Pricing Mortgage-Backed Securities (MBS)

—A Model Describing the Burnout Effect —

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Takeaki KARIYA and Masaaki KOBAYASHI
Institute of Economic Research, Kyoto University
www.kier.kyoto-u.ac.jp/fe/
IBJ-DL Financial Technology, Co. Ltd

(Revised Version)

This paper presents a pricing formula for MBS's and proposes a specific model for MBS prices which describes the so-called burnout phenomenon of prepayments due to refinancing. A numerical example for the model is demonstrated by Monte Carlo simulation. Also an estimation procedure is described.

1. Introduction

A mortgage-backed security(MBS) is a pass-through security so structured that all the payments made by mortgage holders, except for servicing fees, go to the investors who purchase the securities. It is often the case that the payments are protected against the default risk of mortgagors by a guaranty institution.

In this paper, via no-arbitrage pricing theory in discrete-time setting, we first derive a valuing formula for such an MBS with this guaranty as the US GNMA, FNMA, FHLMC etc. for housing loans. A notable feature of such an MBS is that the mortgage holders are given the option of prepaying the loans at any time and hence the investors have to take the prepayment risk in addition to interest risk. The prepayment occurs severely due to refinancing when the mortgage interest rate drops greatly relative to the initial rate. Then investors lose opportunities to allocate the prepaid money in the environment of low interest rate. The prepayment due to refinancing is considered a behavior due to a purely economic incentive of borrowers. On the other hand, prepayment occurs when the mortgage holders sell their houses. They sell their houses either for the economic reason that the prices of their houses are appreciated significantly or for non-economic reasons such as family problems, job problems, etc. The prepayment due to these reasons is difficult to model unless the details of the borrowers are given as data during the loan period. In general, the detailed information on the prepayment reasons in a specific pool of mortgage loans is not available to the investors. Hence they have to model the prepayment behaviors based on various available economic and demographic data or they simply treat it as an additional spread in the discount function, which is typical in the "standard" OAS (option-adjusted spread) model. Prepayment due to defaults is usually ignored because it is small in private housing loans.

In the literature, there is a large body of literature on the US MBS's, both theoretical empirical. Among others, Schwartz and Torous (1989) empirically modeled prepayment as a function of exogenous or explanatory variables in a regression model. This is a typical approach to fit observed prepayment pattern including the so-called burnout phenomenon. The burnout phenomenon is the one that prepayments calm down after a certain period even if the mortgage interest rate is smaller than the initial rate. In modeling this phenomenon, there are a variety of approaches in the Wall Street firms, which are quite different from firm to firm. Dunn and McConnell (1981a,1981b) modeled optimal prepayment strategy of a mortgage holder, where prepayment was regarded as a call option in a continuous-time setting. But as the prepayment behavior was treated as homogeneous in the pool, they did not treat the burnout effect. His approach were followed by Timmis (1985), Dunn and Spatt (1986), Johnston and Van Drunen (1988), who included some frictional factors against prepayment such as cost and lag. Also McConnell and Singh (1994) developed a procedure for evaluating collateralized mortgage obligations. Stanton(1995) introduced the heterogeneity of the prepayment cost in mortgagor's behaviors and presented a more comprehensive model in which mortgage holders make prepayment decisions at discrete time intervals. He stated that these two features of the model endogenously produce the burnout phenomenon.

In our model, we directly explain the burnout phenomenon of prepayment by the heterogeneity of incentives of mortgagors for prepayment. To state our model, note that the prepayment is exclusively grouped into the four categories: 1) prepayment due to refinancing ($d=1$), 2) prepayment due to sales of houses ($d=2$), 3) prepayment due to defaults ($d=3$) and 4) partial prepayment. We ignore the 4 th one in the sequel. Then prepayment is described as the time till an individual mortgagor leaves the pool of mortgages by prepayment. Let τ_k be the exit time of the k-th person :

$$\tau_k = \min\{\tau_{kd} : d = 1,2,3\},$$

where τ_{kd} is the minimum time till the k-th person prepays due to the d-th factor. The reasons and incentives for prepayment of individual mortgage holders in each category are different, and so are τ_k's. In general, the heterogeneity of these reasons and incentives in a loan pool is not considered because of no availability on the data. However, it is important to understand that it forms the main factor for the burnout phenomenon. Thus in this paper we consider the phenomenon the heterogeneity of economic incentives in the pool and model it as the heterogeneity of exit times τ_k's caused by refinancing ($d = 1$). In our model, the heterogeneity is treated as the differences of the boundaries (incentive thresholds) of τ_k's defining exit time in terms of the spread of the initial and current mortgage rates. As a demonstration, we give a numerical example together with a simple interest rate model where the prepayment activity is simulated and an MBS is priced at 0. Also, the effect of the mean level of interest rate and the effect of the mean level of thresholds on the prices of an MBS are studied. In addition, an estimation procedure for unknown parameters in the model is described when the prepayment history up to time n is observed.

As a general reference we cite Fabozzi (1995) on the US MBS's.

2. Cashflow Function of an MBS

In this section, we describe the cashflow function of an MBS as a pass-through security with guaranty against default. We only consider an MBS based on fixed rate loan with equal monthly payment of borrowers. Let R_n be the borrower's mortgage rate at n, C the coupon of the MBS and S the servicing rate. All these rates are annual rates. Also let N be the maturity in month, m the current month for valuing the MBS for the remaining period when the prepayment history up to m is given, and n a future month ($0 \le m \le n \le N$). Further let MB_n be the remaining balance at n when no prepayment occurs. Then as is well known, the constant monthly payment made by all the borrowers in the pool is

(2-1)
$$MP = MB_0 \times \frac{R_0/12\,(1 + R_0/12)^N}{(1 + R_0/12)^N - 1}$$

and the remaining balance at n with no prepayment is

(2-2)
$$MB_n = MB_0 \times \frac{(1 + R_0/12)^N - (1 + R_0/12)^n}{(1 + R_0/12)^N - 1} \quad (n = 1, \cdots, N).$$

Let I_n and P_n be respectively initially scheduled interest and principal payment when no prepayment is assumed. Then they are given by

(2-3)
$$\begin{aligned} P_n &= MB_{n-1} - MB_n \\ &= MB_0 \times \frac{R_0/12 \times (1+R_0/12)^{n-1}}{(1+R_0/12)^N - 1} \qquad (n=1,\cdots,N) \ , \end{aligned}$$

(2-4)
$$\begin{aligned} I_n &= MB_{n-1} \times R_0/12 \\ &= MB_0 \times \frac{R_0/12 \times \left[(1+R_0/12)^N - (1+R_0/12)^{n-1}\right]}{(1+R_0/12)^N - 1} \qquad (n=1,\cdots,N) \ . \end{aligned}$$

Futher let $\overline{MB}_n$ be the actual balance at n when prepayment occurs and let

(2-5)
$$\begin{aligned} Q_n &= (1-SMM_n)\times(1-SMM_{n-1})\times\cdots\times(1-SMM_1) \\ &= \overline{MB}_n / MB_o \qquad\qquad (\overline{MB}_o = MB_o) \end{aligned}$$

be the survival(remaining) rate of the total balance, i.e., the ratio of the actual balance with prepayment and the initial balance. Here SMM_n denotes the Single Monthly Mortality at n or equivalently the marginal monthly mortality rate from $n-1$ to n in terms of the actual balances. Clearly by (2-5)

(2-6)
$$SMM_n = \frac{Q_{n-1} - Q_n}{Q_{n-1}} \ .$$

The unscheduled interest paid at n under prepayment is

(2-7)
$$\bar{I}_n = \overline{MB}_{n-1} \times R_0/12 = I_1 \times Q_{n-1} \ .$$

Now the total cashflow of the MBS paid to investors at n is the change of the actual balances from $n-1$ to n and the interest with the servicing fee deducted ;

(2-8)
$$\begin{aligned} \overline{CF}_n &= \overline{MB}_{n-1} - \overline{MB}_n + \left(\frac{C}{C+S}\right) \times \bar{I}_n \\ &\equiv cQ_n + dQ_{n-1} \ , \end{aligned}$$

where $c = -MB_0$ and $d = MB_0 + [C/(C+S)]\, I_1$.

3. Valuation Formula for an MBS

In this section we derive a theoretical valuation formula for an MBS with heterogeneous prepayments. A basic assumption for doing this is that there are K loan borrowers in the pool, which is a hypothetical assumption, and the loan sizes are equal, say $A_0 = MB_o/K$. We do not need to know the number K of borrowers. This assumption enables us to distinguish individual behaviors with respect to prepayment and to treat the burnout phenomenon caused by their heterogeneous behaviors. It is also assumed that there is no partial prepayment. Let

$$L_n = \text{ the number of borrowers who prepay up to } n.$$

And let

$$PI_n = P_n/K \qquad \text{and} \qquad MBI_n = MB_n/K$$

be the scheduled individual principal payment at n and the individual remaining mortgage balance at n. Clearly

$$PI_n = (MB_{n-1} - MB_n)/K \qquad \text{and} \qquad MBI_n = A_0 - PI_1 - \cdots - PI_n,$$

and by definition the actual mortgage balance at n is expressed as

(3-1) $$\overline{MB}_n = \overline{MB}_{n-1} - PI_n(K - L_n) - MBI_{n-1}(L_n - L_{n-1}),$$

where the second term and third term of the right side are respectively the scheduled amount paid by those who do not prepay at n and the amount paid by those who prepay. Then using these and (2-8) we can show that the remaining balance at n is expressed as

(3-2) $$\boldsymbol{\overline{MB}_n = MB_n\left(1 - \frac{L_n}{K}\right) = A_0(K - L_n)\, MB_n/MB_0}.$$

since by (3-1)

$$\begin{aligned}\overline{MB}_n &= MB_0 - \sum_{j=1}^{n} PI_j(K - L_j) - \sum_{j=1}^{n} MBI_{j-1}(L_j - L_{j-1}) \\ &= MB_0 - \sum_{j=1}^{n} P_j + \frac{L_n}{K}(P_n - MB_{n-1}) + \sum_{j=1}^{n-1} \frac{L_j}{K}(P_j - MB_{j-1} + MB_j) + MB_0\frac{L_0}{K}.\end{aligned}$$

Note $\quad MB_{j-1} = P_j + MB_j \quad$ and $\quad L_0 = 0.$

Therefore, by (2-8) and (3-2), the n-th month cashflow is expressed as a function of secession rate L_n / K;

(3-1)
$$\overline{CF}_n = a_n\left(1-\frac{L_n}{K}\right)+b_n\left(1-\frac{L_{n-1}}{K}\right) ,$$

where a_n and b_n are given by

(3-2a)
$$\begin{aligned} a_n &= P_n - MB_{n-1} \\ b_n &= MB_{n-1} + \frac{C}{C+S}\times I_n \quad . \end{aligned}$$

Note that a_n and b_n are known at 0.

Now to derive a no-arbitrage value at m of the n-th cashflow, let the process of cash be given by

(3-3)
$$B_n = \exp\left(\sum_{j=0}^{n-1} r_j h\right) \qquad (h=1/12) ,$$

where $\{r_j\}$ is an interest rate process. Then by a general no-arbitrage pricing theory in discrete time framework (see Kariya (1997)), we obtain

Theorem 3.1 The no-arbitrage value at m of the MBS with maturity N is given by

(3-4)
$$V(m,N) = \sum_{n=m+1}^{N} CF(m,n) \ ,$$

where

(3-5)
$$\begin{aligned} CF(m,n) &= B_m E_m^*[\overline{CF}_n / B_n] \\ &= E_m^*[\Delta(m,n)(a_n Q_n + b_n Q_{n-1})] \ , \end{aligned}$$

(3-6)
$$\Delta(m,n) = \exp\left(-\sum_{j=m}^{n-1} r_j h\right)$$

and the conditional expectation $E_m^*(\cdot)$ at m is taken with respect to a martingale

(risk

neutrality) measure for $\{r_j\}$ and $\{L_j\}$.

Note that the martingale measure is not unique in our problem because there will be many risk factors for $\{L_j\}$.

Let J_{n-m} be the number of the borrowers who will secede from the pool during the period from the $(m+1)$-th month to the n-th month to get

(3-7) $$L_n = J_{n-m} + L_m \ .$$

Then we obtain

(3-8) $$\begin{aligned} CF(m,n) &= \left(1-\frac{L_m}{K}\right)(a_n+b_n)D(m,n) - a_n E_m^*\left[\Delta(m,n)\frac{J_{n-m}}{K}\right] \\ &\quad - b_n E_m^*\left[\Delta(m,n)\frac{J_{n-1-m}}{K}\right] \ , \end{aligned}$$

where $L_0 = 0$ and

(3-9) $$D(m,n) = E_m^*\left[\Delta(m,n)\right] \ .$$

Therefore to value the MBS, we need to value

(i) the discount bond $D(m,n)$ $(n = m+1,\cdots,N)$

(ii) the conditional expectation

(3-10) $$E_m^*[\Delta(m,n)J_{n-m}] = (1-L_m)E_m^*[\Delta(m,n)] - E_m^*[\Delta(m,n)(1-L_n)] \ .$$

To consider the evaluation of (3-11) a bit further, let τ_k be the exit (secession) time of the k-th borrower and the exit time event $\{\tau_k = j\}$ $(k=1,\cdots,K; j=1,\cdots,N)$. The event generation function is defined by

$$\chi_{k,j} = \chi_{\{\tau_k = j\}} = \begin{cases} 1 & \text{if } \{\tau_k = j\} \\ 0 & \text{otherwise} \end{cases}$$

Then clearly it holds that

1) $\chi_{k,j}\chi_{k,n} = 0$ $(j \neq n)$,

2) $\sum_{j=1}^{N} \chi_{k,j} = 1$,

3) for $L_{k,n} = \sum_{j=1}^{n} \chi_{k,j}$, $L_{k,n} = 1 \to L_{k,n'} = 1 \quad (n' > n)$,

4) $L_n = \sum_{k=1}^{K} L_{k,n}$.

Since L_m is given at m, say $L_m = b$, let these people who prepaid up to m be $k = 1, \cdots, b$. Then

$$J_{n-m} = \sum_{k=b+1}^{K} \left(L_{k,n} - L_{k,m} \right) = \sum_{k=b+1}^{K} \sum_{j=m+1}^{n} \chi_{k,j} \tag{3-11}$$

and hence (3-11) is expressed as

$$\sum_{k=b+1}^{K} E_m^* \left[\Delta(m,n) \left(L_{k,n} - L_{k,m} \right) \right] = \sum_{k=b+1}^{K} \sum_{j=m+1}^{n} E_m^* \left[\Delta(m,n) \chi_{k,j} \right] \tag{3-12}$$

$$= (1-b) E_m^* \left[\Delta(m,n) \right] - \sum_{k=b+1}^{K} E_m^* \left[\Delta(m,n)(1 - L_{k,n}) \right].$$

This is the expression for which the heterogeneous feature of prepayments in the pool is taken into account in the next section. In the sequel we assume that our actual measure which generates interest rates and prepayments is a martingale measure.

4. Interest Incentive Function

In this section, we propose a model to describe the heterogeneity of the incentives of the borrowers for refinancing. We assume for simplicity that a borrower in the pool prepays at n for gains only when the spread of the initial mortgage rate R_0 and the current rate R_n widens more than or equal to his incentive threshold. Then the exit (secession) time of the k-th borrower is expressed as

$$\tau_k = \min \left\{ j : R_0 - R_j \geq c_j(k) \right\}, \tag{4-1}$$

where $c_j(k)$ denotes the incentive threshold of the k-th borrower at j and it can depend on month j. If the demographic information on the k-th person is available, we may be able to include it in the specification of $c_j(k)$. Though with $L_{k,n} = \chi_{\{\tau_k \leq n\}}$,

$$\text{(4-2)} \quad \begin{aligned} E_m[1-L_{k,n}] &= P_m(\tau_k > n) \\ &= P_m\left(\bigcap_{j=1}^{n}\{R_0 - R_j < c_j(k)\}\right), \end{aligned}$$

(3-13) cannot be evaluated independently of the stochastic discount factor $\Delta(m,n)$. In fact, the mortgage rate process $\{R_n\}$ and the interest rate process in $\Delta(m,n)$ are highly correlated. This distinguishes the MBS prepayment model from a credit risk model where the credit spreads of interest rates are often assumed to be independent of nondefaultable rates. To get a basis for the evaluation of (3-13), let us assume for simplicity that the mortgage rate R_n is a linear function of an sh year (long term) interest rate $r_n(sh)$:

$$\text{(4-3)} \quad R_n = \alpha + \beta r_n(sh) \ ,$$

where $h = 1/12$, and that $r_n(sh)$ is a linear function of one-month spot rate r_n (affine model) :

$$\text{(4-4)} \quad r_n(sh) = \gamma(s) + \delta(s) r_n \ .$$

Of course, the assumption for (4-3) and (4-4) can be replaced by a more general set-up associated with forward rates. Under this assumption R_n becomes

$$\text{(4-5)} \quad R_n = \alpha(s) + \beta(s) r_n \ ,$$

where $\beta(s) > 0$. Then

$$\text{(4-6)} \quad E_m\left[(1-L_{k,n})\Delta(m,n)\right] = E_m\left[\left(\prod_{j=1}^{n}\chi_{\{e_j(k)<r_j\}}\right)\Delta(m,n)\right] \ ,$$

where

$$e_j(k) = (R_0 - \alpha(s) - c_j(k))/\beta(s) \ .$$

To get a value from this formula, we need to give a distribution of the thresholds $e_n(k)$ over borrowers $k = 1,\cdots,K$ for each n. This distribution is in fact directly associated with the burnout phenomenon of prepayments in the pool. But we do not have a sufficient knowledge on the distribution. Hence as a most likely case we assume that the threshold $c_n(k)$ is constant over time $n = 1,\cdots,N$ and that the distribution of $c(k)$'s over k is approximated by a normal distribution. To specify it formally, let the spread of the initial and current spot rates be

(4-7) $$r_0 - r_n = u_n$$

Then

(4-8) $$\tau_k = \min\left\{ j : R_0 - R_j \geq d_k \right\} = \min\left\{ j : r_0 - r_j \geq d_k{}' \right\} \ ,$$

where $d_k{}' = d_k / \beta(s)$. In this expression, the thresholds are regarded as d_k's and hence we make

Assumption : Let

$$p_l = \frac{1}{K} \times \left\{ \#\,\text{of } d_k{}'\text{s in } \left[(l-1)\eta,\, l\eta\right) \right\} .$$

Then p_l is approximated by the area of normal distribution $N\left(\mu, \sigma^2\right)$ over $\left[(l-1)\eta,\, l\eta\right)$ $(l = 1, \cdots, q)$.

Under this assumption, the K borrowers are allocated into q groups according to the sizes of their thresholds and then we get the approximation

(4-9) $$\sum_{k=1}^{K} L_{k,n} / K \quad \approx \quad \sum_{l=1}^{q} g_l(u_n) p_l$$

and hence

(4-10) $$\sum_{k=1}^{K} E_0\left[\Delta(0,n) L_{k,n}\right] / K \quad \approx \quad \sum_{l=1}^{q} E_0\left[\Delta(0,n) g_l(u_n)\right] p_l \ ,$$

where q is smaller than K and by assumption,

(4-11) $$\begin{aligned} p_1 &= \Phi((\eta - \mu)/\sigma) \ , \\ p_2 &= \Phi((2\eta - \mu)/\sigma) - \Phi((\eta - \mu)/\sigma) \ , \\ &\vdots \\ p_{q-1} &= \Phi(((q-1)\eta - \mu)/\sigma) - \Phi(((q-2)\eta - \mu)/\sigma) \ , \\ p_q &= 1 - \Phi(((q-1)\eta - \mu)/\sigma) \ . \end{aligned}$$

Here $g_l(u_n)$ denotes the prepayment event of the l-th group and is defined by

(4-12) $$g_l(u_n) = \begin{cases} 1 & \text{if } \ l\eta \leq u_j \ \ (\text{before } n) \\ 0 & \qquad \text{otherwise} \ , \end{cases}$$

where $u_j = r_0 - r_j$ and $q\eta = \infty$. The first group of the borrowers prepays if the spread goes over or is equal to η and the relative portion is p_1, the second group prepays if the spread goes over or is equal to 2η and the portion is p_2, and so forth. Also μ is an average level of the thresholds of the borrowers responding to the spreads, and σ is chosen as such $\mu/3$ so that

$$q\eta = \mu + 3\sigma \quad .$$

For example, when $q = 10$ and $\mu = 0.02$, then $\sigma = 0.02/3$, and $\eta = 0.04/10 = 0.004$. In Figure4-1, the relation of the spread and the distribution of the thresholds is drawn.

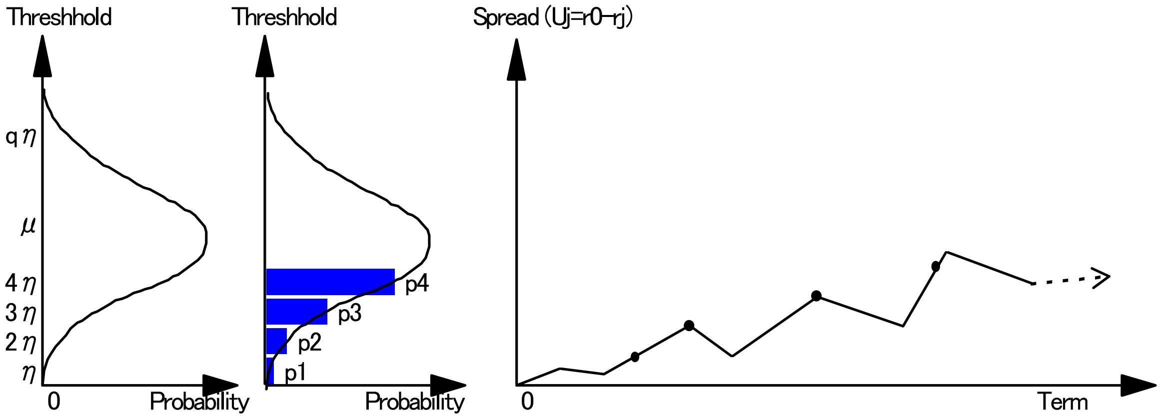

Figure 4-1

5. Monte Carlo (MC) Valuation of an MBS

In this section, we run an MC simulation to value an MBS numerically at 0. We first describe our MC simulation procedure where a simple interest model is assumed. Then theoretical values of an MBS are numerically evaluated by the method.

5.1 Monte Carlo (MC) Simulation Method

Suppose for simplicity that the monthly spot rate process $\{r_n\}$ follows the mean reverting normal model (Vasicek model) ;

(5-1)
$$\Delta r_n = \theta_0 (\theta_1 - r_{n-1}) h + \theta_2 \sqrt{h} \varepsilon_n \qquad (h = 1/12) \ ,$$

$$\varepsilon_n \sim iid \quad N(0,1),$$

where $\Delta r_n = r_n - r_{n-1}$. For $\theta_0, \theta_1, \theta_2$ and r_0 given, generate I paths of N standard normal random numbers ;

(5-2) $$\left(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \cdots, \varepsilon_N^{(i)}\right),$$

where $i = 1, \cdots, I$. Each of (5-2) gives a path of interest rates via (5-1) ;

(5-3) $$\left(r_1^{(i)}, r_2^{(i)}, \cdots, r_N^{(i)}\right),$$

which in turn gives a set of N random discount factors

(5-4) $$\left(\Delta_1^{(i)}, \Delta_2^{(i)}, \cdots, \Delta_N^{(i)}\right) \quad \text{with} \quad \Delta_n^{(i)} = \exp\left(-\sum_{j=0}^{n-1} r_j^{(i)} h\right) .$$

Thus we obtain an estimate of the the discount functions $D(0,n)$'s based on I sets ;

(5-5) $$\hat{D}(0,n) = \frac{1}{I}\sum_{i=1}^{I} \Delta_n^{(i)} \qquad (n = 1, \cdots, N) .$$

Next we evaluate the right side of (4-9). By (4-12), suppose that the i-th path of the spread $u_n^{(i)}$ attains

(5-6) $$v_l = l\eta \qquad (l = 1, \cdots, q)$$

at $m_l^{(i)}$ for the first time, where

(5-7) $$m_1^{(i)} < m_2^{(i)} < \cdots < m_q^{(i)}$$

If the i-th path attains v_q before N, that is , if $m_q^{(i)} < N$, then all the borrowers prepay before the maturity N. If $m_q^{(i)} > N$, the remaining borrowers in the pool pay the last remaining amount at N. Therefore the figures we need to compute the right side of (4-9) are tabulated for $m_q^{(i)} > N$ as follows.

j	0	1	2	$\cdots$	n	$\cdots$	N
$\varepsilon_j{}^{(i)}$		$\varepsilon_1{}^{(i)}$	$\varepsilon_2{}^{(i)}$	$\cdots$	$\varepsilon_n{}^{(i)}$	$\cdots$	$\varepsilon_N{}^{(i)}$
$r_j{}^{(i)}$	r_0	$r_1{}^{(i)}$	$r_2{}^{(i)}$	$\cdots$	$r_n{}^{(i)}$	$\cdots$	$r_N{}^{(i)}$
$\Delta_j{}^{(i)}$		$\Delta_1{}^{(i)}$	$\Delta_2{}^{(i)}$	$\cdots$	$\Delta_n{}^{(i)}$	$\cdots$	$\Delta_N{}^{(i)}$
$u_j{}^{(i)}$		$u_1{}^{(i)}$	$u_2{}^{(i)}$	$\cdots$	$u_n{}^{(i)}$	$\cdots$	$u_N{}^{(i)}$

$g_l\left(u_j{}^{(i)}\right)$

l	p_l \ j	1	2	$m_1{}^{(i)}$			$m_2{}^{(i)}$		N
1	p_1	0	0	1	1	$\cdots$	1	$\cdots$	1
2	p_2	0	0	0	0	$\cdots$	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				$\vdots$
q	p_q	0	0	0	0	$\cdots$	0		1

$$\xi_l{}^{(i)}(j) = \Delta_j{}^{(i)} g_l\left(u_j{}^{(i)}\right) p_l$$

l	p_l			$m_1{}^{(i)}$			$m_2{}^{(i)}$		
1	p_1	0	0	$\Delta_{m_1}{}^{(i)} p_1$	$\Delta_{m_1+1}{}^{(i)} p_1$	$\cdots$	$\Delta_{m_2}{}^{(i)} p_1$	$\cdots$	$\Delta_N{}^{(i)} p_1$
2	p_2	0	0	0	0	$\cdots$	$\Delta_{m_2}{}^{(i)} p_2$	$\cdots$	$\Delta_N{}^{(i)} p_2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				$\vdots$
q	p_q	0	0	0	0	$\cdots$	0	$\cdots$	$\Delta_N{}^{(i)} p_q$
	total	$\lambda^{(i)}(0,1)$	$\lambda^{(i)}(0,2)$						$\lambda^{(i)}(0,N)$

Here

(5-8) $$\xi_l{}^{(i)}(n) = \Delta_n{}^{(i)} g_l\left(u_n{}^{(i)}\right) p_l \quad .$$

Hence summing this up over l to get

(5-9) $$\lambda^{(i)}(0,n)=\sum_{l=1}^{q}\Delta_n{}^{(i)}g_l\left(u_n{}^{(i)}\right)p_l$$

and averaging this over i, we obtain an estimate of the right side of (4-9) ;

(5-10) $$\frac{1}{I}\sum_{i=1}^{I}\lambda^{(i)}(0,n) \quad .$$

Using this, we are able to value an MBS in (3-6) numerically.

5.2 Numerical Valuation

To value an MBS, we first describe our MBS, interest rate model and incentive function. Then we consider the effect of changes of the mean reversion level of interest rates on values of the MBS and the effect of changes of thresholds on values of the MBS.

We consider a 30 year MBS with \$100 face value, and 6.5% coupon made of mortgage loans with 7% rate and equal monthly payment. Here 0.5% is the servicing fee. Thus ,

$$R_0=0.07,\ \ S=0.005,\ \ C=0.065,\ \text{and}\ \ N=360\ \ .$$

In the interest rate model, put

$$\theta_0=0.2,\ \ \theta_1=0.05,\ \ \theta_2=0.008,\ \ h=1/12=0.083,\ \text{and}\ \ r_0=0.05\ \ ,$$

which are respectively the speed of mean reversion, mean reversion level, volatility, time unit for change and initial rate. Thirdly, the parameters of the approximate distribution of thresholds are given as

$$q=10,\ \ \mu=0.02,\ \ \sigma=0.0067,\ \text{and}\ \ \eta=0.004\,(40\,\text{bp})\ \ .$$

Here the mean level of the distribution is 2%, the standard deviation is $\sigma=2/3$ % and η is taken as $q\eta=\mu+3\sigma$. The number of the paths we generate by MC is $I=1{,}000$. In this set-up, we obtained a theoretical value of the MBS as 102.1 dollars.

In figure 5-1, the values of cashflows $CF(0,n)$ without and with prepayments in this MC evaluation are graphed. The cashflows with prepayment are more valued up to about 80 months than the case without prepayment, and they are less valued thereafter. This is the effect of prepayment and changes the value of an MBS.

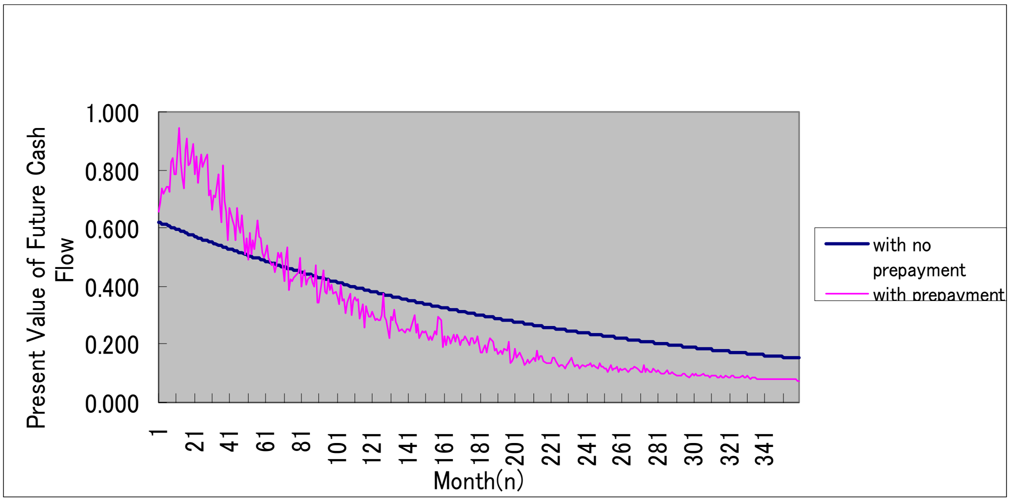

Figure 5-1

Next let us consider the effect of the change of the mean reversion level θ_1 in the interest rate model on values of the MBS, whrere all the other parameters are kept unchanged. The values and the graph are given in Figure 5-2. Clearly it is observed that the greater θ_1 is, the smaller the value of the MBS is. Note that the mean reversion level θ_1 is a long term mean of interest rates and that changes of the level make two effects on the value of the MBS. In fact, on one hand, an increase of θ_1 will make the incentive for prepayment smaller because the spread gets smaller on the average, which will make values of the MBS higher. On the other hand, an increase of θ_1 will make values of the MBS lower because the discount rates in the discount function get larger. The graph shows that the latter effect is much bigger than the former.

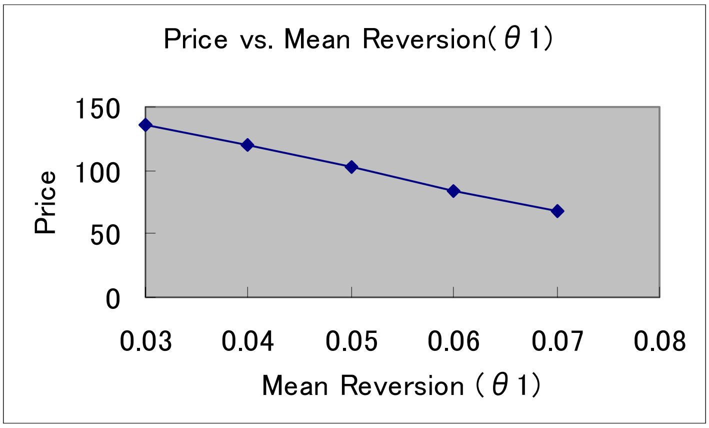

Figure 5-2

Thirdly we consider an effect of changes of the mean μ of thresholds on values of the MBS. Our MC result on this effect is given in Figure 5-3. From the figure, it is observed that the smaller μ is, the lower the value of the MBS is. When μ gets larger, the speed of the increase of the value decreases though the value is increasing. For prepayments occur less when μ gets larger and the value of the MBS will approach to the value with no prepayment.

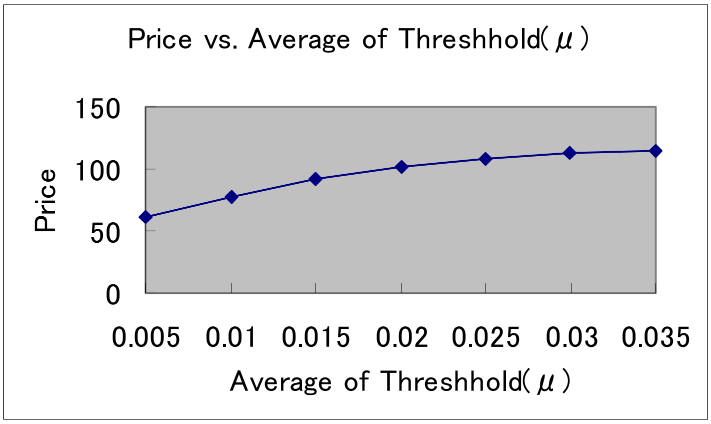

Figure 5-3

Finally we consider the case where the group number q changes. Figure 5-4 summarizes this case. When q is larger, the thresholds for the heterogeneous incentives of borrowers are more divided and hence the MBS will be more accurately valued. But as the graph shows, the values do not change much after $q = 20$.

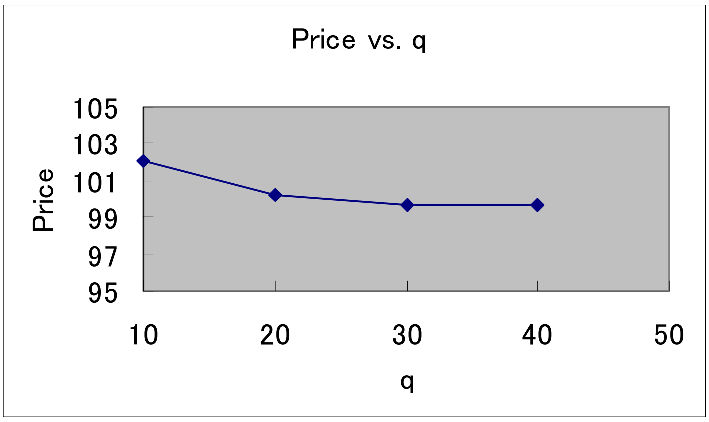

Figure 5-4

6. Estimation Procedure

So far we considered the theoretical price at 0 of an MBS. In this section, we describe the valuation method at m when the prepayment history of the pool is given together with the history of interest rates up to m. Given the two past sequences, the unknown parameters in the incentive function are estimated by the least squares (LS) method that minimizes squared sum of the differences of actual and model prepayments.

To describe the method, we first note that SMM_n's are observable. This information is converted into the secession rates $AMM_n = L_n / K$'s ;

(6-1)

$$\begin{aligned} SMM_n &= \frac{L_n - L_{n-1}}{K - L_{n-1}} = \frac{L_n / K - L_{n-1} / K}{1 - L_{n-1} / K} \\ &= \frac{AMM_n - AMM_{n-1}}{1 - AMM_{n-1}} \end{aligned} \quad .$$

Then from $AMM_0 = L_0 / K = 0$, we obtain the recursion formula

(6-2)

$$AMM_n = AMM_{n-1} + SMM_n (1 - AMM_{n-1}) \quad ,$$

where $n = 1, 2, \cdots, N$. Now suppose that $(AMM_j\ ,\ u_j)$ with $u_j = r_0 - r_j$ are observed for $j = 1, \cdots, m$. Then we estimate the group number q, threshold unit η, threshold mean μ and standard deviation σ by the LS method. Define the objective function to be minimized by

(6-3)

$$\begin{aligned} \Psi(q, \eta, \mu, \sigma) &= \sum_{j=1}^{m} \left[AMM_j - \sum_{l=1}^{q} g_l(u_j) p_l \right]^2 \\ &= \sum_{j=1}^{m} \left[AMM_j - \Phi\left(\frac{l_j \eta - \mu}{\sigma} \right) \right]^2 \\ &\qquad \left(if \ \ l_j = q \, , \quad \Phi\left(\frac{l_j \eta - \mu}{\sigma} \right) = 1 . see \ \ (4-11). \right) \quad , \end{aligned}$$

where

$$l_j = \max\{ l^* \in \mathrm{N} : l^* \eta \le u_k ,\ k = 1, \cdots, j \} .$$

Then the objective function should be minimized under the restrictions

(6-4) $$\eta > 0,\ \ \mu > 0 \text{ and } \sigma > 0 \ .$$

To carry it out, for each q we minimize Ψ with respect to η, μ and σ, where η is assumed to take certain finite number of values η_i's in $[0, \eta^*]$. For example, set $\eta^* = 0.02$ and $\eta_i = i\eta^* / 100$. Also q is assumed to change over $q = 10, \cdots, 30$, from which we find (q, η, μ, σ) minimizing (6-3).

Once the parameters are estimated, the price of an MBS at m is valued through (3-5) in the same way as we discussed in Sections 4 and 5.

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