A Note on Monotonic Assignment Problems*

Minoru Kitahara†

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Abstract

We consider *monotonic* assignment problems, in which efficiency is attained by assigning a "higher rank" position to an agent with a "higher" type, such as queueing problems. The notable feature is that any monotonic assignment problem can be decomposed into homogeneous object problems. As a result, only by referring existing results in homogeneous object problems, we prove the compatibility with Bayesian incentive of (i) envy-freeness if the type distribution is identical, and (ii) ex post individual rationality under equal initial shares if the number of agents is sufficiently large.

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1 Introduction

Consider a community attempting not only to assign positions among its members to minimize the total cost to be engaged in the positions, but also to attain some additional property, such as envy-freeness (EF) and individual rationality (IR).[1] The cost for each member to be engaged in each position is his private information.[2] At the same time, an internal (i.e., budget-balancing) transfer technology as money is available.

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†JSPS Research Fellow, Graduate School of Social Sciences, Tokyo Metropolitan University, 1-1 Minami-Osawa, Hachioji-shi, Tokyo 192-0397, Japan (mkitahar@tmu.ac.jp), and Visiting Research Fellow, Advanced Research Institute for the Sciences and Humanities, Nihon University, 12-5 Goban-cho, Chiyoda-ku, Tokyo 102-8251, Japan.

[1] For assignment problems, see, e.g., Moulin [12].

[2] Here we focus on private value cases. For the literature considering interdependent values, see, e.g., Fieseler et al. [4] and Galavotti et al. [6].

If the efficient assignment is the only concern and hence the transfer can be used only to elicit the private information, or to satisfy Bayesian incentive compatibility (IC), then it suffices, as well known, by making a transfer following d'Aspremont and Gerard-Varet [2]. As suggested by Fujinaka [5], however, such a transfer following d'Aspremont and Gerard-Varet [2] often fails to satisfy additional properties as above.

By focusing on specific assignment problems and developing alternative transfer schemes, on the other hand, some studies find such possibilities. Fujinaka [5] considers the problems of allocating single object,[3] and shows a way to construct a transfer scheme satisfying both EF and IC when the type distribution of each member is (independent and) identical.

Regarding IR under equal initial shares, for which the status quo in terms of IR is to be assigned each position with equal probability,[4] there also exist possibility results assuming the identical type distribution. Gershkov and Schweinzer [7] considers (static) queueing problems with the restriction of linear waiting costs and equal time intervals,[5] and finds the compatibility of the IR with Bayesian incentive. Cramton et al. [1] classically shows the same compatibility in the single object problem, which is recently extended by Galavotti et al. [6] for the stronger requirement of ex post individual rationality (EPIR).

Then, we propose to explore such a possibility within *monotonic* assignment problems, in which the total cost minimization is attained by assigning a "higher rank" position to a member with a "higher" type. It covers generalized versions of the problems as above, homogeneous object problems, in which a number of homogeneous objects are to be allocated (and each member cannot be allocated more than one), and queueing problems without the restriction as above. For example, a higher type may correspond to a higher valuation of the object in homogeneous object problems, and to a higher time impatience in queueing problems.

The notable feature is that any monotonic assignment problem can be decomposed into homogeneous object problems as long as a targeted property satisfies some additivity as IC, EF and EPIR: once such a property is satisfied in each of the homogeneous object problems, then it is also satisfied in the original problem (Proposition 1). Thus, it suffices by showing a possibility only in homogeneous object problems, in which there are essentially only two kinds of positions, allocated an object or nothing, and hence the investigation is in general much easier than with many different positions as in the original problem before the decomposition.

To demonstrate the usefulness, we prove some possibilities only by referring existing results in homogeneous object problems by Fujinaka [5] and Kitahara [10]. By the result of Fujinaka [5],

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[3] See, e.g., Kunreuther et al. [11].

[4] It corresponds to "random order" in Gershkov and Schweinzer [7], and to "equal-share partnership" in Galavotti et al. [6].

[5] See, e.g., Dolan [3].

EF is shown to be compatible with Bayesian incentive in any monotonic assignment problems if the type distribution is identical (Theorem 1). By the result of Kitahara [10], so is EPIR under equal initial shares, even with nonidentical type distributions, if the number of members is sufficiently large (Theorem 2).

These results may add much to the queueing literature. Kayı and Ramaekers [9] shows the compatibility of EF with the stronger incentive requirement of strategy-proofness, but with the same restriction on the cost structure as Gershkov and Schweinzer [7]. The former result shows that the cost structure is allowed to be much general if the incentive requirement is weakened to IC under the identical type distribution. Moreover, as noticed by Galavotti et al. [6], EF implies EPIR under equal initial shares. Thus, the possibility result of Gershkov and Schweinzer [7] is also extended for much general cost structures, even with the stronger requirement of EPIR (Corollary 1). Finally, the latter result may be the first to show a(n exact) possibility regarding IR under equal initial shares with nonidentical type distributions.[6]

The rest of the paper is organized as follows. In the next section, we formalize our model and notations. Then, in Section 3, we provide our results. Finally, Section 4 concludes the paper.

2 Model and notations

There is a set of *agents*, $\{1,\ldots,n\}$. The *type* of each agent i, t_i, is drawn independently from $(0,1)$ according to the cumulative distribution function F_i. Let $\mathbf{t} = (t_1,\ldots,t_n)$, $t_{-i} = (t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n)$, and $(t,t_{-i}) = (t_1,\ldots,t_{i-1},t,t_{i+1},\ldots,t_n)$.

There is also a set of *positions*, $\{1,\ldots,n\}$. In an *assignment problem* $C = (C_1,\ldots,C_n)$, the payoff to an agent i with a type t when a position k is assigned and an amount p of "money" is transferred to him is

$$u^C(k,p|t) = -C_k(t) + p.$$

For regularity, we focus on continuous F and C both of which have continuous and bounded derivatives except for a finite set of points.[7]

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[6] For an approximate possibility, Jackson and Kremer [8] provides a very general result regarding EF.

[7] It still allows discrete type cases. Consider that $n = 2$, the "discrete type" τ_i of each agent is independently drawn from $\{1,2\}$ with $\Pr(\tau_i = k) = p_i^k$, and an agent with type k incurs a cost c_l^k when he is assigned position l. It can be represented as

$$F_i'(t) = \begin{cases} (2/(1-\varepsilon))/p_i^2 & \text{for } t \in (\varepsilon/2, 1/2), \\ (2/(1-\varepsilon))/p_i^1 & \text{for } t \in ((1+\varepsilon)/2, 1), \\ 0 & \text{for } t \in (0,\varepsilon/2) \cup (1/2,(1+\varepsilon)/2), \end{cases}$$

$$C_k'(t) = \begin{cases} (2/\varepsilon)c_k^2 & \text{for } t \in (0,\varepsilon), \\ (2/\varepsilon)(c_k^1 - c_k^2) & \text{for } t \in (1/2,(1+\varepsilon)/2), \\ 0 & \text{for } t \in (2/\varepsilon, 1/2) \cup ((1+\varepsilon)/2). \end{cases}$$

Depending on the type realization of agents, an *assignment scheme* $\kappa = (\kappa_1, \dots, \kappa_n)$ makes a(n one-to-one) position assignment among them,

$$\{\kappa_1(\mathbf{t}), \dots, \kappa_n(\mathbf{t})\} = \{1, \dots, n\},$$

and a *transfer scheme* $\pi = (\pi_1, \dots, \pi_n)$ makes a (budget-balancing) transfer,

$$\sum_i \pi_i(\mathbf{t}) = 0.$$

Let $\bar{t}_k^\kappa(\mathbf{t})$ denote the type of an agent assigned position k (i.e., $\bar{t}_k^\kappa(\mathbf{t}) = t_i$ for $\kappa_i(\mathbf{t}) = k$), and $\bar{I}_m^\kappa(\mathbf{t})$ denote the set of agents assigned no more than mth positions (i.e., $\kappa_i(\mathbf{t}) \leq m$ for $i \in \bar{I}_m^\kappa(\mathbf{t})$).

2.1 Targeted properties

The primary objective is the efficient assignment of positions.

Definition 1. An assignment scheme κ is said to be *efficient* in an assignment problem C if

$$\kappa(\mathbf{t}) \in \arg \min_{(k_1,\dots,k_n):\{k_1,\dots,k_n\}=\{1,\dots,n\}} \sum_i C_{k_i}(t_i). \tag{1}$$

It is also desirable if a kind of ex post dissatisfaction can be resolved.[8]

Definition 2. A transfer scheme π is said to satisfy *envy-freeness* (EF) with an assignment scheme κ in an assignment problem C if

$$u^C(\kappa_i(\mathbf{t}), \pi_i(\mathbf{t})|t_i) \geq u^C(\kappa_j(\mathbf{t}), \pi_j(\mathbf{t})|t_i) \quad \forall i, j. \tag{2}$$

Definition 3. A transfer scheme π is said to satisfy *ex post individual rationality* (EPIR) under *equal initial shares* with an assignment scheme κ in an assignment problem C if

$$u^C(\kappa_i(\mathbf{t}), \pi_i(\mathbf{t})|t_i) \geq -\sum_k \frac{1}{n} C_k(t_i) \quad \forall i. \tag{3}$$

At the same time, the private information must be correctly elicited.

Definition 4. A transfer scheme π is said to satisfy *Bayesian incentive compatibility* (IC) with an assignment scheme κ in an assignment problem C if

$$E_{t_{-i}}[u^C(\kappa_i(\mathbf{t}), \pi_i(\mathbf{t})|t_i)] \geq E_{t_{-i}}[u^C(\kappa_i(t, t_{-i}), \pi_i(t, t_{-i})|t_i)] \quad \forall t \quad \forall i.$$

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[8] EPIR can be interpreted as "no ex post regret of participation" (Galavotti et al. [6]).

2.1.1 Additivity

The key feature of these properties (regarding transfers) is that they all have a kind of additivity.

Definition 5. A property is said to be *additive* if the satisfaction of the property with an assignment scheme κ both by a transfer scheme π in an assignment problem C and by a transfer scheme $\hat{\pi}$ in an assignment problem $\hat{C}$ implies the satisfaction with κ by $\pi + \hat{\pi}$ in $C + \hat{C}$.[9]

Remark 1. Each of EF, EPIR under equal initial shares, and IC is additive.

2.2 Monotonic Assignment Problems

We consider assignment problems with a monotonic structure, where efficiency is attained by monotonically assigning a "higher rank" position to an agent with a "higher" type.

Definition 6. An assignment problem C is said to be *monotonic* if

$$-C_1' \geq \cdots \geq -C_n'.^{10}$$

An assignment scheme κ is said to be *monotonic* if for any i and j, $t_i > t_j$ implies $\kappa_i(\mathbf{t}) < \kappa_j(\mathbf{t})$.

Remark 2. Any monotonic κ satisfies (1) in any monotonic C.

A straightforward interpretation may be that not occupational skills but a general ability mainly matters (or the skills also matter but are correlated enough with the ability), and a lack in the ability is more critical for a higher level job.[11]

2.2.1 Examples

Monotonicity is typically satisfied in queueing problems.

Example 1 (Queueing problem (Kayı and Ramaekers [9], Gershkov and Schweinzer [7]) with general time intervals)**.**

$$-C_k(t) = V - tT_k, \quad 0 \leq T_1 \leq \cdots \leq T_n, \tag{4}$$

where V is the value of some service, an agent must wait T_k units of time to get the service when he is assigned the kth position in the queue, and t is waiting cost per unit of time.[12] Notice that here time intervals are allowed to be different, while the previous literature as Kayı and Ramaekers [9]

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[9] We implicitly use the feature that $\pi + \hat{\pi}$ is also a transfer scheme, i.e., the additivity of budget-balancedness.

[10] Here and below we abuse the notation of derivative as ignoring the points where derivatives are not well determined.

[11] On the other hand, it is not restricted to $-C' \geq 0$. See Example 2 below.

[12] The generalization is not restricted to time intervals. For example, we allow exponential cases as $-C_k(t) = Ve^{-tT_k}$.

and Gershkov and Schweinzer [7] focuses on the equal intervals, $T_2 - T_1 = \cdots = T_n - T_{n-1}$. Thus, for example, it can deal with a situation where the service is provided only on weekdays but stops on weekends, $T_2 - T_1 = \cdots = T_5 - T_4 = 1$ but $T_6 - T_5 = 3$. (An agent assigned the 6th position has to wait until the beginning of next week.)

Notice that the sign of $C_k'(t)$ is allowed to change as t changes, while it is constant in queueing problems. Thus, a spatial interpretation of types and positions is also possible.

Example 2 (Non-constant direction of cost change)**.** An agent incurs the transportation cost of

$$C_k(t) = |t - s_k|^2, \quad s_1 \geq \cdots \geq s_n$$

when he is at t and assigned a task at s_k. Then,

$$C_k'(t) \begin{cases} < 0 & \text{if } t < s_k, \\ = 0 & \text{if } t = s_k, \\ > 0 & \text{if } t > s_k. \end{cases}$$

3 Results

3.1 Decomposability

Any monotonic assignment problem can be represented as a sum of homogeneous object problems.

Remark 3. For any monotonic C,

$$C = \sum_{m=1}^{n} C^m$$

with

$$-C_m^{m\prime} \geq 0 \quad \text{or} \quad C_1^m = \cdots = C_n^m \quad \forall m,$$

where C_k^m is defined as

$$C_k^m = \mathbf{1}_{k \leq m}(C_m - C_{m+1})$$

by abusing the notation as $C_{n+1} = 0$.

Thus, for additive properties, it is enough to find the possibility in homogeneous object problems.

Proposition 1 (Corollary of Remarks 2 and 3)**.** *For any set of additive properties, suppose that for some monotonic κ, for any m, in any C with*

$$C_k = \mathbf{1}_{k \leq m} C_m \quad \forall k \tag{5}$$

and

$$C'_m \geq 0 \quad or \quad C_1 = \cdots = C_n, \tag{6}$$

some transfer scheme $\pi^{m|C,\kappa}$ *satisfies all of the properties with* κ*. Then, in any monotonic* $\hat{C}$*,* κ *is efficient, and* $\sum_m \pi^{m|\hat{C}^m,\kappa}$ *satisfies all of the properties with* κ*.*

3.2 Applications

3.2.1 Envy-freeness with identical type distribution

For identical type distribution cases, Fujinaka [5] shows the compatibility with Bayesian incentive of envy-freeness in homogeneous object problems.[13]

Lemma 1 (Fujinaka [5])**.** *Suppose that* $F_i = F_j$ *for all i and j. Then, for any m, in any C with (5) and (6), for any monotonic* κ*, some transfer scheme satisfies both IC and EF with* κ*. In particular, in the notion of Proposition 1,*

$$\pi_i^{m|C,\kappa} = \left(\mathbf{1}_{\kappa_i \leq m} - \frac{m}{n}\right)\left(C_m(\bar{t}_m^\kappa) - \int_{\bar{t}_{m+1}^\kappa}^{\bar{t}_m^\kappa} C'_m(t)\bar{F}(t)dt\right),$$

where $F_i = \bar{F}$ *for all i.*

Thus, by the additivity of both properties, through Proposition 1, the compatibility is extended for assignment problems satisfying monotonicity.

Theorem 1 (Corollary of Remarks 1 and 2, Lemma 1, and Proposition 1)**.** *Suppose that* $F_i = F_j$ *for all i and j. Then, in any monotonic assignment problem, with some efficient assignment scheme, some transfer scheme satisfies both IC and EF.*

As noticed by Galavotti et al. [6], EF implies EPIR under equal initial shares.

Remark 4 (Galavotti et al. [6]). (2) implies (3).

Thus, the possibility result of Gershkov and Schweinzer [7] is extended for general time intervals with the stronger requirement of ex post individual rationality.

Corollary 1 (Extended version of Proposition 1 in Gershkov and Schweinzer [7])**.** *Suppose that* $F_i = F_j$ *for all i and j. Then, for any C with (4), with some efficient* κ*, some transfer scheme* $\pi^{Q|(T_k)_{k=1}^n,\kappa}$

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[13] Strictly speaking, Fujinaka [5] only considers $C'_m > 0$ and $m \leq n-1$. However, it is clear from his proof that $C'_m(t) = 0$ is allowed, and for $m = n$, it clearly suffices by $\pi = \mathbf{0}$. Kitahara [10] also only considers $m \leq n-1$, but the same discussion applies.

satisfies both IC and EPIR under equal initial shares in C. In particular, consider any monotonic κ *and*

$$\pi_i^{Q|(T_k)_{k=1}^n,\kappa} = \sum_{m=1}^{n-1} \left(\mathbf{1}_{\kappa_i \le m} - \frac{m}{n}\right)(T_m - T_{m+1})\left(\bar{t}_m^\kappa - \int_{\bar{t}_{m+1}^\kappa}^{\bar{t}_m^\kappa} \bar{F}(t)dt\right),$$

where $F_i = \bar{F}$ *for all i.*

3.2.2 Ex post individual rationality under equal initial shares with many agents

The extention of Kitahara [10], showing the compatibility of EPIR under equal initial shares in homogeneous object problems with sufficiently many agents, similarly follows.

Lemma 2 (Kitahara [10]). *For any R, there exists* $\bar{n}$ *such that if* $n \ge \bar{n}$ *and*

$$\sup_t \frac{\max_i \frac{1-F_i(t)}{F_i(t)}}{\min_i \frac{1-F_i(t)}{F_i(t)}} \le R, \tag{7}$$

then for any m, in any C with (5) and (6), for any monotonic κ*, some transfer scheme satisfies both IC and EPIR under equal initial shares with* κ*. In particular, then for any m, for some* (ϕ^m, ψ^m, ξ^m)*,*

$$\begin{aligned} 0 = &- \sum_{I_m:|I_m|=m, i\in I_m} \Pr(\bar{I}_m^\kappa(\mathbf{t}) = I_m, \kappa_i(\mathbf{t}) = m | t_i = t)\left(1 - \frac{m}{n}\right)(1 - \phi_{I_m}^m(t)) \\ &+ \sum_{j \ne i} \Pr(\kappa_i(\mathbf{t}) = m+1, \kappa_j(\mathbf{t}) \ge m+2 | t_i = t)\frac{m}{n}\frac{\xi_j^m(t)}{n-1} \\ &- \Pr(\kappa_i(\mathbf{t}) \le m-1 | t_i = t)\left(1 - \frac{m}{n}\right)\psi_i^m(t) + \Pr(\kappa_i(\mathbf{t}) \ge m+2 | t_i = t)\frac{m}{n}\xi_i^m(t) \\ &- \sum_{j \ne i} \Pr(\kappa_i(\mathbf{t}) = m, \kappa_j(\mathbf{t}) \le m-1 | t_i = t)\left(1 - \frac{m}{n}\right)\frac{\psi_j^m(t)}{n-1} \\ &+ \sum_{I_m:|I_m|=m, i\notin I_m} \Pr(\bar{I}_m^\kappa(\mathbf{t}) = I_m, \kappa_i(\mathbf{t}) = m+1 | t_i = t)\frac{m}{n}\phi_{I_m}^m(t) \quad \forall i \end{aligned}$$

and

$$0 \le \phi_{I_m}^m(t) \le 1 \quad \forall I_m : |I_m| = m \quad \textit{and} \quad 0 \le \psi_i^m(t), \xi_i^m(t) \le 1 \quad \forall i$$

for all t,[14] *by which in the notion of Proposition 1,*

$$\begin{aligned} \pi_i^{m|C,\kappa}(\mathbf{t}) = &\left(\mathbf{1}_{\kappa_i(\mathbf{t}) \le m} - \frac{m}{n}\right)\left(C_m(\bar{t}_m^\kappa(\mathbf{t})) - \int_{\bar{t}_{m+1}^\kappa(\mathbf{t})}^{\bar{t}_m^\kappa(\mathbf{t})} C_m'(t)\phi_{\bar{I}_m^\kappa(\mathbf{t})}^m(t)dt\right) \\ &+ \mathbf{1}_{\kappa_i(\mathbf{t}) \le m-1}\int_{\bar{t}_m^\kappa(\mathbf{t})}^{t_i} C_m'(t)\psi_i^m(t)dt + \mathbf{1}_{\kappa_i(\mathbf{t}) \ge m+2}\int_{t_i}^{\bar{t}_{m+1}^\kappa(\mathbf{t})} C_m'(t)\xi_i^m(t)dt \end{aligned}$$

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[14] In particular, the former condition corresponds to IC as the first order condition, and to the EPIR by implying correspondingly bounded changes in his transfer.

$$-\sum_{j\neq i}\frac{\mathbf{1}_{\kappa_j(\mathbf{t})\leq m-1}\int_{\bar{t}^{\kappa}_m(\mathbf{t})}^{t_j}C'_m(t)\psi^m_j(t)dt+\mathbf{1}_{\kappa_j(\mathbf{t})\geq m+2}\int_{t_j}^{\bar{t}^{\kappa}_{m+1}(\mathbf{t})}C'_m(t)\xi^m_j(t)dt}{n-1}.$$

Theorem 2 (Corollary of Remarks 1 and 2, Lemma 2, and Proposition 1)**.** *For any R, there exists $\bar{n}$ such that if $n \geq \bar{n}$ and (7), then in any monotonic assignment problem, with some efficient assignment scheme, some transfer scheme satisfies both IC and EPIR under equal initial shares.*

4 Conclusion

We notify that if an assignment problem has a monotonic structure and properties considered satisfy a kind of additivity, then the investigation task for the possibility of the properties in the original problem can be decomposed into those in the corresponding homogeneous object problems. Since the latter is in general much easier than the former, not only we may be able to execute the simplified tasks, but also there may exist past (unintended) results to be utilized. In fact, we provide two results only by referring corresponding existing results. Further results are expected in the future through both ways, by executing the simplified tasks and by finding other existing results to be utilized.

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