Valuation of a Default Swap Option

Takeaki Kariya[1]
Research Center for Financial Engineering
Institute of Economic Research, Kyoto University
Sakyoku, Kyoto 606-8501, Japan
kariya@kier.kyoto-u.ac.jp

This paper drives a pricing formula for a default swap option (DSO) that an investment bank in Japan produced on the credit-risk of a convertible bond (CB) issued by a third company C. In this DSO contract, a protection buyer A not only obtains a full hedge for the principal of the CB against a default of C but also owns the option of starting an interest swap between the buyer and a protection seller B when a credit event happens. This option gives A an opportunity to recover the interests from the CB as well. When A starts the swap after a default, the floating rate is associated with the protection premium. After a certain simplification, this paper makes a no-arbitrage valuation for the premium in a discrete time approach. In addition, when the credit quality of the parties A and B is taken into account in the valuation, a fair value of the default swap option is also derived.

1.Introduction

In this paper, we consider a valuation problem on a default swap option (DSO) or equivalently an asset swap option sold in Japan. The product not only guarantees the principal of a specific CB (convertible bond) against a default of the issuer C of the CB, but also it gives an option that gives the right to start an asset swap when one of the default events specified in the contract takes place. A protection buyer A in this contract is the one who pays the protection premiums quarterly till a default of C and who may exercise the swap option at the default. When the swap is started, at the outset the CB issued by company C is exchanged with its principal amount and then an interest swap follows with the coupon rate of the CB as a swap rate. In the swap, the receiver of the fixed rate, who is the protection buyer A for the CB, pays 3 month LIBOR plus the option premium for the right of starting the swap. In this way, the option premium is connected with the swap rate. In other words, the coupon rate minus the option premium acts

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as the swap rate exchanged with 3 month LIBORs. An interesting feature of this product is that the rates of the protection premium and the option premium are common. Here we are interested in valuing the common premium rate in a discrete time setting. In a sense, a discrete time setting is more suitable in credit risk analysis than a continuous time setting. In fact, cash flows such as recovery are delayed some time after a default. In continuous time setting, this fact is not well treated, while in discrete time setting, it is directly incorporated into the model, if necessary together with the uncertainty of the delay.

A typical default swap is a derivative on the credit risk of a corporate bond issued by a third party that guarantees the principal of the bond if the third party gets defaulted. Default swaps of this type are quoted in market for trades of credit risk on government bonds and corporate bonds. The quotation includes the ask-side as well as the offer-side of protection against each bond, from which a market value of the credit risk is found. Compared to this default swap, the above DSO is different in that it includes a swap contract of interest after default.

In Section 2, the details of the DSO are described. The product itself is rather complicated because it includes a cancellation clause and the events specified in the contract that trigger the swap of interests are multiple. In Section 3 we make some simplification and value the simplified DSO in terms of the premium in a discrete time setting. Since the model is incomplete, we use an actual measure in the valuation. In Section 4 the credit quality of the protection buyer and seller is taken into account in the valuation.

2 Description of the Default Swap Option

Here we describe the detailed content of the default swap option.

An investment bank A set up an asset swap (AS) product on the credit of a company C with the following content.

Default Swap Option (97.10.1)

Protection seller of the AS	company B
Protection buyer of the AS	investment bank A
Premium	40bp（quarterly payment、calendar days/360）
Maturity	1999/3/31

Notional Amount 2.5 billion yen

Credit of the trade Company C

Exercise of the right starting a swap and termination of the contract

If a default event specified below happens before or on the maturity day, and the buyer A exercises the right, (1) A pays the accrued premium (corresponding to the days from the last payment day of the premium till the exercise day) and (2) the following asset swap starts for the period from the exercise day till maturity.

Payment of B	2.5billion yen
Delivery by A	the CB's issued by C（coupon 6.5%、maturity 1999/3/31）face value 2.5billion
Delivery day	within ( ) days after the default

Cancellation terms

The protection buyer A can terminate the contract at any time before a default event happens. In such a case A pays the accrued premium from the last payment till the cancellation.

Definition of a default event :

a)The company C fails to make payment of any part of its debts as scheduled with no deferment, which includes the principal and coupon of the CB's as well as other types of debts.

b) The company C takes any legal form of dissolution, bankruptcy, restructuring , etc. along the law.

c) The senior debts of the company C get rating C or the lower in the Moodys' Rating Co. The present rating is B.

In the contract, some more conditions are stated in the definition of default including rather judgmental statement.

The content of the interest swap in the contract

The counter parties of the swap are Co. A and B.

The swap starts at the time when Co A exercises the right of starting the swap after a default of C.

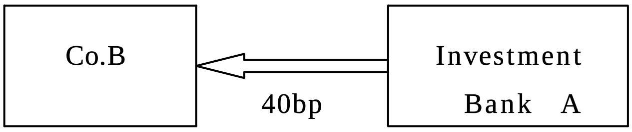
The termination date of the swap is March 31,1999.

The notional principal is 2.5 billion yen.

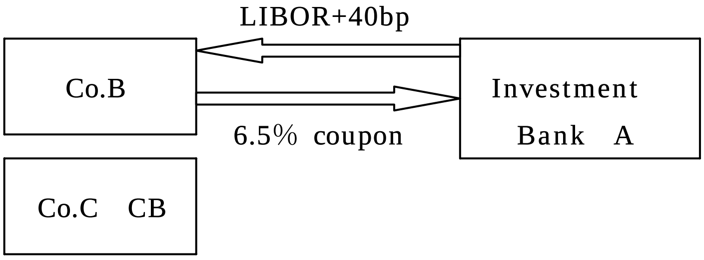
The payment of B is the amount corresponding to the coupon payment (6.5% per year)of the CB's (before tax, full first coupon).

The receipt of B is the amount (LIBOR+40bp) times the notional amount, which A pays quarterly at the end of March, June, September and December).The following diagram describes the situation.

Before default:



After default:



3 Valuation of the DSW

In this section, we will consider a valuation problem of the DSO described in Section 2. By doing so, we can examine the rationality of the premium(0.004) stipulated in the contract for the protection premium and the option premium as a part of the swap floating rate. In general a pricing problem will be solved by first identifying the payoffs of the both counter parties. Before we do this, it is noted that

(1) When Co C comes close to a default situation, the value of the convertibility right of the CB will be almost 0 and the CB may be regarded as a simple corporate coupon bond. After C gets defaulted in fact, the CB loses its liquidity, which becomes 0 at the bankruptcy.
(2) The recovery rate of a CB is usually low at bankruptcy.
(3) Tax may be another element to consider for the protection seller.

First we introduce some notation for modeling and valuation and make some assumptions. Since our valuation problem for the premium is invariant under changes of notional principal, we use 100 yen rather than 2.5 billion yen.

Let 0 denote the contract time of the default swap option and let N denote the maturity (99.3.31), where time unit is daily and it is expressed as an annual basis, meaning that, for example, n=5 is 5/360 year. Also let J

denote the first time to a default. Since the default events acting as a trigger to the start of the swap is multiple, the default time J is defined as the first-to- default time to one of the default events, which is expressed as

(3.1) $J = \min(J_1, \Lambda\ J_K)$,

where the J_k denotes the first time to the occurrence of the k-th event. The total number K of the processes defining event occurrences may be regarded as 3 though the concept of the default events may not completely be distinguishable. However, in this paper we assume $J = J_1$ for simplicity and regard it as bankruptcy because a general case is difficult to treat. This may be regarded as an approximation to the case where the default event processes are highly correlated. But depending on the default events, the cash flows will change to the seller of the DSO and hence it affects the value of the contract. In fact, in the case of the default event c) in the contract, even if Co C is rated as C in the Moodys' rating and if Co A exercises the right of starting the swap, Co. B may not face any loss from the swap unless Co. C fails to pay the interests and principal of the CB's. On the contrary, there is a possibility that Co. C may recover after the swap starts and the value of the CB'S Co B owns becomes much greater because of the convertibility of the CB's. Then Co. B can sell them in the market to get a capital gain. Also a market interest level affects the value of the CB's.

In the sequel, we discuss the valuation problem of the DSO from the standpoint of the protection seller B. Even if we assumed that the default that initiates the swap is simple and bankruptcy, it is still difficult to solve the problem directly and hence we make the following assumptions.

Assumptions

1) When an default event occurs at time J, the protection buyer A necessarily exercises the right and starts the swap.

2) The contract does not allows Co A to cancel the option of starting the swap though the above contract allow A to do so. That is, once the contract is made, the protection buyer has to continue to pay the premiums β (=0.004 in the above case) /4 quarterly even when Co. C does not get bankrupted.

Next, we specify the dates of the cash flows generated by the contract.

Since the quarterly payment dates of the option premiums (protection premiums) are set up or determined by contract in advance, they are known at the time of the contract. Hence let them be denoted by

$$0 < m_1 < m_2 < \Lambda < m_K = N,$$

where m_j's are the numbers of the calendar days counted from the contract day (time 0). The periods between the consecutive dates are measured by annual basis as

$$u_i = {(m_i - m_{i-1})}\big/{360} \qquad (i = 1, \Lambda\ , K).$$

Then before a bankruptcy event happens, a protection premium paid to the seller at m_k is $100\beta u_k$ yen. When the bankruptcy occurs at

(3.2) $J = j$ with $m_{k-1} \le j < m_k$,

the DSO contract stipulates that the seller receives the proportional premium$100\beta u_J$ yen at J and pays 100 yen in exchange for the CB's of face value 100 yen, where $u_J = (J - m_{k-1})/360$. When the default occurs at $J=j$ as in (3.2), then A needs to pay the premium at time j and hence B receives

$$100\beta u_J^k \quad \text{with} \quad u_J^k = (J - m_{k-1})/360.$$

After the bankruptcy of C at $J=j$ as in (3.2), the interest swap starts at m_k,the cash flows of the swap are exchanged at $m_k, \Lambda\ , m_K$. Further it is assumed that for the CB's that B receives from A for 100 yen, B will recover 100γ % of the principal at time m_k as the recovery rate is 100γ%.

Thirdly to formulate the swap triggered by the default of C, let $\{\tilde{r}_n\}$ be the process of 3 month LIBORs. Then at m_k, floating rate $\tilde{r}_{m_{k-1}} + \beta$ is swapped with fixed rate c that is the coupon rate in decimal expression, at m_{k+1}, $\tilde{r}_{m_k} + \beta$ is swapped with c, and so on. Like the vanilla interest swap, in this swap the following exchange is made;

(3.3) $\tilde{r}_{m_{k-1}+i} + \beta \leftrightarrow c$ at m_{k-1+i} $(i = 1, \Lambda\ , K-k)$,

meaning that the floating rates $\{\tilde{r}_n + \beta\}$ are swapped with the fixed rate c.

Now based on these formulations and assumptions, let us specify the payoffs of the DSO and value them at 0. For this purpose, let us define what we call the (accumulated) default generation process by

(3.4)

$$L_n = \sum_{j=1}^{n} \chi_j = \chi_{\{J \le n\}} = \begin{cases} 1 & \text{if} \quad J \le n \\ 0 & \text{otherwise}, \end{cases}$$

where χ_k denotes the default indicator function of a default event $\{J = k\}$. Note that $\chi_0 = L_0 = 0$, $\chi_k \chi_j = 0$ ($k \ne j$) , and $E(\chi_j) = Q(J = j)$. .Hence L_n denotes the accumulated default indicator of $\{J \le k\}$. Clearly $\{L_n\}$ takes 0 or 1 and is nondecreasing and hence a submartingale. Thus one might apply Doob's Representation Theorem to formulate the problem in terms of discrete time intensity.

Further let $\{r_n\}$ be an interest process giving the discount function;

(3.5) $$D(n,N) = E_n^*[d(n,N)]$$

with

$$d(n,N) = \exp\left(-\sum_{j=n}^{N-1} r_j h\right) \quad \text{where } h = 1/360.$$

Of course, $D(n,N)$ discounts a cash flow at N to a value at n. We assume that the default process $\{L_n\}$ and the interest process $\{r_n\}$ are independent.

Ⅰ.Cash flows of the protection seller B until a default

The cash flow income of protection premiums for the protection seller at a time m_k is expressed as the sum of the income $100\beta u_k$ when the company C survives over m_k and the income when it gets defaulted during m_{k-1} and m_k;

(3.6) $$I_k = 100\beta u_k (1 - L_{m_k}) + 100\beta u_{J^C} (1 - L_{m_{k-1}}) L_{J^C} \quad for \quad m_{k-1} \le J^C < m_k .$$

Hence applying the discrete time no-arbitrage theory gives a value of the payoff (3.6) evaluated at 0 as

(3.7) $$V_{Ik} = 100\beta u_k Q(J^C > m_k) D(0, m_k) + 100\beta u_k G_k D(0, m_k) ,$$

where

(3.7b) $$G_k = \sum_{j=1}^{m_k - m_{k-1}} Q(J = m_{k-1} + j) \frac{j}{360} \frac{D(0, m_{k-1} + j)}{D(0, m_k) u_k}.$$

Consequently under the assumption of the independence between interest rates and credit events, this is evaluated as

(3.8) $$V_{\mathrm{I}} = \sum_{k=1}^{K} 100 \left[Q(J^C > m_k) + G_k \right] \beta u_k D(0, m_k).$$

Ⅱ.Valuation of swap

When a default event occurs, the swap described in(3.3) starts. The individual payoff of this swap when a default event happens during the times m_{k-1} and m_k is

(3.9) $$F_{ki} = 100(\tilde{r}_{m_{k-1+i}} + \beta - c) u_{k+i},$$

whose theoretical value at 0 is given by

(3.10) $$V_{ki} = E_0^* \left[F_{ki} d(0, m_{k+i})(1 - L_{m_{k-1}}) L_{m_k} \right].$$

Therefore the swap starting at m_k is valued at 0 as

(3.11) $$V_k = \sum_{i=0}^{K-k} V_{ki} = E_0^* \left[\left(\sum_{i=0}^{K-k} F_{ki} d(0, m_{k+i}) \right) (1 - L_{m_{k-1}}) L_{m_k} \right]$$

and so the total value of the swap part in the DSO product is of the theoretical value;

(3.12) $$V_{\mathrm{II}} = \sum_{k=1}^{K} \sum_{i=0}^{K-k} E_0^* \left[F_{ki} d(0, m_{k+i})(1 - L_{m_{k-1}}) L_{m_k} \right].$$

Again under the assumption of the independence, this is evaluated as

(3.13) $$V_{\mathrm{II}} = \sum_{k=1}^{K} \left\{ \sum_{i=0}^{K-k} E_0^* \left[100(\tilde{r}_{m_{k-1+i}} + \beta - c) u_{k+i} d(0, m_{k+i}) \right] \right\} Q(m_{k-1} < J \le m_k).$$

Ⅲ.The payoff of the CB after default

When the default is bankruptcy, the coupons of the CB's are not paid and the principal is only recovered at the rate γ from the company C. On the other hand, the seller has to buy the CB's for the face value ¥100 and hence the payoff at m_k when the bankruptcy happens during the times m_{k-1} and m_k is given by

(3.14) $\text{III}_k = 100(\gamma - 1)L_{m_k}(1 - L_{m_{k-1}})$.

Therefore the total value at 0 from the bankruptcy for the seller is given by

(3.15) $V_{\text{III}} = \sum_{k=1}^{K} E_0^*[I_k d(0, m_k)] = 100(\gamma - 1)\sum_{k=1}^{K} D(0, m_k)Q(m_{k-1} < J \le m_k)$.

From I,II, and III, the value at 0 of the asset swap for the protection seller is

(3.16) $V = V_{\text{I}} + V_{\text{II}} + V_{\text{III}}$.

Here V_{I} and V_{III} are not dependent on the process of the LIBORs but V_{II} is. Hence we further evaluate V_{II} in (3.13) . Note the inside of { } in V_{II} corresponds to the payoff of the m_{k-1} start forward interest swap. Hence letting the m_{k-1} start forward swap rate be denoted by y_{k-1}, the LIBORs and the swap rate satisfies

(3.17) $\sum_{i=0}^{K-k} E_0^*[(\tilde{r}_{m_{k-1}+1} - y_{k-1})u_{k+i} d(0, m_{k+i})] = 0$.

Hence, noting

$$\tilde{r}_{m_{k-1}+1} + \beta - c = \tilde{r}_{m_{k-1}+1} - y_{k-1} + (y_{k-1} + \beta - c),$$

the inside of $\{\quad\}$ in V_{II} is evaluated in terms of the forward swap rates $\{y_k\}$, which are determined at 0,

(3.18) $\sum_{i=0}^{K-k} E_0^*[(y_{k-1} + \beta - c)u_{k+i} d(0, m_{k+i})] = \sum_{i=0}^{K-k}(y_{k-1} + \beta - c)u_{k+i} D(0, m_{k+i})$.

Thus V_{II} is given by

(3.19) $V_{\text{II}} = \sum_{k=1}^{K}\left[\sum_{i=0}^{K-k} D(0, m_{k+i})\, u_{k+i}\right](y_{k-1} + \beta - c)\, Q(m_{k-1} < J \le m_k)100$.

Combining (3.8), (3.15) and (3.19) and setting V=0 yields the following theorem.

Theorem 3.1 When interest rates and default events are independent, the value of the default swap option (asset swap) at 0 is valued as

(3.20) $V = \sum_{k=1}^{K} 100 D(0, m_k) H_k$,

where

(3.21) $$H_k = \beta u_k\left[1-\sum_{i=1}^{k} q_i^c + G_k\right] + \delta_k(y_{k-1}+\beta-c)q_k^C + (\gamma-1)q_k^C,$$

$$\delta_k = \left[\sum_{i=0}^{K-k} D(0,m_{k+i})u_{k+i}\right] / D(0,m_k), \text{ and}$$

$$q_k^C = Q(m_{k-1} < J^C \le m_k) \ \ with \ \ J^C \equiv J.$$

Thus the fair value of the premium is given by $V^B = 0$ as

(3.22) $$\beta = \frac{\sum_{k=1}^{K} D(0,m_k)\left[(1-\gamma)q_k^C + (c-y_{k-1})\delta_k q_k^C\right]}{\sum_{k=1}^{K} D(0,m_k)\left[u_k(1-\sum_{i=1}^{k} q_i^C + G_k) + \delta_k q_k^C\right]}.$$

where G_k is given by (3.7b).

By this result, the rate β in (3.22) gives the theoretical premium that plays a role in both the protection premium and the spread of the floating rate and the fixed rate for the swap after the bankruptcy of C. It is computed when the followings are given;

1) the term structure of the discount rate $\{D(0,n)\}$ derived at 0 through government bond prices

2) the default probabilities q_k's and G_k's and

3) the forward swap rate $\{y_k\}$ that will start in future for 3 month LIBORs.

Note that the sign of $c - y_{k-1}$ in (3.22) is not definite.

4 Premium with the credit risk of A and B taken into account.

When the credit risks of the parties A and B are taken into account in the valuation of the premium, we need to modify the above result. First, in (3.6) the protection seller A can receive I_k only if the protection buyer B is alive at m_k. Assuming that no payment is made to B if A gets defaulted in $(m_{k-1}, m_k]$, the amount B receives becomes $I_k^B = I_k(1-L_{m_k}^A)$ with I_k in (3.6). On the other hand, if B gets defaulted before the default of C, assume that the contract itself becomes invalid and A loses the amount of the premiums A pays up to m_k. Then the amount A pays at m_k is $I_k^A = -I_k(1-L_{m_k}^B)$, which cannot be retrieved even if B gets defaulted later. In this situation, the value V_I in (3.8) is changed to V_I^A and V_I^B for A and B respectively;

$$
\begin{aligned}
V_I^A &= -\sum_{k=1}^{K} 100(Q_k^C + G_k) Q_k^B \beta\, u_k D(0, m_k), \\
V_I^B &= \sum_{k=1}^{K} 100(Q_k^C + G_k) Q_k^A \beta\, u_k D(0, m_k).
\end{aligned} \tag{4.1}
$$

Here

$$Q_k^D = Q(J^D > m_k)$$

with J^D the default time of D, where $D = A, B, C$.

Next when A and B survive at the time of the default of C, the swap starts. We assume that a default of either of the parties A and B will stop the swap thereafter and no payment needs to be made from either side thereafter. Then in (3.9) the payoff of B given by F_{k_i} is replaced by $F_{k_i}^B = F_{k_i}(1-L_{m_k+i}^A)$ and the payoff of A becomes $F_{k_i}^A = F_{k_i}(1-L_{m_k+i}^B)$. Hence the value V_{II} in (3.13) is changed to V_{II}^B and V_{II}^A for B and A respectively;

$$
\begin{aligned}
V_{\mathrm{II}}^B &= \sum_{k=1}^{K} \left[\sum_{i=0}^{K-k} D(0, m_{k+i}) u_{k+i} \right] (y_{k-1} + \beta - c) Q_k^A 100, \\
V_{\mathrm{II}}^A &= \sum_{k=1}^{K} \left[\sum_{i=0}^{K-k} D(0, m_{k+i}) u_{k+i} \right] (c - \beta - y_{k-1}) Q_k^B 100.
\end{aligned} \tag{4.2}
$$

Finally when C gets defaulted during m_{k-1} and m_k, A and B swap the CB's and 100 when the both survives , and B recovers γ% of the face value at m_k and hence in (3.14) the payoff of A from the exchange is $\mathrm{III}_k^B = \mathrm{III}_k(1-L_{m_k}^A)(1-L_{m_k}^B)$ and the payoff of A is $\mathrm{III}_k^A = -\mathrm{III}_k(1-L_{m_k}^B)(1-L_{m_k}^A)$. Therefore V_{III} in (3.15) is replaced by

$$V_{\mathrm{III}}^{B} = 100(\gamma - 1)\sum_{k=1}^{K} D(0, m_k)\, q_k^C Q_k^A Q_k^B,$$

(4.3)

$$V_{\mathrm{III}}^{A} = 100(1 - \gamma)\sum_{k=1}^{K} D(0, m_k)\, q_k^C Q_k^A Q_k^B.$$

In this setting, the total payoffs of A and B are valued at 0 as

(4.4) $$v^D \equiv V_I^D + V_{\mathrm{II}}^D + V_{\mathrm{III}}^D \quad (D = A, B)$$

Equating these values of A and B , $v^A = v^B$, yields a fair value for β of the DSO with the credit risks of A and B taken into account.

Theorem 4.1 When the counter party's risks of A and B are taken into account, the premium of the DSO is given by

(4.5) $$\beta = \frac{\sum_{k=1}^{K} D(0, m_k)\left\{(c - y_{k-1})\, q_k^c \delta_k (Q_k^A + Q_k^B) + 2(1 - r) q_k^c Q_k^A Q_k^B\right\}(Q_k^A + Q_k^B)}{\sum_{k=1}^{K} D(0, m_k)\left\{(Q_k^C u_k + G_k u_k + \delta_k q_k^c\right\}(Q_k^A + Q_k^B)}.$$

This is a fair value from a viewpoint of a third party. But in view of the party D, D may regard $Q_k^D = 1$ because D does not think he will get defaulted (D=A or B).

Naturally when $Q_k^D = 1$ $(D = A, B)$, the formula is reduced to the one in Theorem 3.1.

5 Concluding Remark.

In this paper we gave a valuation formula for the simplified DSO. Clearly this is only the first step to the problem of valuing the original DSO and the problem calls for a further study. For example, when we include the downward rating in c) in Section 2 as one of the default events in addition to bankruptcy, the problem becomes dramatically difficult to treat theoretically. Then, as we discussed, the value process of the CB will be relevant in the valuation. In addition, J in (3.1) becomes the first time to one of the two events, which makes the problem complicated. One may use a Markov transition model to treat the change of credit quality. But the American nature of the original DSO is here another problem. Even if Co C gets rated as C, the protection buyer A may not exercise the right to start the swap.

Furthermore, the cancellation clause in the contract is difficult to consider without some assumptions.

We leave these problems open.

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