CB - Time Dependent Markov Model for Pricing Convertible Bonds

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2001/March

Abstract

In this paper, we propose what we call the convertible bond (CB)- time dependent Markov model, which prices N given individual convertible bonds simultaneously, and apply it to Japanese convertible bond data. One of the main features of the model is that it makes full use of the correlation structure of convertible bond prices. The empirical results show that the model well describes individual prices in the market.

Keywords convertible bonds, correlation structure, random cash-flow discount function,time-dependent Markov Model.

I Introduction

The Japanese convertible bond (CB) market is the largest in the world and the balance of CB's outstanding in the market was more than 15 trillion yen at the end of 1998. Hence CB is now an important instrument for institutional investment and pension fund portfolios. As is well known, a CB carries two main components; the component as potential bond and the component as potential stock via the convertibility of the bond part into pre-fixed shares of equity. Therefore the price fluctuations of a CB combine those of both main components. In addition, they are affected by other attributes such as coupon rate and the credit risk of the firm. In modeling the stochastic variations of the CB's in the circulation market, it is quite important to model explicitly the two main components since investors take into account the future values of CB's associated with these components of the potential bond and potential stock. In the literature, Ingersoll (1977) and Brennan and Schwartz

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(1977,1979,1980,1988) developed a normative theory on CB pricing based on the process of the value of a firm, and in some literature the process of the convertibility is directly considered by applying an argument of a standard option theory. However, as Takahashi et all.(1990) pointed out, the models based on such theories will not reflect the processes or market values of CB prices in the market because of the idealized setting. For example, it is assumed that bonds are zero-coupon bonds and the market is complete.

In this paper, we propose a statistical model that effectively prices the market values of CB's. The model basically consists of the time-dependent Markov (TDM) model for the potential bond value and a correlated version of the CB option model with the Black-Scholes (BS) model (1973) as a mean for the values of convertibility. A main feature of our approach is to model all the CB prices simultaneously so that our model makes full use of the information contained in the CB prices which are correlated with each other. Another notable feature of our approach is to include attributes of CB's in modeling. It is shown that our CB model well fits the market prices and can be used as a pricing model through which relatively low-priced CB's are found for investment decision-making.

In Section , we describe the basic viewpoint for the CB pricing model and in Section , the CSM(Cross-Sectional Market) and TDM models for pricing the bond-part of CB's are described. These models for individual bond prices, proposed by Kariya (1993), Kariya and Tsuda (1994,1996) and Tsuda and Kariya(1994,1995), price an individual bond by introducing individual attributes into the models. In Section , we describe our approach to pricing the right (option) of the convertibility, which we loosely refer to as the CB call option for simplicity in the sequel. The CB call option is different from the usual call option in that the issuer is the firm that issues the CB and generally it cannot be sold separately from the bond to the third party. In addition, in the CB call the price to convert a CB into stock has been paid as a bond, which is a debt for the firm unless the CB call is exercised. Further the maturity of the CB call is equal to that of the bond and so it is much longer than that of the usual call.

Let us assume the face value of a CB is 100 yen. When it is issued, the exercise (strike) price of the CB call is specified, for example K, and hence the number of shares of the stock when exercised is fixed as

(1) $$n = \frac{100}{K}.$$

As time passes, the stock prices of the firm change and hence the value at t of the CB as a stock when the CB call is exercised at t is given by

(2) $$\pi_t = (\frac{S_t}{K}) \times 100 = nS_t$$

which is often called parity price (as stock) at t, where S_t denotes the stock prices at t of the firm which issued the CB at $t = 0$. The value at t of the CB call is not the parity value because it contains the time value as a call option that is greater than that of a usual call.

After all, the CB call is regarded as a random variable, which is correlated with other CB calls, and the (conditional) mean is assumed to be approximated by the Black-Scholes (BS) formula. The correlation structure among N given CB calls is approximated by the conditional covariance of the pay-offs at maturity of those CB calls, and seems to play an important role in valuing CB's.

In Section , we describe CB-CSM and CB-TDM models. In Section , we present our empirical results on Japanese CB bond prices and show that the CB-TDM model performs well for each individual CB price. In fact, we estimate the model monthly from 1989.4 through 1998.12. The standard deviations of the residuals of the models are mostly less than 4 yen, where the face value of each CB is 100 yen. The standard deviations tend to be affected by a few volatile outliers. It is noted that there are only 6 parameters in the models and the results will be improved further if we include a few parameter to take into account the outlier's movement.

II Modeling CB Prices: Basic Viewpoint

A main motivation to deal with CB's in the circulation market is to seek for returns, in which various expectations are involved on the future potential market values as bond and the CB call option, the two main components of a CB, which stochastically fluctuate with expectations. However, the two components cannot be separated and so they are not directly observable. Hence the stochastic fluctuations of the components inseparably appear as those of CB prices. In other words, the fluctuations of CB prices consist of those of the two different components. Hence let B_t^* denote the latent variables at t of the potential market value as bond, and let C_t^* denote the potential market value as the CB call respectively, and we express the CB market price at t as a function of B_t^* and C_t^*:

$$CB_t = f(B_t^*, nC_t^*), \tag{3}$$

where n is the number of shares obtained when the CB call is exercised at any time. It is clear that

$$CB_t \geq \max(B_t^*, nS_t) \tag{4}$$

because the parity value nS_t is the value of the CB as stock at t. On the other hand, nC_t^* is different from nS_t in that nC_t^* contains no bond value since it corresponds to $\max(nS_t - K, 0)$ when exercised at t, and nC_t^* denotes the value of the CB call of the convertibility which includes the time value reflecting the future movements of stock prices and interest rates. Note the latent variables B_r^* and nC_t^* are random variables. Also note that

$$nS_t \geq nC_t^* \geq n\max(S_t - K, 0) = \max(nS_t - 100, 0). \tag{5}$$

In the sequel, we specify (3) as a linear function of the latent random variables:

$$CB_t = B_t^* + \alpha_t n C_t^*, \tag{6}$$

in which case from (5)

$$\alpha_t n S_t + B_t^* \geq CB_t \geq \alpha_t \max(nS_t - 100, 0) + B_t^*. \tag{7}$$

The parameter α_t in (6) may be interpreted as follows. First (6) can be viewed as a portfolio of the latent bond and the latent CB call and hence

1) When the economy is in recession and stock prices are lower, CB's will be priced low as they carry lower coupon rates. Even in such a situation, the reason why they are traded and priced in the market will be the time value of the CB call. Hence in recessions, people will expect the stock prices to rise in future as the maturity of a CB is relatively long, and so α_t, reflecting the expectations, will be greater than 1. Conversely, in booms, the CB's are highly priced because of the CB call, but people will not find the time values in them and α_t will remain close to 1 or less as people expect the high stock prices to fall. In other words, α_t will reflect the market evaluation of the CB call.
2) When the expected value of C_t^* is approximated by an option formula such as the Black-Scholes formula, it may reflect an adjustment factor for the approximation.

III Modeling the Latent Bond Value

In this section, we describe the TDM model which is applied to modeling the latent bond values B_t^*. Suppose there are N coupon bonds in the market and let P_{it} be the price at the ith bond. Each bond carries as its attributes at least coupon rate and duration to maturity. In addition, such attributes as default risk may be considered in the modeling. In the TDM model each bond is viewed as

$$(C_{it}(\cdot), \mathcal{Z}_{it}, P_{it}), \tag{8}$$

where $C_{it}(s)$ is the cashflow function of the ith bond which is zero unless s is not a duration (term) to a cashflow point of the ith bond, and

$$\mathcal{Z}_{it} = \{z_{ikt} : k = 1, \cdots, q\} \tag{9}$$

is the set of attribute which can be objectively recognized in advance. It can include such past random variables as the past trading volumes, but the variables which are stochastically realized together with the price at t should not be included. In our empirical analysis on CB prices, we simply use as attributes the coupon rate and duration to maturity, because credit risk was not significant as an attribute. Let

$$\mathrm{s}_j = s(i)_j \quad (j = 1, \cdots, M(i) : i = 1, \cdots, N) \quad (10)$$

denote the jth cashflow period(duration) from time t of the ith bond, where N denotes the number of bonds in analysis. Enumerate $s(i)_j$'s in (III) in an ascending order as

$$s_{a1} < s_{a2} < \cdots < s_{aM}, \quad s_{aM} = \max s(i)_{M(i)} \tag{11}$$

and all the cashflow function $C_{it}(s)$'s $(i = 1, \cdots, N)$ are regarded as defined on $0 \leq s \leq s_{aM}$ without loss of generality. In this notation, the ith bond price is expressed as the present value of the cashflows discounted by the random discount factors;

$$P_{it}(0) = \sum_{j=1}^{M} C_{it}(s_{aj}) D_{it}(s_{aj}) = \boldsymbol{C}'_{it} \boldsymbol{D}_{it} \tag{12}$$

with

$$\begin{aligned} \boldsymbol{C}_{it} &= (C_{it}(s_{a1}), \cdots, C_{it}(s_{aM}))' : M \times 1, \\ \boldsymbol{D}_{it} &= (D_{it}(s_{a1}), \cdots, D_{it}(s_{aM}))' : M \times 1. \end{aligned} \tag{13}$$

Then letting $\overline{D}_{it}(s_{aj}) = E[D_{it}(s_{aj})]$ be the mean discount function of the ith bond, the model (12) becomes (14)

$$\begin{aligned} P_{it}(0) &= \boldsymbol{C}'_{it} \overline{\boldsymbol{D}}_{it} + \eta_{it}, \quad where \\ \overline{\boldsymbol{D}}_{it} &= E(\boldsymbol{D}_{it}), \\ \eta_{it} &= \boldsymbol{C}'_{it} \boldsymbol{\nu}_{it}, \quad \boldsymbol{\nu}_{it} = \boldsymbol{D}_{it} - \overline{\boldsymbol{D}}_{it}, \end{aligned} \tag{14}$$

where η_{it} is a part of the present value of the ith bond, which is evaluated by the stochastic discount factor $\boldsymbol{\nu}_{it}$ as deviation from the mean. Here we need to specify the mean discount function (factor) and the stochastic behavior of $\boldsymbol{\nu}_{it}$ so that we can model the N bond prices simultaneously at t. Here we consider the case where the coefficients in a polynomial depend on the characteristics.

$$\begin{aligned} \overline{D}_{it}(s) &= 1 + \sum_{k=1}^{p} \delta_{kt}(\boldsymbol{z}_{it}) s^k, \\ \delta_{kt}(\boldsymbol{z}_{it}) &= \sum_{l=0}^{q} \delta_{klt} z_{ilt}, \end{aligned} \tag{15}$$

where $\boldsymbol{z}_{it} = (z_{i1t}, \cdots, z_{iqt})'$ is a vector of the attribute variables of the ith bond. The $p \times q$ common parameters δ_{ijt} are estimated from N bond prices. In the specification in the covariance structure of η_{it}, the following features of bond prices are taken into account;

1) the price structures between a short term bond and a long term bond are less correlated,

2) random discount factors for cash flows occurring close by are more correlated , and

3) bond prices with shorter maturities fluctuate less.

Hence, η_{it}'s is specified as

$$\begin{aligned} g_{ikt} &= Cov(\eta_{it}, \eta_{kt}) = \lambda_{ikt} \boldsymbol{C}'_{it} \boldsymbol{\Phi}_{ikt} \boldsymbol{C}_{kt}, \\ f_{ikt} &= \boldsymbol{C}'_{it} \boldsymbol{\Phi}_{ikt} \boldsymbol{C}_{kt}, \end{aligned} \tag{16}$$

where λ_{ikt} is the parameter which expresses the covariance structure of the issue attributes between the ith and the kth bond. The specification of λ_{ikt} is

$$\begin{aligned} \lambda_{ikt} &= \sigma^2 a_{iit} \\ &= \sigma^2 \rho a_{ikt} \quad (i \neq k), \\ a_{ikt} &= b(\boldsymbol{z}_{it}, \boldsymbol{z}_{kt}) \exp(-|s_{M(i)} - s_{M(k)}|), \\ \boldsymbol{z}_{it} &= (z_{i1t}, \cdots, z_{iqt})'. \end{aligned} \tag{17}$$

A possible candidate of the function $b(\boldsymbol{z}_{it})$ is specified as min $(s^n_{M(i)}, s^n_{M(k)})$. And

$$\boldsymbol{\Phi}_{ikt} = (\phi_{ikt \cdot jr}) = (\exp(-|s_{aj} - s_{ar}|)), \tag{18}$$

$\lambda_{ikt}\phi_{ikt \cdot jr}$ expresses the covariance structure between the random discount factor $\nu_{it}(s_{aj})$ and $\nu_{kt}(s_{ar})$. $\phi_{ikt \cdot jr}$ corresponds the covariance structure of the discount rate which discounts the cash flow for the term s_{aj} and s_{ar}. The specification in Equation (18) expresses that discount factors for cash flows occurring close by are more correlated. It is important that the specification of η_{it}'s naturally lead the structure that the variances of bond prices with shorter maturities and lower coupon rate become smaller because the size of coupon rate adds the structure in Equation (18). It is called cross-sectional market (CSM) model (Kariya 1993).

Next, we describe the TDM model proposed by Kariya and Tsuda (1994), which takes into account the time series factors carried over from the past. The stochastic discount function of the TDM model is specified as

$$D_{it}(s) = \overline{D}_{it}(s) + \xi_t \nu_{i,t-1}(s+1) + \tau_{it}(s), \tag{19}$$

where $\nu_{it}(s) = D_{it}(s) - \overline{D}_{it}(s)$, and $\tau_{it}(s)$ is white noise. Note $\nu_{i,t-1}(s+1)$ in $D_{i,t-1}(s+1)$ which at $t-1$ discounts the cashflow occurring at $(t-1)+(s+1) = t+s$, while $D_{it}(s)$ at t discounts the cashflow occuring at $t+s$. To identify the starting time points of the periods till the occurrences of cash flows, we use the new notation $\boldsymbol{s}^t_a = (s^t_{a1}, \cdots, s^t_{aM})'$ which is a vector of cash flow of time t.

$$\boldsymbol{D}_{it}(\boldsymbol{s}^t_a) = (D_{it}(s^t_{a1}), \cdots, D_{it}(s^t_{aM}))',$$

$$\begin{aligned}\overline{\boldsymbol{D}}_{it}(\boldsymbol{s}_a^t) &= (\overline{D}_{it}(s_{a1}^t), \cdots, \overline{D}_{it}(s_{aM}^t))', \\ \boldsymbol{\nu}_{it}(\boldsymbol{s}_a^t) &= (\nu_{it}(s_{a1}^t), \cdots, \nu_{it}(s_{aM}^t))', \\ \boldsymbol{\tau}_{it}(\boldsymbol{s}_a^t) &= (\tau_{it}(s_{a1}^t), \cdots, \tau_{it}(s_{aM}^t))'. \end{aligned} \tag{20}$$

The discount function of (19) is expressed as

$$\boldsymbol{D}_{it}(\boldsymbol{s}_a^t) = \overline{\boldsymbol{D}}_{it}(\boldsymbol{s}_a^t) + \xi_t \boldsymbol{\nu}_{i,t-h}(\boldsymbol{s}_a^t + h\mathbf{1}) + \boldsymbol{\omega}_{it}(\boldsymbol{s}_a^t), \tag{21}$$

where $s_{aj}^t + h\mathbf{1} = s_{aj}^{t-h}$ and

$$\boldsymbol{\nu}_{i,t-h}(\boldsymbol{s}_a^t + h\mathbf{1}) = \boldsymbol{\nu}_{i,t-h}(\boldsymbol{s}_a^{t-h}). \tag{22}$$

Therefore, noting $C_{i,t-h}(s+h) = C_{it}(s)$ and using the notation

$$\boldsymbol{C}_{it}(\boldsymbol{s}_a^t) = (C_{it}(s_{a1}^t), \cdots, C_{it}(s_{aM}^t))',$$

by (21) and (22), the TDM model becomes

$$\begin{aligned} P_{it}(0) &= \boldsymbol{C}_{it}(\boldsymbol{s}_a^t)' \boldsymbol{D}_{it}(\boldsymbol{s}_a^t) \\ &= \boldsymbol{C}_{it}(\boldsymbol{s}_a^t)' \overline{\boldsymbol{D}}_{it}(\boldsymbol{s}_a^t) + \xi_t \eta_{i,t-h}(\boldsymbol{s}_a^{t-h}) + \varepsilon_{it}(\boldsymbol{s}_a^t), \end{aligned} \tag{23}$$

where

$$\begin{aligned} \boldsymbol{C}_{i,t-h}(\boldsymbol{s}_a^{t-h}) &= \boldsymbol{C}_{it}(\boldsymbol{s}_a^t), \\ \eta_{i,t-h}(\boldsymbol{s}_a^{t-h}) &= \boldsymbol{C}_{i,t-h}(\boldsymbol{s}_a^{t-h})' \boldsymbol{\nu}_{i,t-h}(\boldsymbol{s}_a^{t-h}), \\ \varepsilon_{it}(\boldsymbol{s}_a^t) &= \boldsymbol{C}_{it}(\boldsymbol{s}_a^t)' \boldsymbol{\tau}_{it}(\boldsymbol{s}_a^t). \end{aligned} \tag{24}$$

We assume the latent bond price B_t^* follows (23).

IV Modeling the Latent CB Call

The mean of a CB call nC_t^* is assumed to be approximated by the option pricing method. In fact, the terminal value at T of nC_t^* is given by

$$nC_T^* = \max(nS_T - 100.0, 0), \tag{25}$$

and hence nC_t^* could be treated as an American option. However the CB option cannot be separated from the CB, the issuer of the CB is the firm and hence the balance of the CB outstanding does not increase and the maturities are in general very long. Here we specify it as

$$n_i C_{it}^* = n_i \overline{C}_{it} + \gamma_{it}, \tag{26}$$

where γ_{it} is the deviation $n_i C_{it}^*$ from the mean, reflecting the volatile market expectations which is random, and it is assumed that the mean is approximated by the Black-Scholes formula :

$$\begin{aligned} c_{it} &\equiv n_i \overline{C}_{it} \\ &= n_i S_{it} \exp(-b_{it} s_{M(i)}) N(e_{it}) \\ &-100 \exp(-s_{M(i)} d_{it}(s_{M(i)})) N(e_{it} - (\sigma_{iit} s_{M(i)})^{1/2}), \end{aligned} \tag{27}$$

$$e_{it} = \left\{\log(n_i S_{it}/100) + s_{M(i)} \left(d_{it}(s_{M(i)}) - b_{it} + \sigma_{iit}/2\right)\right\} / (\sigma_{iit} s_{M(i)})^{1/2}, \tag{28}$$

$$d_{it}(s) = -\log \overline{D}_{it}(s)/s, \quad or \quad \overline{D}_{it}(s) = \exp(-s d_{it}(s)), \tag{29}$$

$$s_{M(i)} = T_{it} - t. \tag{30}$$

Here $N(a)$ is the value of the cumulative normal distribution evaluated at a and b_{it} is an average dividend rate. The conditional covariance between γ_{it} and γ_{kt} under the risk-neutral geometric Brownian motions for $\{S_{it}\}$'s is

$$h_{ikt} = \text{Cov}(\gamma_{it}, \gamma_{kt}) = E(A_{it} A_{kt}) - c_{it} c_{kt}, \tag{31}$$

where

$$\begin{aligned} A_{it} &= \max(n_i S_{iT} - 100, 0), \\ S_{iT} &= S_{it} \exp\left\{\left(d_{it}(s_{M(i)}) - \sigma_{iit}/2\right) s_{M(i)} + (\sigma_{iit} s_{M(i)})^{1/2} Z_{it}\right\}, \\ \text{Cov}(Z_{it}, Z_{kt}) &= \rho_{ikt}, \quad (Z_{1t}, \cdots, Z_{Nt})' \sim N(0, (\rho_{ikt})). \end{aligned} \tag{32}$$

Here σ_{iit} is the variance of the return of the ith stock at t and ρ_{ikt} is the correlation between the ith and kth returns at t. To get the value of h_{ikt} given σ_{iit}'s, we need to evaluate $E(A_{it} A_{kt})$, which is complicated because it involves double integrals. We here evaluate it by a Monte Carlo simulation method.

It should be noted that the latent variables $n_i C_{it}^*$ are treated simultaneously and so the correlation structure of N stock returns is taken into account in modeling through the covariances in (31).

V Convertible Bond Time Dependent Markov Model

Combining CSM model with $P_{it} = B_{it}^*$ and (26) through (32) yields our CB-CSM model;

$$\begin{aligned} CB_{it}(0) &= B_{it}^* + \alpha_t n_i C_{it}^* \\ &= \boldsymbol{C}_{it}' \overline{\boldsymbol{D}}_{it} + \eta_{it} + \alpha_t (c_{it} + \gamma_{it}) \\ &= \boldsymbol{C}_{it}' \overline{\boldsymbol{D}}_{it} + \alpha_t c_{it} + \zeta_{it}, \\ \zeta_{it} &= \eta_{it} + \alpha_t \gamma_{it}. \end{aligned} \tag{33}$$

Here, in the sequel, the stochastic part η_{it} of the latent bond and the stochastic part γ_{it} of the latent CB call are uncorrelated, namely

$$\mathrm{Cov}(\eta_{it}, \gamma_{kt}) = 0 \quad (i, k = 1, \cdots, N). \tag{34}$$

Hence ζ_{it} in (33) satisfies

$$E(\zeta_{it}) = 0, \quad \mathrm{Cov}(\zeta_{it}, \zeta_{kt}) = g_{ikt} + \alpha_t^2 h_{ikt}. \tag{35}$$

Here h_{ikt} is the conditional covariance between $n_i C_{it}^*$ and $n_k C_{kt}^*$ in (31). In this setup, the parameter α_t in the mean of CB_{it} in (33) appears in the covariance of ζ_{it} and ζ_{kt} in (35), which may cause a difficulty in estimation. Therefore, we may replace α_t^2 with ψ_t, leading to the final regression form for empirical analysis.

Let N_t dimensional vector $\boldsymbol{y}_t$ be defined by $\boldsymbol{y}_t = (y_{1t}, \cdots, y_{Nt})', \quad y_{it} = P_{it}(0) - \sum_{j=1}^{M} C_{it}(\boldsymbol{s}_a^t)$. Then the regression model is expressed in vector matrix form:

$$\boldsymbol{y}_t = \boldsymbol{U}_t^* \boldsymbol{\beta}_t^* + \boldsymbol{\zeta}_t, \tag{36}$$

where, the regression coefficient $\boldsymbol{\beta}_t^* = (\boldsymbol{\beta}_t, \alpha_t)'$ consists of the parameters, $\boldsymbol{\beta}_t = (\boldsymbol{\delta}_1', \cdots, \boldsymbol{\delta}_p')', \quad \boldsymbol{\delta}_k = (\delta_{k0t}, \cdots, \delta_{kqt})'$ for the mean discount function $\overline{D}_{it}(s)$ and the coefficient α_t for the CB call option. Furthermore the explanatory variables are

$$\begin{aligned} \boldsymbol{U}_t^* &= (\boldsymbol{U}_t, \boldsymbol{c}_t), \boldsymbol{c}_t = (c_{1t}, \cdots, c_{Nt})', \\ \boldsymbol{U}_t &= (\boldsymbol{u}_{1t}, \cdots, \boldsymbol{u}_{Nt})', \boldsymbol{u}_{it} = (\boldsymbol{u}_{i1t}, \cdots, \boldsymbol{u}_{ipt})', \boldsymbol{u}_{irt} = (u_{i1rt}, \cdots, u_{iqrt})', \\ u_{ikrt} &= \sum_{j=1}^{M} z_{ikt}(s_{aj}^t)^r C_{it}(s_{aj}^t), \quad z_{i0t} = 1. \end{aligned} \tag{37}$$

The variance and covariance of a random factor $\boldsymbol{\zeta}_t$ is expressed by

$$\mathrm{Cov}(\boldsymbol{\zeta}_t) = \boldsymbol{G}_t + \psi \boldsymbol{H}_t, \quad \boldsymbol{G}_t = (g_{ikt}), \quad \boldsymbol{H}_t = (h_{ikt}). \tag{38}$$

The regression coefficient $\boldsymbol{\beta}_t^*$ is estimated by the GLS (generalized least squares) method as

$$\widehat{\boldsymbol{\beta}}_t^* = [\boldsymbol{U}_t^{*'} \{\boldsymbol{G}_t(\hat{\rho}) + \hat{\psi}\boldsymbol{H}_t\}^{-1} \boldsymbol{U}_t^*]^{-1} \boldsymbol{U}_t^{*'} \{\boldsymbol{G}_t(\hat{\rho}) + \hat{\psi}\boldsymbol{H}_t\}^{-1} \boldsymbol{y}_t, \tag{39}$$

which minimizes

$$(\boldsymbol{y}_t - \boldsymbol{U}_t^* \boldsymbol{\beta}_t{}^*)' \{\boldsymbol{G}_t(\rho) + \psi \boldsymbol{H}_t\}^{-1} (\boldsymbol{y}_t - \boldsymbol{U}_t^* \boldsymbol{\beta}_t{}^*).$$

(40)

Given the parameters $\boldsymbol{\theta} = (\boldsymbol{\beta}_t^*, \sigma^2, \rho)$ of the model, the log-likelihood function of the model is given by

$$
\begin{aligned}
l(\boldsymbol{\theta}) \;&= -\frac{N}{2}\log 2\pi - \frac{1}{2}\log|\sigma^2\{\boldsymbol{G}_t(\rho) + \psi \boldsymbol{H}_t\}| \\
&-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{U}_t^*\boldsymbol{\beta}_t^*)[\sigma^2\{\boldsymbol{G}_t(\rho) + \psi\boldsymbol{H}_t\}]^{-1}(\boldsymbol{y}_t - \boldsymbol{U}_t^*\boldsymbol{\beta}_t^*).
\end{aligned} \tag{41}
$$

By maximizing the log likelihood, we can obtain the maximum likelihood estimates $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$. Therefore the maximum likelihood estimate of σ^2 is obtained by

$$
\widehat{\sigma}^2 = \frac{1}{N}(\boldsymbol{y}_t - \boldsymbol{U}_t^*\boldsymbol{\beta}_t^*)'\{\boldsymbol{G}_t(\hat{\rho}) + \hat{\psi}\boldsymbol{H}_t\}^{-1}(\boldsymbol{y}_t - \boldsymbol{U}_t^*\boldsymbol{\beta}_t^*), \tag{42}
$$

and is equivalent to the GLS estimate. Substitution of $\widehat{\sigma}^2$ into (41) yields

$$
l(\boldsymbol{\theta}) \;= -\frac{N}{2}\log 2\pi - \frac{1}{2}\log|\widehat{\sigma}^2\{\boldsymbol{G}_t(\hat{\rho}) + \hat{\psi}\boldsymbol{H}_t\}| - \frac{N}{2}. \tag{43}
$$

As a result, the AIC (Akaike information criterion) for the model is

$$
\begin{aligned}
\text{AIC} \;&= N\log 2\pi + \log|\widehat{\sigma}^2\{\boldsymbol{G}_t(\hat{\rho}) + \hat{\psi}\boldsymbol{H}_t\}| + N \\
&+2(the\ number\ of\ estimated\ parameters\ in\ the\ model).
\end{aligned} \tag{44}
$$

And the CB-TDM model is also specified by the regression model and estimated by the GLS method. We apply the same idea of TDM model to the CB pricing model. The latent bond value B_t^* is obtained by

$$
\begin{aligned}
B_{it}^* \;&= \boldsymbol{C}_{it}(\boldsymbol{s}_a^t)'\boldsymbol{D}_{it}(\boldsymbol{s}_a^t) \\
&= \boldsymbol{C}_{it}(\boldsymbol{s}_a^t)'\overline{\boldsymbol{D}}_{it}(\boldsymbol{s}_a^t) + \xi_t\eta_{i,t-h}(\boldsymbol{s}_a^{t-h}) + \varepsilon_{it}(\boldsymbol{s}_a^t), \\
where \quad \boldsymbol{C}_{i,t-h}(\boldsymbol{s}_a^{t-h}) \;&= \boldsymbol{C}_{it}(\boldsymbol{s}_a^t), \\
\eta_{i,t-h}(\boldsymbol{s}_a^{t-h}) \;&= \boldsymbol{C}_{i,t-h}(\boldsymbol{s}_a^{t-h})'\boldsymbol{\nu}_{i,t-h}(\boldsymbol{s}_a^{t-h}), \\
\varepsilon_{it}(\boldsymbol{s}_a^t) \;&= \boldsymbol{C}_{it}(\boldsymbol{s}_a^t)'\boldsymbol{\omega}_{it}(\boldsymbol{s}_a^t).
\end{aligned} \tag{45}
$$

The part of the CB option for the TDM model is specified as the following:

$$
c_{it}^* = \alpha_t c_{it}(\boldsymbol{s}_a^t) + \xi_t\alpha_{t-h}\gamma_{i,t-h}(\boldsymbol{s}_a^{t-h}) + \varpi_{it}(\boldsymbol{s}_a^t), \tag{46}
$$

where $\varpi_{it}(\boldsymbol{s}_a^t)$ is white noise. Hence The CB-TDM is specified as:

$$
\begin{aligned}
CB_{it}(0) \;&= B_t^* + c_{it}^* \\
&= \boldsymbol{C}_{it}(\boldsymbol{s}_a^t)'\overline{\boldsymbol{D}}_{it}(\boldsymbol{s}_a^t) + \alpha_t c_{it}(\boldsymbol{s}_a^t) + \xi_t\zeta_{i,t-h}(\boldsymbol{s}_a^{t-h}) + \kappa_{it}(\boldsymbol{s}_a^t), \\
\zeta_{i,t-h}(\boldsymbol{s}_a^{t-h}) \;&= \eta_{i,t-h}(\boldsymbol{s}_a^{t-h}) + \alpha_{t-h}\gamma_{i,t-h}(\boldsymbol{s}_a^{t-h}), \\
\kappa_{it}(\boldsymbol{s}_a^t) \;&= \varepsilon_{it}(\boldsymbol{s}_a^t) + \varpi_{it}(\boldsymbol{s}_a^t).
\end{aligned} \tag{47}
$$

Here it is also assumed that $\kappa_{it}(\boldsymbol{s}_a^t)$ has the same covariance structure as (35). Hence the CB-TDM model has an additional variable $\zeta_{i,t-h}(\boldsymbol{s}_a^{t-h})$ in (47) compared to the CB-CSM model in (33). The additional term is estimated by the residual of the CB-CSM model at $t-1$. Therefore (47) is nothing but an extended regression model and the additional explanatory variables takes care of unknown issue attributes revealed in the market. The estimation method of the CB-TDM model is the GLS method, the same as for the CB-CSM model. Let N_t dimensional vector $\boldsymbol{y}_t$ be defined by $\boldsymbol{y}_t = (y_{1t}, \cdots, y_{Nt})', \quad y_{it} = P_{it}(0) - \sum_{j=1}^{M} C_{it}(\boldsymbol{s}_a^t)$.

Then the regression model of the CB-TDM model is expressed in vector matrix form:

$$\boldsymbol{y}_t = \boldsymbol{U}_t^{**}\boldsymbol{\beta}_t^{**} + \boldsymbol{\kappa}_t, \tag{48}$$

where the regressive coefficients $\boldsymbol{\beta}_t^{**} = (\boldsymbol{\beta}_t, \alpha_t, \xi_t)'$ consists of the parameters, $\boldsymbol{\beta}_t = (\boldsymbol{\delta}_1', \cdots, \boldsymbol{\delta}_p')', \quad \boldsymbol{\delta}_k = (\delta_{k0t}, \cdots, \delta_{kqt})'$ for the mean discount function $\overline{D}_{it}(s)$, the coefficient α_t for the CB call option and the coefficient ξ_t for the $\zeta_{i,t-h}(\boldsymbol{s}_a^{t-h})$ at $t-h$ time. And the explanatory variables are

$$\begin{aligned}
\boldsymbol{U}_t^{**} &= (\boldsymbol{U}_t, \boldsymbol{c}_t, \widehat{\boldsymbol{\zeta}}_{t-h}), \boldsymbol{U}_t = (\boldsymbol{u}_{1t}, \cdots, \boldsymbol{u}_{Nt})', \boldsymbol{u}_{it} = (\boldsymbol{u}_{i1t}, \cdots, \boldsymbol{u}_{ipt})', \\
\boldsymbol{u}_{irt} &= (u_{i1rt}, \cdots, u_{iqrt})', u_{ikrt} = \sum_{j=1}^{M} z_{ikt}(s_{aj}^t)^r C_{it}(s_{aj}^t), \\
\boldsymbol{c}_t &= (c_{1t}, \cdots, c_{Nt})', \\
\widehat{\boldsymbol{\zeta}}_{t-h} &= (\widehat{\zeta}_{1,t-h}(\boldsymbol{s}_a^{t-h}), \cdots, \widehat{\zeta}_{N,t-h}(\boldsymbol{s}_a^{t-h}))', \quad z_{i0t} = 1.
\end{aligned} \tag{49}$$

The variance and covariance of a random factor $\boldsymbol{\kappa}_t$ is expressed by

$$\mathrm{Cov}(\boldsymbol{\kappa}_t) = \boldsymbol{G}_t + \psi \boldsymbol{H}_t, \quad \boldsymbol{G}_t = (g_{ikt}), \quad \boldsymbol{H}_t = (h_{ikt}). \tag{50}$$

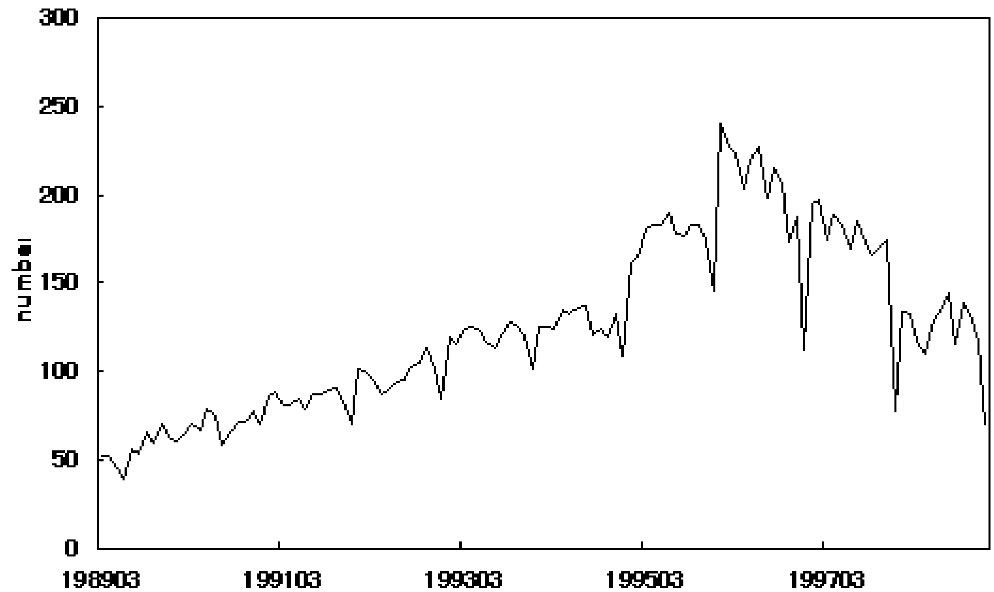
The regression coefficient $\boldsymbol{\beta}_t^{**}$ is estimated by the GLS (generalized least squares) method as

$$\widehat{\boldsymbol{\beta}}_t^{**} = [\boldsymbol{U}_t^{**'}\{\boldsymbol{G}_t(\hat{\rho}) + \hat{\psi}\boldsymbol{H}_t\}^{-1}\boldsymbol{U}_t^{**}]^{-1}\boldsymbol{U}_t^{**'}\{\boldsymbol{G}_t(\hat{\rho}) + \hat{\psi}\boldsymbol{H}_t\}^{-1}\boldsymbol{y}_t, \tag{51}$$

which minimizes

$$(\boldsymbol{y}_t - \boldsymbol{U}_t^{**}\boldsymbol{\beta}_t{}^{**})'\{\boldsymbol{G}_t(\rho) + \psi\boldsymbol{H}_t\}^{-1}(\boldsymbol{y}_t - \boldsymbol{U}_t^{**}\boldsymbol{\beta}_t{}^{**}). \tag{52}$$

In evaluating c_{it}, we also need the volatilities σ_{iit}'s, which should be given prior to estimation. They may be estimated by historical volatilities, or a more sophisticated method such as generalized auto regressive conditional heteroskedasticity model. Also $d_{it}(s_{M(i)})$, the spot rate discounting the CB option value in the mean to a present value, should be given in advance. Otherwise the estimation procedure becomes very complicated. We explain this procedure in our empirical analysis in the next section.



In figure, 198903 means March in the year of 1989.

Figure 1 Number of CB.

VI Empirical Analysis

In this section, the CB-CSM and CB-TDM models are applied to the Japanese CB prices. The data we analyze is monthly data from 1989.3 to 1998.12, listed in the Tokyo Stock Exchange. We only considered the CB's which satisfy the conditions;

1) the prices were formed at the end of each month,

2) the maturities were less than 10 years,

3) the CB's are those of the firms which issued ordinary bonds.

In Figure 1, the graph of sample sizes at each month is shown. As in the graph, the number N_t of the CB's which satisfy the above conditions increases till 1996 and decreases after 1996, which reflects the bad economic conditions for direct equity finances due to the severe recession. The number N_t of the CB's tends to decrease sharply at the end of each December. The reason is that the volume of transaction decreases and the number of the CB's without pricing increases. It is noted that the unknown parameters are δ_{ijt}'s, α_t, ρ_t, ξ_t and $\psi_t = \alpha_t^2$ and hence the degrees of freedom are sufficiently big. In the below we describe our specifications.

.1 The CB call part In valuing the mean CB call c_{it}, we need to set the volatilities δ_{iit} of the stock returns. We computed them based on the one month past daily returns of the N_t stocks corresponding to the CB's of each month t, where the return is the continuously compounded return obtained from the log-price difference. The own discount (spot) rate d_i in the Black-Scholes formula is given by the TDM model for the Japanese Government (JG) bond prices with initial maturities of 10 years, where as attributes the coupon rate and maturity were included in the model. In other words, as an alternative asset to the stocks, the JG bonds with the same attributes are considered in the option-theoretic framework. Further, to obtain the covariances h_{ikt} of γ_{it} and γ_{kt} in (31), we run a Monte Carlo simulation to evaluate the double integrals. Hence first we generated 1000 $Z_t^{(j)} = (Z_{it}^{(j)}, \cdots, Z_{Nt}^{(j)})$ $(j = 1, \cdots, 1000)$ from the multivariate normal $N(0, (\rho_{ikt}))$ and averaged the values of $A_{it}^j A_{kt}^j$ over $j = 1, \cdots, 1000$ to obtain an estimate of $E(A_{it}A_{kt})$.

.2 The bond part In modeling the potential bond value, it is assumed that the mean discount function of each bond is a second order polynomial of the durations to cashflows with the coefficients being functions of coupon rate z_{i1t} and maturity z_{i2t};

$$\overline{D}_{it}(s) = 1 + (\delta_{11t}z_{i1t} + \delta_{12t}z_{i2t})s + (\delta_{21t}z_{i1t} + \delta_{22t}z_{i2t})s^2. \tag{53}$$

We first included an index of credit risk as an additional attribute but it was not significant in the analysis.

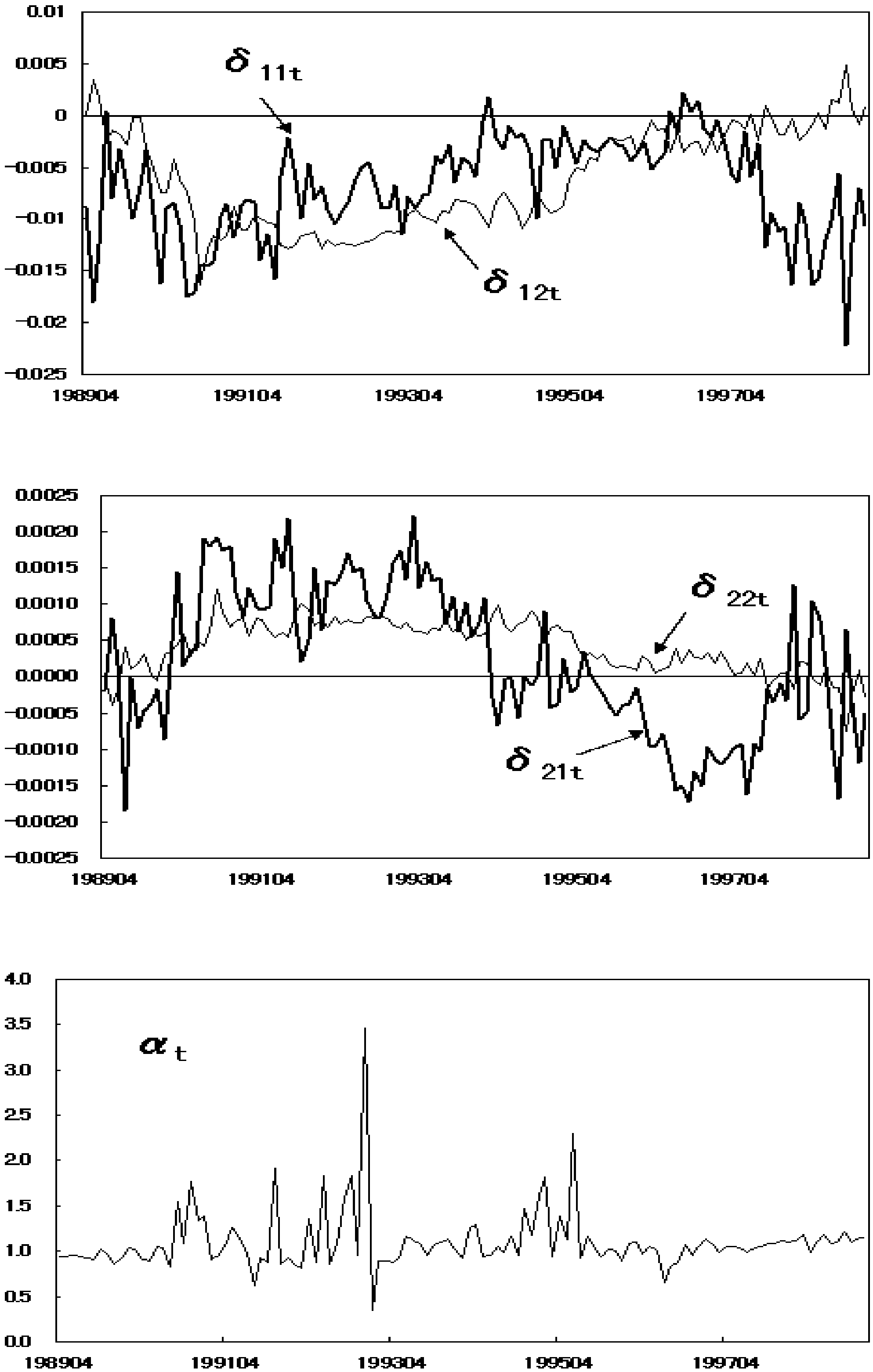
.3 Estimation procedure To minimize the objective function in (40), we use a grid point method as follows. For each given grid point of $\rho_t = 0, 0.01, \cdots, 0.99$, we first set $\psi_t = \alpha_t^2 = 1$ and minimize (40) with (39) to get $\widehat{\delta}_{ijt}$ and $\widehat{\alpha}_t$ for a given ρ_t. Then we set $\psi_t = \widehat{\alpha}_t^2$ to minimize (40) with (39), giving the second estimates of $\widehat{\delta}_{ijt}$ and $\widehat{\alpha}_t$ for the given ρ_t. Repeating this procedure, we minimize (40) for each ρ_t. Finally we choose the set $(\widehat{\delta}_{ijt}, \widehat{\alpha}_t, \widehat{\rho}_t)$ which minimizes (40) over all. In our analysis, when we estimate the CB-TDM model, the coefficient ξ_t is assumed to be equal to 1, based on the empirical result for JG bonds as showed in Kariya and Tsuda (1994).

.4 Empirical Results Figure 2 shows the graphs of the estimated parameter values of δ_{ijt}'s in the mean discount function and the coefficient α_t of the mean value of the CB call for the CB-TDM model. Figure 3 shows the graph of their t-values.

From Figure 2, it is observed:

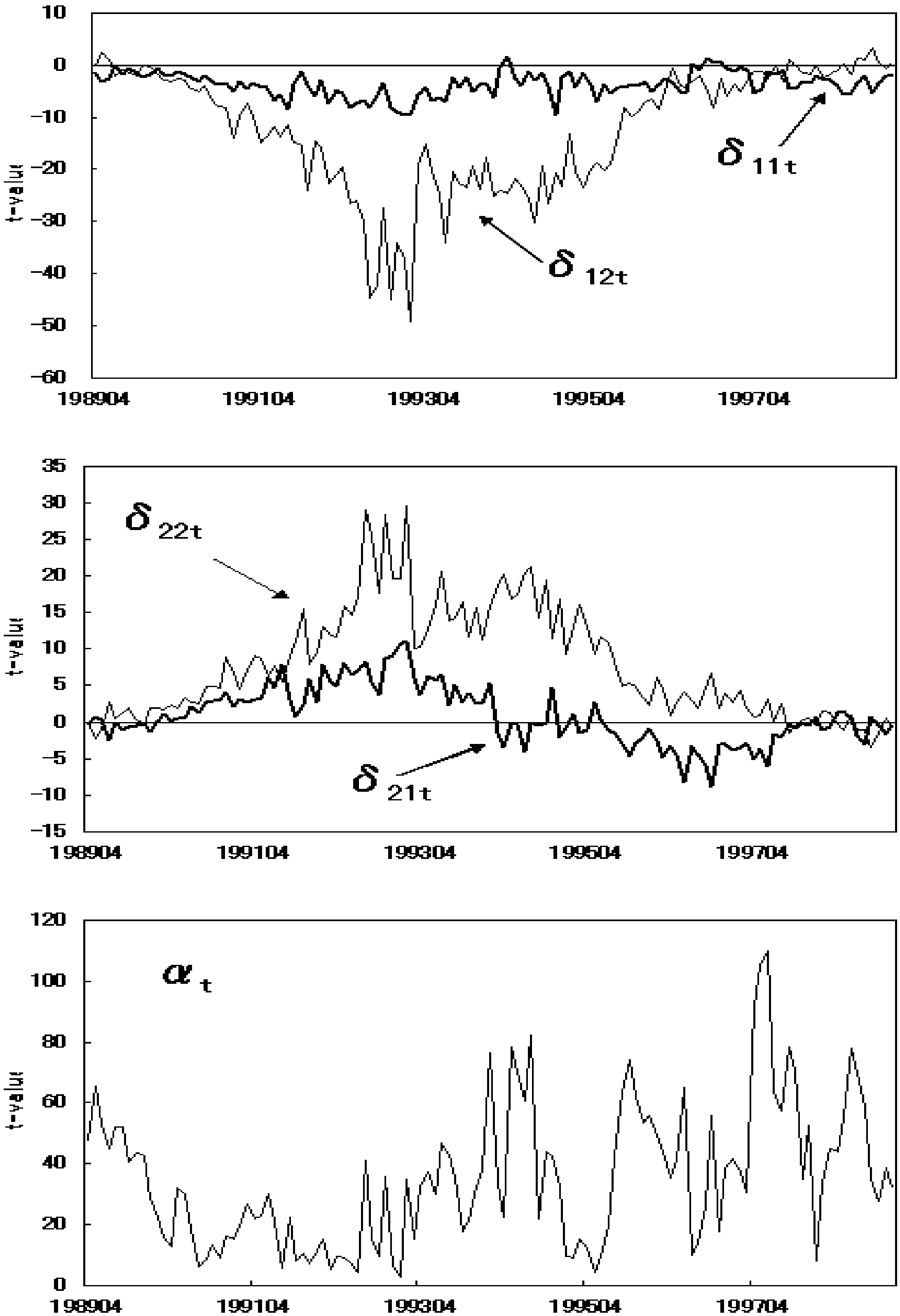
1) The coefficients δ_{11t} and δ_{12t} of the first-order term s in the mean discount function are both mostly negative. The coefficient δ_{11t} of coupon rate gets larger toward zero from 1990 till 1996 and declines after 1996, reflecting an economic movement. The coefficient δ_{12t} of maturity in the s expression moves rather smoothly, declines from 1989 till 1990 and moves upward gradually, when the stock price bubble was crushed and interest rates started to decline.

2) The coefficients δ_{21t} and δ_{22t} of the second-order term s^2 are much smaller than those of the first order term s. The shape of the time series for the coefficients δ_{21t} is almost the opposite of the coefficient δ_{11t}.



In figures, 198904 means April in the year of 1989.

Figure 2 Graphs of Parameters



In figures, 198904 means April in the year of 1989.

Figure 3 T-value of Parameters

3) The movement of α_t is interesting. In 1989, when stock prices were at the peak of the bubble, most of the CB options were in the money. Though it is slightly smaller than 1, α_t stays very close to 1, implying that the CB's would be priced on the average mainly by the mean value of the CB call which is reflected in the t-values. In the year of 1990 when the bubble crushed, it started to fluctuate greatly and more upward though the CB call came out of the money. Afterwards, it fluctuated mostly upward from the value of 1. These movements reflect the market expectations and sentiment on the value of the CB calls. It should be noted that α_t is common to all the CB calls, but it can also be estimated separately according to industry category or some other predetermined or objective category, so that the explanatory power of the model may be improved. The value of α_t rose sharply to 3.4 in 1992.12, and it fell sharply down to 0.3 in 1993.1 because the number of the CB calls without zero is small.

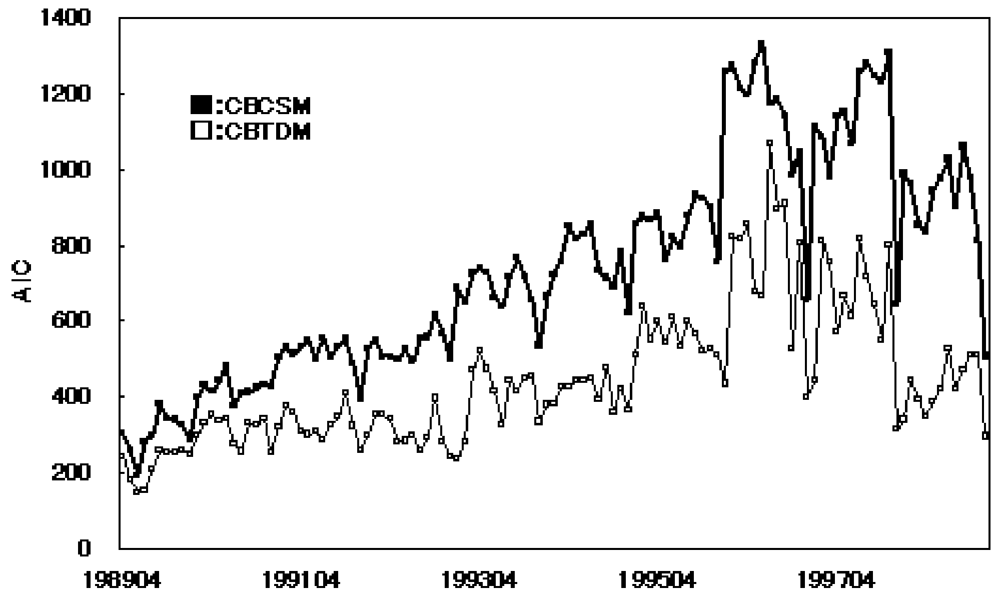
Next it is observed from the graph of the t-values in Figure 3:

1) Though the t-values of δ_{11t} for the coupon rate in the s expression are not relatively significant, they decrease from values of about 0 in the year of 1989 to about -9 in the beginning of 1993. The t-values of δ_{21t} for coupon rate in the s^2 expression increase from about 0 in the year of 1989 to about 11 in the beginning of 1993, and decrease to the values about -9 in late 1996. In contrast to the case of JG bonds, the coupon rate may not be regarded as an important attribute for pricing CB's probably because it is relatively low.

2) The t-values of δ_{12t} in the s expression for maturity are negative and significant for the period from 1991.1 through 1995.12. They decrease from about 0 to about -49 in the middle of 1992, and then increase toward 0. The t-values of δ_{22t} in the s^2 expression for maturity are positive and rather significant for almost the same period as in δ_{12t}, when the economy was in the recession. These imply that maturity as an attribute plays an important role in pricing the CB's.

3) The t-values of α_t are again interesting. When the CB call is deep in the money as in the year of 1989, they exhibit strong significance with α_t remaining stably around 1 as Figure 3. However, they decline sharply and are not significant in late 1989 and in the late 1991 and early 1992, when most of the CB calls are deep out of the money. They became significant again except for the beginning of 1995. The reason is that CB's were newly issued and some of the CB calls were in the money. It is noted that the t-values are constructed based on the covariance matrix of κ_{it} in (47), which includes the covariance matrix of the random parts ϖ_{it} of the CB calls.

.5 Model fitting The results of the fitting of 117 (12months $\times$ 9 years and 9 months)CB-CSM models and CB-TDM models by the GLS method are summarized in terms of AIC values at



In figure, 198904 means April in the year of 1989.

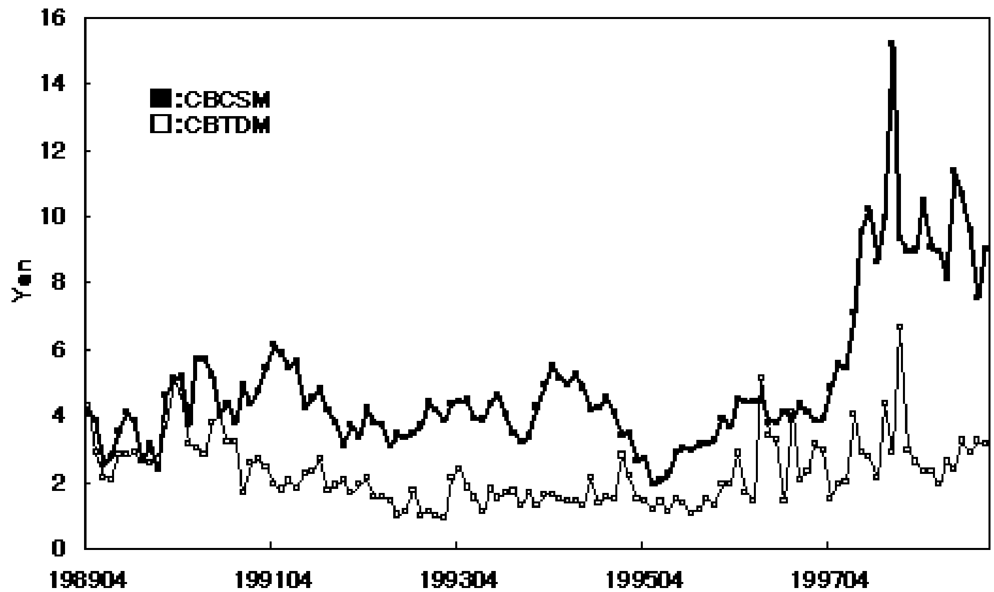
Figure 4 Comparison of AIC.

each time in Figure 4 and residual standard deviations among individual CB's in Figure 5. The residual standard deviations in terms of prices are defined by

(54) $$v_t = \left\{ \frac{1}{N} \sum_{i=1}^{N} (P_{it}(0) - \widehat{P}_{it}(0))^2 \right\}^{1/2}.$$

It is readily seen that the CB-TDM model is better than the CB-CSM model. In fact, the AIC values of the CB-TDM models are uniformly smaller than those of the CB-CSM models. In the year of 1989, where the stock prices are in the bubble and many CB calls are deep in the money, the standard deviations of the CB-TDM model are greater than 2 yen, reflecting the volatile situation of stock prices. In 1989.12 when the bubble is just about to crush, the standard deviations hits the high level and then sharply goes down to a level about 2 yen. From 1990 till 1995 they remains mostly less than 2 yen, reflecting the fact that the bond values matter more though newly issued CB's are involved in the analysis of late years. After 1996, the standard deviations of the CB-TDM model become greater than 2 yen, because some of individual CB prices became very low because of default risk.

To present the performances in more detail, we focus on the periods 1989.4, 1995.3 and 1998.12. The standard deviation of the model of 1989.4 with sample size 41 is about 4 yen and the realized



In figure, 198904 means April in the year of 1989.

Figure 5 Comparison of Residual Standard Deviations.

prices of the CB's and the CB-TDM prices are plotted in Figure 6, while in Figure 7 the realized stock prices and the exercise (strike) prices of the CB calls are plotted. It is seen from Figure 7 that the exercise prices of the CB's are mostly less than the stock prices or equivalently most of the CB calls are in the money in 1989.4. In the newly issued CB's the exercise prices are bid at higher prices and hence the CB calls with longer maturities are out of the money. Hence maturity as an attribute becomes important to describe the prices of CB's. In other words, though we separated the bond part of a CB from the CB call part in model building, the variables corresponding to the bond may show an explanatory power for the term structure of the money positions of the CB calls. From Figure 6 and from the graph of the residuals (realized value minus model value) in Figure 8, the model performance for the individual CB prices is seen to be satisfactory. In other words, the money positions of the CB call, which are supposed to be explained by the BS explanatory variables c_{it} to some extent will affect the analysis.

In the model of 1995.3 when the exercise prices of the CB's are mostly higher than the stock prices, or equivalently, almost all the CB call prices are out of the money as in Figure 7, the model performance for the individual CB prices is much better because the model works more as a bond model, where the realized and model prices are plotted in Figure 6 and the sample size is 150. It can be observed from the graph of the residuals in Figure 8.

In the model of 1998.12 when some of the CB call prices are in the money as in Figure 7, the model performance for the individual CB prices still remains better. The CB data, whose price is around 40 yen because of default risk, affect the analysis as an outlier. It can be observed from the graph of the residuals in Figure 8.

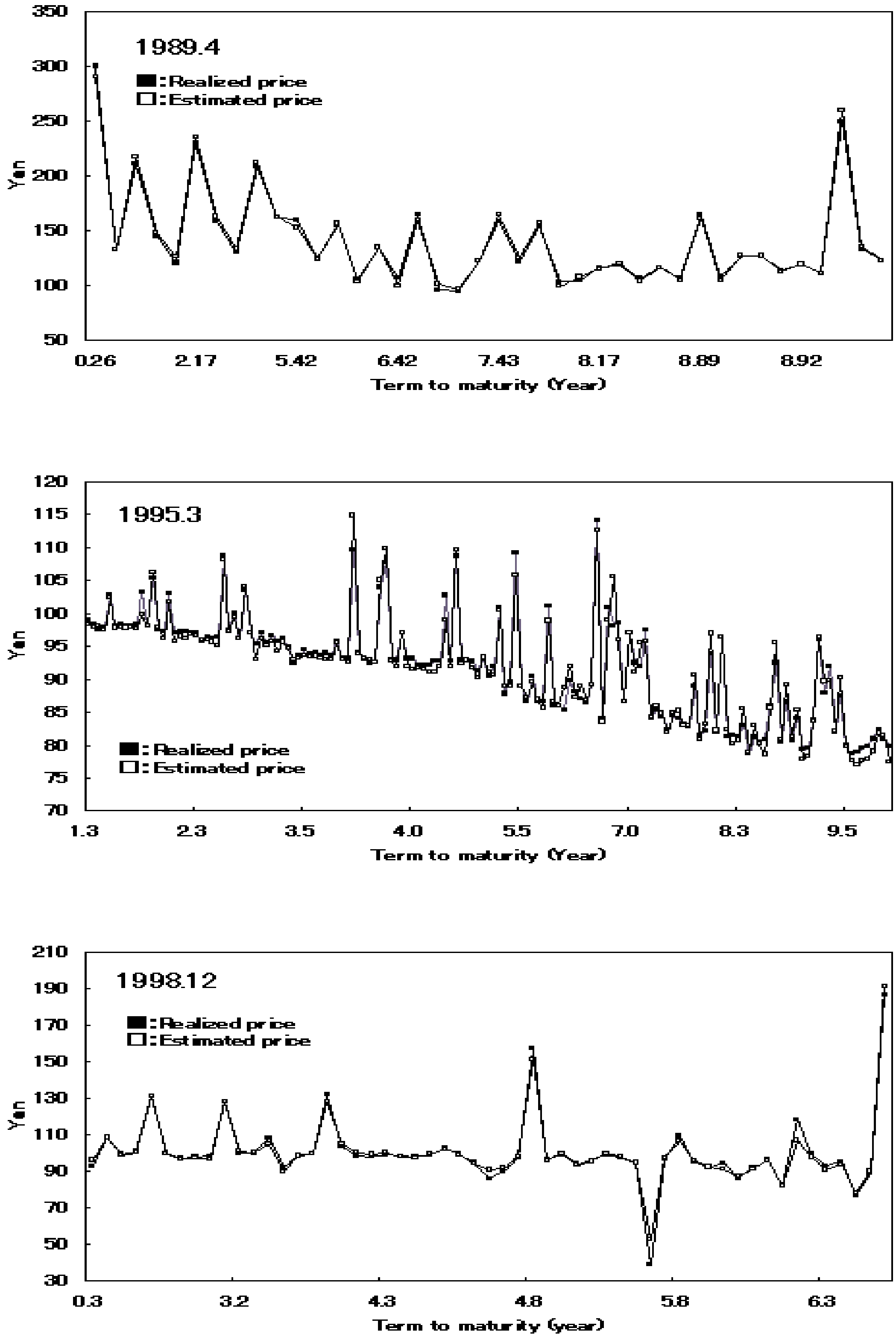


Figure 6 Individual CB Prices.

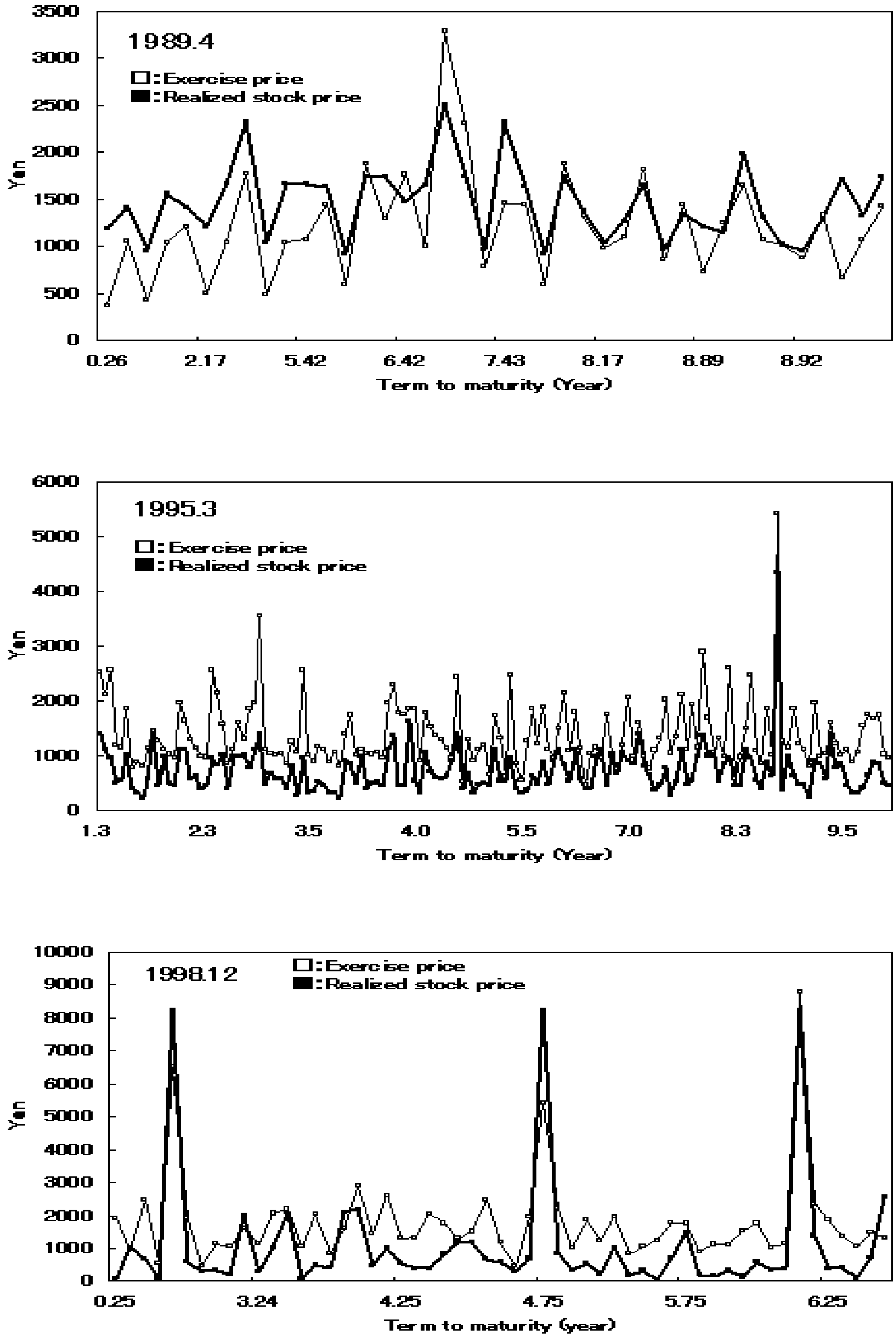


Figure 7 CB Exercise Prices and Stock Prices.

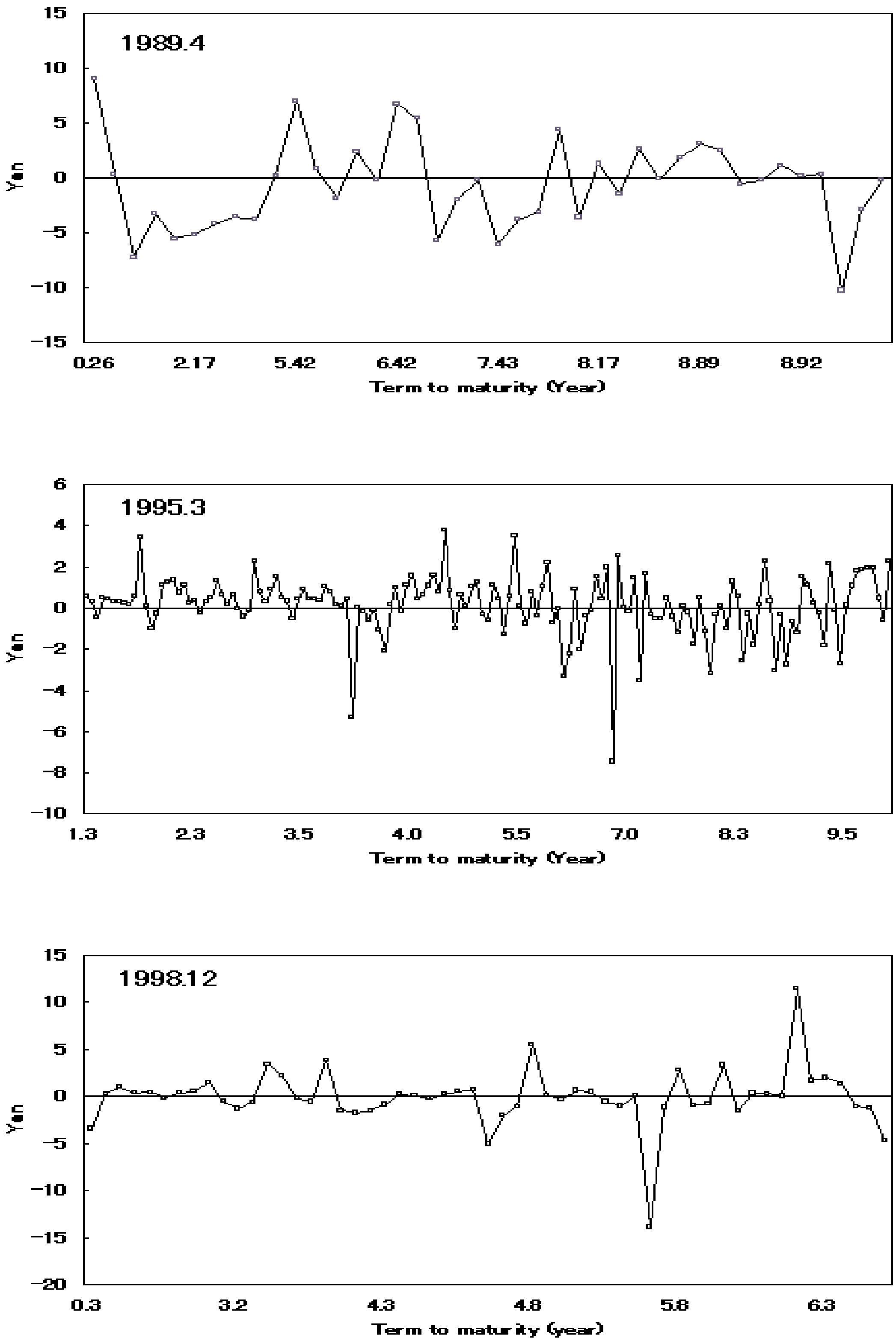


Figure 8 Residuals of Individual CB.

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