Strategy-Proof Cost Sharing under Increasing Returns: Improvement of the Supremal Welfare Loss

Kazuhiko Hashimoto[*]
JSPS Research Fellow, Kobe University

and

Hiroki Saitoh[†]
Faculty of Economics, Kyoto Gakuen University

August 3, 2010

Abstract

We consider an allocation problem of an indivisible good and money. Agents demand the indivisible good and consume it at most one. The indivisible good is produced with a concave cost function. The agents must share the cost among them. We construct a new class of rules, called w-hybrid rules, and characterize them by strategy-proofness, budget-balance, anonymity, envy-freeness, consumer sovereignty and non-bossiness, except for measure zero case. We also show that w-hybrid rules improve the supremal welfare loss compared with the average cost pricing rule [Moulin (1999) and Moulin and Shenker (2001)].

keywords:
JEL Classification numbers:

---

[*] Corresponding address: Graduate School of Economics, Kobe University, 2-1, Rokkodai, Nada, Kobe, Hyogo, 657-8501, JAPAN. E-mail: kazu-hashimoto@econ.kobe-u.ac.jp

[†] Corresponding address: Faculty of Economics, Kyoto Gakuen University, 1-1, Nanjyo Otani, Sogabe-cho, Kameoka, Kyoto 621-8555, JAPAN. E-mail: saito-h@kyotogakuen.ac.jp

1 Introduction

We consider an allocation problem of an indivisible good and money. Agents demand the indivisible good and consume it at most one. The indivisible good is produced with a concave cost function. The agents must share the cost among them.

When the cost function is concave, only one firm survives in the industry, it sets a monopoly price to the good, and the economy gets into inefficiency. We aim to solve this problem by the mechanism design approach.

We consider incentive compatible mechanisms (or rules) that elicit from each agent his preference. We focus on strategy-proof rules, where truthful report of one's preference is always a dominant strategy.

It is well-known that no strategy-proof rule attains Pareto-efficient allocations (except for a trivial case such as a linear cost function) [Holmström (1979)]. So, we must give up the first best efficiency. Holmström (1979) has got a more strong result which suggests that the class of Groves rules is the only one of strategy-proof rules satisfying decision-efficiency. A rule is decision-efficient if it selects an allocation of good that maximizes the aggregate valuation minus production cost. This condition has been often considered as a one of the second best efficiency. However, Groves rules are emerged budget-surplus or budget-deficit, or both. When budget-surplus emerges, we must waste the surplus to keep strategy-proofness. It may be difficult for a society to accept to do this. When budget-deficit emerges, we must offset the shortage from someone. If it is collected from the agents, then strategy-proofness violates. Thus, it is important to consider the rules satisfying budget-balance as the second best efficiency.

The average cost pricing rule is one of the rules satisfying strategy-proofness and budget-balance. It also satisfies coalitionally strategy-proofness, voluntary participation, consumer sovereignty and non-negative payment, [Moulin (1999)]. Among the class of rules satisfying them, the average cost pricing rule achieves the least supremal welfare loss [Moulin and Shenker (2001)]. The supremal welfare loss of a rule is the supremal value of the welfare loss occurring by the rule.

In a binary excludable public good model that is a special case of this problem, Ohseto (2005) has extended the average cost pricing rule (called the serial rule in that context) and constructed a new class of rules, called the augmented serial rules. They satisfy strategy-proofness, budget-balance, envy-freeness, consumer sovereignty (called access independence in that context) and non-bossiness, and improve the supremal welfare loss compared with the serial rule.

We use Ohseto's idea to improve the supremal welfare loss on the general case. We construct a new class of rules, called w-hybrid rules, and characterize them by strategy-proofness, budget-balance, anonymity, envy-freeness, consumer sovereignty and non-bossiness, except for measure zero case. We also show that w-hybrid rules improve the supremal welfare loss compared with the average cost pricing rule.

In Section 2, we set up the model. In Section 3, we define the axioms. In Section 4, we introduce the rules. In Section 5, we state the results. In Appendix, we give all proofs.

2 Model

Let $N = \{1, \ldots, n\}$ be the set of *agents*. There are a *good* with indivisible units and *money*. Each agent wants at most one unit of the good. The cost of producing $k \in \{0, 1, \ldots, n\}$ units of the good is denoted by $c_k \in \mathbb{R}_+$, which is shared among the agents with money. We assume that the cost function is nondecreasing and concave and that there is no fixed cost. Formally, we assume that for all $k, k' \in \{1, \ldots, n\}$, if $k < k'$, then $c_k \leq c_{k'}$ and $c_k - c_{k-1} \geq c_{k'} - c_{k'-1}$, and that $c_0 = 0$. To avoid a trivial cost function, we also assume that $c_1 > 0$.

Each agent $i \in N$ is assigned a good $q_i \in \{0, 1\}$ and his cost share $p_i \in \mathbb{R}$. Each agent $i \in N$ has a *quasi-linear preference* on $\{0, 1\} \times \mathbb{R}$, i.e., if agent i's *valuation* for the good is $v_i \in V \equiv \mathbb{R}_+$, then his preference for $(q_i, p_i) \in \{0, 1\} \times \mathbb{R}$ is represented by

$$u_i((q_i, p_i); v_i) = v_i q_i - p_i.$$

A list $v \equiv (v_i)_{i \in N} \in V^n$ is a *valuation profile.* Let $Z \equiv \{(q, p) = (q_i, p_i)_{i \in N} \in \{0, 1\}^n \times \mathbb{R}^n : \sum_{i \in N} p_i = c_k$, where $k = \sum_{i \in N} q_i\}$ be the set of *feasible allocations.*[1] A *rule* is a function f from V^n to Z. Given a rule f and a valuation profile $v \in V^n$, we denote by $f_i(v) \equiv (q_i(v), p_i(v)) \in \{0, 1\} \times \mathbb{R}$, agent i's allocation under $f(v)$, and by $q(v)$ and $p(v)$, the profile $(q_i(v))_{i \in N}$ and $(p_i(v))_{i \in N}$, respectively. Given $v \in V^n$ and $N' \subseteq N$, $v_{N'} \in V^{\#N'}$ and $v_{-N'} \in V^{\#N \setminus N'}$ denote $(v_j)_{j \in N'}$ and $(v_j)_{j \notin N'}$, respectively.

3 Axioms

We consider rules satisfying the following conditions. At first, we introduce an incentive compatible condition that it is a dominant strategy for any agent to report his true valuation.

---

[1] The condition $\sum_{i \in N} p_i = c_k$ has often been called *budget balance.*

Definition A rule f is *strategy-proof* if for all $v \in V^n$, all $i \in N$ and all $v'_i \in V$, we have $u_i(f_i(v); v_i) \geq u_i(f_i(v'_i, v_{-i}); v_i)$.

Coalitional strategy-proofness[2] says that if no member in $N' \subset N$ is made worse off by changing from $v_{N'}$ to $v'_{N'}$, then no member of N' is made better off.

Definition A rule f is *coalitionally strategy-proof* if for all $v \in V^n$, for all $N' \subset N$ and for all $v'_{N'} \in V^{\#N'}$, we have that if for all $i \in N'$, $u_i(f_i(v'_{N'}, v_{-N'}); v_i) \geq u_i(f_i(v); v_i)$, then for all $i \in N'$, $u_i(f_i(v'_{N'}, v_{-N'}); v_i) = u_i(f_i(v); v_i)$.

Next, we introduce two conditions of equity. *Anonymity* says that a rule is defined independently of the names of the agents. For any permutation (i.e., bijection) $\pi : N \rightarrow N$ and $v \in V^n$, let $v^\pi \in V^n$ be the valuation profile of which the $\pi(i)$-th coordinate is the i-th one of v.

Definition A rule f is *anonymous* if for any $v \in V^n$, any permutation π and any $i \in N$, we have $f_{\pi(i)}(v^\pi) = f_i(v)$.

Envy-freeness says that the rule assigns allocations under which no agent prefer another agent's allocation to his own.

Definition A rule f is *envy-free* if for all $v \in V^n$ and all $i, j \in N$, we have $u_i(f_i(v); v_i) \geq u_i(f_j(v); v_i)$.

Voluntary Participation says that the rule assigns allocations under which no agent prefer the status quo allocation to his own.

Definition A rule f satisfies *voluntary participation* if for all $v \in V^n$ and all $i \in N$, we have $u_i(f_i(v); v_i) \geq u_i((0, 0); v_i)$.

Consumer Sovereignty says that each agent has the option to get the good and the option not to get it regardless of the other agents' preferences.

Definition A rule f satisfies *consumer sovereignty* if for all $i \in N$, there exist $v_i, v'_i \in V$ such that for all $v_{-i} \in V^{n-1}$, we have $q_i(v) = 1$ and $q_i(v'_i, v_{-i}) = 0$.

Definition A rule f satisfies *non-negative payment* if for all $v \in V^n$ and for all $i \in N$, we have $p_i(v) \geq 0$.

---

[2] This condition is often called *strong group strategy-proofness*, for example, by Mutuswami (2005).

Finally, *non-bossiness* says that if the allocation of an agent remains unchanged by changing his preference, then the allocations of the others also do.

Definition A rule f is *non-bossy* if for all $v \in V^n$, all $i \in N$, and all $v'_i \in V$, we have that $f_i(v) = f_i(v'_i, v_{-i})$ implies $f(v) = f(v'_i, v_{-i})$.

4 Rules

Definition For all $t = 1, \ldots, n$, let $L_t(v) = \{i \in N | v_i \geq \frac{c_t}{t}\}$. The *largest unanimous consumers' coalition* at $v \in V^n$, denoted by $A(v)$, is defined by

1. if there exists $a \in \{1, \ldots, n\}$ such that $\# L_a(v) = a$, and for all integer t ($a < t \leq n$), $\# L_t(v) < t$, then $A(v) = L_a(v)$, and
2. otherwise $A(v) = \emptyset$.

Definition A rule f is the *average cost pricing rule* if for all $v \in V^n$, if $i \in A(v)$ then $f_i(v) = (1, \frac{c_a}{a})$, and if $i \notin A(v)$ then $f_i(v) = (0, 0)$, where $a = \# A(v)$.

Definition For all $t = 1, \ldots, n$, let $S_t(v) = \{i \in N | v_i \leq \frac{c_n - c_{n-t}}{t}\}$. The *largest unanimous non-consumers' coalition* at $v \in V^n$, denoted by $B(v)$, is defined by

1. if there exists $b \in \{1, \ldots, n\}$ such that $\# S_b(v) = b$, and for all integer t ($b < t \leq n$), $\# S_t(v) < t$, then $B(v) = S_b(v)$, and
2. otherwise $B(v) = \emptyset$.

Definition A rule f is the *dual rule* if for all $v \in V^n$, if $i \notin B(v)$ then $f_i(v) = (1, \frac{c_n}{n})$, and if $i \in B(v)$ then $f_i(v) = (0, \frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}])$. where $b = \# B(v)$.

Definition Let $w \in \{1, \ldots, n\}$. A rule f is the *w-hybrid rule* if for all $v \in V^n$,

1. when $\# L_n(v) < w$, if $i \in A(v)$ then $f_i(v) = (1, \frac{c_a}{a})$, and if $i \notin A(v)$ then $f_i(v) = (0, 0)$,
2. when $\# L_n(v) \geq w$, if $i \notin B(v)$ then $f_i(v) = (1, \frac{c_n}{n})$, and if $i \in B(v)$ then $f_i(v) = (0, \frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}])$.

where $a = \# A(v)$ and $b = \# B(v)$.

Remark 1 Note that when w equals to n, w-hybrid rule is welfare equivalent to the average cost pricing rule. If some agent's valuation is not larger than or equal to $\frac{c_n}{n}$, that is, $\#L_n(v) < n$, then each agent's allocation is determined by the definition of the average cost pricing rule. If each agent's valuation is larger than or equal to $\frac{c_n}{n}$, that is $\#L_n(v) \geq n$, and some agent's valuation is larger than $\frac{c_n}{n}$, then $B(v) = \emptyset$. Thus each agent's allocation is $(1, \frac{c_n}{n})$ which is the same allocation as the one determined by the average cost pricing rule. If each agent's valuation is equal to $\frac{c_n}{n}$, that is $\#L_n(v) \geq n$, then $B(v) = S_n(v) = N$, and each agent's allocation is $(0, 0)$ which is indifferent to the allocation of the average cost pricing rule. Note also that when w equals to 1, w-hybrid rule is identical to the dual rule.

Remark 2 Let w be such that $1 < w < n$. w-hybrid rule does not necessarily satisfy coalitional strategy-proofness. For example, let $n \geq 3$ and w be such that $1 < w < n$, and assume that $\frac{c_w}{w} > \frac{c_n}{n} > 0$. Let $v_1 = \frac{c_n}{n} + \epsilon$ (for sufficiently small $\epsilon > 0$), $v_2 = \cdots = v_w = \frac{c_n}{n}$, and $v_{w+1} = \cdots = v_n = 0$ Then, $f_1(v) = \cdots = f_w(v) = (1, \frac{c_n}{n})$ and $f_{w+1}(v) = \cdots = f_n(v) = (0, \frac{1}{n-w}[c_w - w\frac{c_n}{n}])$. If agent 2 announces his valuation as 0, then all agent's allocations change to $(0, 0)$. Agent 2 is indifferent in both case, however, agent n strictly prefers $(0, 0)$ to $(0, \frac{1}{n-w}[c_w - w\frac{c_n}{n}])$.

5 Results

We state results. All proofs are in Appendix.

Theorem 1 For each $w \in \{1, \ldots, n\}$, w-hybrid rule satisfies strategy-proofness, anonymity, envy-freeness, consumer sovereignty and non-bossiness.

Theorem 2 If f satisfies strategy-proofness, anonymity, envy-freeness, consumer sovereignty and non-bossiness, then for some $w = 1, \ldots, n$, f coincides with w-hybrid rule on $(V \setminus (\{\frac{c_t}{t}\}_{t=1}^{w-1} \cup \{\frac{c_n - c_{n-t}}{t}\}_{t=1}^{n-w} \cup \{\frac{c_n}{n}\}))^n$.

Definition f is *weakly Pareto-dominated* by $\hat{f}$ if for any $v \in V^n$ and for any $i \in N$, we have $u_i(\hat{f}_i(v); v_i) \geq u_i(f_i(v); v_i)$.

Theorem 3 If f satisfies strategy-proofness, anonymity, envy-freeness, consumer sovereignty and non-bossiness, then for some $w = 1, \ldots, n$, f is weakly Pareto-dominated by w-hybrid rule on $(V \setminus \{\frac{c_n}{n}\})^n$.

Definition An allocation of good q^* is *efficient* at $v \in V^n$ if there exists no q such that $\sum_{i=1}^n v_i q_i - c_k > \sum_{i=1}^n v_i q_i^* - c_{k^*}$, where $k = \sum_{i=1}^n q_i$ and $k^* = \sum_{i=1}^n q_i^*$.

Definition For any rule f, the *supremal welfare loss* of f is defined by

$$\sup_{v\in V^n}[\sum_{i=1}^{n} v_i q_i^* - c_{k^*} - (\sum_{i=1}^{n} v_i q_i(v) - c_k)],$$

where q^* is an efficient allocation of good at v, and $k^* = \sum_{i=1}^{n} q_i^*$ and $k = \sum_{i=1}^{n} q_i(v)$.

Fact (Moulin (1999) and Moulin and Shenker (2001)) Among the rules satisfying coalitionally strategy-proofness, voluntary participation, consumer sovereignty and non-negative payment, the average cost pricing rule attains the minimal value of the supremal welfare loss which can be calculated by

$$\sum_{i=1}^{n} \frac{c_i}{i} - c_n.$$

Theorem 4 Let $w = 1, \ldots, n$. The supremal welfare loss of w-hybrid rule is $\max\{\sum_{i=1}^{w-1} \frac{c_i}{i} + (n-(w-1))\frac{c_n}{n} - c_n, -(w\frac{c_n}{n} + \sum_{i=w+1}^{n} \frac{c_n - c_{i-1}}{n-(i-1)} - c_n)\}$.

Remark 3 Given $w = 1, \ldots, n$. The order between $\sum_{i=1}^{w-1} \frac{c_i}{i} + (n-(w-1))\frac{c_n}{n} - c_n$ and $-(w\frac{c_n}{n} + \sum_{i=w+1}^{n} \frac{c_n - c_{i-1}}{n-(i-1)} - c_n)$ depends on the cost function under consideration. For example, let $n = 3$ and $w = 2$. Then, in the case of $c_1 = c_2 = c_3 = 10$, the former is $\frac{20}{3}$, and the later is $\frac{10}{3}$. In the case of $c_1 = 5$ and $c_2 = c_3 = 10$, the former is $\frac{5}{3}$, and the later is $\frac{10}{3}$.

Remark 4 The order between the supremal welfare losses of the average cost pricing rule and of the dual rule depends on the cost function under consideration. For example, let $n = 3$. Then, in the case of $c_1 = c_2 = c_3 = 10$, the supremal welfare loss of the average cost pricing rule (that is, $w = n = 3$) is $\frac{50}{6}$, and the supremal welfare loss of the dual rule (that is, $w = 1$) is $\frac{40}{6}$. In the case of $c_1 = 5$ and $c_2 = c_3 = 10$, the supremal welfare loss of the average cost pricing rule is $\frac{20}{6}$, and the supremal welfare loss of the dual rule is $\frac{25}{6}$.

Remark 5 The first component of the supremal welfare loss of w-hybrid rule in Theorem 4, that is, $\sum_{i=1}^{w-1} \frac{c_i}{i} + (n-(w-1))\frac{c_n}{n} - c_n$, is non-decreasing in w. On the other hand, the second component, that is, $-(w\frac{c_n}{n} + \sum_{i=w+1}^{n} \frac{c_n - c_{i-1}}{n-(i-1)} - c_n)$, is non-increasing in w.

Corollary 1 The supremal welfare loss of the average cost pricing rule (that is, n-hybrid rule) is larger than or equal to the one of $n-1$-hybrid rule. The supremal welfare loss of the dula rule (that is, 1-hybrid rule) is larger than or equal to the one of 2-hybrid rule.

Remark 6 By Corollary 1, $n-1$-hybrid rule is at least better than the average cost pricing rule (n-hybrid rule) from the point of view of the supremal welfare loss. However, we have not had a simple condition for finding the optimal $w \in \{1, \dots, n\}$ yet.

6 Appendix

6.1 Proof of Theorem 1:

Proof of Theorem 1: Let $w \in \{1, \dots, n\}$, and f be w-hybrid rule.

(1) Strategy-proofness: Let $v \in V^n$, $i \in N$ and $v_i' \in V$. We divide the arguments into four cases.

Case 1. $\#L_n(v) < w$ and $i \in A(v)$.
Let $a = \#A(v)$. This case means that

$$f_i(v) = (1, \frac{c_a}{a}).$$

Since $i \in A(v)$, we must have

$$v_i \geq \frac{c_a}{a} \geq \frac{c_n}{n}.$$

Since $v_i \geq \frac{c_n}{n}$ and $\#L_n(v) < w$, we have

$$\#L_n(v_i', v_{-i}) \leq \#L_n(v) < w$$

We divide this case into two sub-cases.

Sub-case 1-1. $\#L_n(v_i', v_{-i}) < w$ and $i \in A(v_i', v_{-i})$.
Let $a' = \#A(v_i', v_{-i})$. This sub-case means that

$$f_i(v_i', v_{-i}) = (1, \frac{c_{a'}}{a'}).$$

We claim $a = a'$. Assume, to the contrary, that $a \neq a'$. Since $i \in A(v_i', v_{-i})$, we must have

$$v_i' \geq \frac{c_{a'}}{a'}.$$

If $a < a'$, then we have

$$v_i \geq \frac{c_a}{a} \geq \frac{c_{a'}}{a'}.$$

Since $v_i \geq \frac{c_{a'}}{a'}$, $v_i' \geq \frac{c_{a'}}{a'}$ and the other members do not change their valuations, we must have

$$\#L_{a'}(v) = a'.$$

However, from the definition of $A(\cdot)$, for all integer t ($a < t \leq n$), we must have $\#L_t(v) < t$. This is a contradiction. If $a > a'$, then we have

$$v'_i \geq \frac{c_{a'}}{a'} \geq \frac{c_a}{a}.$$

Since $v_i \geq \frac{c_a}{a}$, $v'_i \geq \frac{c_a}{a}$ and the other members do not change their valuations, we must have

$$\#L_a(v'_i, v_{-i}) = a.$$

However, from the definition of $A(\cdot)$, for all integer t ($a' < t \leq n$), we must have $\#L_t(v'_i, v_{-i}) < t$. This is a contradiction. So, we have $a = a'$. Thus the agent i does not manipulate at v by v'_i.

Sub-case 1-2. $\#L_n(v'_i, v_{-i}) < w$ and $i \notin A(v'_i, v_{-i})$.
This sub-case means that

$$f_i(v'_i, v_{-i}) = (0, 0).$$

Since $v_i \geq \frac{c_a}{a}$, we have

$$v_i - \frac{c_a}{a} \geq 0.$$

Thus the agent i does not manipulate at v by v'_i.

Case 2. $\#L_n(v) < w$ and $i \notin A(v)$.
This case means that

$$f_i(v) = (0, 0).$$

We divide this case into four sub-cases.

Sub-case 2-1. $\#L_n(v'_i, v_{-i}) < w$ and $i \in A(v'_i, v_{-i})$.
Let $a' = \#A(v'_i, v_{-i})$. This sub-case means that

$$f_i(v'_i, v_{-i}) = (1, \frac{c_{a'}}{a'}).$$

We claim $v_i < \frac{c_{a'}}{a'}$. Assume, to the contrary, that $v_i \geq \frac{c_{a'}}{a'}$. Since $i \in A(v'_i, v_{-i})$, we must have

$$v'_i \geq \frac{c_{a'}}{a'}.$$

Since $v_i \geq \frac{c_{a'}}{a'}$, $v'_i \geq \frac{c_{a'}}{a'}$ and the other members do not change their valuations, we must have

$$\#L_{a'}(v) = a'.$$

It implies $A(v) \neq \emptyset$. Let $a = \#A(v)$. Since $i \notin A(v)$, it holds that

$$v_i < \frac{c_a}{a}.$$

From the definition of $A(\cdot)$, for all integer t ($a < t \leq n$), we must have $\#L_t(v) < t$. So, it holds that $a \geq a'$. However, it implies

$$v_i \geq \frac{c_{a'}}{a'} \geq \frac{c_a}{a}.$$

This is a contradiction. So, we have $v_i < \frac{c_{a'}}{a'}$. Then it holds that

$$0 > v_i - \frac{c_{a'}}{a'}.$$

Thus the agent i does not manipulate at v by v'_i.

Sub-case 2-2. $\#L_n(v'_i, v_{-i}) < w$ and $i \notin A(v'_i, v_{-i})$.
This sub-case means that

$$f_i(v'_i, v_{-i}) = (0, 0).$$

Thus the agent i does not manipulate at v by v'_i.

Sub-case 2-3. $\#L_n(v'_i, v_{-i}) \geq w$ and $i \notin B(v'_i, v_{-i})$.
This sub-case means that

$$f_i(v'_i, v_{-i}) = (1, \frac{c_n}{n}).$$

Since $\#L_n(v) < w$ and $\#L_n(v'_i, v_{-i}) \geq w$, we must have

$$v_i < \frac{c_n}{n}.$$

Then we have

$$0 > v_i - \frac{c_n}{n}.$$

Thus the agent i does not manipulate at v by v'_i.

Sub-case 2-4. $\#L_n(v'_i, v_{-i}) \geq w$ and $i \in B(v'_i, v_{-i})$.
Since $\#L_n(v) < w$ and $\#L_n(v'_i, v_{-i}) \geq w$, we must have

$$v'_i \geq \frac{c_n}{n},$$

Since $i \in B(v'_i, v_{-i})$, we must have

$$v'_i = \frac{c_n}{n},$$

So, it holds that

$$B(v'_i, v_{-i}) = S_n(v'_i, v_{-i}),$$

that is, $b = n$. Thus $f_i(v'_i, v_{-i}) = (0, 0)$, and the agent i does not manipulate at v by v'_i.

Case 3. $\#L_n(v) \geq w$ and $i \notin B(v)$.
This case means that

$$f_i(v) = (1, \frac{c_n}{n}).$$

We divide this case into four sub-cases.

Sub-case 3-1. $\#L_n(v'_i, v_{-i}) < w$ and $i \in A(v'_i, v_{-i})$.
Since $\#L_n(v) \geq w$ and $\#L_n(v'_i, v_{-i}) < w$, we must have

$$v'_i < \frac{c_n}{n},$$

which contradicts to $i \in A(v'_i, v_{-i})$. Thus this sub-case does not occur.

Sub-case 3-2. $\#L_n(v'_i, v_{-i}) < w$ and $i \notin A(v'_i, v_{-i})$.
This sub-case means that

$$f_i(v'_i, v_{-i}) = (0, 0).$$

Since $\#L_n(v) \geq w$ and $\#L_n(v'_i, v_{-i}) < w$, we must have

$$v_i \geq \frac{c_n}{n}.$$

Then we have

$$v_i - \frac{c_n}{n} \geq 0.$$

Thus the agent i does not manipulate at v by v'_i.

Sub-case 3-3. $\#L_n(v'_i, v_{-i}) \geq w$ and $i \notin B(v'_i, v_{-i})$.
This sub-case means that

$$f_i(v'_i, v_{-i}) = (1, \frac{c_n}{n}).$$

Thus the agent i does not manipulate at v by v'_i.

Sub-case 3-4. $\#L_n(v'_i, v_{-i}) \geq w$ and $i \in B(v'_i, v_{-i})$.
Let $b' = \#B(v'_i, v_{-i})$. This case means that

$$f_i(v'_i, v_{-i}) = (0, \frac{1}{b'}[c_{n-b'} - (n - b')\frac{c_n}{n}]).$$

We claim $v_i > \frac{c_n - c_{n-b'}}{b'}$. Assume, to the contrary, that $v_i \leq \frac{c_n - c_{n-b'}}{b'}$. Since $i \in B(v'_i, v_{-i})$, we must have

$$v'_i \leq \frac{c_n - c_{n-b'}}{b'}.$$

Since $v_i \leq \frac{c_n - c_{n-b'}}{b'}$, $v'_i \leq \frac{c_n - c_{n-b'}}{b'}$ and the other members do not change their valuations, we must have

$$\#S_{b'}(v) = b'.$$

It implies $B(v) \neq \emptyset$. Let $b = \#B(v)$. Since $i \notin B(v)$, it holds that

$$v_i > \frac{c_n - c_{n-b}}{b}.$$

From the definition of $B(\cdot)$, for all integer t ($b < t \leq n$), we must have $\#S_t(v) < t$. So, it holds that $b \geq b'$. However, it implies

$$v_i \leq \frac{c_n - c_{n-b'}}{b'} \leq \frac{c_n - c_{n-b}}{b}.$$

This is a contradiction. So, we have $v_i > \frac{c_n - c_{n-b'}}{b'}$. Then it holds that

$$v_i - \frac{c_n}{n} > -\frac{1}{b'}[c_{n-b'} - (n-b')\frac{c_n}{n}].$$

Thus the agent i does not manipulate at v by v_i'.

Case 4. $\#L_n(v) \geq w$ and $i \in B(v)$.
Let $b = \#B(v)$. This case means that

$$f_i(v) = (0, \frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}]).$$

Since $i \in B(v)$, we must have

$$v_i \leq \frac{c_n - c_{n-b}}{b}.$$

We divide this case into four sub-cases.

Sub-case 4-1. $\#L_n(v_i', v_{-i}) < w$ and $i \in A(v_i', v_{-i})$.
Since $\#L_n(v) \geq w$ and $\#L_n(v_i', v_{-i}) < w$, we must have

$$v_i' < \frac{c_n}{n}$$

which contradicts to $i \in A(v_i', v_{-i})$. Thus this sub-case does not occur.

Sub-case 4-2. $\#L_n(v_i', v_{-i}) < w$ and $i \notin A(v_i', v_{-i})$.
This sub-case means that

$$f_i(v_i', v_{-i}) = (0, 0).$$

We claim $b = n$. Since $\#L_n(v) \geq w$ and $\#L_n(v_i', v_{-i}) < w$, we must have

$$v_i \geq \frac{c_n}{n}.$$

Since $i \in B(v)$, we must have

$$v_i = \frac{c_n}{n}.$$

So, it holds that

$$B(v) = S_n(v),$$

that is, $b = n$. Thus $f_i(v) = (0, 0)$, and the agent i does not manipulate at v by v'_i.

Sub-case 4-3. $\#L_n(v'_i, v_{-i}) \geq w$ and $i \notin B(v'_i, v_{-i})$.
This sub-case means that

$$f_i(v'_i, v_{-i}) = (1, \frac{c_n}{n}).$$

Since $v_i \leq \frac{c_n - c_{n-b}}{b}$, we must have

$$-\frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}] \geq v_i - \frac{c_n}{n}.$$

Thus the agent i does not manipulate at v by v'_i.

Sub-case 4-4. $\#L_n(v'_i, v_{-i}) \geq w$ and $i \in B(v'_i, v_{-i})$.
Let $b' = \#B(v'_i, v_{-i})$. This sub-case means that

$$f_i(v'_i, v_{-i}) = (0, \frac{1}{b'}[c_{n-b'} - (n-b')\frac{c_n}{n}]).$$

We claim $b = b'$. Assume, to the contrary, that $b \neq b'$. Since $i \in B(v'_i, v_{-i})$, we must have

$$v'_i \leq \frac{c_n - c_{n-b'}}{b'}.$$

If $b < b'$, then we have

$$v_i \leq \frac{c_n - c_{n-b}}{b} \leq \frac{c_n - c_{n-b'}}{b'}.$$

Since $v_i \leq \frac{c_n - c_{n-b'}}{b'}$, $v'_i \leq \frac{c_n - c_{n-b'}}{b'}$ and the other members do not change their valuations, we must have

$$\#S_{b'}(v) = b'.$$

However, from the definition of $B(\cdot)$, for all integer t ($b < t \leq n$), we must have $\#S_t(v) < t$. This is a contradiction. If $b > b'$, then we have

$$v'_i \leq \frac{c_n - c_{n-b'}}{b'} \leq \frac{c_n - c_{n-b}}{b}.$$

Since $v_i \leq \frac{c_n - c_{n-b}}{b}$, $v'_i \leq \frac{c_n - c_{n-b}}{b}$ and the other members do not change their valuations, we must have

$$\#S_b(v'_i, v_{-i}) = b.$$

However, from the definition of $B(\cdot)$, for all integer t ($b' < t \leq n$), we must have $\#S_t(v'_i, v_{-i}) < t$. This is a contradiction. So, we have $b = b'$. Thus the agent i does not manipulate at v by v'_i.

(2) Anonymity: Omission.

(3) Envy-freeness: Let $v \in V^n$. We divide the argument into two cases.

Case 1. $\#\{i \in N | v_i \geq \frac{c_n}{n}\} < w$.

Let $A(v)$ be the largest unanimous consumers' coalition at v, and $a = \#A(v)$. Since the payments are equal across the consumers (non-consumers), the consumers (non-consumers) do not envy each other. So, we consider $0 < a < n$. Let $i, j \in N$ be such that $f_i(v) = (1, \frac{c_a}{a})$ and $f_j(v) = (0, 0)$. Since $v_i \geq \frac{c_a}{a}$, it holds that

$$v_i - \frac{c_a}{a} \geq 0.$$

Since $v_j < \frac{c_a}{a}$, it holds that

$$0 > v_j - \frac{c_a}{a}.$$

Thus, no agent envies others.

Case 2. $\#\{i \in N | v_i \geq \frac{c_n}{n}\} \geq w$.

Let $B(v)$ be the largest unanimous non-consumers' coalition at v, and $b = \#B(v)$. Since the payments are equal across the consumers (non-consumers), the consumers (non-consumers) do not envy each other. So, we consider $0 < b < n$. Since the agents which consume (do not consume) the good pay the same amounts, they do not envy each other. Let $i, j \in N$ be such that $f_i(v) = (1, \frac{c_n}{n})$ and $f_j(v) = (0, \frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}])$. Since $v_i > \frac{c_n - c_{n-b}}{b}$, it holds that

$$v_i - \frac{c_n}{n} > -\frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}].$$

Since $v_j \leq \frac{c_n - c_{n-b}}{b}$, it holds that

$$-\frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}] \geq v_j - \frac{c_n}{n}.$$

Thus, no agent envies others.

(4) consumer sovereignty: Omission.

(5) non-bossiness: Omission. □

6.2 Proof of Theorem 2:

Let f be a rule satisfying strategy-proofness, anonymity, envy-freeness, consumer sovereignty and non-bossiness. At first, we prove several lemmas. Lemma 1 states that if for two valuation profiles the rule assigns the same good allocation, then it also assigns the same cost share.

Lemma 1 For all $v, v' \in V^n$, if $q(v) = q(v')$, then $p(v) = p(v')$.

Proof of Lemma 1: Let $v, v' \in V^n$ be such $q(v) = q(v')$. Let $v'' \in V^n$ be such that for all $i \in N$, if $q_i(v) = 1$ then $v''_i = \max\{v_i, v'_i\}$, and if $q_i(v) = 0$ then $v''_i = \min\{v_i, v'_i\}$. By strategy-proofness, we have

$$f_i(v''_i, v_{-i}) = f_i(v) \text{ and } f_i(v''_i, v'_{-i}) = f_i(v').$$

By non-bossiness, we have

$$f(v''_i, v_{-i}) = f(v) \text{ and } f(v''_i, v'_{-i}) = f(v').$$

Repeating this argument for all $i \in N$, we have

$$f(v'') = f(v) \text{ and } f(v'') = f(v').$$

Hence, $p(v) = p(v')$. □

Remark 7 By envy-freeness, it holds that for all $v \in V^n$ and for all $i, j \in N$, if $q_i(v) = q_j(v)$ then $p_i(v) = p_j(v)$.

Lemma 2 states that for two valuation profiles if the rule assigns the good allocations which are the same amount of good, then both consumers (non-consumers) pay the same amount.

Lemma 2 For all $v, v' \in V^n$ and for all $i, j \in N$, if $\#q(v) = \#q(v')$ and $q_i(v) = q_j(v')$, then $p_i(v) = p_j(v')$.

Proof of Lemma 2: Let $v, v' \in V^n$ and $i, j \in N$ be such that $\#q(v) = \#q(v')$ and $q_i(v) = q_j(v')$. Consider a permutation $\pi : N \to N$ as follows:

1. $\pi(i) = j$
2. for each $h \in \{h' \in N \backslash \{i\} | g_{h'}(v) = 0\}$, $\pi(h) \in \{h' \in N \backslash \{j\} | q_{h'}(v') = 0\}$
3. for each $h \in \{h' \in N \backslash \{i\} | g_{h'}(v) = 1\}$, $\pi(h) \in \{h' \in N \backslash \{j\} | q_{h'}(v') = 1\}$.

Then, by anonymity, it hold that

$$q_j(v^\pi) = q_{\pi(i)}(v^\pi) = q_i(v) = q_j(v')$$

and for each $h \in N \setminus \{i\}$,

$$q_{\pi(h)}(v^\pi) = q_h(v) = q_{\pi(h)}(v').$$

That is, $q(v^\pi) = q(v')$. Then, by Lemma 1, we have

$$p(v^\pi) = p(v').$$

By anonymity, we have

$$p_i(v) = p_{\pi(i)}(v^\pi) = p_j(v^\pi) = p_j(v').$$

□

Remark 8 Since, by Lemma 2, consumers' and non-consumers' payments depend only on production level, these are denoted by $p^1(k)$ and $p^0(k)$ when production level is k, respectively.

Lemma 3 states that the larger the number of consumers is, the larger (smaller) the non-consumers' (consumers') payment is.

Lemma 3 For all $k = 0, \ldots, n-2$ and for all $k' = 1, \ldots, n-1$, if $k < k'$, then $p^0(k) \leq p^0(k')$ and $p^1(k+1) \geq p^1(k'+1)$.

Proof of Lemma 3: Let $k = 0, \ldots, n-2$ and $k' = 1, \ldots, n-1$ be such that $k < k'$. At first, we show that $p^0(k) \leq p^0(k+1)$. Assume, to the contrary, that $p^0(k) > p^0(k+1)$. By consumer sovereignty, we have

$$f_{k+1}(v^1_{\{1,\ldots,k+1\}}, v^0_{\{k+2,\ldots,n\}}) = (1, p^1(k+1))$$

and

$$f_{k+1}(v^1_{\{1,\ldots,k\}}, v^0_{\{k+1,\ldots,n\}}) = (0, p^0(k)).$$

By strategy-proofness and consumer sovereignty, it holds that

$$p^1(k+1) - p^0(k) \geq 0.$$

Let $v_{k+1} \in V$ be such that $p^1(k+1) - p^0(k) < v_{k+1} < p^1(k+1) - p^0(k+1)$. Then, by strategy-proofness, it holds that

$$f_{k+1}(v^1_{\{1,\ldots,k\}}, v_{k+1}, v^0_{\{k+2,\ldots,n\}}) = (1, p^1(k+1)).$$

However, we have

$$-p^0(k+1) > v_{k+1} - p^1(k+1),$$

which contradicts to envy-freeness. Thus, we have $p^0(k) \leq p^0(k+1)$. By repeating the same argument, we have $p^0(k) \leq p^0(k')$.

In a similar way, we can show that $p^1(k+1) \geq p^1(k'+1)$. □

Lemma 4, 5 and 6 state the relationship between $p^0(\cdot)$ and $p^1(\cdot)$.

Lemma 4 For any $l = 0, \dots, n-2$, we have either $p^0(l) = p^0(l+1)$ or $p^1(l+1) = p^1(l+2)$.

Proof of Lemma 4: Assume, to the contrary, that for some $l = 0, \dots, n-2$, $p^0(l) < p^0(l+1)$ and $p^1(l+1) > p^1(l+2)$. Let $\hat{v}_1 \in V$ be such that

$$p^1(l+2) - p^0(l+1) < \hat{v}_1 < p^1(l+2) - p^0(l).$$

Let $\tilde{v}_2 \in V$ be such that

$$p^1(l+2) - p^0(l) < \tilde{v}_2 < p^1(l+1) - p^0(l).$$

In the following discussion, we always assume that each agent $i = 3, \dots, n-l$ has v_i^0, and each agent $i = n-l+1, \dots, n$ has v_i^1. For simplicity of notation, for all $v_1, v_2 \in V$, and for each $i = 1, 2$, we use $f_i(v_1, v_2)$ as $f_i(v_1, v_2, v^0_{\{3,\dots,n-l\}}, v^1_{\{n-l+1,\dots,n\}})$.

Since, by consumer sovereignty, $f_1(v_1^0, v_2^0) = (0, p^0(l))$ and $f_1(v_1^1, v_2^0) = (1, p^1(l+1))$, by strategy-proofness, we have

$$f_1(\hat{v}_1, v_2^0) = (0, p^0(l)).$$

By non-bossiness, it holds that

$$f_2(\hat{v}_1, v_2^0) = (0, p^0(l)). \tag{1}$$

Since, by consumer sovereignty, $f_1(v_1^0, v_2^1) = (0, p^0(l+1))$ and $f_1(v_1^1, v_2^1) = (1, p^1(l+2))$, by strategy-proofness, we have

$$f_1(\hat{v}_1, v_2^1) = (1, p^1(l+2)).$$

By non-bossiness, it holds that

$$f_2(\hat{v}_1, v_2^1) = (1, p^1(l+2)). \tag{2}$$

From (1) and (2), by strategy-proofness, we have

$$f_2(\hat{v}_1, \tilde{v}_2) = (1, p^1(l+2)).$$

By non-bossiness, it holds that

$$f_1(\hat{v}_1, \tilde{v}_2) = (1, p^1(l+2)). \tag{3}$$

Since, by consumer sovereignty, $f_2(v_1^0, v_2^0) = (0, p^0(l))$ and $f_2(v_1^0, v_2^1) = (1, p^1(l+1))$, by strategy-proofness, we have

$$f_2(v_1^0, \tilde{v}_2) = (0, p^0(l)).$$

By non-bossiness, it holds that

$$f_1(v_1^0, \tilde{v}_2) = (0, p^0(l)). \tag{4}$$

However, (3) and (4) contradict to strategy-proofness. □

Lemma 5 For any $m = 0, \dots, n-2$ and for any $l = 0, \dots, n-m-2$, we have either $p^0(l) = p^0(l+1)$ or $p^1(l+m+1) = p^1(l+m+2)$.

Proof of Lemma 5: We prove this Lemma with following induction.

1. For all $l = 0, \dots, n-2$, $p^0(l) = p^0(l+1)$ or $p^1(l+1) = p^1(l+2)$.
2. For all $m = 1, \dots, n-2$, if for all $m' = 0, \dots, m-1$ and for all $l' = 0, \dots, n-m'-2$, $p^0(l') = p^0(l'+1)$ or $p^1(l'+m'+1) = p^1(l'+m'+2)$, then for all $l = 0, \dots, n-m-2$, $p^0(l) = p^0(l+1)$ or $p^1(l+m+1) = p^1(l+m+2)$.

The first part of this induction is valid from Lemma 4. So, we prove the second part of this induction.

Let $m = 1, \dots, n-2$. Assume, to the contrary, that for some $l = 0, \dots, n-m-2$,

$$p^0(l) < p^0(l+1) \text{ and } p^1(l+m+1) > p^1(l+m+2).$$

From the hypothesis of the induction, we have[3]

$$p^0(l) < p^0(l+1) = p^0(l+2) = \cdots = p^0(l+m+1)$$

and[4]

$$p^1(l+1) = \cdots = p^1(l+m) = p^1(l+m+1) > p^1(l+m+2).$$

For each $i = 1, \dots, m+1$, let $\hat{v}_i \in V$ be such that

$$p^1(l+m+2) - p^0(l+1) < \hat{v}_i < \min\{p^1(l+m+1) - p^0(l+1), p^1(l+m+2) - p^0(l)\}.$$

Let $\tilde{v}_{m+2} \in V$ be such that

$$p^1(l+m+2) - p^0(l) < \tilde{v}_{m+2} < p^1(l+m+1) - p^0(l).$$

In the following discussion, we always assume that each agent $i = m+3, \dots, n-l$ has v_i^0 and each agent $j = n-l+1, \dots, n$ has v_j^1. For simplicity of notation, for all $v_1, \dots, v_{m+2} \in V$, and for each $i = 1, \dots, m+2$, we use $f_i(v_1, \dots, v_{m+2})$ as $f_i(v_1, \dots, v_{m+2}, v^0_{\{m+3,\dots,n-l\}}, v^1_{\{n-l+1,\dots,n\}})$.

Claim 1: For all $t = 0, \dots, m+1$ and for all $t' = 1, \dots, m+2-t$, the amounts of goods produced by f at $(\hat{v}_{\{1,\dots,t\}}, v^0_{\{t+1,\dots,t+t'\}}, v^1_{\{t+t'+1,\dots,m+2\}})$ is $l+m+2-(t+t')$.

We prove this claim by the following induction.

---

[3] From $p^1(l+m+1) > p^1(l+m+2)$, by setting $l' = l+m-m'$ and $m' = 0$ in the hypothesis, we have $p^0(l+m) = p^0(l+m+1)$. By setting $l' = l+m-m'$ and $m' = 1$, we have $p^0(l+m-1) = p^0(l+m)$. Finally, by setting $l' = l+m-m'$ and $m' = m-1$, we have $p^0(l+1) = p^0(l+2)$.

[4] From $p^0(l) < p^0(l+1)$, by setting $l' = l$ and $m' = 0$ in the hypothesis, we have $p^1(l+1) = p^1(l+2)$. By setting $l' = l$ and $m' = 1$, we have $p^1(l+2) = p^1(l+3)$. Finally, by setting $l' = l$ and $m' = m-1$, we have $p^1(l+m) = p^1(l+m+1)$.

1. For all $t' = 1, \dots, m+2$, the amounts of goods produced by f at $(v^0_{\{1,\dots,t'\}}, v^1_{\{t'+1,\dots,m+2\}})$ is $l+m+2-t'$.
2. For all $t = 1, \dots, m+1$, if for all $t'' = 1, \dots, m+2-(t-1)$, the amounts of goods produced by f at $(\hat{v}_{\{1,\dots,t-1\}}, v^0_{\{t,\dots,t+t''-1\}}, v^1_{\{t+t'',\dots,m+2\}})$ is $l+m+2-(t+t''-1)$, then for all $t' = 1, \dots, m+2-t$, the amounts of goods produced by f at $(\hat{v}_{\{1,\dots,t\}}, v^0_{\{t+1,\dots,t+t'\}}, v^1_{\{t+t'+1,\dots,m+2\}})$ is $l+m+2-(t+t')$.

The first part of this induction is obvious by consumer sovereignty. So, we prove the second part. Let $t \in N$ be such that $1 \leq t \leq m+1$. Let $t' \in N$ be such that $1 \leq t' \leq m+2-t$. Since, by the hypothesis of this induction, we have

$$f_{t+t'}(\hat{v}_{\{1,\dots,t-1\}}, v^0_{\{t,\dots,t+t'-1\}}, v^1_{t+t'}, v^1_{\{t+t'+1,\dots,m+2\}}) = (1, p^1(l+m+2-(t+t'-1)))$$

and[5]

$$f_{t+t'}(\hat{v}_{\{1,\dots,t-1\}}, v^0_{\{t,\dots,t+t'-1\}}, v^0_{t+t'}, v^1_{\{t+t'+1,\dots,m+2\}}) = (0, p^0(l+m+2-(t+t'))),$$

by strategy-proofness, it holds that

$$f_{t+t'}(\hat{v}_{\{1,\dots,t-1\}}, v^0_{\{t,\dots,t+t'-1\}}, \hat{v}_{t+t'}, v^1_{\{t+t'+1,\dots,m+2\}}) = (0, p^0(l+m+2-(t+t'))).$$

By anonymity, we have

$$f_t(\hat{v}_{\{1,\dots,t\}}, v^0_{\{t+1,\dots,t+t'\}}, v^1_{\{t+t'+1,\dots,m+2\}}) = (0, p^0(l+m+2-(t+t'))).$$

Thus, the amounts of goods produced by f at $(\hat{v}_{\{1,\dots,t\}}, v^0_{\{t+1,\dots,t+t'\}}, v^1_{\{t+t'+1,\dots,m+2\}})$ is $l+m+2-(t+t')$.

Claim 2: The amounts of goods produced by f at $(\hat{v}_{\{1,\dots,m+1\}}, v^0_{m+2})$ is l.

Set $t = m+1$ and $t' = 1$ in the claim 1. Then, the amounts of goods produced by f at $(\hat{v}_{\{1,\dots,m+1\}}, v^0_{m+2})$ is l.

Claim 3: The amounts of goods produced by f at $(\hat{v}_{\{1,\dots,m+1\}}, v^1_{m+2})$ is $l+m+2$.

We prove this claim by the following induction.

---

[5] By the hypothesis of this induction, we have

$$\begin{aligned}
& f_{t+t'}(\hat{v}_{\{1,\dots,t-1\}}, v^0_{\{t,\dots,t+t'-1\}}, v^0_{t+t'}, v^1_{\{t+t'+1,\dots,m+2\}}) \\
= \; & f_{t+t'}(\hat{v}_{\{1,\dots,t-1\}}, v^0_{\{t,\dots,t+t'\}}, v^1_{\{t+t'+1,\dots,m+2\}}) \\
= \; & f_{t+t''-1}(\hat{v}_{\{1,\dots,t-1\}}, v^0_{\{t,\dots,t+t''-1\}}, v^1_{\{t+t'',\dots,m+2\}}) \\
= \; & (0, p^0(l+m+2-(t+t''-1))) \\
= \; & (0, p^0(l+m+2-(t+t'))).
\end{aligned}$$

1. The amounts of goods produced by f at $(v^1_{\{1,\dots,m+2\}})$ is $l+m+2$.
2. For all $t = 1,\dots,m+1$, if the amounts of goods produced by f at $(\hat{v}_{\{1,\dots,t-1\}}, v^1_{\{t,\dots,m+2\}})$ is $l+m+2$, then the amounts of goods produced by f at $(\hat{v}_{\{1,\dots,t\}}, v^1_{\{t+1,\dots,m+2\}})$ is $l+m+2$.

The first part of this induction is obvious by consumer sovereignty. So, we prove the second part. Let $t \in N$ be such that $1 \leq t \leq m+1$. By the hypothesis of this induction, we have

$$f_t(\hat{v}_{\{1,\dots,t-1\}}, v^1_t, v^1_{\{t+1,\dots,m+2\}}) = (1, p^1(l+m+2)).$$

By setting t and t' in the claim 1 as $t-1$ and 1 in this induction, respectively we have

$$f_t(\hat{v}_{\{1,\dots,t-1\}}, v^0_t, v^1_{\{t+1,\dots,m+2\}}) = (0, p^0(l+m+2-t)).$$

Then, by strategy-proofness, it holds that

$$f_t(\hat{v}_{\{1,\dots,t-1\}}, \hat{v}_t, v^1_{\{t+1,\dots,m+2\}}) = (1, p^1(l+m+2)).$$

Thus, the amounts of goods produced by f at $(\hat{v}_{\{1,\dots,t\}}, v^1_{\{t+1,\dots,m+2\}})$ is $l+m+2$.

From this induction, we have that the amounts of goods produced by f at $(\hat{v}_{\{1,\dots,m+1\}}, v^1_{m+2})$ is $l+m+2$.

Claim 4: The amounts of goods produced by f at $(\hat{v}_{\{1,\dots,m+1\}}, \tilde{v}_{m+2})$ is $l+m+2$.

From the claims 2 and 3, we have that

$$f_{m+2}(\hat{v}_{\{1,\dots,m+1\}}, v^0_{m+2}) = (0, p^0(l))$$

and

$$f_{m+2}(\hat{v}_{\{1,\dots,m+1\}}, v^1_{m+2}) = (1, p^1(l+m+2)).$$

By strategy-proofness, it holds that

$$f_{m+2}(\hat{v}_{\{1,\dots,m+1\}}, \tilde{v}_{m+2}) = (1, p^1(l+m+2)).$$

Thus, the amounts of goods produced by f at $(\hat{v}_{\{1,\dots,m+1\}}, \tilde{v}_{m+2})$ is $l+m+2$.

Claim 5: The amounts of goods produced by f at $(\hat{v}_{\{1,\dots,m\}}, v^0_{m+1}, \tilde{v}_{m+2})$ is l.

By setting t and t' in the claim 1 as m and 2 in this claim, respectively, we have

$$f_{m+2}(\hat{v}_{\{1,\dots,m\}}, v^0_{m+1}, v^0_{m+2}) = (0, p^0(l)).$$

By setting t and t' in the claim 1 as m and 1 in this claim, respectively, we have

$$f_{m+2}(\hat{v}_{\{1,\ldots,m\}}, v^0_{m+1}, v^1_{m+2}) = (1, p^1(l+1)).$$

Then, by strategy-proofness, it holds that

$$f_{m+2}(\hat{v}_{\{1,\ldots,m\}}, v^0_{m+1}, \tilde{v}_{m+2}) = (0, p^0(l)).$$

Thus, the amounts of goods produced by f at $(\hat{v}_{\{1,\ldots,m\}}, v^0_{m+1}, \tilde{v}_{m+2})$ is l.

Finally, we derive a contradiction. From the claims 4 and 5, we have

$$f_{m+1}(\hat{v}_{\{1,\ldots,m\}}, \hat{v}_{m+1}, \tilde{v}_{m+2}) = (1, p^1(l+m+2))$$

and

$$f_{m+1}(\hat{v}_{\{1,\ldots,m\}}, v^0_{m+1}, \tilde{v}_{m+2}) = (0, p^0(l)).$$

These contradict to strategy-proofness. □

Lemma 6 For all $l = 1, \dots, n-1$, we have either $p^0(l) = 0$ or $p^1(l) = \frac{c_n}{n}$.

Proof of Lemma 6: Let $l = 1, \dots, n-1$. When $p^0(l) > 0$, let $l' = 1, \dots, l$ be the smallest number such that $p^0(l') > 0 = p^0(0)$. Then, by Lemma 5, for all $l'' = l', \dots, n$, we have $p^1(l'') = \frac{c_n}{n} = p^1(n)$. Thus, $p^1(l) = \frac{c_n}{n}$. When $p^1(l) < \frac{c_n}{n}$, by the similar argument, we have $p^0(l) = 0$. □

Remark 9 By Lemma 6, for any $l = 1, \dots, n-1$, if $p^0(l) = 0$ then $p^1(l) = \frac{c_l}{l}$, and if $p^1(l) = \frac{c_n}{n}$ then $p^0(l) = \frac{1}{n-l}[c_l - l\frac{c_n}{n}]$.

Proof of Theorem 2: Let $v \in (V \setminus (\{\frac{c_t}{t}\}_{t=1}^{w-1} \cup \{\frac{c_n - c_{n-t}}{t}\}_{t=1}^{n-w} \cup \{\frac{c_n}{n}\}))^n$. Let $w \in N$ be such that $p^0(w-1) = 0$ and $p^1(w) = \frac{c_n}{n}$. We divide the proof into two cases.

Case 1: $\#L_n(v) < w$.
By anonymity, we can treat as $v_1 \geq \cdots \geq v_n$. Let $a = \#A(v)$. Note that $a < w$. We show that for each $i \in A(v)$, $f_i(v) = (1, \frac{c_a}{a})$, and for each $i \notin A(v)$, $f_i(v) = (0, 0)$.

Step 1-1: $f(v_{\{1, \dots a\}}, v^0_{\{a+1, \dots, n\}}) = f(v^1_{\{1, \dots, a\}}, v^0_{\{a+1, \dots, n\}})$.[6]
By consumer sovereignty and Lemma 6, it holds that

$$f_1(v^1_{\{1, \dots, a\}}, v^0_{\{a+1, \dots, n\}}) = (1, \frac{c_a}{a}).$$

Since the number of agents who get the good at $(v^0_1, v^1_{\{2, \dots, a\}}, v^0_{\{a+1, \dots, n\}})$ is at most $a < w$, by lemma 6, we have

$$f_1(v^0_1, v^1_{\{2, \dots, a\}}, v^0_{\{a+1, \dots, n\}}) = (0, 0).$$

Since $v_1 > \frac{c_a}{a}$, by strategy-proofness, we have

$$f_1(v_1, v^1_{\{2, \dots, a\}}, v^0_{\{a+1, \dots, n\}}) = (1, \frac{c_a}{a}).$$

By non-bossiness, it holds that

$$f(v_1, v^1_{\{2, \dots, a\}}, v^0_{\{a+1, \dots, n\}}) = f(v^1_{\{1, \dots, a\}}, v^0_{\{a+1, \dots, n\}}).$$

Repeating this argument for all $i = 2, \dots, a$, we have

$$f(v_{\{1, \dots a\}}, v^0_{\{a+1, \dots, n\}}) = f(v^1_{\{1, \dots, a\}}, v^0_{\{a+1, \dots, n\}}).$$

Step 1-2: $f(v) = f(v^1_{\{1, \dots, a\}}, v^0_{\{a+1, \dots, n\}})$.
When $v_{a+1} > \frac{c_n}{n}$, since $\#L_n(v) < w$, we have $a + 1 < w$. Since $A(v) = L_a(v)$ is the largest unanimous consumers' coalition at v, we must have

$$v_{a+1} < \frac{c_{a+1}}{a+1}.$$

---

[6] When $a = 0$, go to step 1-2.

Since the number of agents who get the good at $(v_{\{1,\dots a\}}, v^1_{a+1}, v^0_{\{a+2,\dots,n\}})$ is less than w, by lemma 6, for some $t = 1, \dots, a+1$, we have

$$f_{a+1}(v_{\{1,\dots a\}}, v^1_{a+1}, v^0_{\{a+2,\dots,n\}}) = (1, \frac{c_t}{t}).$$

Since $v_{a+1} < \frac{c_{a+1}}{a+1} \leq \frac{c_t}{t}$, by strategy-proofness, we have

$$f_{a+1}(v_{\{1,\dots a\}}, v_{a+1}, v^0_{\{a+2,\dots,n\}}) = (0, 0).$$

By non-bossiness, we have

$$f(v_{\{1,\dots a\}}, v_{a+1}, v^0_{\{a+2,\dots,n\}}) = f(v^1_{\{1,\dots,a\}}, v^0_{\{a+1,\dots,n\}}).$$

When $v_{a+1} < \frac{c_n}{n}$, for some $t = 1, \dots, a+1, n$, it holds that

$$f_{a+1}(v_{\{1,\dots a\}}, v^1_{a+1}, v^0_{\{a+2,\dots,n\}}) = (1, \frac{c_t}{t}).$$

Since $v_{a+1} < \frac{c_n}{n} \leq \frac{c_t}{t}$, by strategy-proofness, we have

$$f_{a+1}(v_{\{1,\dots a\}}, v_{a+1}, v^0_{\{a+2,\dots,n\}}) = (0, 0).$$

By non-bossiness, we have

$$f(v_{\{1,\dots a\}}, v_{a+1}, v^0_{\{a+2,\dots,n\}}) = f(v^1_{\{1,\dots,a\}}, v^0_{\{a+1,\dots,n\}}).$$

Repeating this argument for all $i = a+2, \dots, n$, we have

$$f(v) = f(v^1_{\{1,\dots,a\}}, v^0_{\{a+1,\dots,n\}}).$$

Therefore, for each $i \in A(v)$, $f_i(v) = (1, \frac{c_a}{a})$, and for each $i \notin A(v)$, $f_i(v) = (0, 0)$.

Case 2: $\#L_n(v) \geq w$.

By anonymity, we can treat as $v_1 \leq \cdots \leq v_n$. Let $b = \#B(v)$. Note that $n - b \geq w$. We show that for each $i \in B(v)$, $f_i(v) = (0, \frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}])$, and for each $i \notin B(v)$, $f_i(v) = (1, \frac{c_n}{n})$.

Step 2-1: $f(v_{\{1,\dots b\}}, v^1_{\{b+1,\dots,n\}}) = f(v^0_{\{1,\dots,b\}}, v^1_{\{b+1,\dots,n\}})$.[7]

By consumer sovereignty and Lemma 6, it holds that

$$f_1(v^0_{\{1,\dots,b\}}, v^1_{\{b+1,\dots,n\}}) = (0, \frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}]).$$

Since the number of agents who get the good at $(v^1_1, v^0_{\{2,\dots,b\}}, v^1_{\{b+1,\dots,n\}})$ is at least $n - b \geq w$, by lemma 6, we have

$$f_1(v^1_1, v^0_{\{2,\dots,b\}}, v^1_{\{b+1,\dots,n\}}) = (1, \frac{c_n}{n}).$$

---

[7] When $b = 0$, go to step 2-2

Since $v_1 < \frac{c_n - c_{n-b}}{b}$, by strategy-proofness, we have

$$f_1(v_1, v^0_{\{2,\dots,b\}}, v^1_{\{b+1,\dots,n\}}) = (0, \frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}]).$$

By non-bossiness, it holds that

$$f(v_1, v^0_{\{2,\dots,b\}}, v^1_{\{b+1,\dots,n\}}) = f(v^0_{\{1,\dots,b\}}, v^1_{\{b+1,\dots,n\}}).$$

Repeating this argument for all $i = 2, \dots, b$, we have

$$f(v_{\{1,\dots b\}}, v^1_{\{b+1,\dots,n\}}) = f(v^0_{\{1,\dots,b\}}, v^1_{\{b+1,\dots,n\}}).$$

Step 2-2: $f(v) = f(v^0_{\{1,\dots,b\}}, v^1_{\{b+1,\dots,n\}})$.
When $v_{b+1} < \frac{c_n}{n}$, it holds that $n - (b+1) \geq w$. Since $B(v) = S_b(v)$ is the largest unanimous non-consumers' coalition at v, we must have

$$v_{b+1} > \frac{c_n - c_{n-(b+1)}}{b+1}.$$

Since the number of agents who get the good at $(v_{\{1,\dots b\}}, v^0_{b+1}, v^1_{\{b+2,\dots,n\}})$ is at least w, by lemma 6, for some $t = 1, \dots, b+1$, we have

$$f_{b+1}(v_{\{1,\dots b\}}, v^0_{b+1}, v^1_{\{b+2,\dots,n\}}) = (0, \frac{1}{t}[c_{n-t} - (n-t)\frac{c_n}{n}]).$$

Since $v_{b+1} > \frac{c_n - c_{n-(b+1)}}{b+1}$, by strategy-proofness, we have

$$f_{b+1}(v_{\{1,\dots b\}}, v_{b+1}, v^1_{\{b+2,\dots,n\}}) = (1, \frac{c_n}{n}).$$

By non-bossiness, we have

$$f(v_{\{1,\dots b\}}, v_{b+1}, v^1_{\{b+2,\dots,n\}}) = f(v^0_{\{1,\dots,b\}}, v^1_{\{b+1,\dots,n\}}).$$

When $v_{b+1} > \frac{c_n}{n}$, for some $t = 1, \dots, b+1, n$, we have

$$f_{b+1}(v_{\{1,\dots b\}}, v^0_{b+1}, v^1_{\{b+2,\dots,n\}}) = (0, \frac{1}{t}[c_{n-t} - (n-t)\frac{c_n}{n}]).$$

Since $v_{b+1} > \frac{c_n - c_{n-(b+1)}}{b+1} \geq 0$, by strategy-proofness, we have

$$f_{b+1}(v_{\{1,\dots b\}}, v_{b+1}, v^1_{\{b+2,\dots,n\}}) = (1, \frac{c_n}{n}).$$

By non-bossiness, we have

$$f(v_{\{1,\dots b\}}, v_{b+1}, v^0_{\{b+2,\dots,n\}}) = f(v^0_{\{1,\dots,b\}}, v^1_{\{b+1,\dots,n\}}).$$

Repeating this argument for all $i = b+2, \dots, n$, we have

$$f(v) = f(v^0_{\{1,\dots,b\}}, v^1_{\{b+1,\dots,n\}}).$$

Therefore, for each $i \in B(v)$, $f_i(v) = (0, \frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}])$, and for each $i \notin B(v)$, $f_i(v) = (1, \frac{c_n}{n})$. □

6.3 Proof of Theorem 3:

Proof of Theorem 3: Let f be a rule satisfying strategy-proofness, non-bossiness, consumer sovereignty, anonymity and envy-freeness. Let $v \in (V \setminus \{\frac{c_n}{n}\})^n$. Let $w \in N$ be such that $p^0(w-1) = 0$ and $p^1(w) = \frac{c_n}{n}$. Let f^w be the w-hybrid rule. We divide the argument into two cases.

Case 1: $\#L_n(v) < w$.
By anonymity, without loss of generality, we can treat as $v_1 \geq v_2 \geq \cdots \geq v_n$. Let $a = \#A(v)$ and $k = \sum_{i=1}^{n} q_i(v)$. Note that $a < w$. We divide this case into two sub-cases.

Sub-Case 1-1: $a \geq k$.
When $k = 0$, it is obvious that $f(v)$ is weakly Pareto-dominated by $f^w(v)$. So, we assume that $k > 0$. Let $i = 1, \ldots, a$. Then, we have $f_i^w(v) = (1, \frac{c_a}{a})$. Since $k < w$, we have either

$$f_i(v) = (1, \frac{c_k}{k}) \text{ or } (0, 0).$$

Since $\frac{c_a}{a} \leq \frac{c_k}{k}$ and $v_i \geq \frac{c_a}{a}$, $f_i^w(v)$ is at least as good as $f_i(v)$ in either event.

Let $j = a+1, \ldots, n$. Then, we have $f_j^w(v) = (0, 0)$. Since $k < w$, we have either

$$f_j(v) = (1, \frac{c_k}{k}) \text{ or } (0, 0).$$

Since $v_i < \frac{c_a}{a} \leq \frac{c_k}{k}$, $f_j^w(v)$ is at least as good as $f_j(v)$ in either event.

Sub-Case 1-2: $a < k$.
Let $\hat{v} \in V^n$ be such that for each agent i who gets the good at v, $\hat{v}_i = v_i^1$, and for each agent i who does not get the good at v, $\hat{v}_i = v_i^0$. We divide this sub-case into two sub-sub-cases.

Sub-Sub-Case 1-2-1: $k < w$.
Since $a < k$ and $A(v)$ is the largest unanimous consumers' coalition at v, there exists an agent j who gets the good at v, such that $v_j < \frac{c_k}{k}$. Let j be such a agent. Then, by non-bossiness, it holds that

$$f_j(v_j, \hat{v}_{-j}) = (1, \frac{c_k}{k}). \tag{5}$$

Since $k < w$, by consumer sovereignty, we have

$$f_j(v_j^0, \hat{v}_{-j}) = (0, 0). \tag{6}$$

However, since $v_i < \frac{c_k}{k}$, (5) and (5) contradict to strategy-proofness. Thus, this sub-sub-case does not occur.

Sub-Sub-Case 1-2-2: $k \geq w$.
Since $\#L_n(v) < w$ and $k \geq w$, there exist agents who get the good at v, and

their valuations are less than $\frac{c_n}{n}$. Let $\{j_1, \dots j_h\}$ be the set of all such agents. Since the number of agents who get the good at $(v^0_{\{j_1,\dots,j_h\}}, \hat{v}_{-\{j_1,\dots,j_h\}})$ is less than w, by consumer sovereignty, we have

$$f_{j_1}(v^0_{\{j_1,\dots,j_h\}}, \hat{v}_{-\{j_1,\dots,j_h\}}) = (0, 0).$$

Since, by consumer sovereignty, the agent j_1 gets the good at $(v^1_{j_1}, v^0_{\{j_2,\dots,j_h\}}, \hat{v}_{-\{j_1,\dots,j_h\}})$, and his payment is larger than or equal to $\frac{c_n}{n}$, by strategy-proofness, we have

$$f_{j_1}(v_{j_1}, v^0_{\{j_2,\dots,j_h\}}, \hat{v}_{-\{j_1,\dots,j_h\}}) = (0, 0).$$

By non-bossiness, the number of agents who get the good at $(v_{j_1}, v^0_{\{j_2,\dots,j_h\}}, \hat{v}_{-\{j_1,\dots,j_h\}})$ is less than w. Then, by consumer sovereignty, we have

$$f_{j_2}(v_{j_1}, v^0_{\{j_2,\dots,j_h\}}, \hat{v}_{-\{j_1,\dots,j_h\}}) = (0, 0).$$

Since, by consumer sovereignty, the agent j_2 gets the good at $(v_{j_1}, v^1_{j_2}, v^0_{\{j_3,\dots,j_h\}}, \hat{v}_{-\{j_1,\dots,j_h\}})$, and his payment is larger than or equal to $\frac{c_n}{n}$, by strategy-proofness, we have

$$f_{j_2}(v_{\{j_1,j_2\}}, v^0_{\{j_3,\dots,j_h\}}, \hat{v}_{-\{j_1,\dots,j_h\}}) = (0, 0).$$

By non-bossiness, the number of agents who get the good at $(v_{\{j_1,j_2\}}, v^0_{\{j_3,\dots,j_h\}}, \hat{v}_{-\{j_1,\dots,j_h\}})$ is less than w. Repeating this argument for the remaining agents, we have that the number of agents who get the good at $(v_{\{j_1,\dots,j_h\}}, \hat{v}_{-\{j_1,\dots,j_h\}})$ is less than w. However, by non-bossiness, we must have $f(v) = f(v_{\{j_1,\dots,j_h\}}, \hat{v}_{-\{j_1,\dots,j_h\}})$, that is, the number of agents who get the good at $(v_{\{j_1,\dots,j_h\}}, \hat{v}_{-\{j_1,\dots,j_h\}})$ is k, which is a contradiction. Thus, this sub-sub-case does not occur.

Case 2: $\#L_n(v) \geq w$.

By anonymity, without loss of generality, we can treat as $v_1 \leq v_2 \leq \cdots \leq v_n$. Let $b = \#B(v)$ and $k = \sum_{i=1}^n q_i(v)$. Since for any $i \in N$, $v_i = \frac{c_n}{n}$, note that $n - b \geq w$. We divide this case into two sub-cases.

Sub-Case 2-1: $n - b \leq k$.

When $k = n$, it is obvious that $f(v)$ is weakly Pareto-dominated by $f^w(v)$. So, we assume that $k < n$. Let $i = 1, \dots, b$. Then, we have $f^w_i(v) = (0, \frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}])$. Since $k \geq w$, we have either

$$f_i(v) = (0, \frac{1}{n-k}[c_k - k\frac{c_n}{n}]) \text{ or } (1, \frac{c_n}{n}).$$

Since $\frac{1}{b}[c_{n-b} - (n-b)\frac{c_n}{n}] \leq \frac{1}{n-k}[c_k - k\frac{c_n}{n}]$ and $v_i \leq \frac{c_n - c_{n-b}}{b}$, $f^w_i(v)$ is at least as good as $f_i(v)$ in either event.

Let $j = b+1, \dots, n$. Then, we have $f^w_j(v) = (1, \frac{c_n}{n})$. Since $k \geq w$, we have either

$$f_j(v) = (0, \frac{1}{n-k}[c_k - k\frac{c_n}{n}]) \text{ or } (1, \frac{c_n}{n}).$$

Since $v_i > \frac{c_n - c_{n-b}}{b} \geq \frac{c_n - c_k}{n-k}$, $f_j^w(v)$ is at least as good as $f_j(v)$ in either event.

Sub-Case 2-2: $n - b > k$.

Let $\hat{v} \in V^n$ be such that for each agent i who gets the good at v, $\hat{v}_i = v_i^1$, and for each agent i who does not get the good at v, $\hat{v}_i = v_i^0$. We divide this sub-case into two sub-sub-cases.

Sub-Sub-Case 2-2-1: $k \geq w$.

Since $n - b > k$ and $B(v)$ is the largest unanimous non-consumers' coalition at v, there exists an agent j who does not get the good at v, such that $v_j > \frac{c_n - c_k}{n-k}$. Let j be such a agent. Then, by non-bossiness, it holds that

$$f_j(v_j, \hat{v}_{-j}) = (0, \frac{1}{n-k}[c_k - k\frac{c_n}{n}]). \tag{7}$$

Since $k \geq w$, by consumer sovereignty, we have

$$f_j(v_j^1, \hat{v}_{-j}) = (1, \frac{c_n}{n}). \tag{8}$$

However, since $v_j > \frac{c_n - c_k}{n-k}$, (7) and (8) contradict to strategy-proofness. Thus, this sub-sub-case does not occur.

Sub-Sub-Case 2-2-2: $k < w$.

Since $\#L_n(v) \geq w$, $k < w$ and $v \in (V \setminus \{\frac{c_n}{n}\})^n$, there exist agents who do not get the good at v, and their valuations are larger than $\frac{c_n}{n}$. Let $\{j_1, \ldots j_h\}$ be the set of all such agents. Since the number of agents who get the good at $(v^1_{\{j_1,\ldots,j_h\}}, \hat{v}_{-\{j_1,\ldots,j_h\}})$ is larger than or equal to w, by consumer sovereignty, we have

$$f_{j_1}(v^1_{\{j_1,\ldots,j_h\}}, \hat{v}_{-\{j_1,\ldots,j_h\}}) = (1, \frac{c_n}{n}).$$

Since, by consumer sovereignty, the agent j_1 does not get the good at $(v^0_{j_1}, v^1_{\{j_2,\ldots,j_h\}}, \hat{v}_{-\{j_1,\ldots,j_h\}})$, and his payment is larger than or equal to 0, by strategy-proofness, we have

$$f_{j_1}(v_{j_1}, v^1_{\{j_2,\ldots,j_h\}}, \hat{v}_{-\{j_1,\ldots,j_h\}}) = (1, \frac{c_n}{n}).$$

By non-bossiness, the number of agents who get the good at $(v_{j_1}, v^1_{\{j_2,\ldots,j_h\}}, \hat{v}_{-\{j_1,\ldots,j_h\}})$ is larger than or equal to w. Then, by consumer sovereignty, we have

$$f_{j_2}(v_{j_1}, v^1_{\{j_2,\ldots,j_h\}}, \hat{v}_{-\{j_1,\ldots,j_h\}}) = (1, \frac{c_n}{n}).$$

Since, by consumer sovereignty, the agent j_2 does not get the good at $(v_{j_1}, v^0_{j_2}, v^1_{\{j_3,\ldots,j_h\}}, \hat{v}_{-\{j_1,\ldots,j_h\}})$, and his payment is larger than or equal to 0, by strategy-proofness, we have

$$f_{j_2}(v_{\{j_1,j_2\}}, v^1_{\{j_3,\ldots,j_h\}}, \hat{v}_{-\{j_1,\ldots,j_h\}}) = (1, \frac{c_n}{n}).$$

By non-bossiness, the number of agents who get the good at $(v_{\{j_1,j_2\}}, v^1_{\{j_3,\ldots,j_h\}}, \hat{v}_{-\{j_1,\ldots,j_h\}})$ is larger than or equal to w. Repeating this argument for the remaining agents, we have that the number of agents who get the good at $(v_{\{j_1,\ldots,j_h\}}, \hat{v}_{-\{j_1,\ldots,j_h\}})$ is larger than or equal to w. However, by non-bossiness, we must have $f(v) = f(v_{\{j_1,\ldots,j_h\}}, \hat{v}_{-\{j_1,\ldots,j_h\}})$, that is, the number of agents who get the good at $(v_{\{j_1,\ldots,j_h\}}, \hat{v}_{-\{j_1,\ldots,j_h\}})$ is k, which is a contradiction. Thus, this sub-sub-case does not occur. $\square$

6.4 Proof of Theorem 4:

Proof of Theorem 4: When the cost function is linear (that is, for any $k = \{1, \dots n\}$, $c_k = kc_1$), for any $v \in V^n$, w-hybrid rule assigns an efficient allocation of good at v. So, the supremal welfare loss of w-hybrid rule is 0. Since $\sum_{i=1}^{w-1} \frac{c_i}{i} + (n-(w-1))\frac{c_n}{n} - c_n = -(w\frac{c_n}{n} + \sum_{i=w+1}^{n} \frac{c_n - c_{i-1}}{n-(i-1)} - c_n) = 0$, Theorem ?? is valid.

In what follows, we assume that the cost function is not linear. Without loss of generality, we assume that $v_1 \geq v_2 \geq \cdots \geq v_n$. Let $k, k^* \in \{0, 1, \dots, n\}$ be such that there exist $\hat{v} \in V^n$, $k = \sum_{i=1}^{n} q_i(\hat{v})$ and $k^* = \sum_{i=1}^{n} q_i^*$, where q_i^* is an efficient allocation of good at $\hat{v}$. We consider the supremal welfare loss for such pair (k, k^*), separately. We divide the proof into three cases.

Case 1: We consider the supremal welfare loss when valuation profiles satisfy $\#L_n(v) < w$.

At first, we calculate the supremal welfare loss when $k = 0$ and $k^* = n$. Since the welfare loss is

$$v_1 + \cdots + v_n - c_n,$$

the supremal welfare loss is

$$\sum_{i=1}^{w-1} \frac{c_i}{i} + (n-(w-1))\frac{c_n}{n} - c_n. \tag{9}$$

Next, we claim that the supremal welfare loss of this case is $\sum_{i=1}^{w-1} \frac{c_i}{i} + (n-(w-1))\frac{c_n}{n} - c_n$. Note that $k < w$. We divide the argument into two sub-cases.

Sub-Case 1-1: $k \geq k^*$.
The welfare loss is

$$\sum_{i=1}^{k^*} v_i - c_{k^*} - (\sum_{i=1}^{k} v_i - c_k) = -\sum_{i=k^*+1}^{k} v_i - c_{k^*} + c_k.$$

Since w-hybrid rule produces k units of good, for each $i = 1, \dots, k$, it must be $v_i \geq \frac{c_k}{k}$. So, the supremal welfare loss for this pair (k, k^*) is

$$-(k-k^*)\frac{c_k}{k} - c_{k^*} + c_k \leq 0.$$

That is, the welfare loss does not occur in this sub-case.

Sub-Case 1-2: $k < k^*$.
We divide this sub-case into two sub-sub-cases.

Sub-Sub-Case 1-2-1: $k^* < w$.
Since the welfare loss is

$$\sum_{i=1}^{k^*} v_i - c_{k^*} - (\sum_{i=1}^{k} v_i - c_k) = \sum_{i=k+1}^{k^*} v_i - c_{k^*} + c_k,$$

the supremal welfare loss for this pair (k, k^*) is

$$\sum_{i=k+1}^{k^*} \frac{c_i}{i} - c_{k^*} + c_k. \tag{10}$$

By subtracting (10) from (9), it holds that

$$\sum_{i=1}^{k} \frac{c_i}{i} + \sum_{i=k^*+1}^{w-1} \frac{c_i}{i} + (n - (w-1))\frac{c_n}{n} - c_n + c_{k^*} - c_k \geq 0.$$

Thus, the supremal welfare loss of this sub-sub-case is not larger than $\sum_{i=1}^{w-1} \frac{c_i}{i} + (n - (w-1))\frac{c_n}{n} - c_n$.

Sub-Sub-Case 1-2-2: $k^* \geq w$.
Since the welfare loss is

$$\sum_{i=1}^{k^*} v_i - c_{k^*} - (\sum_{i=1}^{k} v_i - c_k) = \sum_{i=k+1}^{k^*} v_i - c_{k^*} + c_k$$

the supremal welfare loss for this pair (k, k^*) is

$$\sum_{i=k+1}^{w-1} \frac{c_i}{i} + (k^* - (w-1))\frac{c_n}{n} - c_{k^*} + c_k. \tag{11}$$

By subtracting (11) from (9), it holds that

$$\sum_{i=1}^{k} \frac{c_i}{i} + (n - k^*)\frac{c_n}{n} - c_n + c_{k^*} - c_k \geq 0.$$

Thus, the supremal welfare loss of this sub-sub-case is not larger than $\sum_{i=1}^{w-1} \frac{c_i}{i} + (n - (w-1))\frac{c_n}{n} - c_n$.

Case 2: We consider the supremal welfare loss when valuation profiles satisfy $\#L_n(v) \geq w$ and for some $i = 1, \ldots, n$, $v_i > \frac{c_n}{n}$.

At first, we calculate the supremal welfare loss when $k = n$ and $k^* = 0$. Since the welfare loss is

$$-(v_1 + \cdots + v_n - c_n),$$

the supremal welfare loss is

$$-(w\frac{c_n}{n}+\sum_{i=w+1}^{n}\frac{c_n-c_{i-1}}{n-(i-1)}-c_n). \tag{12}$$

Next, we claim that the supremal welfare loss of this case is $-(w\frac{c_n}{n}+\sum_{i=w+1}^{n}\frac{c_n-c_{i-1}}{n-(i-1)}-c_n)$. Note that $k \geq w$. We divide the argument into two sub-cases.

Sub-Case 2-1: $k \leq k^*$.
The welfare loss is

$$\sum_{i=1}^{k^*} v_i - c_{k^*} - (\sum_{i=1}^{k} v_i - c_k) = -\sum_{i=k+1}^{k^*} v_i - c_{k^*} + c_k.$$

Since w-hybrid rule produces k units of good, for each $i = k+1, \ldots, k^*$, it must be $v_i \leq \frac{c_n-c_k}{n-k}$. So, the supremal welfare loss for this pair (k, k^*) is

$$(k^*-k)\frac{c_n-c_k}{n-k} - c_{k^*} + c_k \leq 0.$$

That is, the welfare loss does not occur in this sub-case.

Sub-Case 2-2: $k > k^*$.
We divide this sub-case into two sub-sub-cases.

Sub-Sub-Case 2-2-1: $k^* \geq w$.
Since the welfare loss is

$$\sum_{i=1}^{k^*} v_i - c_{k^*} - (\sum_{i=1}^{k} v_i - c_k) = -\sum_{i=k^*+1}^{k} v_i - c_{k^*} + c_k,$$

the supremal welfare loss for this pair (k, k^*) is

$$-\sum_{i=k^*+1}^{k}\frac{c_n-c_{i-1}}{n-(i-1)} - c_{k^*} + c_k. \tag{13}$$

By subtracting (13) from 12, it holds that

$$-w\frac{c_n}{n} - \sum_{i=w+1}^{k^*}\frac{c_n-c_{i-1}}{n-(i-1)} - \sum_{i=k+1}^{n}\frac{c_n-c_{i-1}}{n-(i-1)} + c_n + c_{k^*} - c_k \geq 0.$$

Thus, the supremal welfare loss of this sub-sub-case is not larger than $-(w\frac{c_n}{n}+\sum_{i=w+1}^{n}\frac{c_n-c_{i-1}}{n-(i-1)}-c_n)$.

Sub-Sub-Case 2-2-2: $k^* < w$.
Since the welfare loss is

$$\sum_{i=1}^{k^*} v_i - c_{k^*} - (\sum_{i=1}^{k} v_i - c_k) = - \sum_{i=k^*+1}^{k} v_i - c_{k^*} + c_k$$

the supremal welfare loss for this pair (k, k^*) is

$$-((w-k^*)\frac{c_n}{n} + \sum_{i=w+1}^{k} \frac{c_n - c_{i-1}}{n-(i-1)}) - c_{k^*} + c_k. \tag{14}$$

By subtracting (14) from 12, it holds that

$$-k^*\frac{c_n}{n} - \sum_{i=k+1}^{n} \frac{c_n - c_{i-1}}{n-(i-1)} + c_n + c_{k^*} - c_k \geq 0.$$

Thus, the supremal welfare loss of this sub-sub-case is not larger than $-(w\frac{c_n}{n} + \sum_{i=w+1}^{n} \frac{c_n - c_{i-1}}{n-(i-1)} - c_n)$.

Case 3: We consider the supremal welfare loss when valuation profiles satisfy $\#L_n(v) \geq w$ and for each $i = 1, \ldots, n$, $v_i \leq \frac{c_n}{n}$.

Since for each $i = 1, \ldots, n$, $v_i \leq \frac{c_n}{n}$, w-hybrid rule produces nothing. Since the welfare loss is

$$\sum_{i=1}^{k^*} v_i - c_{k^*},$$

the supremal welfare loss for this pair $(0, k^*)$ is

$$k^*\frac{c_n}{n} - c_{k^*} \leq 0.$$

That is, the welfare loss does not occur in this case. □

6.5 Proof of Corollary 1:

Proof of Corollary 1: By Remark 5, it is sufficient to show that

$$c_1 + \cdots + \frac{c_{n-1}}{n-1} + \frac{c_n}{n} - c_n \geq -((n-1)\frac{c_n}{n} + \frac{c_n - c_{n-1}}{n-(n-1)} - c_n).$$

By simple calculation, it holds that

$$\begin{aligned} c_1 + \cdots + \frac{c_{n-1}}{n-1} + \frac{c_n}{n} - c_n &+ ((n-1)\frac{c_n}{n} + \frac{c_n - c_{n-1}}{n-(n-1)} - c_n) \\ &= c_1 + \cdots + \frac{c_{n-1}}{n-1} - c_{n-1} \\ &= (c_1 - \frac{c_{n-1}}{n-1}) + \cdots + (\frac{c_{n-1}}{n-1} - \frac{c_{n-1}}{n-1}) \\ &\geq 0. \end{aligned}$$

Thus, $c_1 + \cdots + \frac{c_{n-1}}{n-1} + \frac{c_n}{n} - c_n \geq -((n-1)\frac{c_n}{n} + \frac{c_n - c_{n-1}}{n-(n-1)} - c_n)$.

The comparison of the supremal welfare losses of the dual rule and 2-hybrid rule is similar. □

References

[1] Holmström, B., 1979, Groves Scheme on Restricted Domain. *Econometrica* 47, 1137-1144.

[2] Moulin, H., 1999, Incremental Cost Sharing: Characterization by Coalitional Strategy-Proofness. *Social Choice and Welfare* 16, 279-320.

[3] Moulin, H., S. Shenker, 2001, Strategyproof Sharing of Submodular Costs: Budget Balance versus Efficiency. *Economic Theory* 18, 511-533.

[4] Mutuswami, S., 2005, Strategy-Proofness, Non-Bossiness and Group Strategyproofness in a cost sharing model. *Economics Letters* 89, 83-88.

[5] Ohseto, S., 2005, Augmented Serial Rules for an Excludable Public Good. *Economic Theory* 26, 589-606.

n	AC pricing rule	Dual rule	optimal w-hybrid rule	optimal w
10	1967	1097	1010	2
100	6858	2557	2507	5
1000	17557	5544	5506	20

Table 1: The supremal welfare losses of three rules and the optimal w when the cost function is $c_k = 1000k^{1/3}$.

n	AC pricing rule	Dual rule	optimal w-hybrid rule	optimal w
10	1858	1124	1048	2
100	8589	3835	3702	9
1000	30178	12207	11961	58

Table 2: The supremal welfare losses of three rules and the optimal w when the cost function is $c_k = 1000k^{1/2}$.

n	AC pricing rule	Dual rule	optimal w-hybrid rule	optimal w
10	1578	1029	888	3
100	9906	5155	4870	14
1000	49076	24085	23049	115

Table 3: The supremal welfare losses of three rules and the optimal w when the cost function is $c_k = 1000k^{2/3}$.