Impartial welfare orderings in infinite time horizon

Toyotaka Sakai *†

July 26, 2010

Abstract

Impartial treatment of all generations is a fundamental ethical principle in intertemporal welfare evaluation, espoused by both egalitarianism and utilitarianism. We show that any welfare ordering that fully respects this principle, as well as other standard axioms, is simply represented by a two-dimensional function that only depends on the limit inferior and the limit superior of an infinite utility stream. Among the class of such orderings, the limit inferior appears as an egalitarian criterion, while the sum of the limit inferior and the limit superior appears as a utilitarian criterion. Leximin orderings whose first priority is in the limit inferior and the second is in the limit superior are also characterized.

Keywords: Intergenerational equity, Welfare ordering, Two-dimensional representation, Limit leximin, and Diamond's impossibility theorem.

JEL numbers: D71, D63.

*Department of Economics, Yokohama National University, Yokohama 240-8501, Japan; toyotaka.sakai@gmail.com; +81-45-339-3534/3518 (tel/fax)

†I thank Hiroo Sasaki for his helpful comments. This paper was presented at Waseda University.

1 Introduction

Impartial treatment of all generations is a fundamental ethical principle in utilitarianism and egalitarianism when applied to intertemporal welfare evaluation. In the literature of welfare economics with infinitely many generations, this equal treatment idea is embodied in anonymity axioms of (welfare-)orderings, stating the invariance of ranking of an infinite utility stream up to any permutation of generations.[1] Our purpose is to identify orderings that fully respect this idea. By "fully", we mean with the entire class of permutations in defining anonymity, rather than any restricted class.[2] The anonymity axiom defined is called *full anonymity*.

We also consider the following three axioms on orderings: *weak Pareto* states that a uniform gain of all generations is good; *non-substitution* states that a sacrifice of one generation that yields a uniform gain of an infinite number of generations is not bad[3]; *sup continuity* states that a small misspecification of data does not lead to a big change in the result when measured by the sup norm.

Our study is not the first one that examines implications of *full anonymity*.[4] Furthermore, the other three axioms, or variants of them, have often been discussed in the literature. Our main theorem, however, implies that the combination of these canonical axioms characterizes a novel class of orderings. It shows that an ordering satisfies *full anonymity, weak Pareto, non-substitution*, and *sup continuity* if and only if it is represented by a monotone, continuous, and two-dimensional function that solely depends on the limit inferior and the limit superior of an utility stream.

This result might be surprising because it ensures that an ordering on an infinite dimensional space can be simply described by a two-dimensional function. It shows both the possibility and limitation of ethically ranking utility streams; on the possibility side, it offers a rich class of orderings satisfying the desirable axioms; on the limitation side, any such ordering depends only on two limit concepts that capture long-run movements of a utility stream. There seems to be no previous result similar to ours.

The characterized class contains two particularly interesting orderings: one is the limit inferior ordering, which seems most egalitarian; the other is the ordering represented by the sum of limit inferior and limit superior, which seems most utilitarian. They are characterized by additionally imposing certain versions of Hammond's (1976) equity axiom and d'Aspremont and Gevers' (1977) independence axiom, respectively. We also consider leximin orderings whose first priority is in the limit

[1] This approach was first employed by Diamond (1965).

[2] See, Fleurbaey and Michel (2003) for various such classes.

[3] The idea of *non-substitution* is from Lauwers (1997a), but our definition is somewhat different from his. Non-substitution axioms can be also found in Lauwers (1998), Asheim, Mitra, and Tungodden (2007), and Alcantud and Garcìa-Sanz (2010a,b).

[4] The literature includes Lauwers (1997b, 1998, 2010b), Fleurbaey and Michel (2003), and Sakai (2010b).

inferior and the second is in the limit superior, and characterize them by a set of axioms similar to the axioms characterizing the limit inferior. These egalitarian and utilitarian orderings are quite different from any other existing ordering in the literature[5]. Chambers (2009) shows that a function satisfies "monotonicity," "ordinal covariance", and "reinforcement" if and only if it is either the maximin, the maximax, the limit inferior, or the limit superior. Our and Chambers studies differ in many aspects: "reinforcement" is much stronger than *full anonymity*; he jointly characterizes four functions, while we characterize a family of orderings that depend on two limit concepts; his analysis does not deal with utilitarian or leximin orderings.

Finally, we emphasize our technical contribution in this study. In our model, the set of bounded sequences is employed as the set of utility streams. This set is, however, not analytically tractable due to its richness as an infinite dimensional space. To reduce this difficulty, we first define an easy-to-handle dense subset of it, and concentrate on the ranking of utility streams that belong to the subset. This subset, called the *universal finite domain*, has its own economic meaning, and is a powerful tool for understanding how orderings on an infinite dimensional space behave. We characterize the forms of orderings on the universal finite domain, and then continuously extend the obtained forms to the whole space.

The rest of the paper is organized as follows. Section 2 introduces our model. Section 3 gives the axioms of orderings. Section 4 offers the two-dimensional representation theorem and corollaries to it. Section 5 studies leximin orderings. Section 6 concludes the discussion. Section 7 contains proofs. Examples for the independence of axioms are provided in Appendix.

2 Definitions

Let $\mathbb{N} \equiv \{1, 2, \ldots\}$ be the set of infinite generations. A *utility stream* is a bounded infinite dimensional vector

$$x \in \ell^\infty \equiv \{x \equiv (x_1, x_2, \ldots) : \sup_{t \in \mathbb{N}} |x_t| < \infty\},$$

where each x_t denotes the utility level of generation t. Our interest lies in the ethical ranking of utility streams. A *(welfare-)ordering* is a transitive and complete binary relation $\succsim$ on ℓ^∞.[6] Asymmetric and symmetric parts of $\succsim$ are denoted by $\succ$ and $\sim$, respectively.

We introduce a few notations.

[5]See, for example, Lauwers (1997b), Asheim and Tungodden (2004), Basu and Mitra (2007), Bossert, Sprumont, and Suzumura (2007), Kamaga and Kojima (2010), and Sakai (2010a), among others.

[6]Transitivity: for every $x, y, z \in \ell^\infty$, $x \succsim y$ and $y \succsim z$ imply $x \succsim z$; Completeness: for every $x, y \in \ell^\infty$, either $x \succsim y$ or $y \succsim x$.

- Given $x \in \ell^\infty$, the limit inferior and the limit superior of x are denoted, respectively, by

$$\underline{x} \equiv \lim_{s \to \infty} \inf_{t \geq s} x_t \quad \text{and} \quad \overline{x} \equiv \lim_{s \to \infty} \sup_{t \geq s} x_t.$$

- Given any $a = (a_1, a_2, \ldots, a_L), b = (b_1, b_2, \ldots, b_L) \in \mathbb{R}^L$ with $1 \leq L \leq \infty$, vector inequalities are defined by

$$a > b \iff a_\ell > b_\ell \text{ for all } \ell,$$
$$a \geqq b \iff a_\ell \geq b_\ell \text{ for all } \ell.$$

- For any finite dimensional vector $(a_1, a_2, \ldots, a_L) \in \mathbb{R}^L$ with $1 \leq L < \infty$, we denote by

$$(a_1, a_2, \ldots, a_L)_{rep} \equiv (a_1, a_2, \ldots, a_L, a_1, a_2, \ldots, a_L, \ldots) \in \ell^\infty$$

the utility streams in which the sequence $a_1, a_2, \ldots, a_L$ repeats infinitely. In particular, when a is a (one-dimensional) real-value, $(a)_{rep} = (a, a, \ldots)$ denotes the constant stream consisting of a.

3 Axioms

We define four axioms of orderings. Our central equity axiom requires impartial treatment of all generations. A *permutation* is a bijection π on $\mathbb{N}$ to $\mathbb{N}$. Let Π be the set of permutations. Given $x \in \ell^\infty$ and $\pi \in \Pi$, we write $\pi(x) \equiv (x_{\pi(1)}, x_{\pi(2)}, \ldots)$.

Full anonymity. *For every $x \in \ell^\infty$ and every $\pi \in \Pi$, $x \sim \pi(x)$.*

Our objective is to identify the class of fully anonymous orderings that satisfy some natural requirements. The next efficiency axiom is self-explanatory. Later we will consider its stronger version.

Weak Pareto. *For every $x, y \in \ell^\infty$ and every $\varepsilon > 0$, if $x_t \geq y_t + \varepsilon$ for all $t \in \mathbb{N}$, then $x \succ y$.*

The following axiom states that a "sacrifice" of the first generation that leads to a uniform gain of an infinite number of future generations is not considered bad. It should be noted that this axiom is moderate, since it allows indifference, and so the sacrifice is not strongly recommended.[7]

[7] The idea of *non-substitution* originates with Lauwers (1997a). His non-substitution axiom states that a sacrifice of the first generation that leads to a uniform gain of *all* future generations is considered *good*. His and our non-substitution axioms are independent; his axiom considers only the case for all future generations, but does not allow indifference.

Non-substitution. *For every* $x, y \in \ell^\infty$, *if* $y_1 < x_1$ *and there exist an infinite subset* $\mathbb{N}' \subset \mathbb{N}$ *and* $\varepsilon > 0$ *such that* $x_t + \varepsilon = y_t$ *for all* $t \in \mathbb{N}'$, *and* $x_t = y_t$ *for all* $t \notin \mathbb{N}' \cup \{1\}$, *then* $y \succsim x$.

The next axiom states that a small misspecification of data does not lead to a big change of the results, thereby ensuring certain robustness of ranking.

Sup continuity. *For every* $x \in \ell^\infty$, *the sets* $\{y \in \ell^\infty : x \succsim y\}$ *and* $\{y \in \ell^\infty : y \succsim x\}$ *are closed in the sup norm topology.*

To obtain better understanding of the axioms defined above, we offer three different types of well-known orderings.

- *Overtaking criterion.* Let $\succsim$ be an ordering such that for every $x, y \in \ell^\infty$,

$$\exists T \in \mathbb{N},\ \forall t \geq T,\ \sum_{s=1}^{t} x_s \geq \sum_{s=1}^{t} y_s \Longrightarrow x \succsim y,$$
$$\exists T \in \mathbb{N},\ \forall t \geq T,\ \sum_{s=1}^{t} x_s > \sum_{s=1}^{t} y_s \Longrightarrow x \succ y.$$

  The existence of such an ordering is established by Svensson (1980). This ordering satisfies *weak Pareto* and *non-substitution*, but violates *full anonymity* and *sup continuity*.[8]

- *Maximin ordering.* Let $\succsim$ be the ordering on ℓ^∞ such that for every $x, y \in \ell^\infty$, $x \succsim y \iff \inf_{t\in\mathbb{N}} x_t \geq \inf_{t\in\mathbb{N}} y_t$. This ordering satisfies *full anonymity*, *weak Pareto*, and *sup continuity*, but violates *non-substitution*.

- *Limit inferior ordering.* Let $\succsim$ be the ordering on ℓ^∞ such that for every $x, y \in \ell^\infty$, $x \succsim y \iff \underline{x} \geq \underline{y}$. This ordering satisfies *full anonymity*, *weak Pareto*, *non-substitution*, and *sup continuity*.

Weak Pareto is fairly weak, so it is satisfied by all three orderings. *Non-substitution* is satisfied by both the overtaking criterion and the maximin ordering, and so this axiom does not have a particularly utilitarian or egalitarian flavor. Indeed, *non-substitution* is satisfied by many standard orderings in the literature, including various, not necessarily utilitarian, versions of the overtaking criterion.[9] On the other

[8] It is obvious that this ordering satisfies "strong Pareto" ($x \geqq y$ and $x \neq y \Longrightarrow x \succ y$) and "finite anonymity" ($x \sim \pi(x)$ if $\exists s \in \mathbb{N}, \forall t \geq s, \pi(t) = t$). Results by Fleurbaey and Michel (2003) imply that "strong Pareto" is incompatible with *full anonymity*, and the pair of "strong Pareto" and "finite anonymity" is incompatible with *sup continuity*.

[9] See, for example, Atsumi (1965), von Weizsäcker (1965), Fleurbaey and Michel (2003), Asheim and Tungodden (2004), Asheim, d'Aspremont, and Banerjee (2010), Kamaga and Kojima (2010), and Sakai (2010a).

hand, *full anonymity* and *sup continuity* cannot be satisfied by those overtaking-type orderings, mainly because such orderings usually satisfy certain stronger Pareto axioms and weaker anonymity axioms that together are incompatible with *full anonymity* and *sup continuity* (see, Fleurbaey and Michel 2003). Meanwhile, it is interesting that the limit inferior ordering satisfies all four axioms. Our main theorem will show that this ordering is in fact very representative of such orderings.

4 Two-dimensional representation

Theorem 1. *An ordering* $\succsim$ *on* ℓ^∞ *satisfies full anonymity, weak Pareto, non-substitution, and sup continuity if and only if there exists a continuous function* $W : \mathbb{R}^2 \to \mathbb{R}$ *such that for every* $x, y \in \ell^\infty$,

$$x \succsim y \iff W(\underline{x}, \overline{x}) \geq W(\underline{y}, \overline{y}),$$

and $(\underline{x}, \overline{x}) > (\underline{y}, \overline{y})$ *implies* $W(\underline{x}, \overline{x}) > W(\underline{y}, \overline{y})$.

Proof. See Section 7.1. □

Examples showing the independence of the four axioms in Theorem 1 are provided in Appendix. This result might be somewhat surprising because an ordering on an infinite dimensional space is simply represented by a two-dimensional function. It offers a clear boundary between the possibility and impossibility of ethically ranking utility streams. The characterized orderings depend only on the limit inferior and the limit superior, thereby strongly suggesting the importance of caring lon-run movements of utility streams in normative decision. On the other hand, because of this property, those orderings are insensitive to any change of utility levels of finite generations, which would be counted as a drawback.

A particular advantage of our "two-dimensional orderings" is that they have explicit, constructive, and simple functional representations. This point contrasts with various impossibility results of having such representations or formulas. Indeed, Zame (2007) and Lauwers (2010a) establish impossibilities of having an explicit formula using an argument on the axiom of choice. Basu and Mitra (2003) address an impossibility of having a functional representation. Their argument explains why most desirable orderings obtained in the literature lack an explicit formula or representation (e.g., Svensson 1980, Fleurbaey and Michel 2003, and Sakai 2003). These negative results are built upon "finite anonymity", which is much weaker than *full anonymity*, and "strong Pareto", which is much stronger than *weak Pareto*. This difference among axioms allows us to bypass those impossibilities.

The class of orderings characterized in Theorem 1 is not small. However, whenever $\lim x_t$ and $\lim y_t$ exist, all these orderings agree with the ranking between x and y, since

$$x \succsim y \Longleftrightarrow W(\underline{x}, \overline{x}) \geq W(\underline{y}, \overline{y}) \Longleftrightarrow \lim_{t\to\infty} x_t \geq \lim_{t\to\infty} y_t$$

in that case. This argument gives a justification for the comparison of dynamic economic plans by their achieving steady-states, which is often employed in growth models with multiple steady-states (e.g., Matsuyama 2005).

Among the class of orderings characterized in Theorem 1, we now focus on two orderings that reflect the idea of egalitarianism and utilitarianism. In social choice theory with finite populations, egalitarian and utilitarian orderings are typically characterized on the basis of Hammond's (1976) equity axiom and d'Aspremont and Gevers' (1977) independence axiom, respectively. In our model, versions of these axioms can be stated as below, and they characterize egalitarian and utilitarian two-dimensional representations.

Weak Hammond. *For every $x, y \in \ell^\infty$, if for every odd integer $t \geq 1$, either $x_t > y_t \geq y_{t+1} > x_{t+1}$ or $x_{t+1} > y_{t+1} \geq y_t > x_t$, then $y \succsim x$.*

Weak independence from the origin. *For every $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in \mathbb{R}^2$,*

$$(a_1, a_2)_{rep} \succsim (b_1, b_2)_{rep} \Longleftrightarrow (a_1 + c_1, a_2 + c_2)_{rep} \succsim (b_1 + c_1, b_2 + c_2)_{rep}.$$

The next corollary characterizes the ordering represented by the limit inferior, say, "limit egalitarianism".

Corollary 1. *An ordering $\succsim$ on ℓ^∞ satisfies full anonymity, weak Pareto, non-substitution, sup continuity, and weak Hammond if and only if for every $x, y \in \ell^\infty$, $x \succsim y \Longleftrightarrow \underline{x} \geq \underline{y}$.*

Proof. See Section 7.2. □

The next corollary characterizes the ordering represented by the sum of the limit inferior and the limit superior, say, "limit utilitarianism".

Corollary 2. *An ordering on ℓ^∞ satisfies full anonymity, weak Pareto, non-substitution, sup continuity, and weak independence form the origin if and only if for every $x, y \in \ell^\infty$, $x \succsim y \Longleftrightarrow \underline{x} + \overline{x} \geq \underline{y} + \overline{y}$.*

Proof. See Section 7.2. □

In his study of utilitarianism, Sidgwick (1907, p.414) asserts "the time at which a man exists cannot affect the value of his happiness from a universal point of view."[10] Corollary 2 offers a new way of putting the principle of impartiality across time into the idea of utilitarianism.

[10] This argument is noticed by Bossert, Sprumont, and Suzumura (2007). Sidgwick's view is taken by Ramsey (1928), who has offered an impartial welfare criterion in the context of consumption-saving choice.

Remark 1. A permutation $\pi \in \Pi$ is a *variable step permutation* if there exists a sequence of integers $s_1, s_2, s_3, \ldots$ with $0 < s_1 < s_2 < s_3 < \cdots$ such that for each member s_k of the sequence, $\{1, 2, \ldots, s_k\} = \{\pi(1), \pi(2), \ldots, \pi(s_k)\}$. Then, an ordering $\succsim$ is said to satisfy *variable step anonymity* if for every $x \in \ell^\infty$ and every variable step permutation π, $x \sim \pi(x)$ (Lauwers 1997a). *Variable step anonymity* itself is much weaker than *full anonymity*, but Lauwers (2010b) shows that if a transitive binary relation satisfies *variable step anonymity*, then it in fact satisfies *full anonymity*. Since our study focuses on orderings whose definition includes transitivity, *variable step anonymity* is equivalent to *full anonymity*. This means that all our results hold true even if *full anonymity* is, seemingly, relaxed to *variable step anonymity*.

Remark 2. One may consider that *weak independence form the origin* is too weak as an independence condition, since it applies only to utility streams that consist of two values. A more natural version would be as follows: an ordering $\succsim$ on ℓ^∞ satisfies *independence from the origin* if for every $x, y, z \in \ell^\infty$, $x \succsim y \iff (x + z) \succsim (y + z)$. However, this condition is incompatible with the pair of *weak Pareto* and *full anonymity*. Indeed, if $\succsim$ satisfies *independence from the origin* and *full anonymity*, then a repeated application of these axioms leads to $(0, 3, 3)_{rep} \sim (2, 2, 2)_{rep}$ and $(0, 0, 3)_{rep} \sim (1, 1, 1)_{rep}$. But *full anonymity* implies $(0, 3, 3)_{rep} \sim (0, 0, 3)_{rep}$, and so $(1, 1, 1)_{rep} \sim (2, 2, 2)_{rep}$, which is incompatible with *weak Pareto*.

5 Leximin

It is interesting that the limit utilitarianism in Corollary 2 satisfies the following Pareto axiom that is stronger than *weak Pareto*.[11]

Partial Pareto. *For every $x, y \in \ell^\infty$ with $x \geqq y$, it holds that $x \succsim y$, and whenever x, y also satisfy either $\overline{x} > \overline{y}$ or $\underline{x} > \underline{y}$, it holds that $x \succ y$.*

On the other hand, the limit egalitarianism in Corollary 1 violates *partial Pareto*. This leads us to consider the following leximin notion compatible with this axiom. An ordering $\succsim$ on ℓ^∞ is a *limit leximin* if for every $x, y \in \ell^\infty$,

$$\underline{x} > \underline{y} \Longrightarrow x \succ y,$$
$$\overline{x} > \overline{y} \geq \underline{x} = \underline{y} \Longrightarrow x \succ y,$$
$$\overline{x} = \overline{y} > \underline{x} = \underline{y} \Longrightarrow x \sim y.$$

[11]This axiom should be placed in the literature as a *full anonymity*-consistent relaxation of the following axiom. An ordering $\succsim$ on ℓ^∞ satisfies *Pareto for infinite generations* if for every $x, y \in \mathscr{U}$ with $x \geqq y$, it holds that $x \succsim y$, and whenever $x_t > y_t$ for an infinite number of t, it holds that $x \succ y$ (Sakai 2006). It is easy to see that *Pareto for infinite generations* implies *partial Pareto*, but *Pareto for infinite generations* is incompatible with *anonymity*: *anonymity* requires $(0, 1, 1)_{rep} \sim (0, 0, 1)_{rep}$, while *Pareto for infinite generations* requires $(0, 1, 1)_{rep} \succ (0, 0, 1)_{rep}$.

There are multiple limit leximins, since this definition does not specify the ranking between any pair x, y with $\underline{x} = \underline{y} = \overline{x} = \overline{y}$. Note, however, that $\underline{x} = \underline{y} = \overline{x} = \overline{y}$ is equivalent to $\lim x_t = \lim y_t$, and such "tie cases" are non-generic. In this sense, the notion of limit leximin almost fully specifies the entire ranking of orderings. The following are examples of how to rank x, y with $\underline{x} = \underline{y} = \overline{x} = \overline{y}$.

$$x \sim y, \tag{1}$$
$$x \succsim y \Longleftrightarrow \inf_{t \in \mathbb{N}} x_t \geq \inf_{t \in \mathbb{N}} y_t, \tag{2}$$
$$x \succsim y \Longleftrightarrow \sup_{t \in \mathbb{N}} x_t \geq \sup_{t \in \mathbb{N}} y_t. \tag{3}$$

Since lexicographic orderings are typically discontinuous and our case is no exception, we introduce a weaker continuity axiom.[12]

Sup indifference continuity. *For every* $x \in \ell^\infty$, *the set* $\{y \in \ell^\infty : x \sim y\}$ *is closed in the sup norm topology.*

Recall that the limit egalitarianism in Corollary 1 is characterized by the following set of axioms: *full anonymity, weak Pareto, non-substitution, sup continuity*, and *weak Hammond*. The next theorem shows that if we strengthen *weak Pareto* to *partial Pareto* and weaken *sup continuity* to *sup indifference continuity*, then the limit leximins appear as alternatives.

Theorem 2. *If an ordering* $\succsim$ *on* ℓ^∞ *satisfies full anonymity, partial Pareto, non-substitution, weak Hammond, and sup indifference continuity, then it is a limit leximin.*

Proof. See Section 7.3. □

Examples showing the independence of the five axioms in Theorem 1 are provided in Appendix. There are multiple limit leximins satisfying the axioms in Theorem 2; examples include limit leximins supplementarily defined with tie-breaking rules (1), (2), and (3). Our characterization does not pin down such specific orderings, mainly because our distributional axioms of *partial Pareto* and *weak Hammond* are so weak that they are not applied to any pair x, y with $\underline{x} = \underline{y} = \overline{x} = \overline{y}$. But all limit leximins are identical aside from such non-generic pairs.

6 Concluding remarks

The most particular feature of our analysis is that the orderings we found are entirely or mainly based on the limit inferior and the limit superior. Since these notions

[12] The original idea of this axiom is from Higashi and Hyogo (2010) who study subjective lexicographic expected utility.

are meaningful only in an infinite dimensional space, the welfare judgment given by these orderings are crucially based on infinitely long-run movements of utility streams. This point much contrasts with the approach of constructing an ordering on an infinite dimensional space on the basis of finitely short-run movements of utility streams, as typically observed in the definition of the overtaking criterion, which has played a prominent role in the literature. This gives certain answers to the long-standing question of identifying fully impartial welfare criteria in dynamic economic environments.

7 Proofs

7.1 Proof of Theorem 1

It is obvious that if an ordering $\succsim$ on ℓ^∞ satisfies *full anonymity, weak Pareto, non-substitution*, and *sup continuity*, then it is represented by a two-dimensional function as described in Theorem 1. The proof for the converse statement consists of the following three steps:

Step 1. Define a dense subset $\mathscr{U}$ of ℓ^∞, called the *universal finite domain*.

Step 2. Fully identify ranking of utility streams that belong to $\mathscr{U}$.

Step 3. For every utility stream in $\ell^\infty \setminus \mathscr{U}$, find another utility stream in $\mathscr{U}$ that is indifferent to it. This completes finding the entire ranking.

We begin the proof.

Step 1. Introducing the universal finite domain

Let $\mathscr{S}$ be the set of non-empty, finite subsets of $\mathbb{R}$, that is, $\mathscr{S} \equiv \{X \subset \mathbb{R} : 1 \leq |X| < \infty\}$. Given any finite set of utility parameters $X \in \mathscr{S}$, its infinite product X^∞ denotes the associated domain of utility streams. The *universal finite domain* is the union of all associated domains,

$$\mathscr{U} \equiv \bigcup_{X \in \mathscr{S}} X^\infty.$$

Lemma 1. *$\mathscr{U}$ is a dense subset of ℓ^∞ in the sup norm topology; that is, for every $x \in \ell^\infty$ and every $\varepsilon > 0$, there exists $y \in \mathscr{U}$ such that $\sup_{t\in\mathbb{N}} |x_t - y_t| < \varepsilon$.*

Proof. Pick any $x \in \ell^\infty$ and $\varepsilon > 0$. Let $M > 0$ be a large integer such that $-M\varepsilon < x_t < M\varepsilon$ for all $t \in \mathbb{N}$. For every $t \in \mathbb{N}$, there exists a unique $m \in \{-M+1, -M+2, \ldots, M-1, M\}$ such that $(m-1)\varepsilon < x_t \leq m\varepsilon$, and then let $y_t \equiv m\varepsilon$. Obviously, $\sup_{t\in\mathbb{N}} |x_t - y_t| < \varepsilon$. Since $y = (y_1, y_2, \ldots)$ consists of at most finite number of elements, it follows that $y \in \mathscr{U}$. □

Let $\mathscr{N}$ be the set of infinite subsets of $\mathbb{N}$, that is, $\mathscr{N} \equiv \{\mathbb{N}' \subset \mathbb{N} : |\mathbb{N}'| = \infty\}$. Pick any $x \in \mathscr{U}$, and let $A(x) \equiv \{a \in \mathbb{R} : \exists \mathbb{N}' \in \mathscr{N}, \forall t \in \mathbb{N}', x_t = a\}$ be the set of real values that appear in $x = (x_1, x_2, \ldots)$ infinitely many times. Since x is taken from the universal finite domain, $A(x)$ consists of at most finite number of real values, that is, $1 \leq |A(x)| < \infty$. The properties described by the next lemma should be recognized throughout the subsequent discussion.

Lemma 2. *For every $x \in \mathscr{U}$, $\underline{x} = \min_{a \in A(x)} a$ and $\overline{x} = \max_{a \in A(x)} a$. Furthermore, there exists $s \in \mathbb{N}$ such that $\underline{x} \leq x_t \leq \overline{x}$ for all $t \geq s$.*

Proof. Immediately follows from the definition of the universal finite domain. □

We also remark that for every $x \in \mathscr{U}$, both $\max_{t \in \mathbb{N}} x_t$ and $\min_{t \in \mathbb{N}} x_t$ are well-defined.

Step 2. Identifying ranking on the universal finite domain

We first introduce a mild Pareto axiom.

Dominance. *For every $x, y \in \ell^\infty$, if $x \geqq y$, then $x \succsim y$.*

Note that *partial Pareto* implies *dominance.*

Lemma 3. *If an ordering $\succsim$ on ℓ^∞ satisfies weak Pareto and sup continuity, then it satisfies dominance.*

Proof. Easy. □

The next axiom is a somewhat stronger version of *non-substitution.*

Strong non-substitution. *For every $x, y \in \ell^\infty$, if there exist a finite subset $\mathbb{N}' \subset \mathbb{N}$, an infinite subset $\mathbb{N}'' \subset \mathbb{N}$ and $\varepsilon > 0$ such that*

$$
\begin{aligned}
y_t < x_t & \text{ for all } t \in \mathbb{N}', \\
x_t + \varepsilon \leq y_t & \text{ for all } t \in \mathbb{N}'', \\
x_t = y_t & \text{ for all } t \notin \mathbb{N}' \cup \mathbb{N}'',
\end{aligned}
$$

then $y \succsim x$.

Lemma 4. *If an ordering $\succsim$ on ℓ^∞ satisfies full anonymity, dominance, and non-substitution, then it satisfies strong non-substitution.*

Proof. Consider any $x, y \in \mathscr{U}$ that satisfy the hypothesis of *strong non-substitution*, which associates with $\mathbb{N}' \subset \mathbb{N}$, $\mathbb{N}'' \subset \mathbb{N}$ and $\varepsilon > 0$. Then, define $z \in \ell^\infty$ by

$$z_t \equiv y_t \qquad \text{for all } t \in \mathbb{N}',$$

$$\equiv x_t + \varepsilon \quad \text{for all } t \in \mathbb{N}'',$$
$$\equiv x_t \quad \text{for all } t \notin \mathbb{N}' \cup \mathbb{N}''.$$

By applying *full anonymity* and *non-substitution* $|\mathbb{N}'|$-times, $z \succsim x$. By *dominance*, $y \succsim z$, and so $z \succsim x$. □

Lemma 5. *If an ordering $\succsim$ on ℓ^∞ satisfies full anonymity, dominance, and non-substitution, then for every $x \in \mathscr{U}$ with $\underline{x} < \overline{x}$, it holds that $(\underline{x}, \overline{x})_{rep} \succsim x$.*

Proof. Let $\succsim$ be an ordering on ℓ^∞ satisfying *full anonymity, dominance*, and *non-substitution*. Pick any $x \in \mathscr{U}$ with $\underline{x} < \overline{x}$. Let $\mathbb{N}_1 \equiv \{t \in \mathbb{N} : x_t = \underline{x}\}$. Let $\mathbb{N}_2 \subset \mathbb{N}_1$ be such that $|\mathbb{N}_2| = \infty$ and $|\mathbb{N}_1 \setminus \mathbb{N}_2| = \infty$.

Define y by

$$\begin{aligned} \overline{x} < x_t &\Longrightarrow y_t \equiv \underline{x}, \\ t \in \mathbb{N}_2 &\Longrightarrow y_t \equiv \frac{\underline{x} + \overline{x}}{2}, \\ \text{otherwise } &\Longrightarrow y_t \equiv x_t. \end{aligned}$$

Lemma 4 ensures that $\succsim$ satisfies *strong non-substitution*, which implies $y \succsim x$.

Define z by

$$\begin{aligned} t \in \mathbb{N}_2 &\Longrightarrow z_t \equiv \overline{x}, \\ t \notin \mathbb{N}_2 &\Longrightarrow z_t \equiv x_t. \end{aligned}$$

Since z is obtained from x by a permutation, by *full anonymity*, $x \sim z$. By *dominance*, $z \succsim y$. Therefore, $x \sim y$.

Define w by

$$\begin{aligned} \underline{x} < y_t &\Longrightarrow w_t \equiv \overline{x}, \\ y_t \leq \underline{x} &\Longrightarrow w_t \equiv \underline{x}. \end{aligned}$$

Then by *dominance*, $w \succsim y$, and hence $w \succsim x$. By *full anonymity*, $(\underline{x}, \overline{x})_{rep} \sim w$, and hence $(\underline{x}, \overline{x})_{rep} \succsim x$. □

Lemma 6. *If an ordering $\succsim$ on ℓ^∞ satisfies full anonymity, dominance, and non-substitution, then for every $x \in \mathscr{U}$ with $\underline{x} < \overline{x}$, it holds that $x \succsim (\underline{x}, \overline{x})_{rep}$.*

Proof. Let $\succsim$ be an ordering on ℓ^∞ satisfying *full anonymity, dominance*, and *non-substitution*. Pick any $x \in \mathscr{U}$ with $\underline{x} < \overline{x}$. Let $\mathbb{N}_1 \equiv \{t \in \mathbb{N} : x_t = \overline{x}\}$. Let $\mathbb{N}_2 \subset \mathbb{N}_1$ be such that $|\mathbb{N}_2| = \infty$ and $|\mathbb{N}_1 \setminus \mathbb{N}_2| = \infty$. Define y by

$$t \in \mathbb{N}_2 \Longrightarrow y_t \equiv \frac{\underline{x} + \overline{x}}{2},$$

$$x_t < \underline{x} \Longrightarrow y_t \equiv \underline{x},$$
$$\text{otherwise} \Longrightarrow y_t \equiv x_t$$

Lemma 4 ensures that $\succsim$ satisfies *strong non-substitution*, which implies $x \succsim y$.

Define z by

$$t \in \mathbb{N}_2 \Longrightarrow z_t \equiv \underline{x},$$
$$t \notin \mathbb{N}_2 \Longrightarrow z_t \equiv x_t.$$

Since z is obtained from x by a permutation, by *full anonymity*, $x \sim z$. By *dominance*, $y \succsim z$. Therefore, $x \sim y$.

Define w by

$$y_t < \overline{x} \Longrightarrow w_t \equiv \underline{x},$$
$$\overline{x} \geq y_t \Longrightarrow w_t \equiv \overline{x}.$$

Then by *dominance*, $y \succsim w$, and hence $x \succsim w$. By *full anonymity*, $(\underline{x}, \overline{x})_{rep} \sim w$, and hence $x \succsim (\underline{x}, \overline{x})_{rep}$. □

Lemma 7. *If an ordering* $\succsim$ *on* ℓ^∞ *satisfies full anonymity, dominance, and non-substitution, then for every* $x \in \mathscr{U}$ *with* $\underline{x} < \overline{x}$*, it holds that* $x \sim (\underline{x}, \overline{x})_{rep}$.

Proof. Immediately follows from Lemmas 5 and 6. □

The following lemma is essentially due to Debreu (1954) and Diamond (1965, p. 172).

Lemma 8. *If an ordering* $\succsim$ *on* ℓ^∞ *satisfies weak Pareto and sup continuity, then for every* $x \in \ell^\infty$*, there exists a unique* $a \in \mathbb{R}$ *such that* $x \sim (a)_{rep}$.

Proof. Let $\succsim$ be an ordering on ℓ^∞ satisfying *weak Pareto* and *sup continuity*. Consider any $x \in \ell^\infty$. The set of constant streams $C \equiv \{(a)_{rep} \in \ell^\infty : \exists a \in \mathbb{R}\}$ is closed with respect to the sup norm topology. By *weak Pareto*, both $\{y \in C : y \succsim x\}$ and $\{y \in C : x \succsim y\}$ are non-empty, and they are closed in the sup norm topology. Since no closed set can be partitioned by two disjoint closed sets in the sup norm topology, they have a non-empty intersection, that is, there exists $a \in \mathbb{R}$ such that $x \sim (a)_{rep}$. Uniqueness of such a follows from *weak Pareto*. □

Lemma 9. *If an ordering* $\succsim$ *on* ℓ^∞ *satisfies weak Pareto, dominance, and strong non-substitution, then for every* $x \in \mathscr{U}$ *and* $a \in \mathbb{R}$ *such that* $\underline{x} > a$*, it holds that* $x \succ (a)_{rep}$.

Proof. Let $\succsim$ be an ordering on ℓ^∞ satisfying *weak Pareto, dominance*, and *strong non-substitution.* Pick any $x \in \mathscr{U}$ and $a \in \mathbb{R}$ such that $\underline{x} > a$.

Let $y \in \mathscr{U}$ be such that

$$x_t < \underline{x} \Longrightarrow y_t \equiv x_t,$$
$$\underline{x} \leq x_t \Longrightarrow y_t \equiv \underline{x}.$$

By *dominance*, $x \succsim y$. By *strong non-substitution*, $y \succsim (\frac{\underline{x}}{2} + a)_{rep}$. By *weak Pareto*, $(\frac{\underline{x}}{2} + a)_{rep} \succ (a)_{rep}$. Therefore, $x \succ (a)_{rep}$. □

Lemma 10. *If an ordering $\succsim$ on ℓ^∞ satisfies weak Pareto, strong non-substitution, and sup continuity, then for every $x \in \mathscr{U}$ with $\underline{x} = \overline{x}$, it holds that $x \sim (\underline{x})_{rep}$.*

Proof. Let $\succsim$ be an ordering on ℓ^∞ satisfying *weak Pareto, non-substitution*, and *sup continuity.* For every $x \in \mathscr{U}$ with $\underline{x} = \overline{x}$, Lemma 8 implies that there exists a unique $a \in \mathbb{R}$ such that $x \sim (a)_{rep}$. It suffices to show that $a = \underline{x}$. Note that $\succsim$ satisfies *dominance*, too. By Lemma 9, $a \geq \underline{x}$. There exists $s \in \mathbb{N}$ such that $x_t = \underline{x}$ for all $t > s$.

Suppose, by contradiction, that $a > \underline{x}$. Define y by

$$\begin{aligned} y_t &< \inf_{t\in\mathbb{N}} x_t \quad \text{for all } t \leq s, \\ &\equiv \frac{a + \underline{x}}{2} \quad \text{for all } t > s. \end{aligned}$$

Then by *strong non-substitution*, $y \succsim x$. By *dominance*, $(a)_{rep} \succ y$. Hence, $(a)_{rep} \succ x$, a contradiction. Overall, $a = \underline{x}$ must hold. □

Lemma 11. *If an ordering $\succsim$ on ℓ^∞ satisfies full anonymity, weak Pareto, non-substitution, and sup continuity, then there exists a continuous function $W : \mathbb{R}^2 \to \mathbb{R}$ such that for every $x, y \in \mathscr{U}$, $x \succsim y \Longleftrightarrow W(\underline{x}, \overline{x}) \geq W(\underline{y}, \overline{y})$, and $(\underline{x}, \overline{x}) > (\underline{y}, \overline{y})$ implies $W(\underline{x}, \overline{x}) > W(\underline{y}, \overline{y})$.*

Proof. Let $\succsim$ be an ordering on ℓ^∞ satisfying *full anonymity, weak Pareto, non-substitution*, and *sup continuity.* By Lemmas 7, 8, and 10, for every $x \in \mathscr{U}$, there exists a unique $a \in [\underline{x}, \overline{x}]$ such that $(\underline{x}, \overline{x})_{rep} \sim x \sim (a)_{rep}$. Then, define $W(\underline{x}, \overline{x}) \equiv a$. By *sup continuity*, W is continuous.

By *weak Pareto*, for every $x, y \in \mathscr{U}$, if $(\underline{x}, \overline{x}) > (\underline{y}, \overline{y})$, then $x \succ y$, and so $W(\underline{x}, \overline{x}) > W(\underline{y}, \overline{y})$. □

Step 3. Extending the characterization to ℓ^∞

Lemma 12. *For every $x \in \ell^\infty$ and every $\varepsilon > 0$, there exists $y \in \mathscr{U}$ such that $\underline{x} = \underline{y}$, $\overline{x} = \overline{y}$, and $\sup_{t\in\mathbb{N}} |x_t - y_t| < \varepsilon$.*

Proof. Pick any $x \in \ell^\infty$ and $\varepsilon > 0$. By Lemma 1, there exists $z \in \mathscr{U}$ such that $\sup_{t\in\mathbb{N}} |x_t - z_t| < \varepsilon$.

Consider the case that $a \equiv \lim x_t$ exists. Then, there exists $s \in \mathbb{N}$ such that $|x_t - a| < \varepsilon$ for all $t \geq s$. Define y by

$$\begin{aligned} y_t &\equiv z_t \text{ for all } t < s-1, \\ &\equiv a \text{ for all } t \geq s. \end{aligned}$$

Obviously, $y \in \mathscr{U}$ and it satisfies the desired properties.

Consider the case that $\lim x_t$ does not exist. Then, there exists small $\delta < \varepsilon$ such that $\underline{x} + \delta < \overline{x} - \delta$. Furthermore, by the definitions of the limit inferior and the limit superior, there exists $s \in \mathbb{N}$ such that

$$\underline{x} - \delta < x_t < \overline{x} + \delta \quad \text{for all } t \geq s. \tag{4}$$

Define y by

$$\begin{aligned} y_t &\equiv \underline{x} \;\; \text{if } t \geq s \text{ and } \underline{x} - \delta < x_t < \underline{x} + \delta, \\ &\equiv \overline{x} \;\; \text{if } t \geq s \text{ and } \overline{x} - \delta < x_t < \overline{x} + \delta, \\ &\equiv z_t \;\; \text{otherwise.} \end{aligned}$$

By construction, $y \in \mathscr{U}$ and $\sup_{t\in\mathbb{N}} |x_t - y_t| < \varepsilon$. Note that there exist infinitely many $t \geq s$ with $\underline{x} - \delta < x_t < \underline{x} + \delta$, and infinitely many other $t \geq s$ with $\overline{x} - \delta < x_t < \overline{x} + \delta$. This fact and (4) imply that $\underline{y} = \underline{x}$ and $\overline{y} = \overline{x}$. □

Lemma 13. *If an ordering $\succsim$ on ℓ^∞ satisfies full anonymity, weak Pareto, non-substitution, and sup continuity, then for every $x \in \ell^\infty$, it holds that $x \sim (\underline{x}, \overline{x})_{rep}$.*

Proof. Let $\succsim$ be an ordering on ℓ^∞ satisfying *full anonymity, weak Pareto, non-substitution*, and *sup continuity*. Pick any $x \in \ell^\infty$. By Lemma 12, for every positive integer k, there exists $x^k \in \mathscr{U}$ such that $\underline{x} = \underline{x}^k$, $\overline{x} = \overline{x}^k$, and $\sup_{t\in\mathbb{N}} |x_t - x_t^k| < \frac{1}{k}$. By Lemma 7 or 10, $(\underline{x}, \overline{x})_{rep} \sim x^k$ for all k. Thus by *sup continuity*, $(\underline{x}, \overline{x})_{rep} \sim x$. □

The next lemma completes the proof of Theorem 1.

Lemma 14. *If an ordering $\succsim$ on ℓ^∞ satisfies full anonymity, weak Pareto, non-substitution, and sup continuity, then there exists a continuous function $W : \mathbb{R}^2 \to \mathbb{R}$ such that for every $x, y \in \ell^\infty$, $x \succsim y \iff W(\underline{x}, \overline{x}) \geq W(\underline{y}, \overline{y})$, and $(\underline{x}, \overline{x}) > (\underline{y}, \overline{y})$ implies $W(\underline{x}, \overline{x}) > W(\underline{y}, \overline{y})$.*

Proof. Let $\succsim$ be an ordering on ℓ^∞ satisfying *full anonymity, weak Pareto, non-substitution*, and *sup continuity*. By Lemma 11, there exists a continuous function $W : \mathbb{R}^2 \to \mathbb{R}$ such that for every $z, v \in \mathscr{U}$, $z \succsim v \iff W(\underline{z}, \overline{z}) \geq W(\underline{v}, \overline{v})$, and $(\underline{z}, \overline{z}) > (\underline{v}, \overline{v})$ implies $W(\underline{z}, \overline{z}) > W(\underline{v}, \overline{v})$.

Consider any $x, y \in \ell^\infty$. By Lemma 13, $x \sim (\underline{x}, \overline{x})_{rep}$ and $y \sim (\underline{y}, \overline{y})_{rep}$. Since $(\underline{x}, \overline{x})_{rep}, (\underline{y}, \overline{y})_{rep} \in \mathscr{U}$,

$$x \succsim y \iff (\underline{x}, \overline{x})_{rep} \succsim (\underline{y}, \overline{y})_{rep} \iff W(\underline{x}, \overline{x}) \geq W(\underline{y}, \overline{y}),$$

and $(\underline{x}, \overline{x}) > (\underline{y}, \overline{y})$ implies $W(\underline{x}, \overline{x}) > W(\underline{y}, \overline{y})$. Therefore, W obtained in Lemma 11 is in fact the desired function. □

7.2 Proofs of Corollaries 1 and 2

Proof of Corollary 1. We only prove the "only if" part. It suffices to show that for all $x, y \in \ell^\infty$, ($\star$) $\underline{x} > \underline{y}$ implies $W(\underline{x}, \overline{x}) > W(\underline{y}, \overline{y})$, and ($\star\star$) $\underline{x} = \underline{y}$ implies $W(\underline{x}, \overline{x}) = W(\underline{y}, \overline{y})$.

($\star$) Suppose, by contradiction, that for some $x, y \in \ell^\infty$, $\underline{x} > \underline{y}$ and $W(\underline{y}, \overline{y}) \geq W(\underline{x}, \overline{x})$. Let $a \in \mathbb{R}$ be $\underline{x} > a > \underline{y}$. By *weak Hammond*, $(a)_{rep} \succsim (\underline{y}, \overline{y})_{rep}$, and hence $W(a, a) \geq W(\underline{y}, \overline{y})$. But then $W(a, a) \geq W(\underline{x}, \overline{x})$, so $(a)_{rep} \succ (\underline{x}, \overline{x})_{rep}$. This contradicts *weak Pareto*.

($\star\star$) Suppose, by contradiction, that for some $x, y \in \ell^\infty$, $\underline{x} = \underline{y}$ and $W(\underline{x}, \overline{x}) > W(\underline{y}, \overline{y})$. By Lemma 8, there exists $a \in \mathbb{R}$ such that $x \sim (a)_{rep}$. Since $W(a, a) = W(\underline{x}, \overline{x})$, one has $W(a, a) > W(\underline{y}, \overline{y})$, and then monotonicity of W implies $a > \underline{y}$, which in turn implies $a > \underline{x}$. Then there exists $b \in \mathbb{R}$ with $a > b > \underline{x}$, and by *weak Pareto* and *weak Hammond*, $W(a, a) > W(b, b) \geq W(\underline{x}, \overline{x})$. This contradicts $x \sim (a)_{rep}$. □

Proof of Corollary 2. We only prove the "only if" part. We shall first show that for every $x \in \ell^\infty$,

$$x \sim (\frac{\underline{x} + \overline{x}}{2}, \frac{\underline{x} + \overline{x}}{2})_{rep}.$$

By *full anonymity* and *weak independence from the origin*,

$$(0, \underline{x})_{rep} \sim (\underline{x}, 0)_{rep},\ (0, \frac{\overline{x} + \underline{x}}{2})_{rep} \sim (\underline{x}, \frac{\overline{x} - \underline{x}}{2})_{rep},$$
$$(0, \frac{\overline{x} + \underline{x}}{2})_{rep} \sim (\frac{\overline{x} - \underline{x}}{2}, \underline{x})_{rep},\ (\frac{\overline{x} + \underline{x}}{2}, \frac{\overline{x} + \underline{x}}{2})_{rep} \sim (\overline{x}, \underline{x})_{rep},$$

and hence by Theorem 1,

$$x \sim (\frac{\underline{x} + \overline{x}}{2}, \frac{\underline{x} + \overline{x}}{2})_{rep}. \tag{5}$$

Then, for every $x, y \in \ell^\infty$, by (5) and *weak Pareto*,

$$x \succsim y \iff (\frac{\underline{x} + \overline{x}}{2}, \frac{\underline{x} + \overline{x}}{2})_{rep} \succsim (\frac{\underline{y} + \overline{y}}{2}, \frac{\underline{y} + \overline{y}}{2})_{rep} \iff \underline{x} + \overline{x} \geq \underline{y} + \overline{y},$$

which completes the proof. □

7.3 Proof of Theorem 2

Lemma 15. *If an ordering $\succsim$ on ℓ^∞ satisfies full anonymity, weak Pareto, non-substitution, and sup indifference continuity, then for every $x \in \ell^\infty$ with $\underline{x} < \overline{x}$, it holds that $x \sim (\underline{x}, \overline{x})_{rep}$.*

Proof. This can be proved by the same way as Lemma 13. □

Lemma 16. *If an ordering $\succsim$ on ℓ^∞ satisfies full anonymity, weak Pareto, dominance, non-substitution, and weak Hammond, then for every $x, y \in \mathscr{U}$ with $\underline{x} > \underline{y}$, it holds that $x \succ y$.*

Proof. Let $\succsim$ be an ordering on ℓ^∞ satisfying *full anonymity, weak Pareto, dominance, non-substitution*, and *weak Hammond*. Pick any $x, y \in \mathscr{U}$ with $\underline{x} > \underline{y}$.

Let $x' \in \mathscr{U}$ be a utility stream that is obtained from x by a permutation such that $x'_t = \underline{x}$ for every odd t, and y' be a utility stream that is obtained from y by a permutation such that $y'_t = \underline{y}$ for every odd t. By *full anonymity*, $x \sim x'$ and $y \sim y'$.

Let $a, b, c \in \mathbb{R}$ be such that $\underline{y} < c < b < \underline{x} \leq a \equiv \max\{\max_{t\in\mathbb{N}} x_t, \max_{t\in\mathbb{N}} y_t\}$. By *weak Hammond*, $(b)_{rep} \succsim (c, a)_{rep}$. By *dominance*, $(c, a)_{rep} \succsim y'$.

Let $s \in \mathbb{N}$ be such that $\underline{x} \leq x'_t$ for all $t \geq s$. Define z by

$$\begin{aligned} z_t &< \min\{\min_{t\in\mathbb{N}} x_t, \min_{t\in\mathbb{N}} y_t\} \quad \text{for every } t \leq s, \\ &\equiv \frac{b + \underline{x}}{2} \quad \text{for every } t \geq s + 1. \end{aligned}$$

Lemma 4 ensures that $\succsim$ satisfies *strong non-substitution*, which implies $z \succsim (b)_{rep}$. By *weak Pareto*, $x' \succ z$.

Summarizing, we have

$$x \sim x' \succ z \succsim (b)_{rep} \succsim (c, a)_{rep} \succsim y' \sim y,$$

and hence $x \succ y$. □

Lemma 17. *If an ordering $\succsim$ on ℓ^∞ satisfies partial Pareto and strong non-substitution, then for every $x, y \in \mathscr{U}$ with $\underline{y} = \overline{y} = \underline{x} < \overline{x}$, it holds that $x \succ y$.*

Proof. Let $\succsim$ be an ordering on ℓ^∞ satisfying *partial Pareto* and *non-substitution*. Pick any $x, y \in \mathscr{U}$ with $\underline{y} = \overline{y} = \underline{x} < \overline{x}$. There exists a sufficiently large $s \in \mathbb{N}$ such that $\underline{y} = y_t = \overline{y} = \underline{x} \leq x_t \leq \overline{x}$ for all $t \geq s$.

Let $\mathbb{N}' \equiv \{t \in \mathbb{N} : t \geq s \text{ and } x_t = \overline{x}\}$, and let $\mathbb{N}'' \subset \mathbb{N}'$ be such that $|\mathbb{N}''| = \infty$ and $|\mathbb{N}' \setminus \mathbb{N}''| = \infty$. Define z by

$$z_t < \min\{\min_{t\in\mathbb{N}} x_t, \min_{t\in\mathbb{N}} y_t\} - 1 \text{ for every } t < s,$$

$$\equiv \frac{\underline{x} + \overline{x}}{2} \text{ for every } t \in \mathbb{N}',$$
$$\equiv y_t \text{ for any other } t.$$

By *strong non-substitution*, $z \succsim y$. Since $x \geqq z$ and $\overline{x} > \frac{\underline{x}+\overline{x}}{2} = \overline{z}$, by *partial Pareto*, $x \succ z$. Therefore, $x \succ y$. □

Lemma 18. *If an ordering $\succsim$ on ℓ^∞ satisfies full anonymity, partial Pareto, non-substitution, and weak Hammond, then for every $x, y \in \mathscr{U}$,*

$$\underline{x} > \underline{y} \Longrightarrow x \succ y,$$
$$\overline{x} > \overline{y} \geq \underline{x} = \underline{y} \Longrightarrow x \succ y,$$
$$\overline{x} = \overline{y} > \underline{x} = \underline{y} \Longrightarrow x \sim y.$$

Proof. Consider any $\succsim$ on ℓ^∞ satisfying *full anonymity*, *partial Pareto*, *non-substitution*, and *weak Hammond*. Pick any $x, y \in \mathscr{U}$.

If $\underline{x} > \underline{y}$, then by Lemma 16, $x \succ y$.

If $\overline{x} > \overline{y} > \underline{x} = \underline{y}$, then by Lemma 7 and *partial Pareto*, $x \sim (\underline{x}, \overline{x})_{rep} \succ (\underline{y}, \overline{y})_{rep} \sim y$.

If $\overline{x} > \overline{y} = \underline{x} = \underline{y}$, then by Lemma 17, $x \succ y$.

If $\overline{x} = \overline{y} > \underline{x} = \underline{y}$, then by Lemma 7, $x \sim (\underline{x}, \overline{x})_{rep} \sim (\underline{y}, \overline{y})_{rep} \sim y$. □

The next lemma completes the proof of Theorem 2.

Lemma 19. *If an ordering $\succsim$ on ℓ^∞ satisfies full anonymity, partial Pareto, non-substitution, weak Hammond, and sup indifference continuity, then for every $x, y \in \ell^\infty$,*

$$\underline{x} > \underline{y} \Longrightarrow x \succ y,$$
$$\overline{x} > \overline{y} \geq \underline{x} = \underline{y} \Longrightarrow x \succ y,$$
$$\overline{x} = \overline{y} > \underline{x} = \underline{y} \Longrightarrow x \sim y.$$

Proof. Immediately follows from Lemmas 15 and 18. □

Appendix

We show by examples that the axioms in Theorems 1 and 2 are tight; that is, any one of the axioms is necessary to obtain these characterizations.

For Theorem 1, since any limit leximin shows the necessity of *sup continuity*, the following three examples suffice for our purpose.

Example 1 (*Weak Pareto* for Theorem 1)**.** For every $x, y \in \ell^\infty$, let $x \sim y$.

Example 2 (*Full anonymity* for Theorem 1)**.** For every $x, y \in \ell^\infty$, let $x \succsim y \iff \underline{x'} \geq \underline{y'}$, where $x' = (x_2, x_4, x_6, \ldots)$ and $y' = (y_2, y_4, y_6, \ldots)$.

Example 3 (*Non-substitution* for Theorem 1)**.** For every $x, y \in \ell^\infty$, let $x \succsim y \iff \inf_{t\in\mathbb{N}} x_t \geq \inf_{t\in\mathbb{N}} y_t$.

For Theorem 2, since the limit inferior ordering in Corollary 1 shows the necessity of *partial Pareto* and the limit utilitarianism in Corollary 2 shows the necessity of *weak Hammond*, the following three examples suffice.

Example 4 (*Sup indifference continuity* for Theorem 2)**.** Let $\succsim$ be the limit leximin such that for every $x, y \in \ell^\infty$ with $\underline{x} = \overline{x} = \underline{y} = \overline{y}$,

$$\begin{aligned} x, y \in \mathscr{U} &\Longrightarrow x \sim y, \\ x \in \mathscr{U} \text{ and } y \notin \mathscr{U} &\Longrightarrow x \succ y, \\ x, y \notin \mathscr{U} &\Longrightarrow x \sim y. \end{aligned}$$

Example 5 (*Full anonymity* for Theorem 2)**.** For every $x, y \in \ell^\infty$, let $\succsim$ be such that

$$\begin{aligned} &\underline{x} > \underline{y} \Longrightarrow x \succ y, \\ &\underline{x} = \underline{y} \Longrightarrow (x \succsim y \iff \overline{x'} + 2\overline{x''} \geq \overline{y'} + 2\overline{y''}), \end{aligned}$$

where $x' = (x_1, x_3, x_5, \ldots), x'' = (x_2, x_4, x_6, \ldots), y' = (y_1, y_3, y_5, \ldots)$ and $y'' = (y_2, y_4, y_6, \ldots)$.

Example 6 (*Non-substitution* for Theorem 2)**.** For every $x, y \in \ell^\infty$, let $\succsim$ be such that

$$\begin{aligned} &\inf_{t\in\mathbb{N}} x_t > \inf_{t\in\mathbb{N}} y_t \Longrightarrow x \succ y, \\ &\inf_{t\in\mathbb{N}} x_t = \inf_{t\in\mathbb{N}} y_t \text{ and } \underline{x} > \underline{y} \Longrightarrow x \succ y, \\ &\inf_{t\in\mathbb{N}} x_t = \inf_{t\in\mathbb{N}} y_t \text{ and } \underline{x} = \underline{y} \Longrightarrow (x \succsim y \iff \overline{x} \geq \overline{y}). \end{aligned}$$

References

Alcantud, J.C.R. and M.D. Garcìa-Sanz (2010a) "Paretian Evaluation of Infinite Utility Streams: an Egalitarian Criterion," *Economics Letters*, Vol. 16, 209-211.

Alcantud, J.C.R. and M.D. Garcìa-Sanz (2010b) "Evaluations of Infinite Utility Streams: Pareto-efficient and Egalitarian Axiomatics," mimeo, Universidad de Salamanca.

d'Aspremont, C., G.B. Asheim, K. Banerjee (2010) "Generalized Time-invariant Overtaking," *Journal of Mathematical Economics*, in press.

d'Aspremont, C. and L. Gevers (1977) "Equity and the Informational Basis of Collective Choice," *Review of Economic Studies*, Vol. 46, pp. 199-210.

Asheim, G.B., T. Mitra, and B. Tungodden (2007) "A New Equity Condition for Infinite Utility Streams and the Possibility of Being Paretian," in *Intergenerational Equity and Sustainability*, eds. by J. Roemer and K. Suzumura. Palgrave.

Asheim, G.B. and B. Tungodden (2004) "Resolving Distributional Conflicts Between Generations," *Economic Theory*, Vol. 24, pp. 221-230.

Atsumi, H. (1965) "Neoclassical Growth and the Efficient Program of Capital Accumulation," *Review of Economic Studies*, Vol. 32, pp. 127-136.

Basu, K. and T. Mitra (2003) "Aggregating Infinite Utility Streams with Inter-generational Equity: The Impossibility of Being Paretian," *Econometrica*, Vol. 71, pp. 1557-1563.

Basu, K. and T. Mitra (2007) "Utilitarianism for Infinite Utility Streams: A New Welfare Criterion and Its Axiomatic Characterization," *Journal of Economic Theory*, Vol. 133, pp. 350-373.

Bossert, W., Y. Sprumont, and K. Suzumura (2007) "Ordering Infinite Utility Streams," *Journal of Economic Theory*, Vol. 135, pp. 579-589.

Chambers, C.P. (2009) "Intergenerational Equity: Sup, Inf, Lim sup, and Lim inf," *Social Choice and Welfare*, Vol. 32, pp. 243-252.

Chichilnisky, G. (1996) "An Axiomatic Approach to Sustainable Development," *Social Choice and Welfare*, Vol. 14, pp. 231-257.

Debreu, G. (1954) "Representation of a Preference Ordering by a Numerical Function," in *Decision Processes* (Thrall, R.M., C.H. Coombs, and R.L. Davis, eds), John Wiley.

Diamond, P. (1965) "The Evaluation of Infinite Utility Streams," *Econometrica*, Vol. 33, pp. 170-177.

Fleurbaey, M. and P. Michel (2003) "Intertemporal Equity and the Extension of the Ramsey Criterion," *Journal of Mathematical Economics*, Vol. 39, pp. 777-802.

Hammond, P.J. (1976) "Equity, Arrow's Conditions, and Rawls' Difference Principle," *Econometrica*, Vol. 44, pp. 793-804.

Higashi, Y. and K. Hyogo (2010) "Lexicographic Expected Utility with a Subjective State

Space," *Economic Theory*, online-first.

Kamaga, K. and K. Kojima (2010) "On the Leximin and Utilitarian Overtaking Criteria with Extended Anonymity," *Social Choice and Welfare*, online-first.

Lauwers, L. (1997a) "Continuity and Equity with Infinite Horizons," *Social Choice and Welfare*, Vol. 14, pp. 345-356.

Lauwers, L. (1997b) "Rawlsian Equity and Generalised Utilitarianism with an Infinite Population," *Economic Theory*, Vol. 9, pp. 143-150.

Lauwers, L. (1997c) "Infinite Utility: Insisting on Strong Monotonicity," *Australasian Journal of Philosophy*, Vol. 75, pp. 222-233.

Lauwers, L. (1998) "Intertemporal Objective Functions -Strong Pareto Versus Anonymity," *Mathematical Social Sciences*, Vol.35, pp. 37-55.

Lauwers, L. (2010a) "Ordering Infinite Utility Streams Comes at the Cost of a Non-Ramsey Set," *Journal of Mathematical Economics*, Vol. 20, pp. 32-37.

Lauwers, L. (2010b) "Ordering Infinite Utility Streams: Maximal Anonymity," mimeo, K.U. Leuven.

Matsuyama, K. (2005) "Poverty Traps," in *the New Palgrave Dictionary of Economics*, eds. by L. Blume and S. Durlauf. 2nd Edition, Palgrave Macmillan.

Ramsey F.P. (1928) "A Mathematical Theory of Saving," *Economic Journal*, Vol. 38, pp. 543-559.

Sakai, T. (2003) "An axiomatic approach to intergenerational equity," *Social Choice and Welfare*, Vol. 20, pp. 167-176.

Sakai, T. (2006) "Equitable intergenerational preferences on restricted domains," *Social Choice and Welfare*, Vol. 27, pp. 41-54.

Sakai, T. (2010a) "Intergenerational Equity and an Explicit Construction of Welfare Criteria," *Social Choice and Welfare*, online-first.

Sakai, T. (2010b) "A Characterization and an Impossibility of Finite Length Anonymity for Infinite Generations," *Journal of Mathematical Economics*, forthcoming.

Sidgwick, H. (1907) *The Methods of Ethics*, 7th ed. Macmillan and Co, London.

Svensson, L.-G. (1980) "Equity among Generations," *Econometrica*, Vol. 48, pp. 1951-1956.

von Weizsäcker, C.C. (1965) "Existence of Optimal Programmes of Accumulation for an Infinite Time Horizon," *Review of Economic Studies*, Vol. 32, pp. 85-104.

Zame, W.R. (2007) "Can Utilitarianism be Operationalized?" *Theoretical Economics*, Vol. 2, pp. 187-202.