\mathfrak{I} = \{\{\langle\xi, u\rangle : \langle\xi, u\rangle \in \Xi \text{ and } u = v\} : v \in W\}$. We partition $\mathfrak{I}$ into $\mathfrak{I}_X = \{\{\langle\xi, w\rangle : \langle\xi, w\rangle \in X \cup Z\} : [(w, a)] \prec v \text{ for some } a \text{ and } v\}$ and $\mathfrak{I}_Z = \mathfrak{I} - \mathfrak{I}_X$. Since $X = \{\langle\xi, w\rangle : \langle\xi, w\rangle \in \Xi : [(w, a)] \prec v \text{ for some } a \text{ and } v\}$ and $Z = \Xi - X$, $\mathfrak{I}_X$ and $\mathfrak{I}_Z$ are partitions of X and Z.

Finally, consider K3. Recall $C_{I_v} = \{a \in A : [(v,a)] \prec u \text{ for some } u \in W\}$ for $I_v = \{\langle \xi, u\rangle : \langle \xi, u\rangle \in \Xi \text{ and } u = v\}$ with $v \in W$. Condition K32, i.e., $C_I = \emptyset$ for $I \in \mathfrak{I}_Z$, follows from the definition of Z. The following lemma states that K31 is satisfied.

Lemma 6.6. Let $I \in \mathfrak{I}_X$ and $\langle \xi, v\rangle \in I$. Then there is a bijection from C_I to the set of immediate successors of $\langle \xi, v\rangle$.

Proof. Since $\langle \xi, v\rangle \in I$, $\langle \xi, v\rangle$ is a position. Let $a \in C_I$. Then $[(v,a)] \prec u$ for some u. By Axiom N1, there is another position $\langle \xi, (v,a), w\rangle$ for some $w \in W$. This $\langle \xi, (v,a), w\rangle$ is an immediate successor of $\langle \xi, v\rangle$, and is determined uniquely by Axiom N2. Hence, we can define $\varphi_{\langle \xi, v\rangle}$ by

$$\varphi_{\langle \xi, v\rangle}(a) = \langle \xi, (v,a), w\rangle \text{ for each } a \in C_I.$$

The injectivity of this function follows also from N2. The surjectivity of it can be seen by starting the above argument with any immediate successor of $\langle \xi, v\rangle$. ■

The last step is to verify that $\Gamma(\Pi)$ is infomorphic to Π.

Lemma 6.7. The mapping ψ from $W \cup A$ to $\mathfrak{I} \cup (\bigcup_{I \in \mathfrak{I}} C_I)$ defined by

$$\psi(v) = I_v := \{\langle \xi, u\rangle : \langle \xi, u\rangle \in \Xi \text{ and } u = v\} \text{ for } v \in W; \tag{6.5}$$

$$\psi(a) = a \text{ for all } a \in A \tag{6.6}$$

is an infomorphism from Π to $\Gamma(\Pi)$.

Proof. Lemma 2.3.(b) implies the nonemptiness of I_v for any $v \in W$. Next, $A = \bigcup_{I \in \mathfrak{I}} C_I$ follows from Axiom B2. These two results together with T4 and T5 guarantee that I1 holds for ψ defined by (6.5) and (6.6),

For I2, first we take $\langle \xi, v\rangle \in \prec$, and show $\langle \psi\langle \xi, v\rangle\rangle \in <^*$. If $\langle \xi, v\rangle = \langle v\rangle$, then $\psi\langle \xi, v\rangle = \psi(v) = \langle I_v\rangle$. Since $I_v \in \mathfrak{I}$, it follows by L1 of the definition of $<^*$that $\langle I_v\rangle \in <^*$. Next, suppose $\langle \xi, v\rangle = \langle (v_1, a_1), ..., (v_m, a_m), v_{m+1}\rangle$. Since $\psi\langle \xi, v\rangle = \langle (I_{v_1}, a_1), ..., (I_{v_m}, a_m), I_{v_{m+1}}\rangle$, we prove that $\langle (I_{v_1}, a_1), ..., (I_{v_m}, a_m), I_{v_{m+1}}\rangle \in <^*$. By Lemma 2.3.(a) applied to $\langle \xi, v\rangle$, there is a position $\langle \eta, v\rangle$ and η is a supersequence of $[(v_1, a_1), ..., (v_m, a_m)]$. From η and $[(v_1, a_1), ..., (v_m, a_m)]$, we have the sequence of initial fragments $\langle \eta^1, v_1\rangle, ..., \langle \eta^m, v_m\rangle$ of $\langle \eta, v\rangle$ and $\langle \eta^{m+1}, v_{m+1}\rangle = \langle \eta, v\rangle$, which are positions. Then by T4, we have $\langle \eta^1, v_1\rangle \in I_{v_1}, ..., \langle \eta^{m+1}, v_{m+1}\rangle \in I_{v_{m+1}} = I_v$, and by T3 $\langle \eta^t, v_t\rangle <_{a_t} \langle \eta^{t+1}, v_{t+1}\rangle$ for $t = 1, ..., m$. By L2 of the definition of $<^*$, it follows that $\langle (I_{v_1}, a_1), ..., (I_{v_m}, a_m), I_{v_{m+1}}\rangle \in <^*$.

For the converse, we take $\psi\langle \xi, v\rangle \in <^*$, and show $\langle \xi, v\rangle \in \prec$. Then, either: (a) $\langle \xi, v\rangle = \langle v\rangle$ or (b) $\langle \xi, v\rangle = \langle (v_1, a_1), ..., (v_m, a_m), v_{m+1}\rangle$ for some $v_1, ..., v_{m+1} \in W$ and $a_1, ..., a_m \in A$. If (a) holds, then $\langle \xi, v\rangle = \langle v\rangle \in\prec$ by Axiom B1. Suppose (b). Then $\psi\langle \xi, v\rangle = \langle (I_{v_1}, a_1), ..., (I_{v_m}, a_m), I_{v_{m+1}}\rangle$. By L2, there are nodes $y_1 \in I_{v_1}, ..., y_{m+1} \in I_{v_{m+1}}$ such that $y_t <_{a_t} y_{t+1}$ for $t = 1, ...m$. By T4 these nodes must be expressed as $y_t =$

$\langle \xi_t, v_t \rangle \in \Xi$ for $t = 1, ...m+1$. The last $y_{m+1} = \langle \xi_{m+1}, v_{m+1} \rangle$ is a feasible sequence to $v = v_{m+1}$. Since by T3, $[\xi_t, (v_t, a_t)]$ must be an initial fragment of $\langle \xi_{t+1}, v_{t+1} \rangle$ for $t = 1, ...m$, $\langle \xi_{m+1}, v_{m+1} \rangle$ includes $(v_1, a_1), ..., (v_m, a_m)$ in this order, that is, $\langle \xi_{m+1}, v_{m+1} \rangle$ is a super sequence of $\langle \xi, v \rangle = \langle (v_1, a_1), ..., (v_m, a_m), v_{m+1} \rangle$. Hence, by Axiom B3, $\langle \xi, v \rangle \in \prec$. ■

7. Memory in the two Theories

Among four possible treatments of individual memory discussed in Section 5.1, we have adopted the treatment (a1) by means of a memory function together with a reading device that can read the full memory threads. On the other hand, in the standard treatment by an extensive game, the associated reading device separates different memories only set-theoretically. In Section 7.1, we explore the differences between these two treatments. In Section 7.2, we consider the full cognizance assumption implicitly associated with the standard (classical) interpretation of an extensive game. We show that this is essentially captured by the classical memory function. Finally, we give a characterization of the classical memory function.

7.1. Comparisons between the two Treatments of Memory

Let $\Pi = (W, A, \prec)$ be a full information protocol. We associate a player assignment π with Π. Using the notion of positions, it is easy to formulate the counterparts of the conditions of $\mathfrak{ip}$-perfect information and $\mathfrak{ip}$-perfect recall given in Section 4.2 for an extensive game. Technically, those counterparts are shown to be equivalent to the $\mathfrak{ip}$-memory conditions for the derived game $\Gamma(\Pi, \pi)$ defined in Section 6.1.

Recall that Ξ_i is the set of positions for player i. In the second statement of the following proposition, we use the following notation: For $\langle \xi, w \rangle = \langle (w_1, a_1), ..., (w_m, a_m), w \rangle \in \Xi_i$, ξ_i is defined to be the subsequence of ξ restricted to the pairs (w_t, a_t) with $w_t \in W_i$.

Proposition 7.1.(1). Player i has $\mathfrak{ip}$-perfect information in $\Gamma(\Pi, \pi)$ if and only if for all $\langle \xi, w \rangle, \langle \eta, v \rangle \in \Xi_i$, $w = v$ implies $\xi = \eta$.

(2). Player i has $\mathfrak{ip}$-perfect recall in $\Gamma(\Pi, \pi)$ if and only if for all $\langle \xi, w \rangle, \langle \eta, v \rangle \in \Xi_i$, $w = v$ implies $\xi_i = \eta_i$.

Proof. (1): The $\mathfrak{ip}$-perfect information condition for player i means that every information set of player i in $\Gamma(\Pi, \pi)$ is a singleton set. An information set in $\Gamma(\Pi, \pi)$ is expressed as $I = \{\langle \xi, u \rangle \in \Xi : u = v\}$ for some $v \in W$. The latter part of (1) means that I is singleton.

(2): Lemma 4.1 states that $\mathfrak{ip}$-perfect recall is equivalent to that two nodes in the same information set have exactly the same histories of his own information sets and actions.

Each information set in $\Gamma(\Pi, \pi)$ is represented by an information piece. Thus, the equivalence stated in Lemma 4.1 is translated into: if $\langle \xi, w\rangle, \langle \eta, v\rangle$ in Ξ_i satisfy $w = v$, then $\xi_i = \eta_i$. ■

The above proposition shows that we can capture the standard memory conditions of an extensive game by conditions on an infomorphic information protocol.

Now, let us consider the ability of an extensive game to capture the conditions of memory described by a memory function of a info-memory protocol. Technically, this question may be answered by considering the refinement of the information partition of the derived game $\Gamma(\Pi, \pi)$. It is the conclusion that an information partition does not fully capture a memory function as was already discussed in Section 5.1.

The procedure of refinement is described as follows. We start with an info-memory protocol $(\Pi, \pi, \mathfrak{m})$ and then have the derived extensive game $\Gamma(\Pi, \pi)$ defined in Section 6.2. The game $\Gamma(\Pi, \pi)$ has information partitions $\mathfrak{I}_0, \mathfrak{I}_1, ..., \mathfrak{I}_n$. We replace the information partition $\mathfrak{I}_i$ of (personal) player i in $\Gamma(\Pi, \pi)$ by the refinement $\hat{\mathfrak{I}}_i$ consisting of information sets:

$$\{\langle \eta, u\rangle \in \Xi_i : u = v \text{ and } \mathfrak{m}_i\langle \eta, u\rangle = \mathfrak{m}_i\langle \xi, v\rangle\} \tag{7.1}$$

for $\langle \xi, v\rangle \in \Xi_i$. The newly defined information partition $\hat{\mathfrak{I}}_i$ is the coarsest refinement of $\mathfrak{I}_i$ with the property that player i has the same memory $\mathfrak{m}_i\langle \xi, v\rangle$ in each information set. Let $\hat{\mathfrak{I}}_0 = \mathfrak{I}_0$ and $\hat{\mathfrak{I}} = \bigcup_{i=0}^{n} \hat{\mathfrak{I}}_i$. We call $\Gamma(\Pi, \pi, \mathfrak{m}) = ((X \cup Z, x^0, <), \hat{\mathfrak{I}}_0, \hat{\mathfrak{I}}_1, ..., \hat{\mathfrak{I}}_n, \{C_{\hat{I}}\}_{\hat{I} \in \hat{\mathfrak{I}}})$ the *memory-refined extensive game* derived from info-memory protocol $(\Pi, \pi, \mathfrak{m})$.

Note that $u = v$ is included in (7.1). One interpretation of this addition is that the raw v is still available at $\langle \xi, v\rangle$. This addition is not needed under the assumption that for any $\langle \xi, v\rangle, \langle \eta, u\rangle \in \Xi_i$,

$$v \neq u \text{ implies } \mathfrak{m}_i\langle \xi, v\rangle \neq \mathfrak{m}_i\langle \eta, u\rangle, \tag{7.2}$$

This states that player can perceives differences in current information pieces.

We can find conditions on a memory function $\mathfrak{m}_i$ to guarantee the $\mathfrak{ip}$-memory conditions in the memory-refined derived game $\Gamma(\Pi, \pi, \mathfrak{m})$.

Proposition 7.2.(1). Player i has $\mathfrak{ip}$-perfect information in $\Gamma(\Pi, \pi, \mathfrak{m})$ if and only if for all $\langle \xi, w\rangle, \langle \eta, w\rangle \in \Xi_i$, $\mathfrak{m}_i\langle \xi, w\rangle = \mathfrak{m}_i\langle \eta, w\rangle$ implies $\xi = \eta$.

(2). Player i has $\mathfrak{ip}$-perfect recall in $\Gamma(\Pi, \pi, \mathfrak{m})$ if and only if for all $\langle \xi, w\rangle, \langle \eta, w\rangle, \langle \xi', v\rangle \in \Xi_i$, if $\mathfrak{m}_i\langle \xi, w\rangle = \mathfrak{m}_i\langle \eta, w\rangle$ and $\langle \xi', v\rangle <_a \langle \xi, w\rangle$, then there is a $\langle \eta', v\rangle \in \Xi_i$ with $\langle \eta', v\rangle <_a \langle \eta, w\rangle$ and $\mathfrak{m}_i\langle \xi', v\rangle = \mathfrak{m}_i\langle \eta', v\rangle$.

Proof. The latter part of (1) is the condition that the refined information set of (7.1) is a singleton. We omit the proof of (2). ■

Although these are conditions for $\mathfrak{ip}$-memory conditions, they do not express what are intended by the terms of "perfect information" or "perfect recall". The following simple example illustrates these discrepancies.

Example 7.3 (Forgetfulness in $\mathfrak{ip}$-Perfect Recall). Consider the information protocol Π of Figure 6.1 of Example 6.4. Suppose that this is a one-person protocol with $N = \{1\}$. Consider the Orwell memory function $\mathfrak{m}_1^O$ given by (3.14) in this protocol, i.e., player 1 perceives correctly the present piece and available actions, but recalls only how many times he has moved. For example, for two positions $\langle (w^0, a), w^1 \rangle$ and $\langle (w^0, b), (w^2, a), w^1 \rangle$ in Π,

$$\mathfrak{m}_1^O \langle (w^0, a), w^1 \rangle \quad = \quad \{\langle (\theta, \{\theta\}, \theta), (w^1, \emptyset) \rangle\}$$

$$\mathfrak{m}_1^O \langle (w^0, b), (w^2, a), w^1 \rangle \quad = \quad \{\langle (\theta, \{\theta\}, \theta), (\theta, \{\theta\}, \theta), (w^1, \emptyset) \rangle\}.$$

Although player 1 does not recall the details of previous pieces and actions, he can recall the number of his previous moves. Therefore, he can distinguish between $\mathfrak{m}_1^O \langle (w^0, a), w^1 \rangle$ and $\mathfrak{m}_1^O \langle (w^0, b), (w^2, a), w^1 \rangle$. Thus, each information set in the refined information partition is singleton. This implies that the refined information partition $\hat{\mathfrak{I}}_1$ is of $\mathfrak{ip}$-perfect information, *a fortiori*, it is of $\mathfrak{ip}$-perfect recall. The derived game $\Gamma(\Pi)$ of Figure 6.2 is refined into the game of Figure 7.1.

$$\begin{array}{ccc} x^1 & & x^3 \\ \uparrow_a & & \uparrow_a \\ x^0 & \rightarrow_b & x^2 \end{array}$$

Figure 7.1

The above example points out the limitation of the procedure to express memory by distinguishing two information sets (or information pieces). The Orwell memory function $\mathfrak{m}_1^O$ involves a significant forgetfulness, but as far as restricting our attention to information partitions, we cannot capture that forgetfulness.

7.2. Memory in the Classical Interpretation of a Game

In the game theory literature, it has been typically assumed (cf., Luce-Raiffa [13], Section 3.6) that a player in an extensive game is cognizant of the entire structure of the game. Under this assumption, the player in the game can infer the possible objective histories from his current information set. While this argument remains informal in the literature, it is quite often used. Under this assumption, we can obtain the classical memory function $\mathfrak{m}_i^C$ of (3.13) since when player i receives w, he can infer all positions ending with w.

It may be worth noting that the cognizance of $\mathfrak{m}_i^C$ as a function is equivalent to the cognizance of the extensive game with respect to histories. Thus, if player i has and knows $\mathfrak{m}_i^C$, then the extensive game would provide no additional information about histories. In Example 3.4, the classical memory function $\mathfrak{m}_i^C$ was given in (3.9) for the absentminded driver game. In this example, his memory is subject to forgetfulness, while he knows the game structure and even his own forgetfulness.

We now examine the $\mathfrak{ip}$-perfect information and $\mathfrak{ip}$-perfect recall conditions as attributes of $\mathfrak{m}_i^C$.

Proposition 7.4.(1). Player i has $\mathfrak{ip}$-perfect information in $\Gamma(\Pi, \pi)$ if and only if the memory function $\mathfrak{m}_i^C$ coincides with the perfect information function $\mathfrak{m}_i^{PI}$;

(2). Player i has $\mathfrak{ip}$-perfect recall in $\Gamma(\Pi, \pi)$ if and only if for each $\langle \xi, v \rangle \in \Xi_i$, $\mu, \mu' \in \mathfrak{m}_i^C \langle \xi, v \rangle$ implies $\mu_i = \mu'_i$.

Proof. We prove only (1). Suppose that player i has $\mathfrak{ip}$-perfect information in $\Gamma(\Pi, \pi)$. It follow from Proposition 7.1.(1) that $\langle \xi, v \rangle, \langle \eta, u \rangle \in \Xi_i$, $v = u$ implies $\xi = \eta$. Hence, $\mathfrak{m}_i^C \langle \xi, v \rangle$ consists of a unique thread $\langle \xi, v \rangle^*$. Thus, $\mathfrak{m}_i^C$ coincides with $\mathfrak{m}_i^{PI}$. The converse is proved similarly. ■

Notice that the assertions are about $\Gamma(\Pi, \pi)$ but not about $\Gamma(\Pi, \pi, \mathfrak{m}^C)$. The classical memory function $\mathfrak{m}_i^C$ does not refine the information partition, since it describes essentially the extensive game $\Gamma(\Pi, \pi)$. The argument will be more explicitly materialized in Part II.

We will now use some conditions on memory to characterize the classical memory function. We say that the memory function $\mathfrak{m}_i$ is

(a): *positional* iff for all $\langle \xi, w \rangle \in \Xi_i$, $\mu \in \mathfrak{m}_i \langle \xi, w \rangle$ implies $\mu = \langle \eta, v \rangle^*$ for some $\langle \eta, v \rangle \in \Xi$;

(b): *truthful* iff $\langle \xi, w \rangle^* \in \mathfrak{m}_i \langle \xi, w \rangle$ for all $\langle \xi, w \rangle \in \Xi_i$;

(c): *information-piece determined* iff for all $\langle \xi, w \rangle, \langle \eta, v \rangle \in \Xi_i$, $w = v$ implies $\mathfrak{m}_i \langle \xi, w \rangle = \mathfrak{m}_i \langle \eta, v \rangle$.

Positionality means that all memory threads are complete and correct in the sense of (3.5). Truthfulness is that the true position is included as a memory thread. Information-piece determinacy means that the memory function can be determined by considering information pieces alone. One might interpret the last condition as meaning that the information pieces themselves could be used to describe the memory. We can prove that only the classical memory function satisfies them all.

Proposition 7.5. The memory function $\mathfrak{m}_i$ is the classical memory function $\mathfrak{m}_i^C$ if and only if $\mathfrak{m}_i$ is positional, truthful and information-piece determined.

Proof. We prove the *if* part. Consider the memory at a position $\langle \xi, w \rangle \in \Xi_i$. By positionality and (3.2), $\mathfrak{m}_i \langle \xi, w \rangle \subseteq \{\langle \eta, v \rangle^* : \langle \eta, v \rangle \in \Xi \text{ and } v = w\}$. Consider the converse inclusion. By truthfulness, $\langle \xi, w \rangle^* \in \mathfrak{m}_i \langle \xi, w \rangle$. Consider any other position

$\langle \eta, v \rangle$ with $v = w$. Then, $\langle \eta, v \rangle^* \in \mathfrak{m}_i \langle \eta, v \rangle$ also by truthfulness. Then, by information-piece determinacy and $v = w$, $\langle \eta, v \rangle^* \in \mathfrak{m}_i \langle \xi, w \rangle$. Thus, $\{\langle \eta, v \rangle^* : \langle \eta, v \rangle \in \Xi$ and $v = w\} \subseteq \mathfrak{m}_i \langle \xi, w \rangle$. ■

8. Conclusions and Continuation to Part II

We have developed the theory of information protocols and individual memories. An information protocol is defined based on information pieces and actions as its primitives as well as on a causality relation of histories to new pieces. We also formulated individual memory separately using a memory function. We showed how to consider various types of forgetfulness using memory functions.

The information protocol corresponds to an extensive game and we showed that the set of full information protocols, i.e., satisfying all the axioms, is equivalent, up to information and actions, to the set of extensive games of Kuhn [12]. This means that a full protocol is regarded as an objective description of the game situation in question. The comparison was done without the memory function. We then proceeded to compare our treatment of memory with the standard treatment of memory in an extensive game. We argued that the approach by a memory function has a richer descriptive power than that by an information partition.

In Part II, we will look at the question of how a player may construct his personal view of the society from his memories of experienced information. In this development, basic protocols satisfying only the basic axioms will play important roles. The subject of inductive game theory was initiated by Kaneko-Matsui [10]. The developments in Part I facilitate more extensive considerations of inductive game theory than in [10]. For example, we can consider the relationship between memories and inductively derived views, while only the problem addressed by partial observations was investigated in [10]. A memory function will be crucial in the developments in Part II.

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